Lindelöf spaces $C(X)$ over topological groups

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Abstract. Theorem 1 proves (among the others) that for a locally compact topological group $X$ the following assertions are equivalent: (i) $X$ is metrizable and $\sigma$-compact. (ii) $C_p(X)$ is analytic. (iii) $C_p(X)$ is $K$-analytic. (iv) $C_p(X)$ is Lindelöf. (v) $C_c(X)$ is a separable metrizable and complete locally convex space. (vi) $C_c(X)$ is compactly dominated by irrationals. This result supplements earlier results of Corson, Christensen and Calbrix and provides several applications, for example, it easily applies to show that: (1) For a compact topological group $X$ the Eberlein, Talagrand, Gul’ko and Corson compactness are equivalent and any compact group of this type is metrizable. (2) For a locally compact topological group $X$ the space $C_p(X)$ is Lindelöf iff $C_c(X)$ is weakly Lindelöf. The proofs heavily depend on the following result of independent interest: A locally compact topological group $X$ is metrizable iff every compact subgroup of $X$ has countable tightness (Theorem 2). More applications of Theorem 1 and Theorem 2 are provided.

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1 Introduction

By $C_p(X)$ and $C_c(X)$ we denote the space of continuous real-valued maps on a Tychonoff space $X$ endowed with the pointwise and compact-open topology, respectively.

One of the unsolved problems in the theory of spaces $C_p(X)$ asks ([5, Problem 44, p. 29]) when exactly for a given $X$ the space $C_p(X)$ is Lindelöf. It is well known that, for example, if $X$ is second countable, then $C_p(X)$ is Lindelöf [26, (3.8.D)]. The same conclusion holds also for (not necessarily second countable) Corson compact spaces...
X (Alster-Pol-Gul’ko’s theorem [4, IV.2.22]). We refer the reader to [4, 5, 7, 13, 14, 24] for some other known (positive and negative) results about this problem. Theorem 1 below provides several equivalent conditions for $C_p(X)$ to be Lindelöf when $X$ is a locally compact group. Before its formulation we recall that $C_p(X)$ and $C_c(X)$ are locally convex spaces and that the weak topology of the latter is in general strictly finer than the topology of $C_p(X)$.

**Theorem 1.** For a locally compact topological group $X$ the following assertions are equivalent:

1. $C_p(X)$ is analytic.
2. $C_p(X)$ is $K$-analytic.
3. $C_p(X)$ is Lindelöf.
4. $X$ is metrizable and $\sigma$-compact.
5. $X$ is analytic.
6. $C_p(X)$ is boundedly dominated by irrationals and $X$ is metrizable.
7. $C_c(X)$ is a metrizable, complete and separable locally convex space.
8. $C_c(X)$ is compactly dominated by irrationals.
9. $C_c(X)$ is boundedly dominated by irrationals and $X$ is metrizable.
10. $C_c(X)$ is weakly Lindelöf, i.e. Lindelöf for the weak topology of $C_c(X)$.

This easily implies that for a compact topological group $X$ the Eberlein, Talagrand, Gul’ko and Corson compactness are equivalent and any compact group of this type is metrizable.

A result related with Theorem 1 contained in [19, Theorem 2] states that a locally compact topological group $X$ is metrizable iff the Banach space $C_0(X)$ of continuous, complex valued functions which vanish at infinity is weakly Lindelöf.

It is clear from Theorem 1 that analyticity, $K$-analyticity and the Lindelöf property for $C_c(X)$ over locally compact topological groups are also equivalent conditions to the previous ones. Moreover, Theorem 1 shows that for a locally compact topological group $X$ the space $C_p(X)$ is analytic ($K$-analytic) iff $C_c(X)$ is weakly analytic (weakly $K$-analytic). This provides a variant of Talagrand’s Theorem 3.4 of [42] and Canela’s Proposition 2.2 of [9] (where the $K$-analytic case has been proved for compact Hausdorff spaces and paracompact locally compact spaces $X$, respectively).

