LINEABILITY, SPACEABILITY, AND ADDITIVITY CARDINALS
FOR DARBOUX-LIKE FUNCTIONS

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Abstract. We introduce the concept of maximal lineability cardinal number,
m\mathcal{L}(M), of a subset M of a topological vector space and study its relation to the cardinal numbers known as: additivity A(M), homogeneous lineability H\mathcal{L}(M), and lineability \mathcal{L}(M) of M. In particular, we will describe, in terms of L, the lineability and spaceability of the families of the following Darboux-like functions on \mathbb{R}^n, n \geq 1: extendable, Jones, and almost continuous functions.

1. Preliminaries and background

The work presented here is a contribution to a recent ongoing research concerning the following general question: For an arbitrary subset M of a vector space W, how big can be a vector subspace V contained in M \cup \{0\}? The current state of knowledge concerning this problem is described in the very recent survey article [4]. So far, the term big in the question was understood as a cardinality of a basis of V; however, some other measures of bigness (i.e., in a category sense) can also be considered.

Following [1,23] (see, also, [13]), given a cardinal number \mu we say that M \subset W is \mu-lineable if M \cup \{0\} contains a vector subspace V of the dimension \text{dim}(V) = \mu. Consider the following lineability cardinal number (see [2]):

\mathcal{L}(M) = \min\{\kappa: M \cup \{0\} contains no vector space of dimension \kappa\}.

Notice that M \subset W is \mu-lineable if, and only if, \mu < \mathcal{L}(M). In particular, \mu is the maximal dimension of a subspace of M \cup \{0\} if, and only if, \mathcal{L}(M) = \mu^+. The number \mathcal{L}(M) need not be a cardinal successor (see, e.g., [11]); thus, the maximal dimension of a subspace of M \cup \{0\} does not necessarily exist.

If W is a vector space over the field K and M \subset W, let

st(M) = \{w \in W: (K \setminus \{0\})w \subset M\}.

Notice that

if V is a subspace of W, then V \subset M \cup \{0\} if, and only if, V \subset st(M) \cup \{0\}. \hspace{1cm} (1)

In particular,

\mathcal{L}(M) = \mathcal{L}(st(M)). \hspace{1cm} (2)

Recall also (see, e.g., [15]) that a family M \subset W is said to be star-like provided st(M) = M. Properties (1) and (2) explain why the assumption that M is star-like appears in many results on lineability.

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A simple use of Zorn’s lemma shows that any linear subspace $V_0$ of $M \cup \{0\}$ can be extended to a maximal linear subspace $V$ of $M \cup \{0\}$. Therefore, the following concept is well defined.

**Definition 1.1** (maximal lineability cardinal number). Let $M$ be any arbitrary subset of a vector space $W$. We define

$$mL(M) = \min\{\dim(V) : V \text{ is a maximal linear subspace of } M \cup \{0\}\}.$$ 

Although this notion might seem similar to that of maximal-lineability and maximal-spaceability (introduced by Bernal-González in [3]) they are, in general, not related.

In any case, (1) implies that $mL(M) = mL(st(M))$. Clearly we have

$$HL(M) = mL(M) + 1,$$

where $HL$ is a homogeneous lineability number defined in [2]. (This explains why $HL$ is always a successor cardinal, as shown in [2].) Clearly we have

$$HL(M) = mL(M) + L(M).$$

**Remark 1.2.** It is easy to see that $HL(M) = mL(M) + 1$, where $HL(M)$ is a homogeneous lineability number defined in [2]. (This explains why $HL$ is always a successor cardinal, as shown in [2].) Clearly we have

$$HL(M) = mL(M) + L(M).$$

The inequality may be strict, as shown in [2].

For $M \subset W$ we will also consider the following additivity number (compare [2]), which is a generalization of the notion introduced by T. Natkaniec in [20, 21] and thoroughly studied by the first author [7–11] and F.E. Jordan [18] for $V = \mathbb{R}$ (see, also, [10]):

$$A(M, W) = \min(\{|F| : F \subset W \& (\forall w \in W)(w + F \not\subset M)\} \cup \{|W|^+\}),$$

where $|F|$ is the cardinality of $F$ and $w + F = \{w + f : f \in F\}$. Most of the times the space $W$, usually $W = \mathbb{R}$, will be clear by the context. In such cases we will often write $A(M)$ in place of $A(M, W)$.

We are mostly interested in the topological vector spaces $W$. We say that $M \subset W$ is $\mu$-spaceable with respect to a topology $\tau$ on $W$, provided there exists a $\tau$-closed vector space $V \subset M \cup \{0\}$ of dimension $\mu$. In particular, we can consider also the following spaceability cardinal number:

$$L_\tau(M) = \min\{\kappa : M \cup \{0\} \text{ contains no } \tau \text{-closed subspace of dimension } \kappa\}.$$ 

Notice that $L(M) = L_\tau(M)$ when $\tau$ is the discrete topology.

In what follows, we shall focus on spaces $W = \mathbb{R}^X$ of all functions from $X = \mathbb{R}^n$ to $\mathbb{R}$ and consider the topologies $\tau_u$ and $\tau_p$ of uniform and pointwise convergence, respectively. In particular, we write $L_u(M)$ and $L_p(M)$ for $L_{\tau_u}(M)$ and $L_{\tau_p}(M)$, respectively. Clearly

$$L_p(M) \leq L_u(M) \leq L(M).$$

Recall also a series of definitions that shall be needed throughout the paper.

