Quark Schwinger-Dyson evaluation of the $l_1,l_2$ coefficients in the chiral Lagrangian

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(Received 17 June 2003; published 19 November 2003)

Using a systematic expansion of the quark-antiquark Bethe-Salpeter wave functions in the relativistic quark model and working to $O(P^4)$ in the chiral limit, we are able to derive theoretical expressions relating the coefficients of the chiral Lagrangian $l_1,l_2$ to the underlying quark-antiquark wave functions and interaction kernels. This is accomplished by using a novel technique based on a Ward identity for the quark-antiquark ladder kernel which greatly simplifies the required effort. Numerical evaluations are performed in two simple specific models.


I. INTRODUCTION

It has traditionally been considered a triumph of theoretical physics when the parameters of an effective, low energy theory that correctly describe phenomena at a given scale can be related to those of an underlying, more fundamental scheme of thought that grounds it. Brilliant examples are F. London’s explanation of the quantum nature of the van der Waals forces [1] and the derivation of the atomic relativistic corrections as a consequence of the Dirac’s equation for the electron. Low energy hadronic processes are interpreted with the aid mainly of two types of theories: nucleon-nucleon interactions such as the Nijmegen potentials for the heavier hadrons, and relativistic chiral Lagrangians [4] for the lightest components, the pions.

The deeper quark theories such as QCD or any microscopic models thereof pretend in principle to describe the totality of hadronic physics. They attempt to be complete descriptions of hadronic processes. Unfortunately, the complexity of many body hadronic calculations makes it forbidding to fully exploit the underlying scheme and maintain the validity of the low energy effective theory.

As a consequence, an initial goal for the microscopic theory should be to reproduce in some limit the macroscopic models and to relate their parameters to its own set (hopefully smaller). In this paper we make the case for microscopic quark models inspired in QCD as generating the parameters of the chiral Lagrangian. This Lagrangian, describing the low energy behavior of a pion system, and being able to incorporate the coupling of pions to other mesons [as much as do the low energy theorems of PCAC (partial conservation of axial vector current) [5]] is universal (in the sense that any theory with the same symmetries can be cast in its form) and provides a consistent derivative expansion in powers of the momentum and mass of any pions present in a system, divided by a typical scale of the strong interactions.

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Unfortunately, this derivative expansion has to incorporate new coefficients order by order. These new coefficients absorb the divergences of loops generated by the vertices of smaller order terms, and so their value is generally renormalized. Still, the common usage of this Lagrangian [6] proceeds by fitting these coefficients to some observable set at a given scale.

We show how these coefficients can be related systematically to quark level parameters in the planar approximation. This has been accomplished in the past for the simplest, $O(P^4)$ chiral Lagrangian whose parameters are only 2, in the usual notation, $M_\pi f_\pi$ (the pion mass and decay constant).

To this order, these two parameters are conventionally set to take their physical values. To the next order, the Lagrangian contains six parameters, which generate the $O(P^6)$ vertices $l_1,l_2$ absorbing divergences in the four-pion Green’s function, $l_3,l_4$, which absorb counterterms of the mass and axial current renormalizations, and finally $M_\pi f_\pi$. The complete renormalization scheme is specified in [4]. The parameters $M_\pi f_\pi$ have long been accounted for by relativistic quark models [7,8]. The $l$’s, on the other hand, have not been treated in quark models with noncontact interactions. We emphasize the point that any theory which respects the $SU(2)_L \times SU(2)_R$ chiral symmetry breaking pattern, let it be a Nambu–Jona-Lasinio quark theory [9], a large $N_c$ expansion [10], a string theory, or any other exotic creation, can be cast in the form of the chiral Lagrangian, and the only differences between all of them are the numerical values of the $l_i$ coefficients.

It is therefore of paramount importance to determine them from the theories that we believe correctly describe the physics at the GeV scale, in terms of quarks and antiquarks. Lattice determinations are making progress in that direction [11], but the Schwinger-Dyson (SD) equation formalism should provide an alternative determination in the near future. An interesting paper [12] exists where, at the Lagrangian level, the action for a relativistic quark model is bosonized to obtain an effective meson Lagrangian and then used to calculate pion-pion scattering lengths. We are going to extend this approach theoretically in two directions. First, we will start with the most general chirally symmetric quark...
model, in which the pion is well described by a quark-
antiquark pair after chiral symmetry breaking (encompassing
in this way an ample spectrum of models) and, by using their
chiral properties, reduce the four-pion Green’s functions to a
minimal set of diagrams. In this way, no bosonization is per-
formed, and at all steps the arrangement of the quark inter-
actions to comply with the chiral theorems is explicitly vis-
able. Second, comparing the result with the same calculation
in a chiral Lagrangian formalism, one can immediately read
off the $l_i$ coefficients of the chiral Lagrangian in terms of
diagrams which can be calculated numerically in the quark
model. This rather technical numerical evaluation will be
simplified in this work by confining ourselves to simple, fi-
nite models, although the numerical results will then be lim-
ited. The method used here has already been successfully
exploited to demonstrate how this class of models comply
with the Weinberg theorem in [13–15]. The Weinberg theo-
rem was derived with an expansion to $O(P^2), O(M^2_{\pi})$. We
now concentrate on the $O(P^4), O(M^2_{\pi})$ chiral Lagrangian,
that is, the only parameters are $f_\pi, l_1,$ and $l_2$. We will
perform the same expansion in the quark-antiquark diagrams
and compare the results to read off $l_1, l_2$. The expansion
will be carried out whenever possible in a Feynman diagram lan-
guage to avoid lengthy expressions for the sake of readabil-
ity. The rest of this paper is organized as follows. In Sec. II
we briefly settle the notation for our chiral perturbation
theory discussion and remind the reader of a few well-known
facts in this field. Section III settles the notation of the mi-
icroscopic quark manipulations to follow and provides the
reader with a useful chiral Ward identity recently introduced
[13,14]. Section IV is the core of the paper and presents the
reduction of the pion scattering amplitude, whereas the re-
sulting diagrams are calculated in two simple models in Sec.
V. Some issues clarifying the normalization of the Bethe-
Salpeter equation are relegated to the Appendix.

II. CHIRAL LAGRANGIAN OF ORDER $P^4$

The macroscopic theory one generally writes down for
pion fields alone is to lowest order the nonlinear sigma
model. One can proceed by constructing, from the three pion
fields $\vec{\pi} = (\pi_1, \pi_2, \pi_3)$, a four-vector normalized to 1 (this
normalization is equivalent to eliminating the explicit $\sigma$
degree of freedom from the linear sigma model),

$$U = \begin{pmatrix} 1 - \frac{\vec{\pi}^2/F^2}{\pi/F} \end{pmatrix},$$

and then constructing Lorentz scalar, parity invariant terms.
To $O(P^4)$ that Lagrangian can be extended by terms which
in the chiral limit ($m_q = 0$) have to be of the form [4]

$$L^{(4)} = \frac{1}{F^4} [l_1 (\vec{\pi}_{\mu} \cdot \vec{\pi}_{\nu}) (\vec{\pi}_{\nu} \cdot \vec{\pi}_{\mu}) + l_2 (\vec{\pi}_{\mu} \cdot \vec{\pi}_{\nu})(\vec{\pi}_{\mu} \cdot \vec{\pi}_{\nu})],$$

where the scalar dot products are in isospin space. This La-
grangian is on shell, for massless pions (else the $l_3, l_4$
counterterms should also be present) and contributes at the tree
level to the $O(P^4)$ pion-pion scattering amplitude; and it is
this contribution that we aim to reproduce microscopically.
In the chiral formalism, there are also one-loop contributions
from the $O(P^3)$ Lagrangian which we do not consider in this
work, since our quark level calculation will not be extended
to meson loops. Therefore, to this level, it is fair to compare
our results only with those obtained in chiral perturbation
theory without meson loops. With this caveat in mind, the
pion-pion scattering amplitudes are straightforwardly ob-
ained. By using crossing symmetry, the different isospin
channels can be related in terms of only one amplitude $A$:

$$T_{I=2} = A(t, s, u) + A(u, t, s),$$

$$T_{I=1} = A(t, s, u) - A(u, t, s),$$

$$T_{I=0} = 3A(s, t, u) + A(t, s, u) + A(u, t, s).$$

This amplitude $A(s, t, u)$ can be obtained from the process
$\pi^+ \pi^- \rightarrow \pi^0 \pi_0$. Due to the final state Bose
symmetry, and in the chiral limit when the Mandelstam variables satisfy $s + t + u = 0$, the most general amplitude of order $P^4$ containing
the polynomials $s^2, t^2, u^2, st, su, tu$ reduces to $A_1s^2 + A_2(t - u)^2$. The coefficients obtained from the Lagrangian (2)
above yield

$$A^{(4)}(s, t, u) = \frac{1}{F^4} \left[ \frac{2l_1 + l_2}{2} s^2 + \frac{l_2}{2(t-u)^2} \right].$$

