CHARACTERIZATION OF A BANACH-FINSLER MANIFOLD IN TERMS OF THE ALGEBRAS OF SMOOTH FUNCTIONS

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ABSTRACT. In this note we give sufficient conditions to ensure that the weak Finsler structure of a complete \( C^k \) Finsler manifold \( M \) is determined by the normed algebra \( C^b_k(M) \) of all real-valued, bounded and \( C^k \) smooth functions with bounded derivative defined on \( M \). As a consequence, we obtain: (i) the Finsler structure of a finite-dimensional and complete \( C^k \) Finsler manifold \( M \) is determined by the algebra \( C^b_k(M) \); (ii) the weak Finsler structure of a separable and complete \( C^k \) Finsler manifold \( M \) modeled on a Banach space with a Lipschitz and \( C^k \) smooth bump function is determined by the algebra \( C^b_k(M) \); (iii) the weak Finsler structure of a \( C^k \) uniformly bumpable and complete \( C^k \) Finsler manifold \( M \) modeled on a Weakly Compactly Generated (WCG) Banach space with an (equivalent) \( C^k \) smooth norm is determined by the algebra \( C^b_k(M) \); and (iv) the isometric structure of a WCG Banach space \( X \) with a \( C^1 \) smooth bump function is determined by the algebra \( C^b_1(X) \).

1. Introduction and Preliminaries

In this note, we are interested in characterizing the Finsler structure of a Finsler manifold \( M \) in terms of the space of real-valued, bounded and \( C^k \) smooth functions with bounded derivative defined on \( M \). The problem of the interrelation of the topological, metric and smooth structure of a space \( X \) and the algebraic and topological structure of the space \( C(X) \) (the set of real-valued continuous functions defined on \( X \)) has been largely studied. These results are usually referred as Banach-Stone type theorems. Recall the celebrated Banach-Stone theorem, asserting that the compact spaces \( K \) and \( L \) are homeomorphic if and only if the Banach spaces \( C(K) \) and \( C(L) \) endowed with the sup-norm are isometric. For more information on Banach-Stone type theorems see the survey [10] and references therein.

The Myers-Nakai theorem states that the structure of a complete Riemannian manifold \( M \) is characterized in terms of the Banach algebra \( C^1_b(M) \) of all real-valued, bounded and \( C^1 \) smooth functions with bounded derivative defined on \( M \) endowed with the sup-norm of the function and its derivative. More specifically, two complete Riemannian manifolds \( M \) and \( N \) are equivalent as Riemannian manifolds, i.e. there is a \( C^1 \) diffeomorphism \( h : M \to N \) such that

\[
\langle dh(x)(v), dh(x)(w) \rangle_{h(x)} = \langle v, w \rangle_x
\]
for every $x \in M$ and $v, w \in T_xM$ if and only if the Banach algebras $C^1_b(M)$ and $C^1_b(N)$ are isometric. This result was first proved by S. B. Myers [22] for a compact and Riemannian manifold and by M. Nakai [23] for a finite-dimensional Riemannian manifold. Very recently, I. Garrido, J.A. Jaramillo and Y.C. Rangel [12] gave an extension of the Myers-Nakai theorem for every infinite-dimensional, complete Riemannian manifold. A similar result for the so-called finite-dimensional Riemannian-Finsler manifolds is given in [14] (see also [26]).

Our aim in this work is to extend the Myers-Nakai theorem to the context of Finsler manifolds. On the one hand, we obtain the Myers-Nakai theorem for (i) finite-dimensional and complete Finsler manifolds, and (ii) WCG Banach spaces modeled on a Banach space with a Lipschitz and $C^k$ smooth bump function, and (ii) $C^k$ uniformly bumpy and complete Finsler manifolds modeled on WCG Banach spaces with an equivalent $C^k$ smooth norm. In the proof of these results we will use the ideas of the Riemannian case [12].

The notation we use is standard. The norm in a Banach space $X$ is denoted by $\| \cdot \|$. The dual space of $X$ is denoted by $X^*$ and its dual norm by $\| \cdot \|^*$. The open ball with center $x \in X$ and radius $r > 0$ is denoted by $B(x, r)$. A $C^k$ smooth bump function $b : X \to \mathbb{R}$ is a $C^k$ smooth function on $X$ with bounded, non-empty support, where $\text{supp}(b) = \{ x \in X : b(x) \neq 0 \}$. If $M$ is a Banach manifold, we denote by $T_xM$ the tangent space of $M$ at $x$. Recall that the tangent bundle of $M$ is given by $TM = \{ (x, v) : x \in M \text{ and } v \in T_xM \}$. We refer to [6], [8], [19] and [7] for additional definitions. We will say that the norms $\| \cdot \|_1$ and $\| \cdot \|_2$ defined on a Banach space $X$ are $K$-equivalent ($K \geq 1$) whether $\frac{1}{K} \| v \|_1 \leq \| v \|_2 \leq K \| v \|_1$, for every $v \in X$.

Let us begin by recalling the definition of a $C^k$ Finsler manifold in the sense of Palais as well as some basic properties (for more information about these manifolds see [25], [7], [27], [24], [13] and [18]).

**Definition 1.1.** Let $M$ be a (paracompact) $C^k$ Banach manifold modeled on a Banach space $(X, \| \cdot \|)$, where $k \in \mathbb{N} \cup \{ \infty \}$. Let us consider the tangent bundle $TM$ of $M$ and a continuous map $\| \cdot \|_M : TM \to [0, \infty)$. We say that $(M, \| \cdot \|_M)$ is a $C^k$ Finsler manifold in the sense of Palais if $\| \cdot \|_M$ satisfies the following conditions:

1. **(P1)** For every $x \in M$, the map $\| \cdot \|_x := \| \cdot \|_M|_{T_xM} : T_xM \to [0, \infty)$ is a norm on the tangent space $T_xM$ such that for every chart $\varphi : U \to X$ with $x \in U$, the norm $v \in X \mapsto \| d\varphi^{-1}(\varphi(x))(v) \|_x$ is equivalent to $\| \cdot \|$ on $X$.