The implication $(1) \Rightarrow (5)$ of Theorem 1 is covered by a deep result of Calbrix [8, Theorem 2.3.1], stating that if for a Tychonoff space $X$ the space $C_p(X)$ is analytic, then $X$ must be $\sigma$-compact and analytic (cf., also [18, Theorem 3.7] for a weaker result). Note that $(8) \Rightarrow (2)$ follows also from a recent result of Tkachuk [44, Theorem 2.8]: For a Tychonoff space $X$ the space $C_p(X)$ is $K$-analytic iff it is compactly dominated by irrationals.

Theorem 1 completes the whole picture for spaces $C_p(X)$ and $C_c(X)$ over locally compact groups. The proof of Theorem 1 is transparent, short and elementary. The
key role in the proof belongs to Theorem 2 below: A locally compact topological group \( X \) is metrizable iff every compact subgroup of \( X \) has countable tightness. We provide further interesting applications of Theorem 1 and Theorem 2 (see Proposition 2 and Theorem 3).

2 Definitions and notations

By “a space” we mean “a completely regular Hausdorff space”. A continuous image of the space \( \mathbb{N}^\mathbb{N} \), where the space of integers \( \mathbb{N} \) is endowed with the discrete topology, is called an analytic space. The following order is considered in \( \mathbb{N}^\mathbb{N} : (n_k) \leq (m_k) \) if \( n_k \leq m_k \) for all \( k \in \mathbb{N} \). A continuous image of a space of type \( K_{\alpha\delta} \) is called a \( K \)-analytic space [4, p. 7] (see also [40]). A space \( X \) is \( K \)-analytic iff it is a continuous image of a Lindelöf Čech-complete space. Clearly analytic \( \Rightarrow K \)-analytic \( \Rightarrow \) Lindelöf.

A space \( X \) is compactly dominated by irrationals if it can be covered by an ordered family \( \{K_{\alpha} : \alpha \in \mathbb{N}^\mathbb{N}\} \) of compact sets, i.e. \( K_{\alpha} \subset K_{\beta} \) if \( \alpha \leq \beta \). Every \( K \)-analytic space \( X \) is compactly dominated by irrationals (see [43], or [44, Theorem 2.1(g)]), although the converse fails in general [10], see also Example 4 below. A topological group \( X \) is called trans-separable [25] if \( X \) is covered by countably many translations of each neighborhood of the unit of \( X \).

A subset \( B \) in a (real or complex) topological vector space \( E \) is called bounded if \( B \) is absorbed by each neighborhood of zero of \( E \). A topological vector space \( E \) will be called boundedly dominated by irrationals if \( E \) is covered by a family \( \{B_{\alpha} : \alpha \in \mathbb{N}^\mathbb{N}\} \) of bounded sets with the ordering as above.

A space \( X \) has countable tightness if for every set \( A \subset X \) and every \( x \in A \) there is a countable subset in \( A \) whose closure contains \( x \).

A compact space \( X \) is said to be: Eberlein compact if \( X \) is homeomorphic to a weakly compact subset of a Banach space, Corson compact if \( X \) is homomorphic to a compact subset of a \( \Sigma \)-product of real lines, Talagrand compact if \( C_p(X) \) is \( K \)-analytic and Gul’ko compact if \( C_p(X) \) is countably determined. We refer the reader to [36] for an internal characterization of Eberlein and Corson compacts, to [27] for a good account of relationships among all these notions and to [4, 26, 28] for unexplained terms.

3 Related results and proofs

We shall need the following result.

Fact 1 ([20]; cf. also [15, Theorem 1 and Remark (ii)]). Let \( X \) be a locally compact topological group. Then there exist a compact subgroup \( G \) of \( X \), a number \( n \in \mathbb{N} \cup \{0\} \), and a discrete subset \( D \subset X \) such that \( X \) is homeomorphic to the product \( \mathbb{R}^n \times D \times G \).