A full discussion of this and related issues (for example, the
relation between $f_\pi$ and $F$, which we further ignore in this
paper to the order we are working) can be found in [4,16].

III. NOTATION FOR QUARK MODELS AND CHIRAL
WARD IDENTITIES

A pion with momentum $P$ couples in relativistic models to
fermion lines whose momenta will be denoted by $k, k'$. In
the massless quark limit, whenever $P = 0$, then $k = k'$. We
start by considering the bare fermion propagator from any
standard quark theory,

$$S_0(k) = \frac{i}{k - m + i\epsilon},$$

and, after spontaneous chiral symmetry breaking mediated
by a strong interaction [17,18], the full fermion propagator parameterized as

$$S(k) = \frac{i}{A(k^2)k - B(k^2) + i\epsilon},$$

which we take to be a solution of the planar rainbow
Schwinger-Dyson equation

$$S(k)^{-1} = S_0(k)^{-1} - \int \frac{d^4q}{(2\pi)^4} V^\mu S(k + q)V_\mu K(q).$$
We further define the bare axial vertex which couples a quark-antiquark pair to a pseudoscalar current by means of the shorthand $\gamma^a_A$ (notice that $m_\pi=0=m_q$ in the chiral limit employed in this paper):

$$\gamma^a_A=\frac{\sigma^a}{2}(-iP_\mu\gamma^\mu\gamma_5+2im_\pi\gamma_5),$$

(8)

which satisfies

$$\gamma^a_A(k,k')=\frac{\sigma^a}{2}[S^{-1}_0(k)\gamma_5+\gamma_5S^{-1}(k')]$$

(9)

and the dressed axial vertex, dressed with a planar ladder $\Gamma_A$ is given by

$$\Gamma_A^a(k,k')=\gamma^a_A(k,k')+\int V^aS(k_1+q)\Gamma_A^aS(k_2+q)V_q K(q).$$

(10)

Reconstructing the planar ladder expansion (in graphical form)

$$\begin{array}{c}
\Box = \Box + \Box + \Box + \ldots
\end{array}$$

(11)

Eq. (10) takes the form

$$S\Gamma^a_A S = \begin{array}{c}
\Box
\end{array} \gamma^a$$

(12)

from which one can deduce the axial vector Ward identity

$$\Gamma^a_A(k,k')=\frac{\sigma^a}{2}[S^{-1}(k)\gamma_5+\gamma_5S^{-1}(k')]=\frac{\sigma^a}{2}\Gamma_A.$$ 

(13)

This is analogous to the Abelian vector Ward-Takahashi identity which in terms of the vertex $\Gamma_\mu$ defined by

$$S\Gamma^a_\mu S = \begin{array}{c}
\Box
\end{array} \gamma^a$$

(14)

yields

$$i(k_\mu-k_\mu')\Gamma_\mu(k,k')=S^{-1}(k')-S^{-1}(k).$$

(15)

Next we introduce the bound state formalism for quark-antiquark systems. To this end we remind the reader of the Bethe-Salpeter (BS) amplitude $\chi$ (see [7,12]) for further details) which satisfies a homogeneous Bethe-Salpeter equation:

$$\chi^b(P,k)=\int V^aS\left(k'+\frac{P}{2}\right)\chi^b(P,k')S\left(k'-\frac{P}{2}\right)V^aK(k-k'),$$

(16)

or in graphical form

$$\begin{array}{c}
\chi^b(P,k)
\end{array} = \begin{array}{c}
\chi^b(P,k')
\end{array} \chi^b(P,k')$$

(17)

Each incoming or outgoing pion in a particular process must contribute with one of these $\chi$ functions, which carry pseudoscalar quantum numbers by construction [19]. The BS amplitude for a particular pion depends on the total momentum of the pion $P$ and the momentum of its fermion component $k\pm P/2$. Notice that this equation is the homogeneous part of Eq. (10) above, when we interpret the pion momentum $P$ as $k-k'$ in the vertex definition.

Now let us deepen our study of the vertex $\Gamma_A$. From Eq. (13) it can easily be seen that in the chiral limit ($m_\pi=0$)

$$\Gamma^a_A(0,0)=2iB(k^2)\gamma_5\frac{\sigma^a}{2}.$$ 

(18)

in terms of the SD amplitude $B$ solution of Eq. (7). Equations (16) and (10), homogeneous and not homogeneous, coincide when $\gamma_A=0$. This is satisfied in the limit $m_\pi=0$ when also $P=0$ as can be seen explicitly from Eq. (9); this allows us to identify, up to a normalization constant, $\chi^b(P=0,k)$ with $\Gamma_A(k,k'=k)$. This constant coincides with $f_\pi$, the pion decay constant in the chiral limit (the proof is sketched in the Appendix) and finally entails, in combination with Eq. (18),

$$\chi^b_\pi(P=0,k)=-i\Gamma^a_A(k,k'=k)\frac{2B(k^2)}{f_\pi}\gamma_5\frac{\sigma^a}{2}. $$

(19)

In [20] a proof was given that this BS amplitude, in connection with the axial vector ward identity, makes the pion a Goldstone boson. In terms of our notation this was rewritten in [14].

This discussion suggests a strategy to systematically organize the corrections to the chiral, low momentum limit, in an analogous fashion to that used in chiral perturbation theory. Since the vertex $\Gamma_A$ and diagrams constructed thereof satisfy interesting chiral identities, let us define

$$\chi^a(P,k)=\frac{-i\Gamma^a_A(P,k)+\Delta^a(P,k)}{f_\pi},$$

(20)

where the function $\Delta(P,k)$ so introduced can be expanded in a Taylor series for low $P$. This expansion will organize the momentum corrections to any diagram. One first uses the chiral results for $\Gamma_A$, which provide one with exact low energy theorems, and the numerical corrections as $P$ is increased can then be expressed as overlaps of $\Delta$ functions.

We do not yet specify the color, spin, flavor, or momentum structure of the interaction kernel and vertices $V^aV^aK(q)$, except for one property: they must be chiral symmetry preserving, that is, $V$ commutes with $\gamma_5$. This guarantees the satisfaction of the following chiral ward identity (also discussed in [13,14]), which proved essential:

\[ \int \frac{d^4q}{(2\pi)^4} \gamma_A(q) \gamma^\mu(q) K(q) = 0, \]

where $\gamma_A(q) = \frac{\sigma^a}{2}S^{-1}(q)\gamma_5 + \gamma_5 S^{-1}(q)$.\]
or, for a general vertex not necessarily pseudoscalar,

\[ \frac{A' q + B'}{A^2 q^2 - B^2} \]

This identity allows the reduction of terms with two axial vertices and is the core of the present calculation. [We remark that \( S^{-1} S = I \) is introduced in Eq. (21) and \( S \) completes the ladder in Eq. (11) leaving the explicit \( S^{-1} \).]