**Definition 1.3.** For $X \subseteq \mathbb{R}^n$ a function $f : X \to \mathbb{R}$ is said to be

- **Darboux** if $f[K]$ is a connected subset of $\mathbb{R}$ (i.e., an interval) for every connected subset $K$ of $X$;
- **Darboux** in the sense of Pawlak if $f[L]$ is a connected subset of $\mathbb{R}$ for every arc $L$ of $X$ (i.e., $f$ maps path connected sets into connected sets);
- **almost continuous** (in the sense of Stallings) if each open subset of $X \times \mathbb{R}$ containing the graph of $f$ contains also a continuous function from $X$ to $\mathbb{R}$;

- **continuously extension of a function** $f : X \to \mathbb{R}$ is a function $g : X \to \mathbb{R}$ that extends $f$ in the sense that $f(x) = g(x)$ for all $x \in X$.

- **continuously extension of a function** $f : X \to \mathbb{R}$ is a function $g : X \to \mathbb{R}$ that extends $f$ in the sense that $f(x) = g(x)$ for all $x \in X$.
The above classes of functions are denoted by CIVP, SCIVP, WCIVP, and extendable functions on \( X \). The above functions have been recently partly studied (see, e.g., [2, 14–16]) but here we shall give definitive answers concerning the lineability and spaceability of several previous studied classes.

The above classes of functions are denoted by D(\( X \)), Dp(\( X \)), AC(\( X \)), Conn(\( X \)), Ext(\( X \)), and PC(\( X \)), respectively. The class of continuous functions from \( X \) into \( \mathbb{R} \) is denoted by C(\( X \)). We will drop the domain \( X \) if \( X = \mathbb{R} \).

**Definition 1.4.** A function \( f : \mathbb{R}^n \to \mathbb{R} \) is called
- **everywhere surjective** if \( f[G] = \mathbb{R} \) for every nonempty open set \( G \subset \mathbb{R}^n \);
- **strongly everywhere surjective** if \( f^{-1}(y) \cap G \) has cardinality \( \aleph_0 \) for every \( y \in \mathbb{R} \) and nonempty open set \( G \subset \mathbb{R}^n \); this class was also studied in [4], under the name of \( \aleph_0 \) strongly Darboux functions;
- **perfectly everywhere surjective** if \( f[P] = \mathbb{R} \) for every perfect set \( P \subset \mathbb{R}^n \) (i.e., when \( f^{-1}(r) \) is a Bernstein set for every \( r \in \mathbb{R} \) (compare [6, chap. 7]));
- a **Jones function** (see [17]) if \( f \cap F \not= \emptyset \) for every closed set \( F \subset \mathbb{R}^n \times \mathbb{R} \) whose projection on \( \mathbb{R}^n \) is uncountable.

The classes of these functions are written as ES(\( \mathbb{R}^n \)), SES(\( \mathbb{R}^n \)), PES(\( \mathbb{R}^n \)), and J(\( \mathbb{R}^n \)), respectively. We will drop the domain \( \mathbb{R}^n \) if \( n = 1 \).

**Definition 1.5.** A function \( f : \mathbb{R} \to \mathbb{R} \) has:
- the **Cantor intermediate value property** if for every \( x, y \in \mathbb{R} \) and for each perfect set \( K \) between \( f(x) \) and \( f(y) \) there is a perfect set \( C \) between \( x \) and \( y \) such that \( f[C] \subset K \);
- the **strong Cantor intermediate value property** if for every \( x, y \in \mathbb{R} \) and for each perfect set \( K \) between \( f(x) \) and \( f(y) \) there is a perfect set \( C \) between \( x \) and \( y \) such that \( f[C] \subset K \) and \( f \mid C \) is continuous;
- the **weak Cantor intermediate value property** if for every \( x, y \in \mathbb{R} \) with \( f(x) < f(y) \) there exists a perfect set \( C \) between \( x \) and \( y \) such that \( f[C] \subset (f(x), f(y)) \);
- **perfect roads** if for every \( x \in \mathbb{R} \) there exists a perfect set \( P \subset \mathbb{R} \) having \( x \) as a bilateral (i.e., two sided) limit point for which \( f \mid P \) is continuous at \( x \).

The above classes of functions shall be denoted by CIVP, SCIVP, WCIVP, and FR, respectively.

Notice that all classes defined in the above three definitions are star-like.

The text is organized as follows. In Section 2 we study the relations between additivity and maximal lineability numbers. Sections 3 and 4 focus on the set of extendable functions on \( \mathbb{R} \) and \( \mathbb{R}^n \), respectively. Surprisingly enough, we shall obtain very different results when moving from \( \mathbb{R} \) to \( \mathbb{R}^n \). The lineability of some of the above functions have been recently partly studied (see, e.g., [2,14,16,10]) but here we shall give definitive answers concerning the lineability and spaceability of several previous studied classes.
2. Relation between additivity and lineability numbers

The goal of this section is to examine possible values of numbers \(A(M)\), \(m\mathcal{L}(M)\),
and \(\mathcal{L}(M)\) for a subset \(M\) of a linear space \(W\) over an arbitrary field \(K\). We will concentrate on the cases when \(\emptyset \neq M \subset W\), since it is easy for the cases \(M \in \{\emptyset, W\}\). Indeed, as it can be easily checked, one has \(A(\emptyset) = \mathcal{L}(\emptyset) = 1\) and \(m\mathcal{L}(\emptyset) = 0\); \(A(W) = |W|^+, \mathcal{L}(W) = \dim(W)^+\), and \(m\mathcal{L}(W) = \dim(W)\).