Next we give a result which will become an important tool in the sequel.

Theorem 2. For a locally compact topological group \( X \) the following assertions are equivalent: (1) \( X \) is angelic. (2) Every compact subgroup of \( X \) has countable tightness. (3) \( X \) is metrizable. (4) \( X \) has countable tightness.
Proof. The only non-trivial implication is (2) $\Rightarrow$ (3).

Case 1. $X$ is compact.

First proof. By a result of Kuz’minov [34, Theorem] (see also, [16, Corollary]), every compact Hausdorff group $X$ is dyadic, i.e. $X$ is a continuous image of a space of the form $\{0,1\}^x$, where $x$ is some cardinal number. It is also known that every dyadic Hausdorff space with countable tightness is metrizable [26], 3.12.12(h), p. 231.

Second proof. Assume that $X$ is a non-metrizable compact group. We show that $X$ does not have countable tightness. Since $X$ is a non-metrizable compact group, the weight $w(X)$ of $X$ is uncountable. By [41] (see also, [17, Theorem 3.1]) $X$ contains a subset $Y$ homeomorphic to the Cantor cube $\{0,1\}^{w(X)}$. But $\{0,1\}^\Gamma$ (where $\Gamma$ is an index set with $\text{card}(\Gamma) = w(X)$) does not have countable tightness. Indeed, the constant function $f(\gamma) = 1$, for all $\gamma \in \Gamma$ belongs to the closure of the $\Sigma$-product of the spaces $D_\gamma = \{0,1\}, \gamma \in \Gamma$. If $\{0,1\}^\Gamma$ had countable tightness, $f$ would have countable support, which clearly provides a contradiction.

Case 2. Assume that $X$ is a locally compact group. The proof follows from joint consideration of Fact 1 and Case 1. \hfill $\Box$

Theorem 2 easily applies to deduce that a locally compact topological group $X$ is metrizable if $C_p(X)$ is separable.

Remarks. (1) Arkhangel’skii [3] asked if every compact homogeneous Hausdorff topological space with countable tightness is first countable. Dow in [23], Theorem 6.3, answered this question positively but under PFA. Eberlein compact spaces provide a large class of spaces with countable tightness, and it is known that homogeneous Eberlein compact spaces are first countable [4, III.3.10], but non-metrizable homogeneous Eberlein compact spaces do exist, see [46]. In [30] Gruenhage proved that every Gul’ko compact space contains a dense $G_\delta$ subset which is metrizable. This implies that any compact group which is Gul’ko compact must be metrizable, although there are examples of Corson compact spaces without any dense metrizable subspace [45]. On the other hand, since every Corson compact space has countable tightness [32, Lemma 1.6 (ii)], Theorem 2 applies to show that: In the class of topological groups the Eberlein, Talagrand, Gul’ko, Corson compactness are equivalent properties and each such a compact group is metrizable.

(2) Theorem 2 extends Theorem 12 of [6], where it was shown (under continuum hypothesis (CH)) that a compact group is metrizable if it is sequentially compact. Later, in [22] it was noted that the metrizability of a compact angelic group can be also proved without (CH). The second proof of Theorem 2 presented above was motivated by [22].

(3) For non locally compact topological groups the implication $(1 + 4) \Rightarrow (3)$ of Theorem 2 may fail; in fact, let $X$ be an infinite-dimensional reflexive separable real Banach space equipped with the weak topology. Then $X$ is a $\sigma$-compact angelic locally convex space ([28, p. 39]) whose compact subsets are metrizable and which has countable tightness ([28, Corollary, p. 38]), but $X$ is not metrizable.
The group structure in the implication \((4) \Rightarrow (1)\) of Theorem 2 is essential: the one-point compactification of the space \(\Psi\) of Isbell provides an example of a compact Hausdorff space with countable tightness which is not angelic, see [29, p. 54–55].