### Further ladder properties

We start by observing that the pseudoscalar ladder can be Laurent expanded around its pion vertex. Keeping only the first term, containing the pole, one obtains

\[ \chi_\pi \left( \frac{q}{p} \right) = \chi_\pi \frac{\iota q}{p^2 - m_\pi^2} \]

where

\[ c = \frac{\iota q}{p^2 - m_\pi^2} \]

(in the calculations contained in this paper, \( m_\pi^2 = 0 \)). Combining this together with the definition of \( \Delta \) in Eq. (20), one can use

\[ S(q + P/2)|_{p = 0} = \frac{A q + B}{A^2 q^2 - B^2} \]

\[ \partial^\mu S(q + P/2)|_{p = 0} = \gamma_\mu \frac{A q + B}{A^2 q^2 - B^2} + \frac{2 q^2 A q + B (A + B)}{A^2 q^2 - B^2} \]

\[ \partial^\mu \partial_\mu S(q + P/2)|_{p = 0} = \frac{4 A'}{A^2 q^2 - B^2} + \frac{2 q^2 A q + B (A + B)}{A^2 q^2 - B^2} \]

\[ \frac{A q + B}{A^2 q^2 - B^2} \]

where the following momentum expansion of the propagators is meant by the superscripts (1), (2):
The last diagram in Eq. (26) contains an annoying explicit rung of the interaction. This can be eliminated at the cost of adding a diagram to any expression where it appears: in analogy with

\[
y = \frac{1}{1-y} - 1
\]

one can use

\[
\begin{array}{c}
\includegraphics{diagram1.png}
\end{array}
\]

(30)

IV. PION-PION SCATTERING

A. Generalities

The pion scattering amplitude with the Bethe-Salpeter and planar approximations can be derived from the following Feynman diagrams:

\[
\begin{array}{c}
\includegraphics{diagram2.png}
\end{array}
\]

(31)

where the first two terms provide all possible planar topologies, but upon substitution of Eq. (11) their zeroth order is seen to be double counted; hence we subtract it. The two approximations involved are, first, coupling of the pion to higher Fock space states is not considered and, second, only planar diagrams are utilized. It was shown, using the axial Ward identity, that these two approximations are consistent with past work [13,14]. This is also consistent with past work [21] on resonance exchange, and is equivalent to the lowest order in a \(1/N_c\) expansion. Reduction of this combination of Feynman diagrams to \(O(p^4), O(M^0)\) is our goal. The calculations in this section will treat the \(P\)'s as incoming momenta. Matching the dummy \(P_j\) to the incoming \(q_{i1}, q_{i2}\) and outgoing \(q_{o1}, q_{o2}\) physical pion momenta leads to six different permutations, namely,

\[
(P_1, P_2, P_3, P_4) = \left\{ \begin{array}{c}
(q_{i1}, q_{i2}, -q_{o2}, -q_{o1}) \\
(q_{i1}, q_{i2}, -q_{o1}, -q_{o2}) \\
(q_{i1}, -q_{o1}, -q_{o2}, q_{i2}) \\
(q_{i1}, -q_{o2}, -q_{o1}, q_{i2}) \\
(q_{i1}, -q_{o1}, q_{i2}, -q_{o2}) \\
(q_{i1}, -q_{o2}, q_{i2}, -q_{o1})
\end{array} \right\},
\]

(32)

where the first momentum is fixed to avoid double counting by rotational symmetry of the \(\pi-\pi\) scattering amplitude (31).

We concentrate on \(A(s,t,u)\), the amplitude for \(\pi^+ \pi^- \rightarrow \pi_0 \pi_0\), and use the following isospin wave functions:

\[
\begin{align*}
\pi^+ &= \frac{1}{2} \left( \sigma_1 + i \sigma_2 \right), \\
\pi^- &= \frac{1}{2} \left( -\sigma_1 + i \sigma_2 \right), \\
\pi_0 &= \frac{1}{\sqrt{2}} \sigma_3.
\end{align*}
\]

(33)

From them follow the traces

\[
\text{Tr}(\pi_+ \pi_- \pi_0 \pi_0) = -\frac{1}{2}
\]

(34)

[which multiplies the four first permutations in Eq. (32) above where the two charged pions are adjacent] and

\[
\text{Tr}(\pi_+ \pi_0 \pi_- \pi_0) = \frac{1}{2}
\]

(35)

[multiplying the last two permutations in Eq. (32) where the two charged pions are in opposite corners of the amplitude].

Finally, the Mandelstam variables in the chiral limit satisfy

\[
\begin{align*}
\mathbf{t} &= 2q_{i1}q_{i2} = 2q_{o1}q_{o2}, \\
\mathbf{u} &= -2q_{i1}q_{o1} = -2q_{i2}q_{o2}, \\
\mathbf{s} &= -2q_{i1}q_{o2} = -2q_{i2}q_{o1}.
\end{align*}
\]

(36)

Since isospin has been factored out, we can ignore it in the rest of the calculation. Instead of working with \(\Gamma\), we employ \(\Gamma_A\) as defined in Eq. (13). Accordingly, we define

\[
\chi^A := \frac{\alpha^A}{\sqrt{2}} \chi
\]

(37)

to yield the normalized isospin wave functions in Eq. (33) and the normalization for \(\chi, \Gamma_A\) will be

\[
\chi = \frac{-i\Gamma_A + \Delta}{\sqrt{2f}}.
\]

(38)

We now start evaluating the Feynman amplitude in terms of the \(P\)'s. Start by employing Eq. (38) to treat the product of four BS amplitudes:

\[
\chi_{\pi_1} \chi_{\pi_2} \chi_{\pi_3} \chi_{\pi_4} = \left( \frac{1}{\sqrt{2f}} \right)^4 \left( \Gamma_1 + \Delta_1 \right) \left( \Gamma_2 + \Delta_2 \right) \left( \Gamma_3 + \Delta_3 \right) \left( \Gamma_4 + \Delta_4 \right) + \text{perm}
\]

(39)
where the omitted terms contain an increasing number of
powers of $\Gamma$. Next we proceed to a term by term analysis of
this expansion, further explained in [14].

B. Contribution with four $\Delta$s

The term with four $\Delta$'s is model dependent and no chiral
properties can be used to simplify it, since it contains corrections
to the BS pion wave function beyond the zero-momentum limit. Without evaluating it explicitly in a par-
ticular model yet (but see later), we can parameterize it. To
the order $m^0$ we use here, for on-shell pions, $P_i^2 = 0$ for all $i$.
Therefore, the $4\Delta$ diagrams can only be a combination of
products of different momenta $P_iP_j$. But to order $P^4$, since
each $\Delta$ brings at least one momentum power (the zeroth
power is accounted for already in $\Gamma$), only combinations of
the type $(P_1P_2)(P_3P_4)$, $(P_1P_2)(P_2P_3)$, $(P_1P_3)(P_2P_4)$
can appear. The coefficients of the first two terms have to be
equal because of the cyclic symmetry of Eq. (31). The coef-
cient of the last term is in general independent.

In terms of fictitious momenta, all flowing into the dia-
gram, whose conservation law is $P_1 + P_2 + P_3 + P_4 = 0$, we obtain

$$
\begin{array}{c}
\begin{array}{c}
\Delta_4 \\
\Delta_2 \\
\Delta_1 \\
\Delta_3 \\
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\Delta_4 \\
\Delta_2 \\
\Delta_1 \\
\Delta_3 \\
\end{array}
\end{array} - \begin{array}{c}
\begin{array}{c}
\Delta_4 \\
\Delta_2 \\
\Delta_1 \\
\Delta_3 \\
\end{array}
\end{array} = 3d_1(P_1P_2P_3P_4 + P_1P_4P_2P_3) + 3d_2P_1P_3P_2P_4.
\end{array}
$$

(40)

The two numbers $d_1, d_2$ contain the nontrivial information in
this diagram. We have explicitly pulled out the color factor
(3, as will be shown shortly) from the $d$'s, which contain in
this way only momentum and spin (since flavor will be dealt
with at the end when the external legs are matched to the
physical particles).