Proposition 2.1. Let \(W\) be a vector space over a field \(K\) and let \(\emptyset \neq M \subset W\). Then

(i) \(2 \leq A(M) \leq |W|\) and \(m\mathcal{L}(M) < \mathcal{L}(M) \leq \dim(W)^+\); (ii) if \(A(\text{st}(M)) > |K|\), then \(A(\text{st}(M)) \leq m\mathcal{L}(M)\).

In particular, if \(M\) is star-like, then \(A(M) = |K|\) implies that

(iii) \(A(M) \leq m\mathcal{L}(M) < \mathcal{L}(M) \leq \dim(W)^+\).

Proof. (i) These inequalities are easy to see.

(ii) This can be proved by an easy transfinite induction. Alternatively, notice that A. Bartoszewicz and S. Głąb proved, in [2, corollary 2.3], that if \(M \subset W\) is star-like and \(A(M) > |K|\), then \(A(\text{st}(M)) < \mathcal{L}(\text{st}(M)) = m\mathcal{L}(\text{st}(M))^+ = m\mathcal{L}(M)^+\). Therefore, \(A(\text{st}(M)) \leq m\mathcal{L}(M)\). \(\square\)

In what follows, we will restrict our attention to the star-like families, since, by Proposition 2.1, other cases could be reduced to this situation. Our next theorem shows that, for such families and under assumption that \(A(M) > |K|\), the inequalities (3) constitute all that can be said on these numbers.

Theorem 2.2. Let \(W\) be an infinite dimensional vector space over an infinite field \(K\) and let \(\alpha, \mu,\) and \(\lambda\) be the cardinal numbers such that \(|K| < \alpha \leq \mu < \lambda \leq \dim(W)^+\). Then there exists a star-like \(M \subset W\) containing 0 such that \(A(M) = \alpha\), \(m\mathcal{L}(M) = \mu\), and \(\mathcal{L}(M) = \lambda\).

The proof of this theorem will be based on the following two lemmas. The first of them shows that the theorem holds when \(\alpha = \mu\), while the second shows how such an example can be modified to the general case.

Lemma 2.3. Let \(W\) be an infinite dimensional vector space over an infinite field \(K\) and let \(\mu\) and \(\lambda\) be the cardinal numbers such that \(|K| < \mu < \lambda \leq \dim(W)^+\). Then there exists a star-like \(M \subset W\) containing 0 such that \(A(M) = m\mathcal{L}(M) = \mu\) and \(\mathcal{L}(M) = \lambda\).

Proof. For \(S \subset W\), let \(V(S)\) be the vector subspace of \(W\) spanned by \(S\).

Let \(B\) be a basis for \(W\). For \(w \in W\), let \(\text{supp}(w)\) be the smallest subset \(S\) of \(B\) with \(w \in V(S)\) and let \(c_w: \text{supp}(w) \to K\) be such that \(w = \sum_{b \in \text{supp}(w)} c_w(b) b\). Let \(E\) be the set of all cardinal numbers less than \(\lambda\) and choose a sequence \(\langle B_\eta : \eta \in E\rangle\) of pairwise disjoint subsets of \(B\) such that \(|B_0| = \mu\) and \(|B_\eta| = \eta\) whenever \(0 \neq \eta \in E\). Define

\[M = A \cup \bigcup_{\eta \in E} V(B_\eta),\]

where

\[A = \{w \in W : c_w(b_0) = c_w(b_1) \text{ for some } b_0 \in \text{supp}(w) \cap B_0, b_1 \in \text{supp}(w) \setminus B_0\},\]
We will show that $M$ is as desired.

Clearly, $M$ is star-like and $0 \in M \subseteq W$. Also, $L(M) \geq \lambda$, since for any cardinal $\eta < \lambda$, the set $M$ contains a vector subspace $V(B_\eta)$ with $\dim(V(B_\eta)) \geq \eta$.

To see that $A(M) \geq \mu$, choose an $F \subset W$ with $|F| < \mu$. It is enough to show that $|F| < A(M)$; that is, that there exists a $w \in W$ with $w + F \subset \mathcal{A}$. Let $\pi : V \to V/\mathcal{A}$ be the canonical projection. Then $\pi : V \to V/\mathcal{A}$ is a maximal vector subspace of $M$. Indeed, if $V$ is a vector subspace of $W$ properly containing $V(B_0)$, then there exists a non-zero $v \in V \cap V(B \setminus B_0)$. Choose a $b_0 \in B_0$ and a non-zero $c \in K \setminus \{c_v(b) : b \in \text{supp}(v)\}$. Then $c_b + v \in V \cap M$. So, $V(B_0)$ is a maximal vector subspace of $M$ and indeed $mL(M) \leq \dim(V(B_0)) = \kappa$.

To see that $L(M) \leq \lambda$, choose a vector subspace $V$ of $W$ of dimension $\lambda$. It is enough to show that $V \setminus M \neq \emptyset$. To see this, for every ordinal $\eta \leq \lambda$ let us define $\hat{B}_\eta = \bigcup \{B_\zeta : \zeta \in E \cap \eta\}$. Notice that for every $\eta < \lambda$ there is a non-zero $w \in V$ with $\text{supp}(w) \cap \hat{B}_\eta = \emptyset$.

