ON THE ZETA-FUNCTION OF A POLYNOMIAL AT INFINITY

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ABSTRACT. We use the notion of Milnor fibres of the germ of a meromorphic function and the method of partial resolutions for a study of topology of a polynomial map at infinity (mainly for calculation of the zeta-function of a monodromy). It gives effective methods of computation of the zeta-function for a number of cases and a criterium for a value to be atypical at infinity.

§1.- INTRODUCTION

The main idea of the paper is to bring together methods of [7] and [8] for computing the zeta-function of the monodromy at infinity of a polynomial. Let $P$ be a complex polynomial in $(n + 1)$ variables. It defines a map from $\mathbb{C}^{n+1}$ to $\mathbb{C}$ which also will be denoted by $P$. It is known ([13]) that there exists a finite set $B(P) \subset \mathbb{C}$ such that the map $P$ is a $C^\infty$ locally trivial fibration over its complement. The monodromy transformation $h$ of this fibration corresponding to the loop $z_0 \cdot \exp(2\pi i\tau)$ $(0 \leq \tau \leq 1)$ with $\|z_0\|$ big enough is called the geometric monodromy at infinity of the polynomial $P$. Let $h_*$ be its action in the homology groups of the fibre (the level set) \{\{P = z_0\}\}.

Definition. The zeta-function of the monodromy at infinity of the polynomial $P$ is the rational function

$$
\zeta_P(t) = \prod_{q \geq 0} \left\{ \det \left[ id - t h_* | H_q(\{P = z_0\}; \mathbb{C}) \right] \right\}^{(-1)^q}.
$$

Remark 1. We use the definition from [2], which means that the zeta-function defined this way is the inverse of that used in [1].

The degree of the zeta-function (the degree of the numerator minus the degree of the denominator) is equal to the Euler characteristic $\chi_P$ of the (generic) fibre \{\{P = z_0\}\}. Formulae for the zeta-functions at infinity for certain polynomials were given in particular in [6], [9].
§2.- Zeta-function of a polynomial via zeta-functions of meromorphic germs

A polynomial function \( P : \mathbb{C}^{n+1} \to \mathbb{C} \) defines a meromorphic function \( P \) on the projective space \( \mathbb{CP}^n \). At each point \( x \) of the infinite hyperplane \( \mathbb{CP}^n_\infty \), the germ of the meromorphic function \( P \) has the form \( \frac{F(u,x_1,\ldots,x_n)}{u^d} \) where \( u, x_1, \ldots, x_n \) are local coordinates such that \( \mathbb{CP}^n_\infty = \{ u = 0 \} \), \( F \) is the germ of a holomorphic function, and \( d \) is the degree of the polynomial \( P \).

In [8], for a meromorphic germ \( f = \frac{F}{G} \), there were defined two Milnor fibres (the zero and the infinite ones), two monodromy transformations, and thus two zeta-functions \( \zeta_0^f(t) \) and \( \zeta_\infty^f(t) \). Let \( \zeta_{P,x}^\bullet(t) (\bullet = 0 \text{ or } \infty) \) be the corresponding zeta-function of the germ of the meromorphic function \( P \) at the point \( x \in \mathbb{CP}^n_\infty \).

For the aim of convinience, in [8] we considered only meromorphic germs \( f = \frac{F}{G} \) with \( F(0) = G(0) = 0 \). At a generic point of the infinite hyperplane \( \mathbb{CP}^n_\infty \) the meromorphic function \( P \) has the form \( \frac{1}{u} \).

For a germ of the form \( f = \frac{1}{G} \) with \( G(0) = 0 \), it is reasonable to give the following definition: its infinite Milnor fibre coincides with the (usual) Milnor fibre of the holomorphic germ \( G \) and its zero Milnor fibre is empty. Thus \( \zeta_0^f(t) = 1 \) and \( \zeta_\infty^f(t) = \zeta_G(t) \). According to this definition, for the germ \( \frac{1}{u} \), its infinite zeta-function is equal to \( (1 - t^d) \).