The third term in Eq. (40), to order 4 in momentum, with
no ladder, is simply a wave function overlap given by the
usual Feynman rules [notice an extra $-1$ due to the fer-
mion loop],

$$
\begin{align}
-\int \frac{d^4q}{(2\pi)^4} \frac{i}{A^2q^2 - B^2} & \times [\delta^{(4)}(P_1, q)(A\vec{\gamma} + B) \\
& \times \delta^{(4)}(P_4, q)(A\vec{\gamma} + B) \Delta^{(4)}(P_3, q)(A\vec{\gamma} + B) \\
& \times \delta^{(4)}(P_2, q)(A\vec{\gamma} + B)].
\end{align}
$$

The Dirac traces can easily be computed with FORM. The integral is then reduced by using tensor identities like

$$
\int F(q^2)q_\mu q_\nu q_\alpha q_\beta = \frac{g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}}{24} \int q_4 F(q^2)
$$

to a one-dimensional expression which can then be numeri-
cally evaluated once a specific model (and hence Bethe-
Salpeter wave functions) is chosen. Notice that this simple
momentum routing is correct only to order $P^4$.

The first and second diagrams in Eq. (40) contain a ladder.
A simple way to calculate them is to write an integral equa-
tion for the object

$$
i\mathcal{U} := \begin{array}{c}
\begin{array}{c}
\Delta_2 \\
\Delta_1 \\
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
\Delta_2 \\
\Delta_1 \\
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\Delta_2 \\
\Delta_1 \\
\end{array}
\end{array} i\mathcal{U}
$$

(44)

whose most general expansion up to second order in the
external pion momenta is

$$
i\mathcal{U} = i\mathcal{U}_0(k_1(k_2)k \cdot P_1 \cdot P_2 + \mathcal{U}_1(k_2)k \cdot P_1 \cdot P_2)
$$

(43)

The functions $\mathcal{U}_i$ are obtained by projecting this linear integ-
ral equation (analogous to the Bethe-Salpeter equation)

$$
i\mathcal{U} (P_1, P_2, k) = \begin{array}{c}
\begin{array}{c}
\Delta_2 \\
\Delta_1 \\
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\Delta_2 \\
\Delta_1 \\
\end{array}
\end{array} i\mathcal{U}
$$

(44)

with the matrix projectors $I, \ell, \ell \cdot \ell, \ldots, \ell \cdot \ell \cdot \ell \cdot \ell_2$, which pro-
vides us with a linear system of eight integral equations for
the $\mathcal{U}_i$. Defining a convenient quantity $D := 2k \cdot P_1 \cdot P_2
- k^2 P_2$, the projections are

$$
\begin{align}
4D(P_1, P_2) & = \text{Tr}((2D - 2k \cdot P_1 \cdot \ell \cdot \ell_2 + q^2\ell \cdot \ell_2)\mathcal{U}_1), \\
4D(P_1, P_2) & = \text{Tr}((-2P_1 \cdot P_2 \cdot \ell + 2k \cdot P_1 \cdot \ell \cdot \ell_1 + \ell \cdot \ell_2)\mathcal{U}_2), \\
4P_1, P_2 Dk & = \text{Tr}((2k \cdot P_1 \cdot P_2 - 2(k \cdot P_2)^2\ell_1 + D\ell_2 - k \cdot P_2 \cdot \ell \cdot \ell_2)\mathcal{U}_3), \\
4P_1, P_2 Dk & = \text{Tr}((D\ell_1 - 2(k \cdot P_1)^2 + k \cdot P_1 \cdot \ell \cdot \ell_1 \cdot \ell_2)\mathcal{U}_4), \\
4P_1, P_2 Dk & = \text{Tr}((2k \cdot P_1 \cdot P_2 - 2(k \cdot P_1)^2 + D\ell_2 - k \cdot P_1 \cdot \ell \cdot \ell_2)\mathcal{U}_5), \\
4P_1, P_2 Dk & = \text{Tr}((-2k \cdot P_1 \cdot P_2 + k \cdot P_1 \cdot \ell_1 \cdot \ell_2 + k \cdot P_1 \cdot \ell_2 - k \cdot P_1 \cdot \ell \cdot \ell_2)\mathcal{U}_6), \\
4P_1, P_2 Dk & = \text{Tr}((2k \cdot P_1 \cdot P_2 - 2(k \cdot P_2)^2 + D\ell_2 - k \cdot P_1 \cdot \ell \cdot \ell_2)\mathcal{U}_7), \\
4P_1, P_2 Dk & = \text{Tr}((-2k \cdot P_1 \cdot P_2 + k \cdot P_1 \cdot \ell_1 \cdot \ell_2 + k \cdot P_1 \cdot \ell_2 - k \cdot P_1 \cdot \ell \cdot \ell_2)\mathcal{U}_8).
\end{align}
$$

(45)
and the inhomogeneous part of the equation can easily be written down from the first right-hand diagram in Eq. (44). Once the equation for $\square$ is solved on a computer, the diagram can be closed from the left to give

$$\begin{align*}
\Delta_4 & \quad \Gamma_{1,2} \\
\Delta_3
\end{align*}$$

(46)

and calculated as a simple integral. Here the color factor of 3 in Eq. (40) can easily be seen, since all three vertices are color singlets and carry $\delta_{cc}$ in color space.

C. Contribution with $\Gamma-3\Delta$

We will reduce the $3-\Delta$ contribution fixing the indices $\Gamma_1\Delta_1\Delta_2\Delta_3$ (the other permutations can be easily generated at the end). Employing Eq. (25) in

$$\begin{align*}
\Gamma_4 & \quad \Delta_1 \\
\Delta_3
\end{align*}$$

(47)

in the first diagram fix $j=3$ to substitute Eq. (25), in the second diagram employ $j=1$, we obtain

$$\begin{align*}
\frac{1}{(\sqrt{2f}\pi)^4} P_3 & \quad \Delta_1
\end{align*}$$

(50)

Therefore our next problem is to evaluate diagrams such as

$$\begin{align*}
\frac{1}{(\sqrt{2f}\pi)^4}
\end{align*}$$

(51)

which again explicitly displays the color factor and where the new constants $d_3$ and $d_4$ have to be calculated in a specific model.

To reduce the ladder in this diagram we could attempt to use the vector Ward identity for the $\gamma_\mu$ on the left, but the momenta flowing in the adjoining propagators would require this $\gamma_\mu$ to be contracted with $-P_1-P_2=P_3+P_4$ and not just with $P_3$. Or in the right part of the diagram we could reuse our result for the two-$\Delta$ $\Gamma$ vertex from the previous section. But again the momentum flow is not adequate. The solution to this impasse is to use both ideas, but in a momentum expansion. The ladder in this diagram can be substituted by its momentum expansion (26). Since there are three powers of momentum already committed (one is the explicit $P_3$, the other two need to be one in each $\Delta$) only one more power is needed. Therefore we can use the ladder expansion to order 1, and diagram (51) can be rewritten as

Next one can apply Eq. (22) to the single $\Gamma$ appearing in this expression to generate, after some simple manipulations,

$$\begin{align*}
\gamma_8 \quad \frac{1}{(\sqrt{2f}\pi)^4}
\end{align*}$$

(49)
Now we can use the vector Ward-Takahashi identity, which is satisfied to $O(P^{11})$, on the first diagram and on the left ladder of the second diagram, allowing us to substitute

$$\not p_3 \to iS^{-1}(q) - iS^{-1}(q - P_3).$$

The matrix object with a ladder and two deltas, to $O(P^{12})$, which appears on the second diagram, is just $\Gamma_3$ as defined in Eq. (42). Now it is straightforward to show that Eq. (51) is equal to

$$\begin{pmatrix}
\begin{array}{c}
\not q + \delta_1 \\
\delta_1 + \delta_2 \\
\not q + \delta_2
\end{array}
\end{pmatrix} + \text{permutations}. \quad (53)
$$

where the diagrams have to be evaluated to order $(P^{13})$ since an explicit power of $P$ has already been used. The left diagram in particular gives rise to four simple subdiagrams since two powers of $P$ are committed in the $\Delta$'s, but the other power can be distributed alternatively between these $\Delta$'s or the two propagators which carry a power of $P$.