Indeed, if $\pi_\eta : W = V(\hat{B}_\eta) \oplus V(B \setminus \hat{B}_\eta) \to V(\hat{B}_\eta)$ is the natural projection, then there exist distinct $w_1, w_2 \in V$ with $\pi_\eta(w_1) = \pi_\eta(w_2)$ (as $|V(\hat{B}_\eta)| < \lambda = \dim(V)$). Then $w = w_1 - w_2$ is as required.

Now, choose a non-zero $w_1 \in V$ with $\text{supp}(w_1) \cap B_0 = \text{supp}(w_1) \cap \hat{B}_1 = \emptyset$. Then, $w_1 \notin \mathcal{A}$ and if $\text{supp}(w_1) \notin \hat{B}_\lambda = \bigcup_{\eta \in E} B_\eta$, then also $w_1 \notin \bigcup_{\eta \in E} V(B_\eta)$, and we have $w_1 \notin V \setminus M$. Therefore, we can assume that $\text{supp}(w_1) \subset \hat{B}_\lambda$. Let $\eta < \lambda$ be such that $\text{supp}(w_1) \subset \hat{B}_\eta$ and choose a non-zero $w_2 \in V$ with $\text{supp}(w_2) \cap B_0 = \emptyset$. Then $w = w_2 - w_1 \in V \setminus M$ (since $w \notin \mathcal{A}$, being non-zero with $\text{supp}(w) \cap B_0 = \emptyset$, and $w \notin \bigcup_{\eta \in E} V(B_\eta)$, as its support intersects two different $B_\eta$).

**Lemma 2.4.** Let $W$, $W_0$, and $W_1$ be the vector spaces over an infinite field $K$ such that $W = W_0 \oplus W_1$. Let $M \subseteq W_0$ and

$$F = M + W_1 = \{m + W_1 : m \in M \text{ and } w \in W_1\}.$$  

Then

(i) If $M$ is star-like, then $F$ is also star-like.

(ii) $A(\mathcal{F}, W) = A(M, W_0)$.

(iii) If $0 \in M$, then $mL(F) = mL(M) + \dim(W_1)$.

(iv) If $0 \in M$ and $\dim(W_1) < L(M)$, then $L(F) = L(M) + \dim(W_1)$.

**Proof.** In the following, let $\pi_0 : W = W_0 \oplus W_1 \to W_0$ be the canonical projection.

Let $x \in \mathcal{F}$ and $\lambda \in K \setminus \{0\}$. Since $M$ is star-like and $\pi_0(x) \in M$, we have that $\pi_0(\lambda x) = \lambda \pi_0(x) \in M$, and hence $\lambda x \in M + W_1 = \mathcal{F}$.

Let us see that $A(M, W_0) \leq A(\mathcal{F}, W)$. To this end, let $\kappa < A(M, W_0)$. We need to prove that $\kappa < A(\mathcal{F}, W)$. Indeed, if $F \subset W$ and $|F| = \kappa$, then $|\pi_0[F]| \leq
\(|F| = \kappa\). So, there exists a \(w_0 \in W_0\) such that \(\pi_0[w_0 + F] = w_0 + \pi_0[F] \subseteq M\), that is, \(w_0 + F \subseteq M + W_1 = F\). Therefore, \(\kappa < A(F, W)\).

To see that \(A(F, W) \leq A(M, W_0)\) let \(\kappa < A(F, W)\). We need to show that \(\kappa < A(M, W_0)\). Indeed, let \(F \subseteq W_0\) be such that \(|F| = \kappa\). Since \(|F| < A(F, W)\), there is a \(w \in W\) with \(w + F \subseteq F\). Then \(\pi_0(w) \in W_0\) and \(\pi_0(w) + F = \pi_0[w + F] \subseteq \pi_0[F] = M\), so indeed \(\kappa < A(M)\).

**(III)** First notice that it is enough to show that

\[ V \] is a maximal vector subspace of \(F\) if, and only if, \(V = V_0 + W_1\), where \(V_0\) is a maximal vector subspace of \(M\).

Indeed, if \(V\) is a maximal vector subspace of \(F\) with \(mL(F) = \dim(V)\), then, by (3), \(mL(F) = \dim(V) = \dim(V_0) + \dim(W_1) \geq mL(M) + \dim(W_1)\). Conversely, if \(V_0\) is a maximal vector subspace of \(M\) with \(mL(M) = \dim(V_0)\), then we have \(mL(M) + \dim(W_1) = \dim(V_0) + \dim(W_1) = \dim(V_0 + W_1) \geq mL(F)\).

To see (3), take a maximal vector subspace \(V\) of \(F\). Notice that \(W_1 \subseteq V\), since \(V \subseteq V + W_1 \subseteq F + W_0 = F\) and so, by the maximality, \(V + W_1 = V\). In particular, \(V = V_0 + W_1 \subseteq F = M + W_1\), where \(V_0 = \pi_0[V]\). Thus, \(V_0\) is a vector subspace of \(M\). It must be maximal, since for any its proper extension \(V_0 \subseteq M\), the vector space \(V_0 + W_1 \subseteq F\) would be a proper extension of \(V\).

Conversely, if \(V_0\) is a maximal vector subspace of \(M\), then \(V = V_0 + W_1\) is a vector subspace of \(F\). If cannot have a proper extension \(V \subseteq F\), since then the vector space \(\pi_0[V] \subseteq M\) would be a proper extension of \(V_0\).