(5) For non locally compact topological groups the implication \((1) \Rightarrow (4)\) of Theorem 2 may fail; indeed, the weak topology of any \((DF)\)-space (in the sense of Grothendieck) is angelic, see [11, Theorem 11], but there exist \((DF)\)-spaces whose weak topology does not have countable tightness, [12], p. 514.

We shall need the following applicable interesting facts.

**Fact 2.** (a) Every Baire topological vector space which is boundedly dominated by irrationals is metrizable. (b) Any metrizable topological vector space is boundedly dominated by irrationals.

**Proof.** (a) Assume that \(E\) is a Baire topological vector space covered by a family \(\{K_\alpha : \alpha \in \mathbb{N}^N\}\) of bounded sets such that \(K_\alpha \subset K_\beta\) if \(\alpha \leq \beta\). For a finite sequence of natural numbers \((n_1, \ldots, n_k)\) put

\[C_{n_1, n_2, \ldots, n_k} := \bigcup\{K_\beta : \beta = (m_1), n_j = m_j, j = 1, \ldots, k\}.

**First step.** For a fixed \(\alpha = (n_k) \in \mathbb{N}^N\), set \(W_k := C_{n_1, n_2, \ldots, n_k}, k \in \mathbb{N}\). For every neighbourhood of zero \(U\) in \(E\) there exists \(k \in \mathbb{N}\) such that \(W_k \subset 2kU\).

Indeed, otherwise there exists a neighbourhood of zero \(U\) in \(E\) such that for every \(k \in \mathbb{N}\) there exists \(x_k \in W_k\) such that \(2^{-k}x_k \notin U\). Since \(x_k \in W_k\) for every \(k \in \mathbb{N}\), there exists \(\beta_k = (m^n_{nk}) \in \mathbb{N}^N\) such that \(x_k \in K_{\beta_k}, n_j = m^n_{j,k}\) for \(j = 1, 2, \ldots, k\). Set \(a_n = \max\{m^n_{nk} : k \in \mathbb{N}\}\) for \(n \in \mathbb{N}\) and \(\gamma = (a_n)\). As \(\gamma \geq \beta_k\) for every \(k \in \mathbb{N}\), we have \(K_{\beta_k} \subset K_\gamma\), and \(x_k \notin K_\gamma\) for all \(k \in \mathbb{N}\). Since \(K_\gamma\) is bounded, then \(2^{-k}x_k \to 0\) in \(E\) which provides a contradiction.

**Second step.** In order to prove that \(E\) is metrizable, we will construct recursively a countable basis of neighborhoods of zero in \(E\). Write \(E = \bigcup_{n \in \mathbb{N}} C_n\) with \(C_n\) as above (with just one subindex). Since \(E\) is Baire, there exists \(n_1 \in \mathbb{N}\) such that \(C_{n_1}\) is of the second category and \(\text{Int } C_{n_1} \neq \emptyset\). So there exists \(x_1 \in C_{n_1}\) and a zero neighborhood \(U_1\) such that \(x_1 + U_1 \subseteq C_{n_1}\).

Now \(C_{n_1} = \bigcup_{n_2 \in \mathbb{N}} C_{n_1, n_2}\) and at least for one natural number \(n_2\) the set \(C_{n_1, n_2}\) is of the second category and \(\text{Int } C_{n_1, n_2} \neq \emptyset\). There exist thus \(x_2 \in C_{n_1, n_2}\) and a zero neighborhood \(U_2\) so that \(x_2 + U_2 \subseteq C_{n_1, n_2}\). Proceeding on and on we obtain sequences \((n_k) \in \mathbb{N}^N, (x_k)_k\) in \(E\) and a sequence \((U_k)_k\) of neighbourhoods of zero in \(E\) such that \(x_k \in \text{Int } W_k\) and \(x_k + U_k \subseteq W_k\) for all \(k \in \mathbb{N}\).