These diagrams can all be evaluated easily as a simple loop integral on the computer to obtain $d_5, d_6$. The contribution from Eq. (47) is finally

$$\frac{3}{(\sqrt{2} \pi)^{2}} \left[ d_5 (P_1 \cdot P_3 P_1 \cdot P_2 - P_1 \cdot P_2 P_2 \cdot P_3) ight. + d_4 (P_2 \cdot P_3 P_1 \cdot P_2 - P_1 \cdot P_3 P_2 \cdot P_3) + \text{permutations} \bigg]. \quad (54)
$$

D. Contribution with $\Delta \Gamma \Gamma$

The contribution

\begin{equation}
\begin{split}
\Gamma_4 & + \Gamma_4 \\
\Gamma_3 & + \Gamma_3 \\
\Gamma_2 & - \Gamma_2
\end{split}
\end{equation}

(55)

can be reduced as follows. (1) apply Eqs. (20) and (23) to the $\Delta_1$ in the first term, and Eq. (12) to the $\Gamma_3$ in the second diagram to obtain an expression similar to Eq. (48) in which $\Gamma_4$ is isolated. (2) Employ Eq. (22) to eliminate $\Gamma_4$. (3) Repeat the operation to eliminate $\Gamma_2$. We obtain

$$\left(2k \cdot P_3 (A' \not f - B') + A' \not P_3 \right) \cdot (q + \delta_1 + \delta_2) \quad (56)$$

In this expression, two explicit powers of $P$ are present, namely, $P_1$ and $P_3$ in the vertices. The other two powers have to be produced from a propagator expansion. This is the third (and last) different diagram type that we need to parametrize:

$$\Gamma_4 (P_1 \cdot P_3 P_1 \cdot P_2 - P_1 \cdot P_2 P_2 \cdot P_3) \cdot (q + \delta_1 + \delta_2) = 3 d_5 P_1 \cdot P_3 P_1 \cdot P_4 + 3 d_6 P_3 \cdot P_4 \cdot P_1 \cdot P_4. \quad (57)$$

Next we show how to calculate $d_5, d_6$. Since there is a ladder that contains powers of $P$, we need to recall the ladder expansion to second order in Eq. (26). Eliminating the loose rung with the help of Eq. (30), and employing the vector Ward identity to generate a vertex

$$V(P) = 2 p \cdot P (A'(q) \not f - B'(q)) + A(q) \not p,$$

we can show that

$$\begin{pmatrix}
\begin{array}{c}
V(P_1) \\
V(P_3)
\end{array}
\end{pmatrix} \cdot (q + \delta_1 + \delta_2) + \begin{pmatrix}
\begin{array}{c}
V(P_1) \\
V(P_3)
\end{array}
\end{pmatrix} \cdot (k + \delta_1 + \delta_2) - \begin{pmatrix}
\begin{array}{c}
V(P_1) \\
V(P_3)
\end{array}
\end{pmatrix} \cdot (q + \delta_1 + \delta_2) \quad (58)
$$

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The first and third diagrams are again straightforward traces and integrals. Only the middle diagram contains a ladder. We can interpret this ladder as "dressing" either of the vertices and immediately write an integral equation for one of them. Taking, for example,

\[ 2kP_1(A'(k) - B'(k)) + A(k)P_1 \]

\[ = \sqrt{P_1, k + \frac{P_1 + P_4}{2}, P_1 + P_4} \]

\[ = \{2kP_1[A'(k) - B'(k)] + A(k)P_1\} \times S(k + P_1 + P_4)^{(1)}S(k)^{-1} \]

\[ \sqrt{\frac{q}{2}} x \]

(60)

The function vertex \( \sqrt{\cdot} \) so defined satisfies a linear inhomogeneous integral equation (the first argument is the momentum entering the diagram through the vertex; the second is the relative and the third the total momentum between the fermion lines at the vertex):

As is evident, \( \sqrt{\cdot} \) admits an expansion up to second order in momentum identical to Eq. (43) in terms of a new set of functions \( \sqrt{\cdot} \theta(k^2), \ldots, \sqrt{\cdot} \theta(k^4) \). The integral system of equations (60) is very similar to (44), the only difference being the inhomogeneous term. Therefore the linear projections in Eq. (45) still apply, and both systems can be solved with basically the same iterative computer code.

Finally, the middle diagram in Eq. (58) can be closed to read

\[ \sqrt{\frac{q}{2}} x \]

(61)

which is easy to evaluate with the help of a symbolic manipulation program. Finally, we give the expression for Eq. (55) in terms of the \( d' \) s:

\[ 3d_5P_1 \cdot P_3(P_1 \cdot P_4 + P_1 \cdot P_2) + 3d_6(P_1 \cdot P_2P_3 \cdot P_4) + P_1 \cdot P_4P_3 \cdot P_4 + \text{permutations}. \]  

(62)

**E. Contribution with \( \Delta \Gamma \Delta \Gamma \)**

With two \( \Delta \) corrections, there are two topologically distinct diagrams that can contribute. They are different because while reading around the fermion loop one can find the external legs in the order \( \Gamma \Delta \Delta \Delta \) or in the order \( \Gamma \Delta \Gamma \Delta \).

We start with the first term, namely,

\[ \frac{-3}{(\sqrt{2} f_\pi)^4}[d_5(P_1 \cdot P_3P_1 \cdot P_4 + P_1 \cdot P_3P_3 \cdot P_4 + \text{permutation}) + d_6(P_3 \cdot P_4P_1 \cdot P_4 + P_1 \cdot P_4P_3 \cdot P_4 + \text{permutation})]. \] 

(63)

\[ + \text{permutations} \]

and the corresponding three more permutations. Again, by applying Eq. (23) to the \( \Delta_1 \) in the first diagram and Eq. (10) to the \( \Gamma_3 \) in the second, then using Eq. (13) to simplify the remaining \( \Gamma_4 \), neglecting the positive parity ladders when they are divided by a \( c_j \) containing the pion pole, reabsorbing the ladders and simplifying, one obtains

\[ \frac{-3}{(\sqrt{2} f_\pi)^4}[d_5(P_1 \cdot P_3P_1 \cdot P_4 + P_1 \cdot P_3P_3 \cdot P_4 + \text{permutation}) + d_6(P_3 \cdot P_4P_1 \cdot P_4 + P_1 \cdot P_4P_3 \cdot P_4 + \text{permutation})]. \] 

(64)

**F. \( \Delta \Gamma \Delta \Gamma \)**

Two permutations contribute: \( \Delta_1 \Gamma_2 \Delta_3 \Gamma_4 \) and \( \Gamma_1 \Delta_2 \Gamma_3 \Delta_4 \). The reduction is in all respects analogous to the previous ones, yielding a contribution

\[ \frac{-3}{(\sqrt{2} f_\pi)^4}[d_5(P_1 \cdot P_3P_1 \cdot P_4 + P_1 \cdot P_3P_3 \cdot P_4 + \text{permutation}) + d_6(P_3 \cdot P_4P_1 \cdot P_4 + P_1 \cdot P_4P_3 \cdot P_4 + \text{permutation})]. \] 

(65)

\[ + \text{permutations} \]