2. **(P2)** For every $x_0 \in M$, every $\varepsilon > 0$ and every chart $\varphi : U \to X$ with $x_0 \in U$, there is an open neighborhood $W$ of $x_0$ such that if $x \in W$ and $v \in X$, then

\[
(1.1) \quad \frac{1}{1 + \varepsilon} \| d\varphi^{-1}(\varphi(x_0))(v) \|_{x_0} \leq \| d\varphi^{-1}(\varphi(x))(v) \|_x \leq (1 + \varepsilon) \| d\varphi^{-1}(\varphi(x_0))(v) \|_{x_0}.
\]
In terms of equivalence of norms, the above inequalities yield the fact that the norms \( |d\varphi^{-1}(\varphi(x))(\cdot)|_x \) and \( |d\varphi^{-1}(\varphi(x_0))(\cdot)|_{x_0} \) are \((1 + \varepsilon)\)-equivalent.

Let us recall that Banach spaces and Riemannian manifolds are \( C^\infty \) Finsler manifolds in the sense of Palais [25].

Let \( M \) be a Finsler manifold, we denote by \( T_x M^* \) the dual space of the tangent space \( T_x M \). Let \( f : M \to \mathbb{R} \) be a differentiable function at \( p \in M \). The norm of \( df(p) \in T_p M^* \) is given by

\[
|df(p)|_p = \sup \{|df(p)(v)| : v \in T_p M, |v|_p \leq 1\}.
\]

Let us consider a differentiable function \( f : M \to N \) between Finsler manifolds \( M \) and \( N \). The norm of the derivative at the point \( p \in M \) is defined as

\[
|df(p)|_p = \sup \{|df(p)(v)| : v \in T_p M, |v|_p \leq 1\} = \sup \{ |\xi(df(p)(v))| : \xi \in T_{f(p)} N^*, v \in T_p M \text{ and } |v|_p = 1 = |\xi|_{T_{f(p)}}^* \},
\]

where \( |\cdot|_{T_{f(p)}}^* \) is the dual norm of \( |\cdot|_{T_{f(p)}} \). Recall that if \((M,|\cdot|,|\cdot|_M)\) is a Finsler manifold, the length of a piecewise \( C^1 \) smooth path \( c : [a, b] \to M \) is defined as \( \ell(c) = \int_a^b |c'(t)|_{c(t)} dt \). Besides, if \( M \) is connected, then it is connected by piecewise \( C^1 \) smooth paths, and the associated Finsler metric \( d_M \) on \( M \) is defined as

\[
d_M(p, q) = \inf \{ \ell(c) : c \text{ is a piecewise } C^1 \text{ smooth path connecting } p \text{ and } q \}.
\]

It was shown in [25] that the Finsler metric is consistent with the topology given in \( M \). The open ball of center \( p \in M \) and radius \( r > 0 \) is denoted by \( B_M(p, r) := \{ q \in M : d_M(p, q) < r \} \). The Lipschitz constant \( \text{Lip}(f) \) of a Lipschitz function \( f : M \to N \), where \( M \) and \( N \) are Finsler manifolds, is defined as \( \text{Lip}(f) = \sup \{ \frac{d_N(f(x), f(y))}{d_M(x, y)} : x, y \in M, x \neq y \} \). We shall only consider connected manifolds. Let us recall the following “mean value inequality” for Finsler manifolds [1, 18].

**Lemma 1.2.** Let \( M \) and \( N \) be \( C^1 \) Finsler manifolds (in the sense of Palais) and \( f : M \to N \) a \( C^1 \) smooth function. Then, \( f \) is Lipschitz if and only if \( |df|_\infty := \sup \{||df(x)||_x : x \in M\} < \infty \). Furthermore, \( \text{Lip}(f) = |df|_\infty \).

We will also need the following result related to the \((1 + \varepsilon)\)-bi-Lipschitz local behavior of the charts of a \( C^1 \) Finsler manifold in the sense of Palais [18, Lemma 2.4].

**Lemma 1.3.** Let us consider a \( C^1 \) Finsler manifold \( M \) (in the sense of Palais). Then, for every \( x_0 \in M \) and every chart \((U, \varphi)\) with \( x_0 \in U \) satisfying inequality (1.1), there exists an open neighborhood \( V \subset U \) of \( x_0 \) satisfying

\[
\frac{1}{1 + \varepsilon} d_M(p, q) \leq ||| \varphi(p) - \varphi(q) ||| \leq (1 + \varepsilon) d_M(p, q), \text{ for every } p, q \in V,
\]

where \( ||| \cdot ||| \) is the (equivalent) norm \(|d\varphi^{-1}(\varphi(x_0))(\cdot)|_{x_0} \) defined on \( X \).

Now, let us recall the concept of uniformly bumpable manifold, introduced by D. Azagra, J. Ferrera and F. López-Mesas [1] for Riemannian manifolds. A natural extension to Finsler manifolds is defined in the same way [18].

**Definition 1.4.** A \( C^k \) Finsler manifold in the sense of Palais \( M \) is \( C^k \) uniformly bumpable whenever there are \( R > 1 \) and \( r > 0 \) such that for every \( p \in M \) and \( \delta \in (0, r) \) there exists a \( C^k \) smooth function \( b : M \to [0, 1] \) such that:
(1) \( b(p) = 1 \),
(2) \( b(q) = 0 \) whenever \( d_M(p, q) \geq \delta \),
(3) \( \sup_{q \in M} ||db(q)|| \leq R/\delta \).

Note that this is not a restrictive definition: D. Azagra, J. Ferrera, F. Lopéz-Mesas and Y. Rangel [3] proved that every separable Riemannian manifold is \( C^\infty \) uniformly bumpable. This result was generalized in [18], where it was proved that every \( C^1 \) Finsler manifold (in the sense of Palais) modeled on a certain class of Banach spaces (such as Hilbert spaces, Banach spaces with separable dual, closed subspaces of \( c_0(\Gamma) \) for every set \( \Gamma \neq \emptyset \)) is \( C^1 \) uniformly bumpable. In particular, every Riemannian manifold (either separable or non-separable) is \( C^1 \) uniformly bumpable.

It is straightforward to verify that if a \( C^k \) Finsler manifold \( M \) is modeled on a Banach space \( X \) and \( M \) is \( C^k \) uniformly bumpable, then \( X \) admits a Lipschitz \( C^k \) smooth bump function. Besides, a separable \( C^k \) Finsler manifold \( M \) is modeled on a Banach space with a Lipschitz, \( C^k \) smooth bump function if and only if \( M \) is \( C^k \) uniformly bumpable [18]. Nevertheless, we do not know whether this equivalence holds in the non-separable case.