We claim that \((2^{-k}U_k)_k\) forms a countable basis of neighbourhoods of zero in \(E\). Indeed, take an arbitrary zero neighborhood \(M\) in \(E\), and another one \(V\) closed and balanced such that \(V - V \subseteq M\). By the first step, for \(V\) and \((n_k)\) thus fixed, there exists \(j \in \mathbb{N}\) such that \(W_j \subset 2^jV\). Therefore \(x_j + U_j \subseteq W_j\), implies \(U_j \subseteq W_j - W_j \subseteq 2^jV - 2^jV \subseteq 2^jM\). Hence \(2^{-j}U_j \subseteq M\), and the assertion follows.
(b) If $E$ is a metrizable topological vector space with a countable basis of balanced neighbourhoods of zero $(U_n)_{n=1}^\infty$, set $K_x := \bigcap_{n=1}^\infty n_k U_k$ for every $x = (n_k) \in \mathbb{N}^\mathbb{N}$. It is clear that the family $\{K_x : x \in \mathbb{N}^\mathbb{N}\}$ is as required. \hfill \Box

**Fact 3.** If $X$ is a paracompact locally compact space for which $C_p(X)$ is boundedly dominated by irrationals, then $X$ is $\sigma$-compact.

**Proof.** Since $X$ is a topological direct sum of a family $\{X_i : i \in I\}$ of locally compact $\sigma$-compact spaces [26, (5.1.27)], then $C_p(X) = \prod_{i \in I} C_p(X_i)$. But $C_p(X)$ is boundedly dominated by irrationals, so $I$ is countable. Indeed, otherwise $\prod_{i \in I} C_p(X_i)$ would contain a closed subspace of the type $\mathbb{R}^A$ for some uncountable $A$. But $\mathbb{R}^A$ is Baire and Fact 2(a) applies to deduce that $\mathbb{R}^A$ is metrizable, hence $A$ is countable. Thus $X$, being a countable topological direct sum of $\sigma$-compact spaces, is $\sigma$-compact. \hfill \Box

**Fact 4.** Let $X$ be a hemicompact space whose compact subsets are metrizable. Then: (a) $C_c(X)$ is a metrizable separable locally convex space. (b) If moreover $X$ is a $k$-space, then $C_c(X)$ is a complete metrizable separable locally convex space.

**Proof.** (a) is [49, Corollary, p. 271], and (a) together with a well-known assertion on $k$-spaces gives (b).

Now we are ready to prove Theorem 1.

**Proof of Theorem 1.** Clearly (1) implies (2), (2) implies (3), see [40]. Assume that $C_p(X)$ is Lindelöf. By Asanov theorem, see [4], I.4.1, the space $X$ has countable tightness. By Theorem 2 the space $X$ is metrizable. Therefore $X$ is a metrizable space for which $C_p(X)$ is Lindelöf. Hence, by [4, I.4.7], $X$ is separable. This shows that (3) implies (4). Clearly (4) implies (5).

(5) implies (4): Since $X$ is an analytic Baire topological group, Theorem 5.4 of [18] applies to show that $X$ is metrizable (and Lindelöf).

(4) implies (6): Since $X$ is locally compact and $\sigma$-compact, it is hemicompact and therefore $C_c(X)$ is metrizable. Now Fact 2(b) applies to show that $C_c(X)$ is boundedly dominated by irrationals. Hence $C_p(X)$ is boundedly dominated by irrationals as well.

(6) implies (4): Since, by assumption, $X$ is a locally compact topological group, it is paracompact. Now $X$ is $\sigma$-compact by Fact 3.

(4) implies (7): Since $X$ is locally compact metrizable and $\sigma$-compact, by Fact 4 $C_c(X)$ is a separable, metrizable and complete locally convex space, so (7) holds.