**G. \( 4-\Gamma \) contribution**

The last piece stems from the term with four powers of \( \Gamma \). By repeated use of Eq. (22) it can be shown to contribute

\[ \frac{-3}{(\sqrt{2} f_\pi)^4}[d_5(P_1 \cdot P_3P_3P_1 \cdot P_4 + P_1 \cdot P_3P_3P_4 + \text{permutation}) + d_6(P_3 \cdot P_4P_1 \cdot P_4P_3 \cdot P_4 + \text{permutation})]. \] 

(66)

[Here the cyclicity of Eq. (31) can be recovered by using \( \Sigma_i p_i = 0 \).]
Combining the results from Secs. IV A–IV F and summing all permutations (32) with the isospin factors (24), (35), the standard amplitude \( A(s,t,u) \), takes a form identical to Eq. (4), and we deduce

\[
l_1 = \frac{3}{32}(-2d_1 + d_2 - 6d_5 + 2d_4 + 5d_5 + d_6),
\]

\[
l_2 = -\frac{3}{16}(d_2 - 2d_3 + 2d_4 + d_5 - d_6).
\]  

(68)

Therefore, to obtain the \( l \)'s numerically, one must (1) solve the Schwinger-Dyson equations for the propagator, Eq. (7); (2) employ the functions obtained \( A, B \) as input to the Bethe-Salpeter equations (16) and solve them; (3) use the \( F_0, G_0, \ldots \) obtained as input for Eqs. (44) and (60) to obtain \( \cup \) and \( \vee \); (4) perform the integrals (46), (53), (61); and (5) assemble Eqs. (40), (54), (62), (65), (66), (67). In this external momentum expansion all integrals and integral equations are functions only of internal variables of quark momenta \( q^2, k^2, k \cdot q \). The diagrams could alternatively be evaluated on the lattice.

V. MODEL EVALUATIONS

We would like to provide simple model evaluations of all these calculations, well aware of model limitations and that a thorough phenomenological analysis can be carried out only with more sophisticated interactions such as those employed in [7,12]. We employ two Feynman gauge models featuring the interaction

\[
VK(q)V = \gamma_\mu K(q) \gamma^\mu.
\]  

(69)

This simple choice of a vector-vector interaction (as opposed to the more popular Landau gauge transverse tensor kernel) simplifies the Gamma matrix traceology (in this calculation carried out with the help of two independent computer codes, one written in MATHEMATICA and one in FORM [22]) so that the standard Llewelyn-Smith BS wave function for the pion [7] reduces to

\[
\chi(P,k) = \gamma_5(E(P,k) + F(P,k)\not{q} + G(P,k)\not{k} \cdot P)
= \gamma_5\left(E0(k^2) + \frac{(kP)^2}{2}E2(k^2) + F0(k^2)\not{q}
+ G0(k^2)\not{k} \cdot P + \ldots\right),
\]  

(70)

where the standard \( H \) function decouples from the rest of the system and therefore is ignored, and the momentum expansion shown is complete up to second order for a symmetric momentum routing \( \chi \) (that is, the fermion lines out of \( \chi \) carry \( k + P/2 \) and \( k - P/2 \). The \( E0 \) term is determined by chiral symmetry to be \( \Gamma_A(P=0)/\sqrt{2iF_\pi} \). The rest of the expansion constructs the function \( \Delta(P,k) \). The same power series (70) can be written down for the axial vertex

\[
\Gamma_A(P,q) = \left[ A(q-P/2)\left(\not{q} - \frac{\not{p}}{2}\right) - A(q+P/2)\left(\not{q} + \frac{\not{p}}{2}\right)\right] \frac{\gamma_5}{i},
\]  

(71)

expanding in powers of \( P \), and up to a normalization we recover the equivalent to Eq. (70):

\[
E0_A = 2B(q^2),
\]

\[
F0_A = -A(q^2),
\]

\[
G0_A = -2A'(q^2),
\]

\[
E2_A = 2B''(q^2),
\]

(72)

Subtracting Eq. (72) from Eq. (70) we obtain some new functions of \( q^2 \) which provide the needed expansions for \( \Delta \):

\[
\Delta^{(1)}(q+P/2,P_1) = \Delta^{(1)}(q,P_1) = F0(q^2)\not{q} + G0(q^2)q \cdot P_1\not{q},
\]

\[
\Delta^{(2)}(q+P/2,P_1) = E2(q^2)\left(\frac{q \cdot P_1}{2}\right)^2 + F0'(q^2)\not{q}q \cdot P_1
+ G0'(q^2)\not{q}q \cdot P_1\not{q}
\]  

(73)

(valid for symmetric momentum routing when \( E1,F1,G1 \) all vanish). Further, with the \( \gamma_\mu \gamma^\mu \) kernel another trace-related simplification occurs in Eqs. (44) and (60), and the functions \( \cup_6, \cup_7, \cup_8, \vee_6, \vee_7, \) and \( \vee_8 \) equal the inhomogeneous term in their respective equations, the homogeneous (integral) parts of the equations being zero.

All that remains is to consider some specific form for \( K(q^2) \). We will look at two models whose Euclidean angular integrals can be done analytically, leaving only one-dimensional integral equations to solve numerically.

<table>
<thead>
<tr>
<th>( \Lambda )</th>
<th>( g )</th>
<th>( M(k^2=0) )</th>
<th>( -\langle \Psi \Psi \rangle^{1/3} )</th>
<th>( f_\pi )</th>
<th>( l_1 )</th>
<th>( l_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>500</td>
<td>6.3</td>
<td>385</td>
<td>222</td>
<td>76</td>
<td>-0.02</td>
<td>0.066</td>
</tr>
<tr>
<td>600</td>
<td>6.3</td>
<td>462</td>
<td>267</td>
<td>91</td>
<td>-0.02</td>
<td>0.060</td>
</tr>
<tr>
<td>500</td>
<td>6.5</td>
<td>468</td>
<td>237</td>
<td>83</td>
<td>-0.018</td>
<td>0.062</td>
</tr>
<tr>
<td>800</td>
<td>5.5</td>
<td>125</td>
<td>213</td>
<td>51</td>
<td>-0.019</td>
<td>0.080</td>
</tr>
</tbody>
</table>
The first is a simple Gaussian kernel (whose Euclidean angular integrals are Bessel functions [23])

\[ K(q) = g^2 \exp(-q^2/\Lambda^2), \tag{74} \]

where \( g \) provides the coupling strength and \( \Lambda \) the scale of the interaction (the results are sketched in Table I).

The second is a rational kernel

\[ K(q) = g^2 \left[ \frac{1}{(q^2-\lambda^2)} - \frac{1}{(q^2-\Lambda^2)} \right], \tag{75} \]

where \( g, \lambda, \) and \( \Lambda \) represent some quark model parameters fitted to provide a good condensate and constituent quark interaction.

The angular integrals are straightforward using relations such as

\[ \int_{-1}^{1} \frac{\sqrt{1-x^2}}{a-x} = \pi(a - \sqrt{a^2-1}), \]

\[ \int_{-1}^{1} x \frac{\sqrt{1-x^2}}{a-x} = \pi(a^2 - 1/2 - a \sqrt{a^2-1}). \]

Hence all integral equations are one dimensional in momentum space.

When the dimensionful parameters, of the order of the strong interaction scale, are close in value, and for large enough \( g \), the potential is infrared enhanced, supporting chiral symmetry breaking. Notice that the rainbow-ladder fermion loop diagrams constructed with these interactions are finite due to the exponential or \( 1/q^4 \) high-energy behavior, and no renormalization program is needed. This is associated with a gluon propagator scale, which determines the scale of the Bethe-Salpeter wave functions, which in turn regulate the meson loops that would appear in chiral perturbation theory. We are currently investigating these issues.