From now on, we shall refer to \( C^k \) Finsler manifolds in the sense of Palais as \( C^k \) Finsler manifolds, and \( k \in \mathbb{N} \cup \{\infty\} \). We shall use the standard notation of \( C^k(U, Y) \) for the set of all \( k \)-times continuously differentiable functions defined on an open subset \( U \) of a Banach space (Finsler manifold) taking values into a Banach space (Finsler manifold) \( Y \). We shall write \( C^k(U) \) whenever \( Y = \mathbb{R} \).

Now, let us recall the concept of weakly \( C^k \) smooth function.

**Definition 1.5.** Let \( X \) and \( Y \) be Banach spaces and consider a function \( f : U \to Y \), where \( U \) is an open subset of \( X \). The function \( f \) is said to be a **weakly \( C^k \) smooth** at the point \( x_0 \) whenever there is an open neighborhood \( U_{x_0} \) of \( x_0 \) such that \( y^* \circ f \) is \( C^k \) smooth at \( U_{x_0} \), for every \( y^* \in Y^* \). The function \( f \) is said to be a **weakly \( C^k \) smooth** on \( U \) whenever \( f \) is weakly \( C^k \) smooth at every point \( x \in U \).

On the one hand, J. M. Gutiérrez and J. L. G. Llavona [15] proved that if \( f : U \to Y \) is weakly \( C^k \) smooth on \( U \), then \( g \circ f \in C^k(U) \) for all \( g \in C^k(Y) \). They also proved that if \( f : U \to Y \) is weakly \( C^k \) smooth on \( U \), then \( f \in C^{k-1}(U) \). For \( k = 1 \), the above yields that every weakly \( C^1 \) smooth function on \( U \) is continuous on \( U \). Also, for \( k = \infty \), every weakly \( C^\infty \) smooth function on \( U \) is \( C^\infty \) smooth on \( U \). M. Bachir and G. Lancien [4] proved that, if the Banach space \( Y \) has the Schur property, then the concept of weakly \( C^k \) smoothness coincides with the concept of \( C^k \) smoothness.

On the other hand, there are examples of weakly \( C^1 \) smooth functions that are not \( C^1 \) smooth (see [15] and [4]).

**Definition 1.6.** Let \( M \) and \( N \) be \( C^k \) Finsler manifolds and \( U \subset M \), \( O \subset N \) open subsets of \( M \) and \( N \), respectively. A function \( f : U \to N \) is said to be **weakly \( C^k \) smooth** at the point \( x_0 \) if there exist charts \((W, \varphi)\) of \( M \) at \( x_0 \) and \((V, \psi)\) of \( N \) at \( f(x_0) \) such that \( \psi \circ f \circ \varphi^{-1} \) is weakly \( C^k \) smooth at \( \varphi(W) \). We say that \( f : U \to N \) is **weakly \( C^k \) smooth** in \( U \) if \( f \) is weakly \( C^k \) smooth at every point \( x \in U \). We say that a bijection \( f : U \to O \) is a **weakly \( C^k \) diffeomorphism** if \( f \) and \( f^{-1} \) are weakly \( C^k \) smooth on \( U \) and \( O \), respectively. Notice that these definitions do not depend on the chosen charts.
Let us note that there are homeomorphisms which are weakly $C^1$ smooth but not differentiable. Indeed, we follow [15, Example 3.9] and define $g : \mathbb{R} \to c_0(\mathbb{N})$ and $h : c_0(\mathbb{N}) \to c_0(\mathbb{N})$ by $g(t) = (0, \frac{1}{t} \sin(2t), \ldots, \frac{1}{n} \sin(nt), \ldots)$ and $h(x) = x + g(x_1)$ for every $t \in \mathbb{R}$ and $x = (x_1, \ldots, x_n, \ldots) \in c_0$. The function $h$ is an homeomorphism, $h^{-1}(y) = y - g(y_1)$ for every $y \in c_0$, and $h$ is weakly $C^1$ smooth on $c_0(\mathbb{N})$. Notice that if $h$ were differentiable at a point $x \in c_0$ with $x_1 = 0$, then

$$h'(x)(1,0,0,\ldots) = (1,1,1,\ldots) \in \ell_\infty \setminus c_0,$$

which is a contradiction.

Now, let us consider different definitions of isometries between $C^k$ Finsler manifolds.

**Definition 1.7.** Let $(M, || \cdot ||_M)$ and $(N, || \cdot ||_N)$ be $C^k$ Finsler manifolds and a bijection $h : M \to N$.

1. **(MI)** We say that $h$ is a **metric isometry** for the Finsler metrics, if

   $$d_N(h(x), h(y)) = d_M(x, y), \quad \text{for every } x, y \in M.$$

2. **(FI)** We say that $h$ is a $C^k$ **Finsler isometry** if it is a $C^k$ diffeomorphism satisfying

   $$||dh(x)(v)||_{h(x)} = ||(h(x), dh(x)(v))||_N = ||(x, v)||_M = ||v||_x,$$

   for every $x \in M$ and $v \in T_xM$. We say that the Finsler manifolds $M$ and $N$ are $C^k$ **equivalent as Finsler manifolds** if there is a $C^k$ Finsler isometry between $M$ and $N$.

3. **(w-FI)** We say that $h$ is a weak $C^k$ **Finsler isometry** if it is a weakly $C^k$ diffeomorphism and a metric isometry for the Finsler metrics. We say that the Finsler manifolds $M$ and $N$ are **weakly $C^k$ equivalent as Finsler manifolds** if there is a weak $C^k$ Finsler isometry between $M$ and $N$.

**Proposition 1.8.** Let $M$ and $N$ be $C^k$ Finsler manifolds. Let us assume that there is a $C^k$ diffeomorphism and metric isometry (for the Finsler metrics) $h : M \to N$. Then $h$ is a $C^k$ Finsler isometry.

**Proof.** Let us fix $x \in M$ and $y = h(x) \in N$. For every $\varepsilon > 0$, there are $r > 0$ and charts $\varphi : B_M(x, r) \subset M \to X$ and $\psi : B_N(y, r) \subset N \to Y$ satisfying inequalities (1.1) and (1.2). Since $h : M \to N$ is a metric isometry, $h$ is a bijection from $B_M(x, r)$ onto $B_N(y, r)$.

Let us consider the equivalent norms on $X$ and $Y$ defined as $||\cdot||_{x} := ||d\varphi^{-1}(\varphi(x))(\cdot)||_{x}$ and $||\cdot||_{y} := ||d\psi^{-1}(\psi(y))(\cdot)||_{y}$, respectively.