**(IV)** To see that \(\mathcal{L}(F) \leq \dim(W_1) + \mathcal{L}(M)\), choose a vector space \(V \subseteq F\). We need to show that \(\dim(V) < \dim(W_1) + \mathcal{L}(M)\). Indeed, \(V = V_0 + W_1\) is a vector subspace of \(F + W_1 = F\) and \(\dim(V) \leq \dim(V_0) + \dim(W_1) + \dim(\pi_0[V_1])\), since \(V_1 = W_1 + \pi_0[V_1]\). Therefore, \(\dim(V) \leq \dim(W_1) + \dim(\pi_0[V_1]) \leq \dim(W_1) + \mathcal{L}(M)\), since \(\dim(W_1) < \mathcal{L}(M)\) and \(\dim(\pi_0[V_1]) < \dim(W_1) + \mathcal{L}(M)\), as \(\pi_0[V_1]\) is a vector subspace of \(M = \pi_0[F]\). So, \(\mathcal{L}(F) \leq \dim(W_1) + \mathcal{L}(M)\).

To see that \(\dim(W_1) + \mathcal{L}(M) \leq \mathcal{L}(F)\), choose a \(\kappa < \dim(W_1) + \mathcal{L}(M)\). We need to show that \(\kappa < \mathcal{L}(F)\), that is, that there exists a vector subspace \(V\) of \(F\) with \(\dim(V) \geq \kappa\). First, notice that \(\dim(W_1) < \mathcal{L}(M)\) and \(\kappa < \dim(W_1) + \mathcal{L}(M)\) imply that there exists a \(\mu < \mathcal{L}(M)\) such that \(\kappa \leq \dim(W_1) + \mu < \dim(W_1) + \mathcal{L}(M)\). (For finite value of \(\mathcal{L}(M)\), take \(\mu = \max\{\kappa - \dim(W_1), 0\}\); for infinite \(\mathcal{L}(M)\), the number \(\mu = \max\{\kappa, \dim(W_1)\}\) works.) Choose a vector subspace \(V_0\) of \(M\) with \(\dim(V_0) \geq \mu\). Then the vector subspace \(V_0 + W_1 = V_0 + W_1 \subseteq F\) is as desired, since we have \(\dim(V) = \dim(W_1) + \dim(V_0) \geq \dim(W_1) + \mu \geq \kappa\).

**Proof of Theorem 2.2.** Represent \(W\) as \(W_0 \oplus W_1\), where \(\dim(W_0) = \lambda\) and \(\dim(W_1) = \mu\). Use Lemma 2.3 to find a star-like \(M \subseteq W_0\) containing 0 such that \(A(M, W_0) = mL(M) = \alpha\) and \(\mathcal{L}(M) = \lambda\). Let \(F = M + W_1 \subseteq B\). Then, by Lemma 2.4 \(F \ni 0\) is star-like such that \(A(F) = A(M, W_0) = \alpha\), \(mL(F) = mL(M) + \dim(W_1) = \alpha + \mu = \mu\), and \(\mathcal{L}(F) = \mathcal{L}(M) + \dim(W_2) = \lambda + \alpha = \lambda\), as required.

A. Bartoszewicz and S. Głąb have asked [2] open question 1] whether the inequality \(A(F^+) \geq mL(F)\) (which is equivalent to \(A(F) \geq mL(F)\)) holds for any family \(F \subseteq \mathbb{R}^\mathbb{R}\). Of course, for the star-like families \(F\) with \(A(F) > c\), a positive answer to this question would mean that, under these assumptions, we have \(A(F) = mL(F)\). Notice that Theorem 2.2 gives, in particular, a negative answer to this question.
We do not have a comprehensive example, similar to that provided by Theorem 2.2, for the case when $A(M) \leq |K|$. However, the machinery built above, together with the results from [2], lead to the following result.

**Theorem 2.5.** Let $W$ a vector space over an infinite field $K$ with $\dim(W) \geq 2^{|K|}$. If $2 \leq \kappa \leq |W|$, there exists a star-like family $F \subseteq W$ containing $0$ such that $A(F) = \kappa$ and $mL(F) = \dim(W)$ (so that $L(F) = \dim(W)^+$).

**Proof.** Represent $W$ as $W = W_0 \oplus W_1$, where $\dim(W_0) = 2^{|K|}$ and $\dim(W_1) = \dim(W)$. If $2 \leq \kappa \leq |K|$, then, by [2, Theorem 2.5], there exists a star-like family $M \subset W_0$ such that $A(M, W_0) = \kappa$. Notice that the family constructed in that result contains $0$. Then, by Lemma 2.4, the family $F = M + W_1$ satisfies that $A(F) = A(M, W_0) = \kappa$ and $mL(F) = mL(M) + \dim(W_1) = \dim(W)$. \hfill $\Box$

3. **Spaceability of Darboux-like functions on $\mathbb{R}$**

Recall (see, e.g., [3, chart 1] or [7]) that we have the following strict inclusions, indicated by the arrows, between the Darboux-like functions from $\mathbb{R}$ to $\mathbb{R}$. The next theorem, strengthening the results presented in the table from [4, page 14], determines fully the lineability, $L$, and spaceability, $L_p$, numbers for these classes.

![Relations between the Darboux-like classes of functions from $\mathbb{R}$ to $\mathbb{R}$. Arrows indicate strict inclusions.](image)

**Theorem 3.1.** $L_p(Ext) = (2^c)^+$. In particular, all Darboux-like classes of functions from Figure 1, except $C$, are $2^c$-spaceable with respect to the topology of pointwise convergence.