VI. RESULTS AND DISCUSSION

We have worked out the pion-pion scattering amplitude at low energy in the chiral limit to \( O(P^4) \) from a general microscopic quark Schwinger-Dyson approach. Upon comparison with the chiral Lagrangian formalism, the main result of this paper is a pair of relations that would in principle allow us to directly evaluate the coefficients \( l_1, l_2 \) in any specific model (i.e., after choosing the Lorentz structure of the quark-antiquark interaction and the strength and shape of any potential or dressed gluon propagator) provided it supports the standard mechanism of chiral symmetry breaking [17,18,24].

The symmetry properties of the \( \Gamma \) pion vertex are model dependent and are of order \( P^4 \) (therefore vanishing at low energy). Lorentz invariance restricts their form and allows for only two coefficients.

On a first glance, the diagrams (40), etc., seem as difficult to calculate as the full pion-pion scattering amplitude, but one needs to remember that each \( \Delta \) pion wave function entering the calculation can be taken to have only one power of the external \( P \) to this order, since there are four of them and they vanish by definition at \( P = 0 \), and the propagators can also be taken at \( P = 0 \), leaving just one momentum \( k \) around the loop. This is a major simplification.

In the previous section we showed how the numerical evaluation can be performed with two very simple (too simple) kernels. Our results for \( l_1 \) and \( l_2 \) have phenomenologically correct signs and ratios, especially for the Gaussian toy model (see Tables I and II). Possibly because our results are computed at \( P = 0 \) and without pion loops, our \( l_1 \) and \( l_2 \) seem too large in absolute value. This can be appreciated [4,6,9,10,21,25–29,31] in Tables III, IV, and V, which summarize the present status of knowledge of these coefficients.

<table>
<thead>
<tr>
<th>Authors</th>
<th>( l_1'(m_\rho) \times 10^3 )</th>
<th>( l_2'(m_\rho) \times 10^3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gasser and Leutwyler [4]</td>
<td>-4.2±3.9</td>
<td>9.0±2.7</td>
</tr>
<tr>
<td>Bijnen, Colangelo, Talavera; Colangelo, Gasser, Leutwyler [26]</td>
<td>-2.2±0.6</td>
<td>9.0±2.7</td>
</tr>
<tr>
<td>Yndurain [28]</td>
<td>-4.1±0.7</td>
<td>9.8±0.5</td>
</tr>
<tr>
<td>Gómez Nicola and Peláez [6]</td>
<td>-3.3±0.7</td>
<td>4.8±0.6</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Authors</th>
<th>( l_1'(m_\rho) \times 10^3 )</th>
<th>( l_2'(m_\rho) \times 10^3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gasser and Leutwyler [4]</td>
<td>-8.4</td>
<td>8.4</td>
</tr>
<tr>
<td>Pham and Truong [27]</td>
<td>-5.5 to -24</td>
<td>14</td>
</tr>
<tr>
<td>Ecker et al. [21]</td>
<td>-6.1±3.9</td>
<td>5.3±2.7</td>
</tr>
<tr>
<td>Pennington and Portolés</td>
<td>-2.1±1.2</td>
<td>5.9±1.0</td>
</tr>
</tbody>
</table>

TABLE IV. Phenomenological determinations of the \( l \) parameters (based on \( \rho \) meson resonance saturation).
The various conventions used in the literature \([SU(3), SU(2)],\) renormalized, barred, etc.] have been unified to therenormalized \(I\)'s at the scale of the \(\rho\) meson. Some of those determinations carry information relative to finite quark (and therefore pion) masses, to kaons and etas, or to pion loops, none of which have been taken into account in this work. The fairest comparison therefore is with the oldest results of Andrianov \([10]\) in the large \(N_c\) limit (no gluon corrections) and those of Pham and Truong \([27]\). The latter authors obtain the interesting relations

\[
l_2 = \frac{f_\pi^2}{m_\rho},
\]

and

\[
l_1 = \frac{1}{3} \frac{f_\pi^2}{m_\sigma} - 2l_2.
\]

If the \(\sigma\) mass is sent to infinity, then the large \(N_c\) ratio is recovered. For finite \(\sigma\) masses between 350 and 750 MeV we obtain the range given in Table IV. This demonstrates the importance of repeating the model evaluations with kernels whose meson excitations in various channels are known.

Our approach is potentially superior to the resonance saturation approximations since it includes the full vertex and ladder structures, that is, effects of continuua and higher resonances. In particular, we include the four-pion direct interaction, and the exchange of the full series of excited vector and scalar mesons. We also stress that the masses of our \(\rho\) and \(\sigma\) mesons are expected to be of the right order of magnitude, because our constituent quark mass has reasonable values of the order of 300 to 400 MeV (see Tables I and II). Our approach is also potentially superior to Nambu–Jona-Lasinio determinations in allowing us to lift the approximation of contact interactions between quarks. Thus we pave the way to calculating the \(l_i\) coefficients from the lattice or from accurate Schwinger-Dyson solutions. Finite current quark masses and meson loops remain to be incorporated to improve the precision of our calculation. We are currently considering some of these issues \([32]\).

### Acknowledgments

The authors acknowledge useful discussions with S. Cotanch, F. Kllefeld, B. Hiller, P. Maris, J. R. Pelaez, A. Dobado, A. Gomez Nicola, and specially E. Ribeiro. This work was partially supported by Univ. Complutense on a travel grant, and by grants FPA 2000-0956 and BFM 2002-01003 (Spain), F.J.L.-E. is thankful for the hospitality and scholarly atmosphere at IST Lisbon.

### Appendix: Norm and \(f_\pi\)

In this appendix we present the proof of the well-known fact that the normalization of the Bethe-Salpeter wave function is the pion decay constant, rewritten in terms of our Ward identity techniques. At null external pion momentum, \(\chi^a(P=0)\) and \(\Gamma^a\) are proportional because Eq. (10) reduces to the homogeneous BS equation (16). We introduce an arbitrary proportionality constant \(n_\pi\) by means of

\[
\chi^a(P=0) = \frac{\Gamma^a}{in_\pi};
\]  

(A1)

the \(i\) guarantees \(\chi\) to be real because of the explicit form of \(\Gamma\) in Eq. (18).

The normalizing condition for the Bethe-Salpeter solution is, following Llewelyn-Smith \([30]\),

\[
\chi^a_{-\rho} \partial_{\rho \mu} \left( \frac{\chi^b}{\chi^b_{\rho}} \right) \chi^b_{\rho} = 2i P_\mu \delta^{ab} \cdot \quad \text{(A2)}
\]

Because the right hand side of the normalization condition is of first order in \(P\), we can expand in \(\chi\) and in \(\Gamma^a\) as in Eq. (20). There are two terms, depending on whether the derivative is applied to the upper or lower propagator. The first term can be written as

\[
\chi^a_{-\rho} \partial_{\rho \mu} S^{-1}_{\rho} = \Gamma^a_{\rho} \partial_{\rho \mu} S^{-1}_{\rho} \quad \text{(A3)}
\]

The term with \(\Gamma\) can be shown to be zero. To see it, one needs to take a derivative of Eq. (7) that gives

\[
S \partial_{\mu} S^{-1} \quad = \quad \partial_{\mu} S_0^{-1} \quad \text{(A4)}
\]

which applied to the \(\Gamma\) term in Eq. (A3) and employing Eq. (22) reduces it to
Reabsorbing the remaining ladder, and employing the free propagator and bare axial vertex in Eqs. (5),(9) this is proportional (because of the isospin factor not included) to

$$\text{Tr} \int \left( i \not\!p \partial \mu S^{-1} - \Gamma_{A_{-p}} \not\!\gamma_\mu \not\!\gamma_5 \frac{\gamma_\mu \gamma_5}{2i} S \right).$$  \hspace{1cm} (A6)

Eliminating $\Gamma$ with its Ward identity (13), and after some elementary operations, this equals

$$-i \text{Tr} \left( (\gamma_\nu \partial \mu S - \partial \nu S \gamma_\mu) \Gamma_{A_{-p}} \right) = 0. \hspace{1cm} (A7)$$

Returning to Eq. (A3), once the term with $\Gamma\Gamma$ has been shown to vanish, we need to evaluate the terms with a $\chi$ and a $\Gamma$.