Since $h$ is a metric isometry, we obtain from Lemma 1.3, for $p, q$ in an open neighborhood of $\varphi(x)$,

$$||\psi \circ h \circ \varphi^{-1}(p) - \psi \circ h \circ \varphi^{-1}(q)||_y \leq (1 + \varepsilon)d_N(h \circ \varphi^{-1}(p), h \circ \varphi^{-1}(q)) = (1 + \varepsilon)d_M(\varphi^{-1}(p), \varphi^{-1}(q)) \leq (1 + \varepsilon)^2||p - q||_x.$$
Thus, \( \sup\{||d(\psi \circ h \circ \varphi^{-1})(\varphi(x))(w)||_y : ||w||_x \leq 1\} \leq (1 + \varepsilon)^2 \). Now, for every \( v \in T_x M \) with \( v \neq 0 \), let us write \( w = d\varphi(x)(v) \in X \). We have
\[
||dh(x)(v)||_y = ||d\psi^{-1}(\psi(y))d\psi(y)dh(x)(v)||_y = ||d(\psi \circ h)(x)||_y = \\
= ||d(\psi \circ h)(x)df^{-1}(\varphi(x))(w)||_y = ||d(\psi \circ h \circ \varphi^{-1})(\varphi(x))(w)||_y \leq \\
\leq (1 + \varepsilon)^2||w||_x = (1 + \varepsilon)^2||v||_x.
\]

Since this inequality holds for every \( \varepsilon > 0 \) and the same argument works for \( h^{-1} \), we conclude that \( ||dh(x)(v)||_y = ||v||_x \) for all \( v \in T_x M \). Thus, \( h \) is a \( C^k \) Finsler isometry.

Let us now turn our attention to the Banach algebra \( C^1_b(M) \), the algebra of all real-valued, \( C^1 \) smooth and bounded functions with bounded derivative defined on a \( C^1 \) Finsler manifold \( M \), i.e.
\[
C^1_b(M) = \{ f : M \to \mathbb{R} : f \in C^1(M), ||f||_\infty < \infty \text{ and } ||df||_\infty < \infty \},
\]
where \( ||f||_\infty := \sup\{|f(x)| : x \in M\} \) and \( ||df||_\infty := \sup\{|df(x)|_x : x \in M\} \).
The usual norm considered on \( C^1_b(M) \) is \( ||f||_{C^1} = \max\{||f||_\infty, ||df||_\infty\} \) for every \( f \in C^1_b(M) \) and \( (C^1_b(M), ||\cdot||_{C^1}) \) is a Banach space. Let us notice that, by Lemma 1.2, we have \( ||df||_{\infty} = \text{Lip}(f) \). Recall that \( (C^1_b(M), 2||\cdot||_{C^1}) \) is a Banach algebra.

For \( 2 \leq k \leq \infty \) and a \( C^k \) Finsler manifold \( M \), let us consider the algebra \( C^k_b(M) \) of all real-valued, \( C^k \) smooth and bounded functions that have bounded first derivative, i.e.
\[
C^k_b(M) = \{ f : M \to \mathbb{R} : f \in C^k(M), ||f||_\infty < \infty \text{ and } ||df||_\infty < \infty \} = C^k(M) \cap C^1_b(M).
\]
with the norm \( ||\cdot||_{C^1} \). Thus, \( C^k_b(M) \) is a subalgebra of \( C^1_b(M) \). Nevertheless, it is not a Banach algebra.

A function \( \varphi : C^k_b(M) \to \mathbb{R} \) is said to be an algebra homomorphism whether for all \( f, g \in C^k_b(M) \) and \( \lambda, \eta \in \mathbb{R} \),
(i) \( \varphi(\lambda f + \eta g) = \lambda \varphi(f) + \eta \varphi(g) \), and
(ii) \( \varphi(f \cdot g) = \varphi(f) \varphi(g) \).

Let us denote by \( H(C^k_b(M)) \) the set of all nonzero algebra homomorphisms, i.e.
\[
H(C^k_b(M)) = \{ \varphi : C^k_b(M) \to \mathbb{R} : \varphi \text{ is an algebra homomorphism and } \varphi(1) = 1 \}.
\]
Let us list some of the basic properties of the algebra \( C^k_b(M) \) and the algebra homomorphisms \( H(C^k_b(M)) \). They can be checked as in the Riemannian case (see [11], [12] and [17]).

(a) If \( \varphi \in H(C^k_b(M)) \), then \( \varphi \neq 0 \) if and only if \( \varphi(1) = 1 \).
(b) If \( \varphi \in H(C^k_b(M)) \), then \( \varphi \) is positive, i.e. \( \varphi(f) \geq 0 \) for every \( f \geq 0 \).
(c) If the \( C^k \) Finsler manifold \( M \) is modeled on a Banach space that admits a Lipschitz and \( C^k \) smooth bump function, then \( C^k_b(M) \) is a unital algebra that separates points and closed sets of \( M \). Let us briefly give the proof for completeness. Let us take \( x \in M \), and \( C \subset M \) a closed subset of \( M \) with \( x \notin C \). Let us take \( r > 0 \) small enough so that \( C \cap B_M(x, r) = \emptyset \) and a chart \( \varphi : B_M(x, r) \to X \) satisfying inequality (1.1). Let us take \( s > 0 \) small enough so that \( \varphi(x) \in B(\varphi(x), s) \subset \varphi(B(x, r/2)) \subset X \) and a Lipschitz and
Let $M$ be a complete $C^k$ Finsler manifold that is $C^k$ uniformly bumpable. Then, $\varphi \in H(C^k_b(M))$ has a countable neighborhood basis in $H(C^k_b(M))$ if and only if $\varphi \in M$.  

2. A Myers-Nakai Theorem

Our main result is the following Banach-Stone type theorem for a certain class of Finsler manifolds. It states that the algebra structure of $C^k_b(M)$ determines the $C^k$ Finsler manifold. Recall that two normed algebras $(A,|| \cdot ||_A)$ and $(B,|| \cdot ||_B)$ are equivalent as normed algebras whenever there exists an algebra isomorphism $T : A \to B$ satisfying $||T(a)||_B = ||a||_A$ for every $a \in A$. Let us begin by defining the class of Banach spaces where the Finsler manifolds shall be modeled.