**Proof.** In [11, corollary 3.4] it is shown that there exists an $f \in Ext$ and an $F_\sigma$ first category set $M \subset \mathbb{R}$ such that

if $g \in \mathbb{R}^\mathbb{R}$ and $g \upharpoonright M = f \upharpoonright M$, then $g \in Ext$. \hfill (4)

It is easy to see that for any real number $r \neq 0$ the function $rf$ satisfies the same property.

Notice also that there exists a family $\{ h_\xi \in \mathbb{R}^\mathbb{R} : \xi < c \}$ of increasing homeomorphisms such that the sets $M_\xi = h_\xi[M]$, $\xi < c$, are pairwise disjoint. (See, e.g., [11, lemma 3.2].) It is easy to see that each function $f_\xi = f \circ h_\xi^{-1}$ satisfies (4) with the set $M_\xi$. Increasing one of the sets $M_\xi$, if necessary, we can also assume that $\{M_\xi : \xi < c\}$ is a partition of $\mathbb{R}$. Let $\bar{f} = (f_\xi \upharpoonright M_\xi : \xi < c)$ and define

$$V(\bar{f}) = \left\{ \bigcup_{\xi < c} t(\xi)(f_\xi \upharpoonright M_\xi) : t \in \mathbb{R}^c \right\}.$$ \hfill (5)
It is easy to see that \( V(\vec{f}) \) is \( 2^c \)-dimensional \( \tau_p \)-closed linear subspace of \( \text{Ext} \). □

As the cardinality of the family \( \text{Bor} \) of Borel functions from \( \mathbb{R} \) to \( \mathbb{R} \) is \( c \), Theorem 3.1 easily implies that \( \text{Ext} \setminus \text{Bor} \) is \( 2^c \)-lineable: \( L(\text{Ext} \setminus \text{Bor}) = (2^c)^+ \). Actually, we have an even stronger result:

**Proposition 3.2.** \( L_p(\text{Ext} \cap \text{SES} \setminus \text{Bor}) = (2^c)^+ \).

*Proof.* The function \( f \mid M \) satisfying (4) may also have the property that \( M \) is \( c \)-dense in \( \mathbb{R} \) and \( f \mid M \) is SES non-Borel. (6)

Indeed, this can be ensured by enlarging \( M \) by a \( c \)-dense first category set \( N \subset \mathbb{R} \setminus M \) and redefining \( f \) on \( N \) so that \( f \mid N \) is non-Borel and SES.

Now, if \( f \) satisfies both (4) and (6) and \( \vec{f} = \langle f_\xi \mid M_\xi : \xi < c \rangle \) is defined as in Theorem 3.1 then the space \( V(\vec{f}) \) given in \( (5) \) is as required. □

Notice also that \( \text{Ext} \cap \text{PES} = \text{PR} \cap \text{PES} = \emptyset \). In particular, the space \( V \) from Proposition 3.2 is disjoint with PES.

**Remark 3.3.** Clearly, Theorem 3.1 implies that \( \text{Ext} \) is \( 2^c \)-lineable. This result has been also independently proved by T. Natkaniec. (See preprint [22].) The technique used in [22] is similar, but different from that used in the proof of Theorem 3.1.

Recall, that it is known that \( L(\text{AC} \setminus \text{Ext}) = (2^c)^+ \). See [15] or [4, page 14]. However, we do not know what the exact values of the following cardinals are.

**Problem 3.4.** Determine the following numbers:

\[ L_p(\mathcal{F} \setminus \mathcal{G}), \quad L_u(\mathcal{F} \setminus \mathcal{G}), \quad \text{and} \quad L_u(\mathcal{F} \setminus \mathcal{G}) \]

for \( \mathcal{F} \in \{ \text{Conn} \setminus \text{AC}, \text{D} \setminus \text{Conn}, \text{PC} \setminus \text{D} \} \) and \( \mathcal{G} \in \{ \text{SCIVP}, \text{CIVP}, \text{PR} \} \).

**Problem 3.5.** Is it consistent with the axioms of set theory ZFC that either \( A(\mathcal{F}) < mL(\mathcal{F}) \) or \( mL(\mathcal{F})^+ < L(\mathcal{F}) \) for any of the classes \( \mathcal{F} \in \{ \text{Ext}, \text{AC}, \text{Conn}, \text{D}, \text{PC} \} \)?

Notice, that the generalized continuum hypothesis GCH implies that \( A(\mathcal{F}) = mL(\mathcal{F}) \) and \( mL(\mathcal{F})^+ = L(\mathcal{F}) \) for every \( \mathcal{F} \in \{ \text{Ext}, \text{AC}, \text{Conn}, \text{D}, \text{PC} \} \).

4. **Spaceability of Darboux-like functions on \( \mathbb{R}^n \), \( n \geq 2 \)**

Recall (see, e.g., [8, chart 2] or [7]) that we have the following strict inclusions, indicated by the arrows, between the Darboux-like functions from \( \mathbb{R}^n \) to \( \mathbb{R} \) for \( n \geq 2 \).

\[
\begin{align*}
\text{Conn}(\mathbb{R}^n) & \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \q
Proposition 4.1. Let \( n > 0 \) and let \( f : \mathbb{R}^n \to \mathbb{R} \) be a peripherally continuous function. Then for any \( x_0 \in \mathbb{R}^n \) and any open set \( W \) in \( \mathbb{R}^n \) containing \( x_0 \), there exists an open set \( U \subseteq W \) such that \( x_0 \in U \) and the restriction of \( f \) to \( \text{bd}(U) \) is continuous. Moreover, given any \( \varepsilon > 0 \), the set \( U \) can be chosen so that \( |f(x_0) - f(y)| < \varepsilon \) for every \( y \in \text{bd}(U) \).

