We explore the viscosities of a photon gas by means of the Euler-Heisenberg effective theory and quantum electrodynamics at zero electron chemical potential. We find parametric estimates that show a very large shear viscosity and an extremely small bulk viscosity (reflecting the very weak coupling simultaneously with a very approximate dilatation invariance). The system is of some interest because it exemplifies very neatly the influence of the breaking of scale invariance on the bulk viscosity.

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I. INTRODUCTION

Relativistic Heavy-Ion Collider (RHIC) experiments have drawn much attention to the quark and gluon plasma as a nearly perfect fluid [1]. The outcome of an extensive body of investigations is that, near the phase transition to a hadron medium, the fluid is very strongly coupled, featuring a minimum in the shear viscosity [2,3] (normalized to the entropy density). Under current discussion is the behavior of the bulk viscosity [4] and whether [5,6] or not [7] there is a maximum of the bulk viscosity in the same crossover region. Also well known are the transport coefficients in perturbative QCD [8], as well as the low-energy pion gas [9,10] and strongly coupled Fermi systems [11].

In this brief report we examine quite an opposite example: a photon gas at a very low temperature. Classical electrodynamics is a linear theory, in which light beams cross each other without interacting. Thus, all transport is affected by interactions with the cavity walls such as in a waveguide, and one cannot really talk of a fluid in infinite matter.

However, in QED a photon can fluctuate instantly into an electron-positron pair, and another photon can Compton-scatter off the virtual charge thus created. Therefore, QED is compatible and its inversion can then be carried out. An effective chemical potential could be introduced and additional terms appear in the source function (see discussion and the extra terms for a massive pion gas in Ref. [10]). In addition to the energy-zero mode in the collision term, an extra zero mode related to the particle conservation—when \( A(p) \) in Eq. (17) is a constant—is also present. However, this mode is also orthogonal to the source function. Thus, the Boltzmann equation is again compatible and its inversion can then be carried out.

The two parametric estimates for the bulk viscosity are given in Eqs. (33) and (35), for the case with a dominant inelastic/elastic interaction, respectively.

The effective Lagrangian of Euler-Heisenberg [14] is not sufficient for these estimates of the bulk viscosity due to thermal nonanalyticities. The effective theory is constructed as a derivative power series and by standard counting, \( T \sim p \), so that one expects observables to be organized in terms of \( (T/m_e)^\gamma \), where \( m_e \) is the electron mass. The exponentials in Eqs. (33) and (35) are singularities that are not captured in such a framework. This is due to the need to break scale invariance with a thermal photon mass, which is not possible in a one-loop perturbative calculation in the effective theory. Thus, we turn to a full
calculation of the thermal photon mass within QED—and thus the bulk viscosity—in Sec. III, where happily we can draw much from the literature. Our conclusions and outlook are presented in Sec. IV.

II. PHOTON-PHOTON SCATTERING IN THE EULER-HEISENBERG EFFECTIVE THEORY

For low-frequency radiation, the box diagram (left panel of Fig. 1) necessary to calculate the photon-photon scattering amplitude in QED reduces to the effective Lagrangian density of Euler-Heisenberg, expressible in a gauge-invariant way in terms of the electric and magnetic fields as

\[
\mathcal{L}_{EH} = \frac{1}{2} \left( \mathbf{E}^2 - \mathbf{B}^2 \right) + \frac{e^4}{360 \pi^2 m_e^2} \left[ \left( \mathbf{E}^2 - \mathbf{B}^2 \right)^2 + 7 (\mathbf{E} \cdot \mathbf{B})^2 \right].
\]

(1)

The Feynman rule for the photon-photon vertex (depicted in the right panel of Fig. 1) is extracted from the interaction part of this Lagrangian that, when written as a function of the electromagnetic tensor, reads

\[
\mathcal{L} = a F_{\alpha \beta} F^{\alpha \beta} F_{\mu \nu} F^{\mu \nu} + b F_{\alpha \beta} F_{\mu \nu} F^{\alpha \mu} F^{\beta \nu},
\]

(2)

with low-energy constants \( a = - \frac{\alpha^2}{36 m_e^2} \) and \( b = \frac{2 \alpha^2}{90 m_e^2} \) and \( \alpha \approx 1/137 \) the electromagnetic coupling constant.