Definition 2.1. A Banach space $(X,|| \cdot ||)$ is said to be $k$-admissible if for every equivalent norm $| \cdot |$ and $\varepsilon > 0$, there are an open subset $B \supset \{x \in X : |x| \leq 1\}$ of $X$ and a $C^k$ smooth function $g : B \to \mathbb{R}$ such that

(i) $|g(x) - |x|| < \varepsilon$ for $x \in B$, and

(ii) $\text{Lip}(g) \leq (1 + \varepsilon)$ for the norm $| \cdot |$.

It is easy to prove the following lemma.

Lemma 2.2. Let $X$ be a Banach space with one of the following properties:

(A.1) Density of the set of equivalent $C^k$ smooth norms: every equivalent norm on $X$ can be approximated in the Hausdorff metric by equivalent $C^k$ smooth norms [6].

(A.2) $C^k$-fine approximation property ($k \geq 2$) and density of the set of equivalent $C^1$ smooth norms: For every $C^1$ smooth function $f : X \to \mathbb{R}$ and every

$C^k$ smooth bump function $b : X \to \mathbb{R}$ with $b(\varphi(x)) = 1$ and $b(z) = 0$ for every $z \notin B(\varphi(x), s)$. Let us define $h : M \to \mathbb{R}$ as $h(p) = b(\varphi(p))$ for every $p \in B_M(x, r)$ and $h(p) = 0$ otherwise. Then $h \in C^k_b(M)$, $h(x) = 1$ and $h(c) = 0$ for every $c \in C$.

(d) The space $H(C^k_b(M))$ is closed as a topological subspace of $\mathbb{R}^{C^k_b(M)}$ with the product topology. Moreover, since every function in $C^k_b(M)$ is bounded, it can be checked that $H(C^k_b(M))$ is compact in $\mathbb{R}^{C^k_b(M)}$.

(e) If $C^k_b(M)$ separates points and closed subsets, then $M$ can be embedded as a topological subspace of $H(C^k_b(M))$ by identifying every $x \in M$ with the point evaluation homomorphism $\delta_x$ given by $\delta_x(f) = f(x)$ for every $f \in C^k_b(M)$. Also, it can be checked that the subset $\delta(M) = \{\delta_x : x \in M\}$ is dense in $H(C^k_b(M))$. Therefore, it follows that $H(C^k_b(M))$ is a compactification of $M$.

(f) Every $f \in C^k_b(M)$ admits a continuous extension $\hat{f}$ to $H(C^k_b(M))$, where $\hat{f}(\varphi) = \varphi(f)$ for every $\varphi \in H(C^k_b(M))$. Notice that this extension $\hat{f}$ coincides in $H(C^k_b(M))$ with the projection $\pi_f : \mathbb{R}^{C^k_b(M)} \to \mathbb{R}$, given by $\pi_f(\varphi) = \varphi(f)$, i.e. $\pi_f|_{H(C^k_b(M))} = \hat{f}$. In the following, we shall identify $M$ with $\delta(M)$ in $H(C^k_b(M))$.

The next proposition can be proved in a similar way to the Riemannian case [12].

Proposition 1.9. Let $M$ be a complete $C^k$ Finsler manifold that is $C^k$ uniformly bumpable. Then, $\varphi \in H(C^k_b(M))$ has a countable neighborhood basis in $H(C^k_b(M))$ if and only if $\varphi \in M$. 
\[ \varepsilon > 0, \text{there is a } C^k \text{ smooth function } g : X \to \mathbb{R} \text{ satisfying } |f(x) - g(x)| < \varepsilon \]
\[ \text{and } ||f'(x) - g'(x)|| < \varepsilon \text{ for all } x \in X \text{ (see [16], [2] and [20]); also, every equivalent norm defined on } X \text{ can be approximated in the Hausdorff metric by equivalent } C^1 \text{ smooth norms (see [6, Theorem II 4.1]).} \]

Then \( X \) is \( k \)-admissible.

Banach spaces satisfying condition (A.2) are, for instance, separable Banach spaces with a Lipschitz \( C^k \) smooth bump function. Banach spaces satisfying condition (A.1) for \( k = 1 \) are, for instance, Weakly Compactly Generated (WCG) Banach spaces with a \( C^1 \) smooth bump function.

**Theorem 2.3.** Let \( M \) and \( N \) be complete \( C^k \) Finsler manifolds that are \( C^k \) uniformly bumpy and are modeled on \( k \)-admissible Banach spaces. Then \( M \) and \( N \) are weakly \( C^k \) equivalent as Finsler manifolds if and only if \( C^k_b(M) \) and \( C^k_b(N) \) are equivalent as normed algebras. Moreover, every normed algebra isomorphism \( T : C^k_b(N) \to C^k_b(M) \) is of the form \( T(f) = f \circ h \) where \( h : M \to N \) is a weak \( C^k \) Finsler isometry. In particular, \( h \) is a \( C^{k-1} \) Finsler isometry whenever \( k \geq 2 \).

In order to prove Theorem 2.3, we shall follow the ideas of the Riemannian case [12]. Let us divide the proof into several propositions.

**Proposition 2.4.** Let \( M \) and \( N \) be \( C^k \) Finsler manifolds such that \( N \) is modeled on a \( k \)-admissible Banach space \( Y \). Let \( h : M \to N \) be a map such that \( T : C^k_b(N) \to C^k_b(M) \) given by \( T(f) = f \circ h \) is continuous. Then \( h \) is \( ||T|| \)-Lipschitz for the Finsler metrics.

**Proof.** For every \( y \in N \), let us take a chart \( \psi_y : V_y \to Y \) with \( \psi_y(y) = 0 \). Let us consider the equivalent norm on \( Y \), \( ||| \cdot |||_y := ||d\psi_y^{-1}(0)(\cdot)||_y \) and fix \( \varepsilon > 0 \). Let us define the ball \( B_{||| \cdot |||_y}(z,t) := \{w \in Y : |||w - z|||_y < t\} \).

**Fact.** For every \( r > 0 \) such that \( B_{||| \cdot |||_y}(0,r) \subset \psi_y(V_y) \) and every \( \bar{\varepsilon} > 0 \), there exists a \( C^k \) smooth and Lipschitz function \( f_y : Y \to \mathbb{R} \) such that

(1) \( f_y(0) = r \),
(2) \( ||f_y||_\infty := \sup\{||f_y(z)|| : z \in Y\} = r \),
(3) \( \text{Lip}(f_y) \leq (1 + \varepsilon)^2 \) for the norm \( ||| \cdot |||_y \),
(4) \( f_y(z) = 0 \) for every \( z \in Y \) with \( |||z|||_y \geq r \), and
(5) \( ||z||_y \leq r - f_y(z) + \bar{\varepsilon} \) for every \( ||z||_y \leq r \).

