GROTHENDIECK’S THEOREM FOR ABSOLUTELY SUMMING MULTILINEAR OPERATORS IS OPTIMAL

D. PELLEGRINO AND J. B. SEOANE-SEPÚLVEDA

Abstract. Grothendieck’s theorem asserts that every continuous linear operator from \( \ell_1 \) to \( \ell_2 \) is absolutely \((1;1)\)-summing. In this note we prove that the optimal constant \( g_m \) so that every continuous \( m \)-linear operator from \( \ell_1 \times \cdots \times \ell_1 \) to \( \ell_2 \) is absolutely \((g_m;1)\)-summing is \( 2^{m+1} \). We also show that if \( g_m < 2^{m+1} \) there is \( c \)-dimensional linear space composed by continuous non absolutely \((g_m;1)\)-summing \( m \)-linear operators from \( \ell_1 \times \cdots \times \ell_1 \) to \( \ell_2 \). In particular, our result solves (in the positive) a conjecture posed by A.T. Bernardino in 2011.

1. Introduction

A celebrated result of Grothendieck asserts that every continuous linear operator from \( \ell_1 \) to \( \ell_2 \) is absolutely \((1;1)\)-summing. It was recently proved [3] that this result can be lifted to multilinear operators in the following fashion:

Every continuous \( m \)-linear operator from \( \ell_1 \times \cdots \times \ell_1 \) to \( \ell_2 \) is absolutely \( \left( \frac{2^{m+1}}{m+1}, 1 \right) \)-summing.

In the same paper the author conjectured that the value \( 2^{m+1} \) is optimal. A particular case of our main result gives a positive solution to this conjecture:

Theorem 1.1. The estimate \( 2^{m+1} \) is optimal. Moreover, if \( g_m < 2^{m+1} \) then there exists a \( c \)-dimensional linear space formed (except by the null vector) by continuous non absolutely \((g_m;1)\)-summing \( m \)-linear operators. This result is optimal in terms of dimension.

Above, \( c \) denotes the cardinality of the continuum. In other words, our main result shows that if \( g_m < 2^{m+1} \), the set of continuous non absolutely \((g_m;1)\)-summing multilinear operators is \( c \)-lineable and moreover, maximal lineable. For the theory of lineability we refer to [1, 2] and the references therein.

Our proof of the optimality of \( 2^{m+1} \) is inspired on ideas that date back to the classical work of Lindenstrauss and Pelczyński [9] and, later, explored in a series of papers (see, e.g., [4, 5, 6, 10]).

Throughout this note, \( X \) and \( Y \) shall stand for Banach spaces over \( \mathbb{K} = \mathbb{R} \) or \( \mathbb{C} \). The closed unit ball of \( X \) is denoted by \( B_X \) and the topological dual of \( X \) by \( X^* \). The closed unit ball of \( X \) is denoted by \( B_X \) and the topological dual of \( X \) by \( X^* \). Also, recall that a continuous linear operator \( u : X \to Y \) is absolutely \((q;1)\)-summing (see [7]) if there exists \( C \geq 0 \) such that

\[
\left( \sum_{j=1}^{n} \| u(x_j) \|^{q} \right)^{\frac{1}{q}} \leq C \sup_{\varphi \in B_{X^*}} \left( \sum_{j=1}^{n} | \varphi(x_j) | \right)
\]

for every \( n \in \mathbb{N} \) and \( x_1, \ldots, x_n \in X \). The nonlinear theory of absolutely summing operators was designed by Pietsch in 1983 ([11]) and since then has been intensively studied. One of the possible polynomial generalizations of absolutely summing operators is the concept of absolutely summing polynomial. The space of continuous \( m \)-homogeneous polynomials from \( X \) to \( Y \) will be henceforth denoted by \( \mathcal{P}^{(m)}(X;Y) \). Given a positive integer \( m \) and \( q \geq \frac{1}{m} \), a continuous \( m \)-homogeneous polynomial \( P : X \to Y \) is absolutely \((q;1)\)-summing if there exists a constant

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$C \geq 0$ such that
\[
\left( \sum_{j=1}^{n} \|P(x_j)\|^q \right)^{\frac{1}{q}} \leq C \left( \sup_{\varphi \in B_{X^*}} \sum_{j=1}^{n} |\varphi(x_j)| \right)^{\frac{m}{q}}
\]
for every $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in X$. If $q < 1$, the infimum of the constants $C$ satisfying the above inequality is a Banach quasinorm for the space of absolutely $(q,1)$-summing polynomials from $X$ to $Y$, and it is denoted by $\pi_{q,1}$. For multilinear mappings the definition is similar:

A continuous $m$-linear operator $T : X \times \cdots \times X \to Y$ is absolutely $(q;1)$-summing (with $q \geq \frac{1}{m}$) if there is a constant $C \geq 0$ such that
\[
\left( \sum_{j=1}^{n} \|T(x_j^{(1)}, \ldots, x_j^{(m)})\|^q \right)^{\frac{1}{q}} \leq C \prod_{k=1}^{m} \left( \sup_{\varphi \in B_{X^k}} \sum_{j=1}^{n} |\varphi(x_j^{(k)})| \right)
\]
for every $n \in \mathbb{N}$ and $x_1^{(k)}, \ldots, x_n^{(k)} \in X$, and $k = 1, \ldots, m$. If $q < 1$, the infimum of the constants $C$ satisfying the above inequality is a Banach quasinorm for the space of absolutely $(q;1)$-summing $m$-linear operators from $X \times \cdots \times X$ to $Y$.

2. THE PROOF OF THEOREM 1.1

Let $1 \leq g_m < \frac{2}{m+1}$. The first part of our argument is mentioned en passant, without proof, in [6], but since we have a more self-contained approach, we present the details for the sake of completeness. Let $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in \ell_1$ be non null vectors. Consider $x_1^*, \ldots, x_n^* \in B_{\ell_\infty}$ so that $x_j^*(x_j) = \|x_j\|$ for every $j = 1, \ldots, n$. Let $a_1, \ldots, a_n$ be scalars such that $\sum_{j=1}^{n} |a_j|^{\frac{2}{m+1}} = 1$ and define
\[
P_n : \ell_1 \to \ell_2, \quad P_n(x) = \sum_{j=1}^{n} a_j x_j^*(x)^{m} e_j,
\]
where $e_j$ is the $j$-th canonical vector of $\ell_2$. For every $x \in \ell_1$,
\[
\|P_n(x)\| = \left( \sum_{j=1}^{n} \left| a_j \frac{1}{m} x_j^*(x)^m \right|^2 \right)^{\frac{1}{2}} \leq \left( \sum_{j=1}^{n} |a_j|^{\frac{2}{m+1}} \right)^{\frac{1}{2}} \|x\|^m = \|x\|^m.
\]

Since $P_n$ is a polynomial of finite type, then it is plain that $P_n$ is absolutely $(g_m;1)$-summing. Note that for $k = 1, \ldots, n$, we have
\[
\|P_n(x_k)\| = \left| \sum_{j=1}^{n} a_j \frac{1}{m} x_j^*(x_k)^m e_j \right| \geq \left| a_k \frac{1}{m} x_k^*(x_k)^m \right| = |a_k|^{\frac{1}{m}} \|x_k\|^m.
\]

To simplify the notation we write $\|\cdot\|_{w,1} := \sup_{\varphi \in B_{X^*}} \sum_{j=1}^{n} |\varphi(x_j^{(k)})|$. Thus, we have
\[
\left( \sum_{j=1}^{n} \|x_j\|^{mg_m} |a_j| \right)^{\frac{1}{mg_m}} \leq \left( \sum_{j=1}^{n} \|P_n(x_j)\|_{w,1} \right)^{\frac{1}{mg_m}} \leq \pi_{g_m,1}(P_n) \|\cdot\|_{w,1}^{\frac{m}{g_m}}.
\]
Since this last inequality holds whenever \( \sum_{j=1}^{n} |a_j| \frac{2}{g_m} = 1 \), denoting \( \left( \frac{2}{g_m} \right)^* \) to the conjugate of \( \frac{2}{g_m} \)
we obtain
\[
\left( \sum_{j=1}^{n} \|x_j\|^{mg_m} \left( \frac{2}{g_m} \right)^* \right)^{\frac{1}{mg_m}} \leq \sup \left\{ \sum_{j=1}^{n} |a_j| \|x_j\|^{mg_m}; \sum_{j=1}^{n} |a_j|^{\frac{2}{g_m}} = 1 \right\}
\leq \left( \pi_{gm,1}(P_n) \|\langle x_j \rangle_{j=1}^{n} \|^m \right)^{gm}
\]
and, then,
\[
\left( \sum_{j=1}^{n} \|x_j\|^{mg_m} \left( \frac{2}{g_m} \right)^* \right)^{\frac{1}{mg_m}} \leq \pi_{gm,1}(P_n).
\]