The photon four-point function at tree level \( \mathcal{M}(p_1, p_2, p_3, p) \) was obtained not too long ago in Ref. [15]. Although we only use its parametric dependence in this brief paper, since we have detected some typos and to provide all necessary information should the numeric details on the corresponding Chapman-Enskog expansion can be found in Ref. [3]).

To check known results once more, we take Eq. (8) to the center-of-mass frame,

\[
[s] = \frac{139}{2025} \frac{\alpha^4}{m_e^8} \frac{e^8}{16 \omega^6 (\cos^2 \theta + 3)^2},
\]

(9)

where \( \omega \) is the center-of-mass energy, and then calculate the differential cross section,

\[
\frac{d\sigma_{\gamma \gamma \rightarrow \gamma \gamma}}{d\Omega_{CM}} = \frac{1}{2} \frac{1}{64 \pi^2} \frac{1}{(2 \omega)^2} \frac{1}{|\mathcal{M}|^2},
\]

(10)

where a factor of 1/2 has been included because the two photons are identical in the final state). The result

\[
\frac{d\sigma_{\gamma \gamma \rightarrow \gamma \gamma}}{d\Omega_{CM}} = \frac{139}{64800 \pi^2} \frac{\alpha^4 \omega^6}{m_e^8 (\cos^2 \theta + 3)^2}
\]

(11)

can be integrated to obtain the total cross section [16,17],

\[
\sigma_{\gamma \gamma \rightarrow \gamma \gamma} = \frac{973}{10125 \pi} \frac{\alpha^4 \omega^6}{m_e^8}.
\]

(12)

The photons are very nearly free. Thus, quasiparticle kinetic theory is a very accurate starting point. The shear viscosity of the photon gas is then expressible as an integral over the shear function \( \delta f_p \propto B(p) \) (all details are given e.g. in Ref. [18]) that characterizes the separation from the equilibrium Bose-Einstein function \( f_p = n_p \) in the Landau-Lifschitz reference frame,

\[
\eta = \frac{2}{15 T^3} \int \frac{d^3 p}{(2 \pi)^3} E_p (1 + n_p) p^6 B(p).
\]

(13)
Parametrizing a small separation from equilibrium, \( B(p) \) satisfies a linearized Boltzmann-like equation (the Uehling-Uhlenbeck equation),

\[
n_p(n_p) p^i p^j = \frac{1}{2 T^2} \int (1 + n_1)(1 + n_2) n_p \left( M^2 \right)^3 \prod_{k=1}^{3} \frac{d^3 p_k}{2 E_k (2 \pi)^3} (2 \pi)^4 \delta^4(p_1 + p_2 - p_3 - p)(p^i p^j B(p)) + p^i p^j B(p_3) - p^i p^j B(p_1) - p^i p^j B(p_2)).
\]

A full numerical evaluation is beyond our present scope. Nevertheless, the parametric dependence of the shear viscosity can already be obtained by examining these two equations, or by employing the relaxation-time approximation with a thermally averaged cross section (12),

\[
\eta \sim \frac{T}{\sigma_{\gamma \gamma}} - \frac{1}{\alpha^4 T^5}.
\]

This establishes Eq. (31) below. We now proceed to the bulk viscosity.