Let \( S = \{ \Xi \} \) be a prestratification of the infinite hyperplane \( \mathbb{CP}^n_\infty \) (that is a partitioning of \( \mathbb{CP}^n_\infty \) into semi-analytic subspaces without any regularity conditions) such that, for each stratum \( \Xi \) of \( S \), the infinite zeta-function \( \zeta_{\Xi,x}^\infty(t) \) does not depend on \( x \), for \( x \in \Xi \). Let us denote this zeta-function by \( \zeta_{\Xi}^\infty(t) \) and by \( \chi_{\Xi}^\infty \) its degree \( \deg \zeta_{\Xi}^\infty(t) \). A straightforward repetition of the arguments from the proof of Theorem 1 in [7] gives

**Theorem 1.**

\[
\zeta_P(t) = \prod_{\Xi \in S} [\zeta_{\Xi}^\infty(t)]^{\chi(\Xi)},
\]

\[
\chi_P = \sum_{\Xi \in S} \chi_{\Xi}^\infty \cdot \chi(\Xi).
\]

**Remark 2.** One can write the formula for \( \chi_P \) in the form of an integral with respect to the Euler characteristic

\[
\chi_P = \int_{\mathbb{CP}^n_\infty} \chi_{P,x}^\infty \, d\chi
\]

in the sense of Viro ([14]).

**Remark 3.** Let \( P_d \) be the (highest) homogeneous part of degree \( d \) of the polynomial \( P \). Then at each point \( x \in \mathbb{CP}^n_\infty \setminus \{ P_d = 0 \} \) the germ of the meromorphic function \( P \) is of the form \( \frac{1}{u^d} \). The set \( \Xi^n = \mathbb{CP}^n_\infty \setminus \{ P_d = 0 \} \) can be considered as the \( n \)-dimensional stratum of a partitioning. It brings the factor \( (1 - t^d)^{\chi(\Xi^n)} \) into the zeta-function \( \zeta_P(t) \).

§3.- Examples

3.1. Yomdin-at-infinity polynomials. This name was introduced in [4]. For a polynomial \( P \in \mathbb{C}[z_0, z_1, \ldots, z_n] \), let \( P_d \) be its homogeneous part of degree \( d \). Let a
polynomial $P$ be of the form $P = P_d + P_{d-k} + \text{terms of lower degree}$, $k \geq 1$. Let us consider hypersurfaces in $\mathbb{CP}^n$ defined by $\{P_d = 0\}$ and $\{P_{d-k} = 0\}$. Let $\text{Sing}(P_d)$ be the singular locus of the hypersurface $\{P_d = 0\}$ (including all points where $\{P_d = 0\}$ is not reduced). One says that $P$ is a Yomdine-at-infinity polynomial if $\text{Sing}(P_d) \cap \{P_{d-k} = 0\} = \emptyset$ (in particular it implies that $\text{Sing}(P_d)$ is finite).

Y. Yomdin ([15]) has considered critical points of holomorphic functions which are local versions of such polynomials. He gave a formula for their Milnor numbers. The generic fibre (level set) of a Yomdine-at-infinity polynomial is homotopy equivalent to the bouquet of $n$-dimensional spheres ([5]). Its Euler characteristic $\chi_P$ (or rather the (global) Milnor number) has been determined in [4]. For $k = 1$, the zeta-function of such a polynomial has been obtained in [6].

Let $P(z_0, z_1, \ldots, z_n) = P_d + P_{d-k} + \ldots$ be a Yomdine-at-infinity polynomial. Let $\text{Sing}(P_d)$ consist of $s$ points $Q_1, \ldots, Q_s$. One has the following natural stratification of the infinite hyperplane $\mathbb{CP}^n_{\infty}$:

1. the $n$-dimensional stratum $\Xi^n = \mathbb{CP}^n_{\infty} \setminus \{P_d = 0\}$;
2. the $(n-1)$-dimensional stratum $\Xi^{n-1} = \{P_d = 0\} \setminus \{Q_1, \ldots, Q_s\}$;
3. the 0-dimensional strata $\Xi^0_i$ ($i = 1, \ldots, s$), each consisting of one point $Q_i$.

The Euler characteristic of the stratum $\Xi^n$ is equal to

$$\chi(\mathbb{CP}^n_{\infty}) - \chi(\{P_d = 0\}) = (n + 1) - \chi(n, d) + (-1)^{n-1} \sum_{i=1}^s \mu_i,$$

where $\chi(n, d) = (n + 1) + \frac{(1-d)^{n+1}-1}{d}$ is the Euler characteristic of a non-singular hypersurface of degree $d$ in the complex projective space $\mathbb{CP}^n_{\infty}$. $\mu_i$ is the Milnor number of the germ of the hypersurface $\{P_d = 0\} \subset \mathbb{CP}^n_{\infty}$ at the point $Q_i$. At each point of the stratum $\Xi^n$, the germ of the meromorphic function $P$ has (in some local coordinates $u, y_1, \ldots, y_n$) the form $\frac{1}{w^d} (\mathbb{CP}^n_{\infty} = \{u = 0\})$ and its infinite zeta-function $\zeta_{\infty}^n(t)$ is equal to $(1 - t^d)$.