Since \( 1 \leq g_m < \frac{2}{m-1} \), we have \( mg_m \left( \frac{2}{g_m} \right)^* < 2 \) and from a weak version of the Dvoretzky-Rogers Theorem we know that \( id_{\ell_1} \) is not \( \left( mg_m \left( \frac{2}{g_m} \right)^*; 1 \right) \)-summing. Combining this fact with (1) we conclude that we can find \( x_j \) in \( \ell_1 \) for all positive integer \( j \) so that
\[
\lim_{n \to \infty} \pi_{gm,1}(P_n) = \infty \text{ and } \|P_m\| = 1.
\]

We thus conclude that the space of all absolutely \( (g_m; 1) \)-summing \( m \)-homogeneous polynomials from \( \ell_1 \) to \( \ell_2 \) is not closed in \( P^n \ell_1; \ell_2 \). In fact, otherwise, since the quasinorm \( \pi_{gm,1} \) is complete, the Open Mapping Theorem to \( F \)-spaces would contradict (2).

Now, let \( P : \ell_1 \to \ell_2 \) be a continuous non \( (g_m; 1) \)-summing \( m \)-homogeneous polynomial. Split \( \mathbb{N} \) into a countable union of pairwise disjoint countable sets \( \mathbb{N}_1, \mathbb{N}_2, \ldots \). For all \( j \), let
\[
\mathbb{N}_j = \left\{ a_{1}^{(j)} < a_{2}^{(j)} < \cdots \right\},
\]
and define \( P^{(j)} : \ell_1 \to \ell_2 \) by \( (P^{(j)}(x))_{k}^{(j)} = (P(x))_{k} \) and \( (P^{(j)}(x))_{k} = 0 \) if \( k \notin \mathbb{N}_j \). It is simple to prove that \( P^{(j)} \) is also a continuous non \( (g_m; 1) \)-summing \( m \)-homogeneous polynomial and the set \( \{ P^{(1)}, P^{(2)}, \ldots \} \) is linearly independent. Finally, we note that the linear operator \( \Phi : \ell_1 \to P^n \ell_1; \ell_2 \) given by \( (\beta_j)_{j=1}^{\infty} \mapsto \sum_{j=1}^{\infty} \beta_j P^{(j)} \) is injective and it is simple to verify that
\( \Phi (\ell_1) \) is composed (except by the null vector) exclusively by non absolutely \( (g_m; 1) \)-summing \( m \)-homogeneous polynomials. We thus conclude that the set of continuous non \( (g_m; 1) \)-summing \( m \)-homogeneous polynomials is \( \Phi (\ell_1) \)-lineable. Since \( \dim \Phi (\ell_1) = \dim \ell_1 = c \), we conclude that this set is \( c \)-lineable.

Since \( P^n (\ell_1; \ell_2) \) is isomorphic to the space of symmetric \( m \)-linear operators from \( \ell_1 \times \cdots \times \ell_1 \) to \( \ell_2 \) and since \( P \) is absolutely \( (g_m; 1) \)-summing if and only if its associated symmetric \( m \)-linear operator is absolutely \( (g_m; 1) \)-summing, our result is translated to the multilinear setting.

The above result is optimal in terms of dimension. In fact, it is well known that \( \ell_1 \) is isometric to the completion of its projective tensor product, i.e., \( \ell_1 = \ell_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi \ell_1 \). Thus
\[
\dim L \left( \ell_1; \ell_2 \right) = \dim L \left( \ell_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi \ell_1; \ell_2 \right) = \dim L (\ell_1; \ell_2) = c.
\]

We remark that if \( \pi_{gm,1} \) was locally convex, since we have proved that the space of all absolutely \( (g_m; 1) \)-summing \( m \)-homogeneous polynomials from \( \ell_1 \) to \( \ell_2 \) is not closed in the space of all continuous \( m \)-homogeneous polynomials from \( \ell_1 \) to \( \ell_2 \), then from a result due to Drewnowski (see [8, Theorem 5.6 and its reformulation]) we would conclude that the set of all continuous \( m \)-homogeneous polynomials from \( \ell_1 \) to \( \ell_2 \) that fail to be absolutely \( (g_m; 1) \)-summing is spaceable, i.e., contains (except for the null vector) a closed infinite-dimensional subspace.
References