III. BULK VISCOSITY AND THERMAL NONANALYTICITY

A. Bulk viscosity in the effective theory

An evaluation of the bulk viscosity along the lines of Sec. II is bound to fail. To see it, we write down the equivalent of Eqs. (13) and (14), where the disturbance from equilibrium is \( \delta f_p \propto A(p) \),

\[
\zeta = \frac{2}{T} \int \frac{d^3 p}{(2 \pi)^3} E_p n_p (1 + n_p) A(p) \frac{E_p (p \cdot \nabla_p) E_p}{3} + p^2 B(p_3) - p^2 B(p_1) - p^2 B(p_2)).
\]

and

\[
n_p(1 + n_p) \left( \frac{p \cdot \nabla_p E_p}{3} - v_s^2 \frac{\partial (\beta E_p)}{\partial \beta} \right)
= \int (1 + n_1)(1 + n_2) n_p \left[ A(p) + A(p_3) - A(p_1) \right]

- A(p_2) \left[ \frac{1}{2 E_p} \left( M^2 \right)^3 \prod_{k=1}^{3} \frac{d^3 p_k}{2 E_k (2 \pi)^3} (2 \pi)^4 \right]

\times \delta^4(p_1 + p_2 - p_3 - p),
\]

where \( \beta = 1/T \) and \( v_s^2 \) denotes the adiabatic speed of sound squared.

In a dilatation-invariant theory (such as Maxwell’s theory) \( \zeta \) vanishes directly because the factor \( \left( \frac{p \cdot \nabla_p E_p}{3} - v_s^2 \frac{\partial (\beta E_p)}{\partial \beta} \right) \) averages to zero upon integrating Eq. (17) over \( p \). The electron mass—which breaks dilatation invariance—does enter into the constants \( a \) and \( b \) in the low-energy theory (2), which appear in the squared, average amplitude \( M^2 \) in Eq. (17), but as we now examine it does not affect the vanishing of the factor in question in the low-energy effective theory, because this is controlled by the quasiparticle (photon) mass, not by the electron mass.

Indeed, the one-loop correction to the photon vacuum polarization using the Euler-Heisenberg theory was calculated in Ref. [19]. It leads to a dispersion relation for transverse photons at low \( T \) that is linear in \( p \),

\[
E_p = \sqrt{1 - \gamma p},
\]

with the speed of light reduced by

\[
\gamma = \frac{44 \pi^2 a^2 T^4}{2025 m_e^4}.
\]

Therefore, the low-energy theory does not generate a thermal mass for the photon. In fact, this one-loop masslessness is valid at any order in the perturbative expansion [20].

To see the vanishing of the bulk viscosity in a bit more detail, it is convenient to reduce Eq. (16) to

\[
\zeta = \frac{2}{T} \int \frac{d^3 p}{(2 \pi)^3} E_p n_p (1 + n_p) A(p) p^2 \frac{\partial (\beta E_p)}{\partial \beta}.
\]

But in the Landau-Lifshitz formalism, in addition to the Landau-Lifshitz condition that fixes the reference frame, there is a so-called “condition of fit” that fixes the energy content of the system. It involves the 00-component of the stress-energy tensor [21],

\[
0 \equiv \tau_{00} = \int \frac{-2d^3 p}{(2 \pi)^3} E_p n_p (1 + n_p) A(p) E_p \frac{\partial (\beta E_p)}{\partial \beta}.
\]

Using the dispersion relation in Eq. (18), without a photon thermal mass, this yields

\[
\int \frac{d^3 p}{(2 \pi)^3} E_p n_p (1 + n_p) A(p) p^2 = 0.
\]

Since the bulk viscosity in Eq. (20) is proportional to the integral in Eq. (22), it must vanish. This happens even as the speed of sound differs from the conformal value \( v_s^2 = 1/3 = -4\gamma \) because each of the terms in \( \left( \frac{p \cdot \nabla_p E_p}{3} - v_s^2 \frac{\partial (\beta E_p)}{\partial \beta} \right) \) separately yields a zero integral!

B. One-loop QED

We have just found that the bulk viscosity is predicted to vanish within the Euler-Heisenberg effective Lagrangian at the lowest order. This is a surprising feature, since the Lagrangian is supposed to represent photon-photon interactions in QED at low momentum and zero electron chemical potential (that is, in practice, at zero electron density).