At each point of the stratum $\Xi^{n-1}$, the germ of the polynomial $P$ has (in some local coordinates $u, y_1, \ldots, y_n$) the form $\frac{1}{w}$. Its infinite zeta-function $\zeta_{\infty}^{n-1}(t)$ is equal to 1 and thus it does not contribute a factor to the zeta-function of the polynomial $P$.

At a point $Q_i$ ($i = 1, \ldots, s$), the germ of the meromorphic function $P$ has the form $\varphi(u, y_1, \ldots, y_n) = g_i(y_1, \ldots, y_n) + u^k$, where $g_i$ is a local equation of the hypersurface $\{P_d = 0\} \subset \mathbb{CP}^n_{\infty}$ at the point $Q_i$. Thus $\mu_i$ is its Milnor number.

To compute the infinite zeta-function $\zeta_{\varphi}(t)$ of the meromorphic germ $\varphi$, let us consider a resolution $\pi : (\mathcal{X}, \mathcal{D}) \to (\mathbb{C}^n, 0)$ of the singularity $g_i$, i.e., a proper modification of $(\mathbb{C}^n, 0)$ which is an isomorphism outside the origin in $\mathbb{C}^n$ and such that, at each point of the exceptional divisor $\mathcal{D}$, the lifting $g_i \circ \pi$ of the function $g_i$ to the space $\mathcal{X}$ of the modification has (in some local coordinates) the form $y_1^{m_1} \cdot \ldots \cdot y_n^{m_n} (m_i \geq 0)$.

Let us consider the modification $\tilde{\pi} = id \times \pi : (\mathbb{C}_u \times \mathcal{X}, 0 \times \mathcal{D}) \to (\mathbb{C}^{n+1}, 0) = (\mathbb{C}_u \times \mathbb{C}^n, 0)$ of the space $(\mathbb{C}^{n+1}, 0)$ – the trivial extension: $(u, x) \mapsto (u, \pi(x))$. Let $\tilde{\varphi} = \varphi \circ \tilde{\pi}$ be the lifting of the meromorphic function $\varphi$ to the space $\mathbb{C}_u \times \mathcal{X}$ of the modification $\tilde{\pi}$. Let $\mathcal{M}^\infty_\varphi = \tilde{\pi}^{-1}(\mathcal{M}^\infty_\varphi)$ ($\mathcal{M}^\infty_\varphi$ is the infinite Milnor fibre of the germ $\varphi$) be the local level set of the meromorphic function $\tilde{\varphi}$ (close to the infinite one). In the natural way one has the (infinite) monodromy $h^\infty_\varphi$ acting on $\mathcal{M}^\infty_\varphi$ and its zeta-function $\zeta^\infty_\varphi(t)$.
Theorem 2.
\[ \zeta^\infty_{\tilde{\phi}}(t) = (1 - t^{d-k})\chi(D)^{-1}\zeta^\infty_{\hat{\phi}}(t). \]