Clearly (7) implies (8). If fact, any separable and complete metric space $Y$ is compactly dominated by irrationals. A direct proof: For a countable and dense sequence $(x_n)_{n=1}^\infty$ in $C_c(X)$, set $K_x := \bigcap_{k=1}^\infty \bigcup_{j=1}^{n_k} B(x_j, k^{-1})$, where $B(x_j, k^{-1})$ is the closed ball in $C_c(X)$ with the center at point $x_j$ and radius $k^{-1}$ for $x = (n_k) \in \mathbb{N}^\mathbb{N}$ and all $j, k \in \mathbb{N}$. Then $\{K_x : x \in \mathbb{N}^\mathbb{N}\}$ is as required.

(8) implies (9): Clearly the first part of (9) holds. Moreover, by assumption on $C_c(X)$ the space $C_c(X)$ is trans-separable, [39]. Therefore every compact subset of $X$
is metrizable (see again [39] remark after Corollary 2), hence by Theorem 2 the space \( X \) is metrizable, and the second part of (9) holds.

Evidently (7) implies (1). We show now that (9) implies (4). Assume (9). Since \( X \) is paracompact and locally compact, the space \( C_c(X) \) is Baire, [37, 10.1.26]. Then by Fact 2(a) the space \( C_c(X) \) is metrizable. Hence \( X \) is hemicompact, therefore (4) holds too.

Finally (7) implies (10) and (10) implies (3) are evident. \( \square \)

It is worth noting that in (6) of Theorem 1 the condition “\( C_p(X) \) is boundedly dominated by irrationals” cannot be replaced by the following one “\( C_p(X) \) is \( \sigma \)-bounded”, i.e. \( C_p(X) \) is covered by a sequence of bounded subsets.

**Example 1.** For the additive group \( \mathbb{Z} \) of integers with the discrete topology the space \( C_p(\mathbb{Z}) \) is boundedly dominated by irrationals but \( C_p(\mathbb{Z}) \) is not \( \sigma \)-bounded.

Indeed, note that \( C_p(\mathbb{Z}) \), as a metrizable locally convex space, is boundedly dominated by irrationals, see Fact 2. Since \( \mathbb{R}^\mathbb{Z} = C_p(\mathbb{Z}) = C_c(\mathbb{Z}) \), we note that \( C_p(\mathbb{Z}) \) is a Baire locally convex space. But \( C_p(\mathbb{Z}) \) is not normable, it cannot be \( \sigma \)-bounded. \( \square \)

The implication \((1) \Rightarrow (4)\) of Theorem 1 fails for topological groups which are not locally compact as the following example shows.

**Example 2.** Let \( X \) be the strong dual of \( \mathbb{R}^\mathbb{N} \). Then \( X \) is a non-metrizable Montel (DF)-space and yet \( C_p(X) \) is analytic.

Indeed, since \( \mathbb{R}^\mathbb{N} \) is a Frechet-Montel space, \( X \) is hemicompact \( k \)-space [49, p. 267] whose compact subsets are metrizable. Hence, by Fact 4, \( C_c(X) \) is a complete separable metrizable space. Now \( C_p(X) \) is a continuous image of \( C_c(X) \), therefore it is analytic. Clearly \( X \) is not metrizable, nor locally compact. \( \square \)

Theorem 1 characterizes those locally compact topological groups \( X \) for which \( C_c(X) \) is compactly dominated by irrationals. It turns out that the compact domination of locally compact topological groups can be characterized as follows.

**Proposition 1.** A locally compact topological group \( X \) is compactly dominated by irrationals iff it is Lindelöf.

**Proof.** If \( X \) is a locally compact topological space which is Lindelöf, then clearly \( X \) is \( \sigma \)-compact, so it is compactly dominated by irrationals. Indeed, let \( (K_n)_n \) be an increasing sequence of compact subsets of \( X \) covering \( X \). Put \( K_x := K_{a_1} \) for \( x = (a_n) \in \mathbb{N}^\mathbb{N} \). If \( X \) is a locally compact group compactly dominated by irrationals, then \( X \) (according to [39]) is trans-separable. If \( U \) is a compact neighbourhood of zero, then countably many translations of \( U \) cover \( X \). Hence, \( X \) is \( \sigma \)-compact. \( \square \)

Note that in general Proposition 1 fails for locally compact topological spaces.