We now show that this is not the case in QED, and that the bulk viscosity is actually calculably finite (although, at
CMB temperatures, it is tiny). In one-loop QED, the dispersion relation has been calculated \cite{20,22,23},

\[ E_p^2 = p^2 + m^2, \]

with a photon thermal mass

\[ m^2 = \frac{8\alpha}{\sqrt{2\pi}} m_e^3/2 T^{1/2} e^{-m_e/T}. \]

We immediately see the difficulty with the effective theory. This thermal mass contains the factor \( e^{-m_e/T} \) that admits no Taylor expansion around \( T = 0 \). Therefore, the usual counting \( p \sim T \) does not lead to a polynomial behavior upon expanding. Since in QED we do know the microscopic theory, we can proceed with the mass in Eq. (23) and repeat the reasoning of Sec. III A.

We use the condition of fit (21) to introduce an extra term in the bulk viscosity (16) so that the integrand coincides with the left-hand side of the Boltzmann equation (17). This is

\[ \xi = \frac{2}{T} \int \frac{d^3p}{(2\pi)^3} \frac{1}{E_p} n_p(1 + n_p) A(p) \times \left[ E_p^2 \left( \frac{1}{3} - u^2 \right) - \frac{m^2}{3} + \frac{u^2 T}{2} \frac{\partial m^2}{\partial T} \right], \]

where Eq. (23) has been used.

To reduce this equation further we need to calculate the speed of sound \( v_s \). We thus turn to the thermodynamics of a noninteracting (ideal) quasiparticle gas \cite{24} following the same methodology of QCD exposed at length in Refs. [8,25,26].

The entropy density of an ideal photon gas with dispersion relation \( E_p^2 = p^2 + m^2 \) is, to order \( \alpha \),

\[ s(T) = \frac{4\pi^2}{45} T^3 - \frac{4\alpha}{3\sqrt{\pi}} m_e^3/2 T^{3/2} e^{-m_e/T}. \]

From this expression we can compute the speed of sound,

\[ v_s^2 = \frac{s(T)}{T \frac{ds(T)}{dT}} = \frac{1}{3} - \frac{5\sqrt{2} m_e^2}{24 \pi^2} \frac{m^2}{T^2} \]

\[ = \frac{1}{3} - \frac{5\alpha}{3\sqrt{\pi}} \frac{m_e^3/2}{T^{5/2}} e^{-m_e/T}. \]

At \( O(\alpha) \), the source function in the bulk viscosity—the last bracket in Eq. (25)—reads

\[ \left[ E_p^2 \left( \frac{1}{3} - u^2 \right) - \frac{m^2}{3} + \frac{u^2 T}{2} \frac{\partial m^2}{\partial T} \right] \]

\[ = m^2 \left( \frac{5\sqrt{2} m_e}{24 \pi^2} \frac{E_p}{T^2} - \frac{1}{4} \right), \]

which is proportional to the square of the photon thermal mass, and therefore is nonzero for one-loop QED at finite temperature.

If only the parametric dependence is desired, straightforward algebra starting in Eq. (25), combined with Eqs. (17) and (28), leads to

\[ \xi \sim \frac{1}{\alpha^4} \frac{m_e^4}{T^{10}} e^{-m_e/T}, \]

which can also be obtained from the simple estimate

\[ \xi \sim \frac{T}{\bar{\sigma}} \left( \frac{1}{3} - u_s^2 \right)^2 \]

and which immediately yields Eq. (35) in Sec. IV.

Returning now to infinitely slow relaxation, Furry’s theorem forbids a finite five-point function. The six-point function is suppressed with respect to Eq. (7) by \( e^2/m^2 \). This is because of the two additional photons attached to it, and the consequent two additional fermion propagators. Squaring this suppression factor, we immediately obtain the estimate in Eq. (33) below.

\[ \eta = \eta_0 \frac{1}{\alpha^4} \frac{m_e^8}{T^5} \approx \eta_0 \frac{1.4 \times 10^{19} \text{ eV}^3}{T^5 (\text{K})}, \]

which is typical of such weakly coupled boson gases \cite{27}. The estimate follows trivially from the extensive discussion above in Sec. II where we have given enough detail to calculate the numeric coefficient if it was ever necessary. Normalized to the entropy density in Eq. (26), the coefficient reads at leading order

\[ \frac{\eta}{s} \approx \frac{1}{\alpha^4} \frac{m_e^8}{T^8}. \]

We show this coefficient as a function of temperature (using \( \eta_0 = 1 \)) in Fig. 2.