Proof. The infinite monodromy transformation of the function \( \tilde{\phi} \) can be described in the following way. Let \( h_{\tilde{\phi}}^\infty : M_{\tilde{\phi}}^\infty \to M_{\hat{\phi}}^\infty \) be the infinite monodromy transformation of the germ \( \varphi \). One can suppose that it preserves the intersection of the Milnor fibre \( M_{\tilde{\phi}}^\infty \) with the line \( \mathbb{C}_u \times \{0\} \). There it coincides with the infinite monodromy transformation of the restriction \( \varphi|_{\mathbb{C}_u \times \{0\}} = u^k\frac{\partial f}{\partial u} \) of the germ \( \varphi \) to this line, i.e., with a cyclic permutation of \((d - k)\) points. The zeta-function of a cyclic permutation of \((d - k)\) points is equal to \((1 - t^{d-k})\). The projection \( \tilde{\pi} : M_{\tilde{\phi}}^\infty \to M_{\hat{\phi}}^\infty \) is an isomorphism outside \( M_{\tilde{\phi}}^\infty \cap (\mathbb{C}_u \times \{0\}) \), the preimage of each point from \( M_{\tilde{\phi}}^\infty \cap (\mathbb{C}_u \times \{0\}) \) is isomorphic to the exceptional divisor \( D \). This means that the transformation (the diffeomorphism) \( h_{\tilde{\phi}}^\infty : M_{\tilde{\phi}}^\infty \to M_{\hat{\phi}}^\infty \) can be constructed in such a way that it preserves \( \tilde{\pi}^{-1}(M_{\tilde{\phi}}^\infty \cap (\mathbb{C}_u \times \{0\})) \) and acts on it by a cyclic permutation of \((d - k)\) copies of \( D \). The zeta-function of this transformation of \((\{d - k\} \text{ points}) \times D\) is equal to \((1 - t^{d-k})\chi(D)\). The result follows from the multiplication property of the zeta-function of a transformation (see [2] p. 94). \( \square \)

For \( \tilde{m} = (m_1, m_2, \ldots, m_n) \) with integer \( m_1 \geq m_2 \geq \ldots \geq m_n \geq 0 \), let \( S_{\tilde{m}} \) be the set of points of the exceptional divisor \( D \) of the resolution \( \pi \) at which the lifting of the germ \( g_i \) has the form \( y_1^{m_1} \cdot \ldots \cdot y_n^{m_n} \); for \( m \geq 1 \), let \( S_m \) be \( S_{(m,0,\ldots,0)} \). By the formula of A’Campo ([1])
\[ \zeta_{g_i}(t) = \prod_{m \geq 1} (1 - t^m)\chi(S_m). \]  

At a point \( x \in \{0\} \times S_{\tilde{m}} \subset \{0\} \times D \), the lifting \( \varphi = \varphi \circ \tilde{\pi} \) of the function \( \varphi \) has the local form \( \frac{\partial f}{\partial u} u^d \). Thus, for fixed \( \tilde{m} \), the infinite zeta-function \( \zeta^\infty_{\tilde{\phi},x}(t) \) of the germ of the meromorphic function \( \tilde{\phi} \) at a point \( x \) from \( \{0\} \times S_{\tilde{m}} \) is one and the same. It can be determined by the Varchenko type formula from [8]. If there are more than one integers \( m_i \) different from zero, \( \zeta^\infty_{\tilde{\phi},x}(t) = (1 - t^{d-k}) \). For \( x \in \{0\} \times S_m \),
\[ \zeta^\infty_{\tilde{\phi},x}(t) = (1 - t^{d-k})(1 - t^{\frac{m(d-k)}{\gcd(m,k)}})\chi(S_m). \]

According to Theorem 1
\[ \zeta^\infty_{\tilde{\phi}}(t) = (1 - t^{d-k})\chi(D) \prod_{m \geq 1} (1 - t^{\frac{m(d-k)}{\gcd(m,k)}})\chi(S_m) \]

and by Theorem 2
\[ \zeta^\infty_{\hat{\phi}}(t) = (1 - t^{d-k}) \prod_{m \geq 1} (1 - t^{\frac{m(d-k)}{\gcd(m,k)}})\chi(S_m). \]  

The zeta-function \( \zeta_h(t) \) of a transformation \( h : X \to X \) of a space \( X \) into itself determines the zeta-function \( \zeta^k_h(t) \) of the \( k \)-th power \( h^k \) of the transformation \( h \). In particular, if \( \zeta_h(t) = \prod_{m \geq 1} (1 - t^m)^{a_m} \), then
\[ \zeta^k_h(t) = \prod_{m \geq 1} \left(1 - t^{\frac{m}{\gcd(k,m)}}\right)^{\gcd(k,m)\cdot a_m}. \]
The formulae (1) and (2) mean that

\[ \zeta_\varphi^\infty(t) = (1 - t^{d-k}) \left( \zeta_{\varphi_i}^k(t^{d-k}) \right)^{-1} \]

(3).