**Example 3** ([44, Example (3.5)]). There exists a locally compact topological space compactly dominated by irrationals which is not Lindelöf, so neither \( \sigma \)-compact.
4 More about metrizability of locally compact groups

This section deals with other applications of Theorem 2. Let \( \mathbb{T} \) be the multiplicative group of complex numbers modulus one endowed with the topology induced from \( \mathbb{C} \). For a topological Abelian group \( X \) we denote by \( X^\wedge \) the set of all continuous group homomorphisms (characters) \( \varphi : X \to \mathbb{T} \). The coarsest group topology on \( X \) for which all elements of \( X^\wedge \) are continuous is called the Bohr topology and is denoted by \( \sigma(X, X^\wedge) \). The symbol \( X_c^\wedge \) stands for \( X^\wedge \) endowed with the compact-open topology, see [6] for details.

Proposition 2. Let \( X \) be a separable and metrizable Abelian topological group. Then \( X_c^\wedge \) is locally compact if and only if \( X_c^\wedge \) is metrizable.

Proof. It is straightforward to prove that the evaluation map \( e : X \times X_c^\wedge \to \mathbb{T} \) defined by \( e(x, \phi) = \phi(x), \phi \in X_c^\wedge, x \in X \) is sequentially continuous for any topological abelian group \( X \). If furthermore \( X \) and \( X_c^\wedge \) are metrizable, then \( e \) is continuous. By [35, Proposition 1.2], the group \( X_c^\wedge \) is locally compact. For the converse, note that every compact subset of \( X_c^\wedge \) is metrizable (since \( X \) is separable). Now Theorem 2 applies.

We provide another application of Theorem 2. It is well known that there are non-metrizable absolutely convex weakly compact sets in Banach spaces over the field \( \mathbb{K} \) of real or complex numbers. It turns out that the situation is much better if the valued field \( \mathbb{K} \) is different from \( \mathbb{R} \) and \( \mathbb{C} \) (Theorem 3(b)).

Recall that a non-trivially valued field \( \mathbb{K} := (\mathbb{K}, | |) \) is non-archimedean if \( |t + s| \leq \max\{|t|, |s|\} \) for all \( t, s \in \mathbb{K} \), see [47]. A subset \( B \) of a vector space \( E \) over a non-archimedean non-trivially valued field \( \mathbb{K} \) is called absolutely \( \mathbb{K} \)-convex, if from \( x, y \in B, t, s \in \mathbb{K}, \) and \( |t| \leq 1, |s| \leq 1 \) it follows that \( tx + sy \in B \).

If \( E \) is a (Hausdorff) topological vector space over a non-archimedean non-trivially valued complete field \( \mathbb{K} \), and \( E \) contains a non-zero compact absolutely \( \mathbb{K} \)-convex set, then \( \mathbb{K} \) must be locally compact, see [21].

Now we are ready to apply Fact 1 and Theorem 2 to get the following statement.

Theorem 3. Let \( E \) be a Hausdorff topological vector space over a locally compact non-trivially valued field \( \mathbb{K} \). Then:

(a) If \( \mathbb{K} \) is archimedean, then every locally compact subgroup \( X \) of \( E \) is metrizable.

(b) If \( \mathbb{K} \) is non-archimedean and \( E \) is a metrizable dually separating space, then every absolutely \( \mathbb{K} \)-convex locally compact subset \( X \) of the topological vector space \((E, \sigma(E, E'))\) is metrizable in \( \sigma(E, E') \).