Diagrammatically again, one has [recall Eqs. (22), (23), and (A4)]

$$\Gamma_{A_{-p}}^{\mu} = \gamma_5 \not\!\gamma_\mu \not\!\gamma_5 \frac{\gamma_\mu \gamma_5}{2i} S$$

We get two terms. The first is of order $P^2$ because of the $c$ in the denominator, and vanishes. The second is indeed non-zero. Going back now to Eq. (A2), we obtain

$$2i P_\mu = \left( \frac{1}{16\pi^2} \frac{2i}{2i} \right) \gamma_5 \not\!\gamma_\mu \not\!\gamma_5 \frac{\gamma_\mu \gamma_5}{2i} S$$

The definition of the weak decay constant $f_\pi$ yields (with no pion loops)

$$\chi_{-p}^{\mu} = i P_\mu f_\pi \delta^{ab}$$

and therefore we must have

$$n_\pi = f_\pi,$$

which immediately leads to Eq. (20). Finally, direct calculation of Eq. (A10) leads to

$$i f_\pi^2 = 3 \int \frac{d^4 q}{(2\pi)^4} \frac{1}{(A^2 q^2 - B^2)^2} \times \left[ E0 \left[ 4AB + 2q^2 \left( B - \frac{dA}{dq^2} - A \frac{dB}{dq^2} \right) \right] \right.$$}

$$\left. + F0(2q^2 A^2 + 4B^2) + q^2 G0(B^2 - q^2 A^2) \right] \right). \hspace{1cm} (A11)$$

Notice the explicit color factor of 3. All through the paper the Bethe-Salpeter wave functions have been taken proportional to the identity $\delta_{\nu,\mu}$ in color space. Had we normalized them in a different way, say $\delta_{\nu,\mu}/\sqrt{3}$, this would immediately affect the formula above, reducing the factor to $\sqrt{3}$, the rest being absorbed by the functions $E0$, $F0$, and $G0$. The low energy theorems (Gell-Mann–Oakes–Renner theorem, Weinberg’s amplitude, etc.) are unchanged by this choice since the explicit form of these functions is never used to prove them: they are always eliminated in terms of $f_\pi$. But the $l_1,l_2$ constants of the chiral Lagrangian would indeed have to be rewritten in terms of the modified Bethe-Salpeter wave functions. This, of course, would not affect its numerical value.

The normalizing condition (A2),

$$\chi_{-p}^{\mu} \not\!\gamma_5 \not\!\gamma_\mu \not\!\gamma_5 \frac{\gamma_\mu \gamma_5}{2i} S = 2i P_\mu \hspace{1cm} (A12)$$

which yields Eq. (A10), can also be directly evaluated without using Ward identities. By taking the derivative of this equation with respect to $P^\mu$ (and contracting over $\mu$ as usual), we get the following (derivatives with respect to $P$ act only on the function immediately behind them and the color factor is explicit):

$$8i = 3 \int \frac{d^4 q}{(2\pi)^4} \text{Tr} \left[ 2 \partial \mu \chi_{-p}(P) \partial \nu S(q + P/2) \chi_{-p}(-P) S(q - P/2) \right.$$}

$$\left. + 2 \chi_{-p}(P) \partial \mu S(q + P/2) \chi_{-p}(-P) S(q - P/2) \right.$$}

$$\left. + 2 \chi_{-p}(P) \partial \mu S(q + P/2) \chi_{-p}(-P) \partial \nu S(q - P/2) \right.$$}

$$\left. + 2 \chi_{-p}(P) \partial \mu S(q + P/2) \chi_{-p}(-P) \partial \mu S(q - P/2) \right]. \hspace{1cm} (A13)$$
To calculate this normalization one needs to make the wave function explicit,
\[
\chi_\pi(P) = \gamma_5 [E_0(q^2) + F_0(q^2)\phi + G_0(q^2)q \cdot p \phi + \cdots],
\]
(A14)
which evaluated at \(P = 0\) yields
\[
\chi_\pi(0) = \gamma_5 E_0(q^2), \quad \partial_\mu \chi_\pi(P = 0) = \gamma_5 (F_0 \gamma_\mu + G_0 \phi q_\mu),
\]
and make use of the propagator expansion (27), (28), (29) above. A simple check on the resulting expression is to substitute \(\chi\) by \(\Gamma\), yielding
\[
-3 \int \frac{d^4q}{(2\pi)^4} \left[ -2B(\partial_\rho B) \partial_\rho \rho \left( \frac{1}{A^2q^2 - B^2} \right) - B^2\partial_\rho \partial_\rho \left( \frac{1}{A^2q^2 - B^2} \right) \right] = 0,
\]
which vanishes upon employing Green’s first identity. This checks the zero in Eq. (A7) with a completely independent calculation and is also approximately observed in our computer codes. The result for \(n_\pi\) is
\[
\begin{align*}
n_\pi^2 &= 3i \int \frac{d^4q}{(2\pi)^4} \frac{1}{A^2q^2 - B^2} [(AB' - BA')] \\
&\times 2q^2B(\bar{F}_0 + q^2\bar{G}_0) - AB^2(4\bar{F}_0 + q^2\bar{G}_0)],
\end{align*}
\]
(A15)
which, upon comparison with Eq. (A11) provides an integral constraint between the Schwinger-Dyson and Bethe-Salpeter solutions. The barred quantities have been defined in Eq. (73).

To conclude this discussion we recall the derivation of the Gell-Mann–Oakes–Renner (GMOR) relation in the Bethe-Salpeter formalism. This has been presented in [13,14], together with a discussion of the Weinberg theorem. At zero external pion momentum, the axial Ward identity reads
\[
2i\Gamma^a_\pi(k,k) = \frac{\sigma^a}{2} (\gamma_5 S^{-1} + S^{-1}\gamma_5)
\]
(A16)
and, as previously discussed,
\[
\Gamma^a_\pi(k) = \frac{B_k}{m} \frac{\sigma^a}{2} \gamma_5.
\]
(A17)
To start the simple demonstration of the GMOR relation, “undress” the vertex \(\Gamma^a_\pi\) to write
\[
S \Gamma^a_\pi S = \frac{\gamma \gamma^a}{2} \chi^a \chi^0 \gamma^0 \chi^a \chi^0 /
\]
(A18)
and, neglecting the contribution of higher pion states (which exactly decouple in the chiral limit, as in Ref. [24]), we can saturate the ladder using the pion pole, yielding
\[
\Gamma^a_\pi \approx \frac{\gamma \gamma^a}{2} \chi^a \chi^0 \gamma^0 \chi^a \chi^0
\]
(A19)
Substituting now Eq. (A1) in the form
\[
\chi^a = \frac{S^{-1} \gamma^5 + \gamma^5 S^{-1} \sigma^a}{in_\pi \gamma^5}
\]
(A20)
and comparing with Eq. (A17), we immediately obtain
\[
-2m \text{Tr } S = n_\pi^2 M^2
\]
corresponding to the GMOR relation upon identification of \(n_\pi = f_\pi\), consistent with the Llewelyn-Smith normalization condition.

We finally remind the reader of the explicit expression for the BCS planar condensate:
\[
\langle \bar{\Psi}_u \Psi_u \rangle = \langle \bar{\Psi}_d \Psi_d \rangle = \text{Tr } S = i \text{Tr } \int \frac{A_k \vec{k} + B_k}{A_k \vec{k}^2 - B_k^2 + i\varepsilon}.
\]
(A21)


[32] A short preliminary version of this work was reported at the Vth International Conference on Quark Confinement and the Hadron Spectrum; see F.J. Llanes-Estrada and P. Bicudo, hep-ph/0212182.