Combining the computations for the stratification \{\Xi^n, \Xi^{n-1}, \Xi^0\} of the infinite hyperplane \(\mathbb{CP}^n_\infty\), one has

**Theorem 3.** For a Yomdin-at-infinity polynomial \(P = P_d + P_{d-k} + \ldots\), its zeta-function at infinity is equal to

\[ \zeta_P(t) = (1 - t^d) \chi(\Xi^n) (1 - t^{d-k})^s \left( \prod_{i=1}^s \zeta_{\varphi_i}^k(t^{d-k}) \right)^{-1}, \]

where \(\chi(\Xi^n) = \frac{1 - (1-d)^{n+1}}{d} + (-1)^{n-1} \sum_{i=1}^s \mu(g_i)\) and \(g_i\) is a local equation of the hypersurface \(\{P_d = 0\} \subset \mathbb{CP}^n_\infty\) at its singular point \(Q_i\).

### 3.2.

Let \((n + 1)\) be equal to 3, \(P = P_d + P_{d-k} + \ldots\), \(\{P_d = 0\}\) is a curve in \(\mathbb{CP}^2_\infty\). Let \(C_i^{q_1} + \ldots + C_i^{q_r}\) be its decomposition into irreducible components. Let \(\{P_d = 0\}_{red}\) be the reduced curve \(C_1 + \ldots + C_r\) and let \(\text{Sing}(\{P_d = 0\}_{red})\) consist of \(s\) points \(\{Q_1, \ldots, Q_s\}\). Suppose that:

1. the curve \(\{P_d = 0\}_{red}\) is reduced;
2. \(Q_i \not\in \{P_d = 0\}_{red}, (i = 1, \ldots, s)\);
3. for each \(j\) with \(q_j > 1\), the curves \(C_j\) and \(\{P_d = 0\}_{red}\) intersect transversally, i.e., the set \(C_j \cap \{P_d = 0\}_{red}\) consists of \(d_j(d - k)\) different points (\(d_j = \deg C_j\)).

The generic fibre of the polynomial \(P\) is homotopy equivalent to the bouquet of 2-dimensional spheres. In this case the number of these spheres is equal to

\[ \mu(P) = \dim_\mathbb{C} \mathbb{C}[x, y, z]/\text{Jac}(P) \]

and is equal to

\[ (d - 1)^3 - k \cdot \left( \chi(\{P_d = 0\}) + d(2d - \tilde{d} - 3) \right) + k^2 \cdot (d - \tilde{d}), \]

where \(\tilde{d} = d_1 + \ldots + d_r\) is the degree of the (reduced) curve \(\{P_d = 0\}_{red}, [4]\). Let us consider the following partitioning of the infinite hyperplane \(\mathbb{CP}^2_\infty\):

1. the 0-dimensional stratum \(\Xi_0^1\) consisting of one point \(Q_i\) each \((i = 1, \ldots, s)\);
2. the 0-dimensional stratum \(\Lambda_j^0 = C_j \cap \{P_d = 0\}\), for each \(j = 1, \ldots, r\);
3. the 1-dimensional stratum \(\Xi_j^1 = C_j \setminus (\{Q_i\} \cup \Lambda_j^0)\), for each \(j = 1, \ldots, r\);
4. the 2-dimensional stratum \(\Xi_2^2 = \mathbb{CP}^2_\infty \setminus \{P_d = 0\}\).

At each point of the stratum \(\Xi_2^2\), the germ of the meromorphic function \(P\) has the form \(\frac{1}{w} (\mathbb{CP}^2_\infty = \{u = 0\})\). Its infinite zeta-function is equal to \((1 - t^d)\). The Euler characteristic \(\chi(\Xi_2^2)\) of the stratum \(\Xi_2^2\) is equal to

\[ \chi(\mathbb{CP}^2_\infty) - \chi(\{P_d = 0\}) = 3 - 3\tilde{d} + \tilde{d}^2 - \sum_{i=1}^s \mu_i, \]

where \(\mu_i\) is the Milnor number of the (reduced) curve \(\{P_d = 0\}_{red}\) at the point \(Q_i\).
At each point of the stratum $\Xi^1_j$, the germ of the meromorphic function $P$ has the form $\frac{g_j^q + u^k}{u^d}$. Its infinite zeta-function can be determined by the Varchenko type formula from [8] and is equal to
\[
(1 - t^{d-k})(1 - t^{g\cdot c.d.(q_j,k) - g.c.d.(q_j,k)}).
\]
The Euler characteristic of the stratum $\Xi^1_j$ is equal to
\[
\chi(C_j) - d_j(d - k) - \#\{C_j \cap \{Q_i : i = 1, \ldots, s\}\}.
\]
At each point of the stratum $\Lambda^0_i$, the germ of the meromorphic function $P$ has the form $\frac{g_i(y_1^q + u^k)\cdot y_2}{u^d}$. Its infinite zeta-function is equal to 1.