Theorem 4.2. For \( n \geq 2 \), \( \mathcal{L}_p(\text{Ext}(\mathbb{R}^n)) = \mathcal{L}_u(\text{Ext}(\mathbb{R}^n)) = \mathcal{L}(\text{Ext}(\mathbb{R}^n)) = \mathfrak{c}^+ \). In particular, the classes \( C(\mathbb{R}^n) \) and \( \text{Ext}(\mathbb{R}^n) \) are \( \mathfrak{c} \)-spaceable with respect to the pointwise convergence topology \( \tau_p \) but are not \( \mathfrak{c}^+ \)-lineable.

Proof. First, notice that \( \mathcal{L}_p(C(\mathbb{R}^n)) = \mathfrak{c}^+ \) is justified by the space \( C_0 \) of all continuous functions linear on the interval \([k, k+1]\) for every integer \( k \in \mathbb{Z} \). Indeed, \( C_0 \) is linearly isomorphic to \( \mathbb{R}^\mathbb{Z} \).

Now, since \( \mathfrak{c}^+ = \mathcal{L}_p(C(\mathbb{R}^n)) \leq \mathcal{L}_p(\text{Ext}(\mathbb{R}^n)) \leq \mathcal{L}_u(\text{Ext}(\mathbb{R}^n)) \leq \mathcal{L}(\text{Ext}(\mathbb{R}^n)) \), it is enough to show that \( \mathcal{L}(\text{Ext}(\mathbb{R}^n)) \leq \mathfrak{c}^+ \), that is, that \( \text{Ext}(\mathbb{R}^n) \) is not \( \mathfrak{c}^+ \)-lineable. To see this, by way of contradiction, assume that there exists a vector space \( V \subseteq \text{Ext}(\mathbb{R}^n) \) of cardinality \( \mathfrak{c} \). Fix a countable dense set \( D \subseteq \mathbb{R}^n \) and let \( \{x_k, \varepsilon_k\} : k < \omega \) be an enumeration of \( D \times \{2^{-m} : m < \omega \} \). By Proposition 4.1, for every function \( f \in \text{Ext}(\mathbb{R}^n) \) and \( k \) we may choose an open neighborhood \( U_k^f \) of \( x_k \) at most \( \varepsilon_k \) such that \( f \mid \text{bd}(U_k^f) \) is continuous. Consider the mapping \( \text{Ext}(\mathbb{R}^n) \ni f \to T_f = \{f \mid \text{bd}(U_k^f) : k < \omega \} \). Since its range has cardinality \( \mathfrak{c} \), there are distinct \( f_1, f_2 \in \text{Ext}(\mathbb{R}^n) \) with \( T_{f_1} = T_{f_2} \). In particular, \( f = f_1 - f_2 \in V \) is equal zero on the set \( M = \bigcup_{k < \omega} \text{bd}(U_k^{f_1}) \). Notice that the complement \( M^c \) of \( M \) is zero-dimensional. We will show that \( f \) is not extendable, by showing that it does not satisfy Proposition 4.1.

Indeed, since \( f_1 \neq f_2 \), there is an \( x \in \mathbb{R}^n \) with \( f(x) \neq 0 \). Let \( \varepsilon = |f(x)| \) and let \( W \) be any bounded neighborhood of \( x \). Then, there is no set \( U \) as required by Proposition 4.1.

To see this, notice that for any open set \( U \subseteq W \) with \( x \in U \), its boundary is of dimension at least 1. In particular, \( M \cap \text{bd}(U) \neq \emptyset \) and, for \( y \in M \cap \text{bd}(U) \), we have \( |f(x) - f(y)| = |f(x)| = \varepsilon \).

Theorem 4.2 determines the values of the numbers \( \mathcal{L}_p(\mathcal{F}), \mathcal{L}_u(\mathcal{F}), \) and \( \mathcal{L}(\mathcal{F}) \) for \( \mathcal{F} \in \{C(\mathbb{R}^n), \text{Ext}(\mathbb{R}^n), \text{Conn}(\mathbb{R}^n), \text{PR}(\mathbb{R}^n)\} \) and \( n \geq 2 \). In the remainder of this section we will examine these cardinal numbers for the remaining classes from the diagram in Figure 2. For this, we will need the following fact, improving a recent result of the second author, see [13] Theorem 2.2.

Proposition 4.3. \( \mathcal{L}_p(J(\mathbb{R}^n)) = (2^\mathfrak{c})^+ \) for every \( n \geq 1 \). In particular, the families \( J(\mathbb{R}^n), \text{PES}(\mathbb{R}^n), \text{SES}(\mathbb{R}^n), \) and \( \text{ES}(\mathbb{R}^n) \) are \( 2^\mathfrak{c} \)-spaceable with respect to the topology of pointwise convergence.