The bulk viscosity, on the contrary, is tiny in infinite matter at temperatures much below the electron mass,

\[ \zeta = \zeta_0 \frac{1}{\alpha^4} \frac{m_e^{17}}{T^{14}} e^{-2m_e/T}, \]

in the bona fide hydrodynamic limit for arbitrarily long wave modes out of equilibrium, a regime where inelastic \( \gamma \gamma \rightarrow 4 \gamma \) processes dominate the relaxation. \( \zeta_0 \) is the constant coefficient for the bulk viscosity dominated by inelastic processes. Normalized to the entropy density, the bulk viscosity reads

\[ \frac{\zeta}{s} \approx \frac{1}{\alpha^4} \frac{m_e^{17}}{T^{17}} e^{-2m_e/T}. \]
FIG. 2. Shear viscosity over entropy density (solid line) and the bulk viscosity over entropy density due to inelastic (dashed) and elastic (dotted) processes. The numerical coefficients \( \eta_0 = \zeta_0 = \zeta_0' \) are set to one.

If one considers times that are not so large, so that the modes relaxing are purely kinetic, the bulk viscosity is then dominated in effect by elastic photon-photon scattering,

\[
\zeta = \eta_0 \frac{1}{\alpha^2} \frac{1}{T^{10}} e^{-2m_c/T},
\]

with \( \eta_0' \) the numerical coefficient independent of \( m_c \) and \( T \). Normalized to the entropy density,

\[
\zeta_s = \frac{1}{\alpha^2} \frac{1}{T^{13}} e^{-2m_c/T}.
\]

Both bulk viscosities are tiny when compared to the shear viscosity. For example, by evaluating Eq. (35) we find \( \zeta \sim (T/10 \text{K}) \cdot e^{2127.3 - m_c/T} \) \( \text{cm}^3 \). Since \( m_c / T \approx 2.2 \times 10^9 \) for the CMB temperature today, the exponent is huge and negative, so that the bulk viscosity is incredibly small (it of course gets bigger as we proceed back in time towards recombination). This smallness is because the photon gas is very nearly a relativistic, conformally invariant gas, whose bulk viscosity is well known to be zero [28].

When the temperature becomes comparable, or if an electron chemical potential is introduced (finite electron density), then the population of electrons dominates the physics and the situation is very different, which we hope to address in a future publication. Compton (eventually Thomson) scattering slows down diffusive momentum transfer at tree level, instead of the one loop in QED that is necessary if only photons are present.

The huge ratio \( \eta / \zeta \) is typical of very weakly coupled systems, and quite the opposite of what is being seen at RHIC, where, near the phase transition, the bulk viscosity could actually play an important role. In the photon gas, the bulk viscosity turns out to be so small because, in spite of the weak (polynomial) coupling, the breaking of conformal invariance is exponentially suppressed. To see this we have resorted to the microscopic QED theory, since the effective theory of Euler-Heisenberg for the photons alone seems to miss the photon thermal mass due to its nonanalytic behavior.

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APPENDIX: PHOTON FOUR-POINT FUNCTION

We provide the corrected four-point function at tree level from the Euler-Heisenberg Lagrangian (2) (to be compared with the result of Ref. [15]):

\[
iM^{\alpha\beta\mu\nu}(p_1, p_2, p_3, p) = \frac{1}{\alpha^2} \frac{1}{T^{10}} e^{-2m_c/T} \epsilon_{\alpha} \epsilon_{\beta} \epsilon_{\mu} \epsilon_{\nu} \frac{1}{\alpha^2} \frac{1}{T^{10}} e^{-2m_c/T}.
\]

\[
\eta = \frac{1}{\alpha^2} \frac{1}{T^{10}} e^{-2m_c/T},
\]

\[
\zeta_s = \frac{1}{\alpha^2} \frac{1}{T^{13}} e^{-2m_c/T}.
\]