At a point $Q_i$, the germ of the meromorphic function $P$ has the form $\frac{g_i(y_1^q + u^k)}{u^d}$, where $\{g_i = 0\}$ is the local equation of the (non-reduced) curve $\{P_d = 0\}$ at the point $Q_i$. Its infinite zeta-function is equal to
\[
(1 - t^{d-k})\left(\epsilon_i^k(t^{d-k})\right)^{-1}.
\]

Remark 4. We can not apply the formula (3) directly since the singularity of the germ $g_i$ is, in general, not isolated. However, it is not difficult to see that, actually, the proof of this formula uses only the fact that the singularity of the germ $g_i$ can be resolved by a modification which is an isomorphism outside the origin. This is so for a curve singularity.

Thus one obtains
\[
\zeta_P(t) = (1 - t^d)^\chi(\Xi)\sum_{i}^{\pm}(-1)^i c_i \left(\frac{g_i(q_j,\ldots,q_{j+n})}{u^d}\right)^{\frac{g_i(q_j,\ldots,q_{j+n})}{u^d}} \cdot \prod_{j=1}^{\infty} \left(1 - t^\delta\right)^{-\frac{\mu_i}{\delta}}.
\]

§4.- On the bifurcation set of a polynomial map

As we have mentioned, a polynomial map $P : \mathbb{C}^{n+1} \to \mathbb{C}$ defines a locally trivial fibration over the complement to a finite set in $\mathbb{C}$. The minimal set $B(P)$ with this property is called the bifurcation set of $P$. The bifurcation set consists of critical values of the polynomial $P$ (in the affine part) and of atypical (“critical”) values at infinity.

In order to consider a level set $\{P = c\}$, one can substitute the polynomial $P$ by the polynomial $(P - c)$ and consider the zero level set. Thus let us consider the zero level set $V_0 = \{P = 0\} \subset \mathbb{C}^{n+1}$ of the polynomial $P$. Let us suppose that the level set $V_0$ of the polynomial $P$ has only isolated singular points (in the affine part $\mathbb{C}^{n+1}$). For $\rho > 0$, let $B_\rho$ be the open ball of radius $\rho$ centred at the origin in $\mathbb{C}^{n+1}$ and $S_\rho = \partial B_\rho$ be the $(2n + 1)$-dimensional sphere of radius $\rho$ with the centre at the origin. There exists $R > 0$ such that, for all $\rho \geq R$, the sphere $S_\rho$ is transversal to the level set $V_0 = \{P = 0\}$ of the polynomial map $P$. The restriction $P|_{\mathbb{C}^{n+1} \setminus B_R} : \mathbb{C}^{n+1} \setminus B_R \to \mathbb{C}$ of the function $P$ to the complement of the ball $B_R$ defines a $C^\infty$ locally trivial fibration over a punctured neighbourhood of the origin in $\mathbb{C}$. The loop $\varepsilon_0 \cdot \exp(2\pi i \tau)$ ($0 \leq \tau \leq 1$, $\|\varepsilon_0\|$ small enough) defines the monodromy transformation $h : V_{\varepsilon_0} \setminus B_R \to V_{\varepsilon_0} \setminus B_R$. Let us denote its zeta-function $\zeta_h(t)$ by $\zeta(t)$. We use the following definition.
**Definition.** The value 0 is *atypical at infinity* for the polynomial $P$ if the restriction $P|_{C^{n+1}\setminus B_R}$ of the function $P$ to the complement of the ball $B_R$ is not a $C^\infty$ locally trivial fibration over a neighbourhood of the origin in $\mathbb{C}$.

**Remark 5.** This definition does not depend on a choice of coordinates, i.e., it is invariant with respect to polynomial diffeomorphisms of the space $\mathbb{C}^{n+1}$. One can see that an atypical at infinity value is atypical, i.e., it belongs to the bifurcation set $B(P)$ of the polynomial $P$. Moreover, the bifurcation set $B(P)$ is the union of the set of critical values of the polynomial $P$ (in $\mathbb{C}^{n+1}$) and of the set of values atypical at infinity in the described sense. If the singular locus of the level set $V_0 = \{P = 0\}$ is not finite, the value 0 hardly can be considered as typical at infinity. Thus, one should consider this definition as a (possible) general definition of a value atypical at infinity. In fact, the same definition was used in [10].