Proof. Let \( \{B_\xi : \xi < \mathfrak{c}\} \) be a decomposition of \( \mathbb{R}^n \) into pairwise disjoint Bernstein sets. For every \( \xi < \mathfrak{c} \), let \( f_\xi : B_\xi \to \mathbb{R} \) be such that \( f_\xi \cap F \neq \emptyset \) for every closed set \( F \subseteq \mathbb{R}^n \times \mathbb{R} \) whose projection on \( \mathbb{R}^n \) is uncountable. (All of this can be easily constructed by transfinite induction. See, e.g., [14].) Notice that

\[
\text{if } g \in \mathbb{R} \text{ and } g \upharpoonright M_\xi = r \cdot f_\xi \text{ for some } \xi < \mathfrak{c} \text{ and } r \neq 0, \text{ then } g \in J(\mathbb{R}^n). 
\]

Now, if \( f = (f_\xi \upharpoonright M_\xi) : \xi < \mathfrak{c} \) and \( V(f) \) is given by \( \mathfrak{c} \), then \( V(f) \) is \( 2^\mathfrak{c} \)-dimensional \( \tau_p \)-closed linear subspace of \( J(\mathbb{R}^n) \).
Every function in \( J(\mathbb{R}^n) \) is surjective. In particular, the above result implies that the class of surjective functions is \( 2^c \)-lineable. One could also wonder about the lineability of the family of one-to-one functions from \( \mathbb{R}^n \) to \( \mathbb{R} \), given below.

**Remark 4.4.** The family of one-to-one functions from \( \mathbb{R}^n \) to \( \mathbb{R} \) is 1-lineable but not 2-lineable.

**Proof.** Clearly the family is 1-lineable. To see that it is not 2-lineable, choose two injective linearly independent functions \( f \) and \( g \) generating a linear space \( Z \). Take arbitrary \( x \neq y \) in \( \mathbb{R}^n \) and consider the function \( h = f + \alpha g \in Z \setminus \{0\} \), where \( \alpha = (f(x) - f(y))/(g(y) - g(x)) \in \mathbb{R} \). Then, we have \( h(x) = h(y) \), so \( Z \) contains a function which is not one-to-one. \( \square \)

Other examples of 1-lineable but not 2-lineable sets and, in general, not lineable sets can be found in [4, 5].

**Theorem 4.5.** For \( n \geq 2 \), \( J(\mathbb{R}^n) \subset AC(\mathbb{R}^n) \setminus D(\mathbb{R}^n) \). In particular, the class \( AC(\mathbb{R}^n) \setminus D(\mathbb{R}^n) \) is \( 2^c \)-spaceable and \( L_p(AC(\mathbb{R}^n) \setminus D(\mathbb{R}^n)) = (2^c)^+ \).

**Proof.** By Proposition 4.3 it is enough to show that \( J(\mathbb{R}^n) \subset AC(\mathbb{R}^n) \setminus D(\mathbb{R}^n) \). Clearly, \( J(\mathbb{R}^n) \subset AC(\mathbb{R}^n) \cap PES(\mathbb{R}^n) \) for any \( n \geq 1 \). Thus, it is enough to show that \( PES(\mathbb{R}^n) \cap D(\mathbb{R}^n) = \emptyset \) for \( n \geq 2 \). But this follows immediately from the fact that, under \( n \geq 2 \), every Bernstein set in \( \mathbb{R}^n \) is connected. \( \square \)

**Remark 4.6.** Notice that, since \( AC(\mathbb{R}^n) \subset D_p(\mathbb{R}^n) \), then, for \( n \geq 2 \), we have \( L_p(D_p(\mathbb{R}^n) \setminus D(\mathbb{R}^n)) = (2^c)^+ \). So, \( D_p(\mathbb{R}^n) \setminus D(\mathbb{R}^n) \) is also \( 2^c \)-spaceable.

**Theorem 4.7.** For \( n \geq 2 \), \( L_p(D(\mathbb{R}^n) \setminus AC(\mathbb{R}^n)) = (2^c)^+ \). In particular, the class \( D(\mathbb{R}^n) \setminus AC(\mathbb{R}^n) \) is \( 2^c \)-spaceable.

**Proof.** Let \( \pi_1 : \mathbb{R}^n \to \mathbb{R} \) the projection of \( \mathbb{R}^n \) on its first coordinate. Let \( W = V(f) \subset J \) be the vector space of cardinality \( 2^c \) build in Proposition 4.3. Then the vector space \( V = \{ f \circ \pi_1 : f \in W \} \)

is obviously contained in \( D(\mathbb{R}^n) \) and has dimension \( 2^c \). On the other side, if \( f \in W \) then \( f \circ \pi_1 \) cannot be in \( AC(\mathbb{R}^n) \), because then \( f \) would be continuous. (See [19].) This is not possible, because \( J \cap C = \emptyset \). Therefore, \( V \subset D(\mathbb{R}^n) \setminus AC(\mathbb{R}^n) \). To finish, let us remark that the space \( V \) is also closed by pointwise convergence. \( \square \)

**Remark 4.8.** Notice that, in \( \mathbb{R}^n \) (for every \( n \in \mathbb{N} \)), we have that \( AC \setminus \text{Ext} \) is \( 2^c \)-spaceable (since this class contains the Jones functions). Also, in \( \mathbb{R} \), \( J \subset AC \setminus SCIVP \subset AC \setminus \text{Ext} \) and, since \( L_p(J) = (2^c)^+ \), we have (from the previous results) that

\( L_p(AC \setminus \text{Ext}) = L_u(AC \setminus \text{Ext}) = (2^c)^+ \).

**Problem 4.9.** For \( n \geq 2 \), determine the values of the numbers \( L_p(AC(\mathbb{R}^n) \cap D(\mathbb{R}^n)) \), \( L_u(AC(\mathbb{R}^n) \cap D(\mathbb{R}^n)) \), and \( L_p(AC(\mathbb{R}^n) \cap D(\mathbb{R}^n)) \).

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