Let us prove the Fact. First of all, let us take \( r > 0, \bar{\varepsilon} > 0 \) and \( 0 < \alpha < \min\{1, \frac{\bar{\varepsilon}}{4}, \frac{\bar{\varepsilon}}{2r}\} \).

Since \( N \) is a \( C^k \) Finsler manifold modeled on a \( k \)-admissible Banach space \( Y \), there are an open subset \( B \supset \{x \in Y : |||x|||_y \leq 1\} \) of \( Y \) and a \( C^k \) smooth function \( g : B \to \mathbb{R} \) such that

(i) \( g(x) - |||x|||_y < \alpha/2 \) on \( B \), and
(ii) \( \text{Lip}(g) \leq (1 + \alpha/2) \) for the norm \( ||| \cdot |||_y \).

Now, let us take a \( C^\infty \) smooth and Lipschitz function \( \theta : \mathbb{R} \to [0,1] \) such that

(i) \( \theta(t) = 0 \) whenever \( t \leq \alpha \),
(ii) \( \theta(t) = 1 \) whenever \( t \geq 1 - \alpha \),
(iii) \( \text{Lip}(\theta) \leq (1 + \varepsilon) \), and
(iv) \( |\theta(t) - t| \leq 2\alpha \) for every \( t \in [0,1 + \alpha] \).
Let us define
\[ f(x) = \begin{cases} \theta(g(x)) & \text{if } x \in B, \\ 1 & \text{if } x \in Y \setminus B. \end{cases} \]
It is straightforward to verify that \( f \) is well-defined, \( C^k \) smooth, \( f(x) = 1 \) whenever \( |||x|||_y \geq 1 \) and \( f(x) = 0 \) whenever \( |||x|||_y \leq \alpha/2 \). Let us now consider \( f_y : Y \to [0, r] \) as \( f_y(z) = r(1 - \frac{f(z)}{r}) \), which is \( C^k \) smooth, Lipschitz and satisfies:

(i) \( f_y(0) = r \),
(ii) \( \|f_y\|_\infty = r \),
(iii) \( |f_y(z) - f_y(x)| \leq (1 + \varepsilon)(1 + \alpha/2)||z - x||_y \leq (1 + e)^2||z - x||_y \),
(iv) \( f_y(z) = 0 \) for every \( z \in Y \) with \( |||z|||_y \geq r \),
(v) \( \|\frac{z}{r}|||z|||_y \leq \frac{\alpha}{2} + g(\frac{z}{r}) \leq \frac{\alpha}{2} + 2\alpha + f(\frac{z}{r}) \) for every \( |||z|||_y \leq r \). Thus, \( |||z|||_y \leq r(\frac{\alpha}{2} + 2\alpha) + r - f_y(z) \leq \varepsilon + r - f_y(z) \) for every \( |||z|||_y \leq r \).

Let us now prove Proposition 2.4. Let us fix \( p_1, p_2 \in M \) and \( \varepsilon > 0 \). Let us consider \( \sigma : [0, 1] \to M \) a piecewise \( C^1 \) smooth path in \( M \) joining \( p_1 \) and \( p_2 \), with \( \ell(\sigma) \leq d_M(p_1, p_2) + \varepsilon \). Since \( h : M \to N \) is continuous, the path \( \tilde{\sigma} := h \circ \sigma : [0, 1] \to N \), joining \( h(p_1) \) and \( h(p_2) \), is continuous as well. For every \( q \in \tilde{\sigma}([0, 1]) \), there is \( 0 < r_q < 1 \) and a chart \( \psi_q : V_q \to Y \) such that \( \psi_q(q) = 0 \), \( B_N(q, r_q) \subset V_q \) and the bijection \( \psi_q : V_q \to \psi_q(V_q) = (1 + \varepsilon)\)-bi-Lipschitz for the norm \( ||d\psi_q^{-1}(0)(\cdot)||_q \) in \( Y \) (see Lemma 1.3). Since \( \tilde{\sigma}([0, 1]) \) is a compact set of \( N \), there is a finite family of points \( 0 = t_1 < t_2 < \ldots < t_m = 1 \) and a family of open intervals \( \{I_k\}_{k=1}^m \) covering the interval \([0, 1]\) so that, if we define \( q_k := \tilde{\sigma}(t_k) \) and \( r_k := r_{q_k} \), for every \( k = 1, \ldots, m \), we have

(a) \( \tilde{\sigma}(I_k) \subset B_N(q_k, r_k/(1 + \varepsilon)) \),
(b) \( I_j \cap I_k \neq \emptyset \) if and only if \( |j - k| \leq 1 \).

It is clear that \( \tilde{\sigma}([0, 1]) \subset \bigcup_{k=1}^m B_N(q_k, \frac{r_k}{1 + \varepsilon}) \). Now, let us select a point \( s_k \in I_k \cap I_{k+1} \) such that \( t_k < s_k < t_{k+1} \), for every \( k = 1, \ldots, m - 1 \). Let us write \( a_k := \tilde{\sigma}(s_k) \), for every \( k = 1, \ldots, m - 1 \), \( \psi_k := \psi_{q_k} \), \( V_k := V_{q_k} \) and \( ||| \cdot |||_k := ||d\psi_k^{-1}(0)(\cdot)||_q \) for every \( k = 1, \ldots, m \). Notice that \( a_k \in B_N(q_k, \frac{r_k}{1 + \varepsilon}) \cap B_N(q_{k+1}, \frac{r_k}{1 + \varepsilon}) \), for every \( k = 1, \ldots, m - 1 \). Since \( \psi_k : V_k \to \psi_k(V_k) = (1 + \varepsilon)\)-bi-Lipschitz for the norm \( ||| \cdot |||_k \) in \( Y \), we deduce that \( \psi_k(a_k) \in B_{||| \cdot |||_k}(0, r_k) \), for every \( k = 1, \ldots, m - 1 \).