Let $\mathcal{S}$ be a prestratification of the infinite hyperplane $\mathbb{CP}^\infty$ such that, for each stratum $\Xi$ of $\mathcal{S}$, the zero zeta-function $\zeta^0_\mathcal{P}(t)$ of the germ of the meromorphic function $P$ at a point $x \in \mathbb{CP}^n$ does not depend on the point $x$, for $x \in \Xi$ (let it be $\zeta^0_\Xi(t)$ and let its degree be $\chi^0_\Xi$).

**Theorem 4.**

$$
\zeta^0_\mathcal{P}(t) = \prod_{\Xi \in \mathcal{S}} [\zeta^0_\Xi(t)]^{\chi(\Xi)},
$$

$$
\chi(V_{\varepsilon_0} \setminus B_R) = \sum_{\Xi \in \mathcal{S}} \chi^0_\Xi \cdot \chi(\Xi).
$$

The proof is essentially the same as that of Theorem 1. Since the Euler characteristic of the set $V_0 \setminus B_R$ is equal to 0, one has

**Corollary 1.** If $\zeta^0_\mathcal{P}(t) \neq 1$, then the value 0 is atypical at infinity for the polynomial $P$.

In several papers (see, e.g., [3], [11], [12]) there was considered an integer $\lambda_\mathcal{P}(c)$ ($c \in \mathbb{C}$) such that

$$
\chi(\{P = c\}) = \chi(\{P = c + \varepsilon\}) + (-1)^n + \sum \mu_i + \lambda_\mathcal{P}(c),
$$

where $\mu_i$ are the Milnor numbers of the (isolated) singular points of the level set $\{P = c\} \subset \mathbb{C}^{n+1}$. Theorem 4 gives the following formula for this invariant:

**Corollary 2.**

$$
\lambda_\mathcal{P}(0) = (-1)^n \deg \zeta^0_\mathcal{P}(t) = (-1)^n \sum_{\Xi \in \mathcal{S}} \chi^0_\Xi \cdot \chi(\Xi) \left( = (-1)^n \int_{\mathbb{CP}^n} \chi^0_{\mathcal{P},x} d\chi \right).
$$

**Example.** Let $P(x, y, z) = x^a y^b (x^c y^d - z^{c+d}) + z$, $(ad - bc) \neq 0$, and let $D = \deg(P) = a + b + c + d$. The curve $\{P_D = 0\} \subset \mathbb{CP}^2_\infty$ consists of three components: the line $C_1 = \{x = 0\}$ with multiplicity $a$, the line $C_2 = \{y = 0\}$ with multiplicity $b$, and the reduced curve $C_3 = \{x^c y^d - z^{c+d} = 0\}$. Let $Q_1 = C_2 \cap C_3 = (1 : 0 : 0)$, $Q_2 = C_1 \cap C_3 = (0 : 1 : 0)$, $Q_3 = C_1 \cap C_2 = (0 : 0 : 1)$. At each point $x$ of the infinite hyperplane $\mathbb{CP}^2_\infty$ except $Q_1$ and $Q_2$, one has $\zeta^0_{\mathcal{P},x}(t) = 1$. At the point $Q_1$, the germ of the meromorphic function $P$ has the form $y^b(y^d - z^{c+d}) + z u^{D-1}$.
Its zero zeta-function can be obtained by the Varchenko type formula from [8]. If \((ad - bc) < 0\), then \(\zeta^0_{P,Q_1}(t) = 1\). If \((ad - bc) > 0\), then

\[
\zeta^0_{P,Q_1}(t) = (1 - t^{\frac{ad - bc}{G.C.D.}.})^{G.C.D.},
\]

where \(G.C.D. = g.c.d(c, d) \cdot g.c.d.(\frac{ad - bc}{g.c.d(c, d)}, D - 1)\). At the point \(Q_2\), we have just the symmetric situation. Finally

\[
\zeta^0_P(t) = (1 - t^{\frac{|ad - bc|}{G.C.D.}})^{G.C.D.}.
\]

It means that the value 0 is atypical at infinity. In the same way \(\zeta^0_{P-c}(t) = 1\), for \(c \neq 0\).

References


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