Proof. (a) As \( \mathbb{K} \) is archimedean, from Ostrowski’s theorem, [47], Theorem 1.2, it follows that \( \mathbb{K} \) is either the field of real or complex numbers. Since \( X \) is locally
compact it is homeomorphic to the product $\mathbb{R}^n \times D \times G$, where $D$ and $G$ are as in Fact 1. The conclusion follows now taking into account that any compact subgroup in a (real or complex) topological vector space is trivial.

(b) Since $(\mathbb{K}, +)$ is a locally compact Abelian group, $\mathbb{K}^\wedge$ separates the points of $\mathbb{K}$. Fix a non-constant $\chi \in \mathbb{K}^\wedge$. Then $E^\wedge = \{ \chi \circ x' : x' \in E' \}$ (cf. [50, Theorem 2]). Since $E'$ separates points of $E$, we deduce that the Bohr topology of the group $(E, +)$ is Hausdorff and it is coarser than the weak topology $\sigma(E, E')$. Since $E$ is metrizable, by Theorem 11 of [6] the group $(E, \sigma(E, E'))$ is angelic and from [28, Theorem p. 31] the space $(E, \sigma(E, E'))$ is also angelic. On the other hand, since $\mathbb{K}$ is non-archimedean and $X$ is an absolutely $\mathbb{K}$-convex subset of $E$, it is an additive subgroup of $E$. Consequently, $X$ is an angelic locally compact group, and by Theorem 2, we note that $X$ is metrizable.

$\square$

**Remark.** Theorem 3(a) fails for a non-archimedean $\mathbb{K}$. In fact, let $\mathbb{K} := \mathbb{Q}_2$ and $B := \{ t \in \mathbb{Q}_2 : |t|_2 \leq 1 \}$. Then $B^c$ is a non-metrizable compact additive subgroup of the topological vector space $\mathbb{K}^c$ (here $c$ stands for the cardinality of continuum). The same example shows also that the metrizability of $E$ is essential in Theorem 3(b).

Recall that a *locally $\mathbb{K}$-convex* topological vector space $E$ over a locally compact $\mathbb{K}$, i.e. a topological vector space with a basis of absolutely $\mathbb{K}$-convex neighbourhoods of zero, is dually separating, see [33]. If $E$ is not locally $\mathbb{K}$-convex, the above property fails in general. Indeed, the space $E := L^p([0,1], \mathbb{K})$ of all Borel measurable functions on $[0,1]$ with values in a non-archimedean non-trivially valued field $(K, | |)$ for which the Lebesgue integral $\|f\|_p := (\int_0^1 |f(x)|^p \, dx)^{1/p}$ is finite ($p \geq 1$) is not locally $\mathbb{K}$-convex and $E$ does not admit a nontrivial $\mathbb{K}$-convex subset. Therefore $E^\prime = \{ 0 \}$, see [38], p. 131, or [47], p. 89.

Since the weak topology of a metrizable locally convex space over the real or complex numbers is angelic, and has countable tightness see [28, p. 39 and p. 38] (see also [11] and [12] for a large class of spaces enjoying this property), Theorem 2 and Theorem 3 may suggest the following problem (inspired also by a related problem mentioned by Wallace [48, p. 96]: “What spaces admit what algebraic structures?”).

**Problem.** Let $E$ be a real locally convex space. For which compact subsets $X$ of $E$ does there exist a compact topological group which is homeomorphic to $X$?

Clearly every such $X$ must be homogeneous. Also $X$ cannot be convex since the Schauder fixed point theorem fails for compact topological groups (translations do not have a fixed point).

**Remark.** After sending the paper:

1) R. Buzyakova informed us that the results of Theorem 2 were known. We independently observed that 4) $\Rightarrow$ 3) of Theorem 2 is covered by [31, Theorem 2.3].

2) A. Arhangels’kii informed us that a statement similar to 3) $\Rightarrow$ 4) in Theorem 1 also holds in the following context: if $X$ is a topological group which is a $p$-space in the sense of [2] and $C_p(X)$ is Lindelöf, then $X$ is metrizable and separable.
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References

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