Now, let us apply the above Fact to \( r_k, \varepsilon \) and \( \varepsilon = \varepsilon/2m \) to obtain functions \( f_k : Y \to [0, r_k] \) satisfying properties (1)–(5), \( k = 1, \ldots, m \). Let us define the \( C^k \) smooth and Lipschitz functions \( g_k : N \to [0, r_k] \) as \( g_k(z) = f_k(\psi_k(z)) \) for every \( z \in V_k \) and \( g_k(z) = 0 \) for \( z \not\in V_k, k = 1, \ldots, m \). Then,

(i) \( g_k \in C^k_b(N) \),
(ii) \( g_k(q_k) = r_k \),
(iii) \( |g_k(z) - g_k(x)| \leq (1 + \varepsilon)^3 d_N(z, x) \) for all \( z, x \in N \);
(iv) If \( z \in \psi_k^{-1}(B_{||| \cdot |||_k}(0, r_k)) \), then \( |||\psi_k(z)|||_k \leq r_k \) and from condition (5) on the Fact, we obtain
\[ d_N(z, q_k) \leq (1 + \varepsilon)|||\psi_k(z) - \psi_k(q_k)|||_k = (1 + \varepsilon)|||\psi_k(z)|||_k \leq (1 + \varepsilon)(r_k - g_k(z) + \varepsilon/2m). \]

The Lipschitz constant of \( g_k \circ h \), for \( k = 1, \ldots, m \), is the following
\[ \text{Lip}(g_k \circ h) \leq ||g_k \circ h||_{C^k_b(M)} = ||T(g_k)||_{C^k_b(M)} \leq ||T|| ||g_k||_{C^k_b(N)} = ||T|| \max\{||g_k||_{\infty}, ||dg_k||_{\infty}\} \leq ||T||(1 + \varepsilon)^3. \]
Now, since $r_k = g_k(q_k) = g_k(h(\sigma(t_k)))$ and $\psi_k(h(\sigma(s_k))) \in B_{||\cdot||}_k(0,r_k)$, we have

$$d_N(h(p_1),h(p_2)) \leq \sum_{k=1}^{m-1} [d_N(h(\sigma(t_k)), h(\sigma(s_k))) + d_N(h(\sigma(s_k)), h(\sigma(t_{k+1}))))] \leq$$

$$\leq \sum_{k=1}^{m-1} (1 + \varepsilon)[g_k(q_k) - g_k(h(\sigma(s_k)))+$$

$$+ g_{k+1}(q_{k+1}) - g_{k+1}(h(\sigma(s_k))) + \varepsilon/m] \leq$$

$$\leq \sum_{k=1}^{m-1} (1 + \varepsilon)\text{Lip}(g_k \circ h)d_M(\sigma(t_k), \sigma(s_k))+$$

$$+ \text{Lip}(g_{k+1} \circ h)d_M(\sigma(t_{k+1}), \sigma(s_k)) + \varepsilon/m] \leq$$

$$\leq \sum_{k=1}^{m-1} ||T||(1 + \varepsilon)^4[d_M(\sigma(t_k), \sigma(s_k)) + d_M(\sigma(t_{k+1}), \sigma(s_k))] + \varepsilon(1 + \varepsilon) \leq$$

$$\leq \sum_{k=1}^{m-1} ||T||(1 + \varepsilon)^4d_\sigma(\varepsilon_0,\varepsilon_1) + \varepsilon(1 + \varepsilon) = ||T||(1 + \varepsilon)^4\ell(\sigma) + \varepsilon(1 + \varepsilon) \leq$$

$$\leq ||T||(1 + \varepsilon)^4(d_M(p_1,p_2) + \varepsilon) + \varepsilon(1 + \varepsilon)$$

for every $\varepsilon > 0$. Thus, $h$ is $||T||$-Lipschitz.

\[\square\]

**Lemma 2.5.** Let $M$ and $N$ be $C^k$ Finsler manifolds such that $N$ is modeled on a Banach space with a Lipschitz $C^k$ smooth bump function. Let $h : M \to N$ be a homeomorphism such that $f \circ h \in C^k_b(N)$ for every $f \in C^k_b(N)$. Then, $h$ is a weakly $C^k$ smooth function on $M$.

**Proof.** Let us fix $x \in M$ and $\varepsilon = 1$. There are charts $\varphi : U \to X$ of $M$ at $x$ and $\psi : V \to Y$ of $N$ at $h(x)$ satisfying inequalities (1.1) and (1.2) on $U$ and $V$, respectively. We can assume that $h(U) \subset V$. Since $Y$ admits a Lipschitz and $C^k$ smooth bump function and $\psi(h(U))$ is an open neighborhood of $\psi(h(x))$ in $Y$, there are real numbers $0 < s < r$ such that $B(\psi(h(x)), s) \subset B(\psi(h(x)), r) \subset \psi(h(U))$ and a Lipschitz and $C^k$ smooth function $\alpha : Y \to \mathbb{R}$ such that $\alpha(y) = 1$ for $y \in B(\psi(h(x)), s)$ and $\alpha(y) = 0$ for $y \notin B(\psi(h(x)), r)$. Let us define $U_0 := h^{-1}(\psi^{-1}(B(\psi(h(x)), s))) \subset U$, which is an open neighborhood of $x$ in $M$.

Let us check that $y^* \circ (\psi \circ h \circ \varphi^{-1})$ is $C^k$ smooth on $\varphi(U_0) \subset X$ for all $y^* \in Y^*$. Following the proof of [9, Theorem 4], we define $g : N \to \mathbb{R}$ as $g(y) = 0$ whenever $y \notin V$ and $g(y) = \alpha(\psi(y)) \cdot y^*(\psi(y))$ whenever $y \in V$. It is clear that $g \in C^k_b(N)$ and, by assumption, $g \circ h \in C^k_b(M)$. Now, it follows that $\psi(h(\varphi^{-1}(z))) \in B(\psi(h(x)), s)$ for every $z \in \varphi(U_0)$. Thus

$$y^* \circ (\psi \circ h \circ \varphi^{-1})(z) = y^*(\psi(h(\varphi^{-1}(z)))) = \alpha(\psi(h(\varphi^{-1}(z))))y^*(\psi(h(\varphi^{-1}(z)))) =$$

$$= g(h(\varphi^{-1}(z))) = g \circ h \circ \varphi^{-1}(z),$$

for every $z \in \varphi(U_0)$. Since $(g \circ h) \circ \varphi^{-1}$ is $C^k$ smooth on $\varphi(U_0)$, we have that $y^* \circ (\psi \circ h \circ \varphi^{-1})$ is $C^k$ smooth on $\varphi(U_0)$. Thus $\psi \circ h \circ \varphi^{-1}$ is weakly $C^k$ smooth on $\varphi(U_0)$ and $h$ is weakly $C^k$ smooth on $M$. \[\square\]
Proof of Theorem 2.3. If \( h : M \to N \) is a weak \( C^k \) Finsler isometry, we can define the operator \( T : C^k_b(N) \to C^k_b(M) \) by \( T(f) = f \circ h \). Let us check that \( T \) is well defined. For every \( x \in M \), there are charts \( \varphi : U \to X \) of \( M \) at \( x \) and \( \psi : V \to Y \) of \( N \) at \( h(x) \), such that \( h(U) \subset V \) and \( \psi \circ h \circ \varphi^{-1} \) is weakly \( C^k \) smooth on \( \varphi(U) \subset X \). Also, \( f \circ \psi^{-1} \) is \( C^k \) smooth on \( \psi(V) \subset Y \). Thus, by [15, Proposition 4.2], \((f \circ \psi^{-1}) \circ (\psi \circ h \circ \varphi^{-1}) = f \circ h \circ \varphi^{-1} \) is \( C^k \) smooth on \( \varphi(U) \). Therefore, \( f \circ h \) is \( C^k \) smooth on \( U \). Since this holds for every \( x \in M \), we deduce that \( f \circ h \) is \( C^k \) smooth on \( M \). Moreover, \( T \) is an algebra isomorphism with \(||T(f)|||_{C^k_b(M)} = ||f \circ h|||_{C^k_b(M)} = ||f|||_{C^k_b(N)} \) for every \( f \in C^k_b(N) \).

Conversely, let \( T : C^k_b(N) \to C^k_b(M) \) be a normed algebra isometry. Then, we can define the function \( h : H(C^k_b(M)) \to H(C^k_b(N)) \) by \( h(\varphi) = \varphi \circ T \) for every \( \varphi \in H(C^k_b(M)) \). The function \( h \) is a bijection. Moreover, \( h \) is an homeomorphism.

Recall that we identify \( x \in M \) with \( \delta_x \in C^k_b(M) \). Thus, \( h(x) = h(\delta_x) = \delta_x \circ T \). Since \( h \) is an homeomorphism, by Proposition 1.9, we obtain for every \( p \in N \) a unique point \( x \in M \) such that \( h(\delta_x) = \delta_p \). Let us check that \( T(f) = f \circ h \) for all \( f \in C^k_b(N) \).

Indeed, for every \( x \in M \) and every \( f \in C^k_b(N) \),

\[
T(f)(x) = \delta_x(T(f)) = (\delta_x \circ T)(f) = h(\delta_x)(f) = \delta_{h(x)}(f) = f(h(x)) = f \circ h(x).
\]

Now, from Proposition 2.4 and Lemma 2.5 we deduce that \( h \) is a weak \( C^k \) Finsler isometry.

Remark 2.6. It is worth mentioning that, for Riemannian manifolds, every metric isometry is a \( C^1 \) Finsler isometry. This result was proved by S. Myers and N. Steenrod [21] in the finite-dimensional case and by I. Garrido, J.A. Jaramillo and Y.C. Rangel [12] in the general case. Also, S. Deng and Z. Hou [5] obtained a version for finite-dimensional Riemannian-Finsler manifolds. Nevertheless, there is no a generalization, up to our knowledge, of the Myers-Steenrod theorem for all Finsler manifolds. Thus, for \( k = 1 \) we can only assure that the metric isometry obtained in Theorem 2.3 is weakly \( C^1 \) smooth.

Let us finish this note with some interesting corollaries of Theorem 2.3. First, recall that every separable Banach space with a Lipschitz \( C^k \) smooth bump function satisfies condition (A.2) and every WCG Banach space with a \( C^1 \) smooth bump function satisfies condition (A.1) for \( k = 1 \).

Corollary 2.7. Let \( M \) and \( N \) be complete, \( C^1 \) Finsler manifolds that are \( C^1 \) uniformly bumpable and are modeled on WCG Banach spaces. Then \( M \) and \( N \) are weakly \( C^1 \) equivalent as Finsler manifolds if, and only if, \( C^1_b(M) \) and \( C^1_b(N) \) are equivalent as normed algebras. Moreover, every normed algebra isomorphism \( T : C^1_b(N) \to C^1_b(M) \) is of the form \( T(f) = f \circ h \) where \( h : M \to N \) is a weak \( C^1 \) Finsler isometry.

Corollary 2.8. Let \( M \) and \( N \) be complete, separable \( C^k \) Finsler manifolds that are modeled on Banach spaces with a Lipschitz and \( C^k \) smooth bump function. Then \( M \) and \( N \) are weakly \( C^k \) equivalent as Finsler manifolds if and only if \( C^k_b(M) \) and \( C^k_b(N) \) are equivalent as normed algebras. Moreover, every normed algebra isomorphism \( T : C^k_b(N) \to C^k_b(M) \) is of the form \( T(f) = f \circ h \) where \( h : M \to N \) is a weak \( C^k \) Finsler isometry. In particular, \( h \) is a \( C^{k-1} \) Finsler isometry whenever \( k \geq 2 \).
Since every weakly $C^k$ smooth function with values in a finite-dimensional normed space is $C^k$ smooth and every finite-dimensional $C^k$ Finsler manifold is $C^k$ uniformly bumpable [18], we obtain the following Myers-Nakai result for finite-dimensional $C^k$ Finsler manifolds.

**Corollary 2.9.** Let $M$ and $N$ be complete and finite dimensional $C^k$ Finsler manifolds. Then $M$ and $N$ are $C^k$ equivalent as Finsler manifolds if, and only if, $C^k_b(M)$ and $C^k_b(N)$ are equivalent as normed algebras. Moreover, every normed algebra isomorphism $T : C^k_b(N) \to C^k_b(M)$ is of the form $T(f) = f \circ h$ where $h : M \to N$ is a $C^k$ Finsler isometry.

We obtain an interesting application of Finsler manifolds to Banach spaces. Recall the well-known Mazur-Ulam Theorem establishing that every surjective isometry between two Banach spaces is affine.

**Corollary 2.10.** Let $X$ and $Y$ be WCG Banach spaces with $C^1$ smooth bump functions. Then $X$ and $Y$ are isometric if, and only if, $C^1_b(X)$ and $C^1_b(Y)$ are equivalent as normed algebras. Moreover, every normed algebra isomorphism $T : C^1_b(Y) \to C^1_b(X)$ is of the form $T(f) = f \circ h$ where $h : X \to Y$ is a surjective isometry. In particular, $h$ and $h^{-1}$ are affine isometries.

**References**


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