INTEGRABLE DEFORMATIONS OF
ALGEBRAIC CURVES. *

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Abstract

A general scheme for determining and studying integrable deformations of algebraic curves, based on the use of Lenard relations, is presented. We emphasize the use of several types of dynamical variables: branches, power sums and potentials.

Key words: Algebraic curves. Integrable systems. Lenard relations.
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1 Introduction

The theory of algebraic curves is a fundamental ingredient in the analysis of integrable nonlinear differential equations as it is shown, for example, by its relevance in the description of the finite-gap solutions or the formulation of the Whitham averaging method [1]-[7]. A particularly interesting problem is characterizing and classifying integrable deformations of algebraic curves. In [6]-[7] Krichever formulated a general theory of dispersionless hierarchies of integrable models associated with the deformations of algebraic curves arising in the Whitham averaging method. In [8]-[9] a different approach for determining integrable deformations of algebraic curves curves $C$ defined by monic polynomial equations

$$F(p, k) := p^N - \sum_{n=1}^{N} u_n(k)p^{N-n} = 0, \quad u_n \in \mathbb{C}[k],$$

was proposed. It applies for finding the deformations $C(x, t)$ consistent with the degrees of the polynomials $u_n$ and characterized by the existence of an action function $S = S(k, x, t)$ verifying

1. The multiple-valued function $p = p(k)$ determined by (1) can be expressed as
   $$p = S_x.$$

2. The function $S_t$ represents, like $p = S_x$, a meromorphic function on $C(x, t)$ with poles only at $k = \infty$.

As a consequence of these conditions $p$ obeys the equation

$$\partial_t p = \partial_x Q,$$

where $Q := S_t$ is of the form [8, 10] (i.e. $Q \in \mathbb{C}[k, p]/$),

$$Q = \sum_{r=1}^{N} a_r(k, x, t)p^{N-r}, \quad a_r \in \mathbb{C}[k].$$

The form of equation (2) motivates the use of several sets of dynamical variables to formulate these deformations conveniently. One should also note that the equation (2) provides an infinite number of conservation laws, when one expands $p$ and $S$ in the Laurent series in $k$. In this sense, we say that the equation (2) is integrable.
2 Algebraic curves and dynamical variables

In order to describe deformations of the curve \( C \) defined by (1) we will use not only the coefficients \( u_n \) (potentials) but also some other alternative sets of dynamical variables. Firstly, we have the \( N \) branches \( p_i = p_i(k) \) \((i = 1, \ldots, N)\) of the multiple-valued function function \( p = p(k) \) satisfying

\[
F(p, k) = \prod_{i=1}^{N} (p - p_i(k)) = 0. \tag{3}
\]

There is an important result concerning the branches \( p_i \). Let \( \mathbb{C}((\lambda)) \) denote the field of Laurent series in \( \lambda \) with at most a finite number of terms with positive powers

\[
\sum_{n=-\infty}^{N} c_n \lambda^n, \quad N \in \mathbb{Z}.
\]

Then we have \([11-12]\) :

**Theorem 1. (Newton Theorem)**

There exists a positive integer \( l \) such that the \( N \) branches

\[
p_i(z) := \left( p_i(k) \right) \bigg|_{k=z^l}, \tag{4}
\]

are elements of \( \mathbb{C}((z)) \). Furthermore, if \( F(p, k) \) is irreducible as a polynomial over the field \( \mathbb{C}((k)) \) then \( l_0 = N \) is the least permissible \( l \) and the branches \( p_i(z) \) can be labelled so that

\[
p_i(z) = p_N(\epsilon^l z), \quad \epsilon := \exp \frac{2\pi i}{N}.
\]

**Notation convention** Henceforth, given an algebraic curve \( C \) we will denote by \( z \) the variable associated with the least positive integer \( l_0 \) for which the substitution \( k = z^{l_0} \) implies \( p_i \in \mathbb{C}((z)), \forall i \).

The potentials can be expressed as elementary symmetric polynomials \( s_n \) \([13-14]\) of the branches \( p_i \)

\[
u_n = (-1)^{n-1} s_n(p_1, p_2, \ldots) = (-1)^{n-1} \sum_{1 \leq i_1 < \ldots < i_n \leq N} p_{i_1} \cdots p_{i_n}, \tag{5}\]

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Notice that, according to the famous Abel theorem \cite{13}, for \( N > 4 \) the branches \( p_i \) of the generic equation (1) cannot be written in terms of the potentials \( u_n \) by means of rational operations and radicals.

Our second set of alternative dynamical variables is provided by the power sums \cite{13-14}

\[
P_k = \frac{1}{k} (p_1^k + \cdots + p_N^k), \quad k \geq 1,
\]

where we have added a factor \( 1/k \) to their standard definition. One can relate potentials and power sums through Newton recurrence formulas, the solution of which is given by Waring’s formula \cite{14}

\[
P_k = \sum_{1 \leq i \leq k} \frac{1}{i} (u_1 + \cdots + u_N)^i,
\]

where the superscript \((k)\) in the summation sign indicates that only the terms of weight \( k \) are retained in the expansion of \((u_1 + \cdots + u_N)^i\) (it is assumed that the weight of the potential \( u_n \) is given by \( n \)). The corresponding inverse formula is \cite{12}

\[
u_n = -S_n(-P_1, -P_2, \ldots)
\]

\[
= -\sum_{i_1+i_2+\cdots+i_k=n} \frac{1}{i_1!i_2!\cdots i_k!} (-P_1)^{i_1} \cdots (-P_k)^{i_k},
\]

where \( S_n \) are the Schur’s polynomials defined by the identity \( \exp(\sum_{n \geq 1} \lambda^n x_n) = \sum_{n \geq 0} \lambda^n S_n(x) \). The formula (8) is the consequence of the identity \cite{15}

\[
\exp \left( -\sum_{n \geq 1} \lambda^n P_n \right) = \sum_{n \geq 0} (-\lambda)^n s_n(p_1, p_2, \ldots).
\]

**Examples**

For \( N = 2 \), the equation for the curve is

\[
F := p^2 - u_1 p - u_2 = 0,
\]

and the first power sums are

\[
P_1 = u_1, \quad P_2 = \frac{1}{2} u_1^2 + u_2.
\]
For $N = 3$

$$F := p^3 - u_1 p^2 - u_2 p - u_3 = 0,$$

we have

$$P_1 = u_1, \quad P_2 = u_2 + \frac{1}{2}u_1^2, \quad P_3 = u_3 + u_1 u_2 + \frac{1}{3}u_1^3,$$

$$P_4 = u_1 u_3 + \frac{1}{2}u_2^2 + u_2 u_1^2 + \frac{1}{4}u_1^4,$$

Power sums have also the meaning of moments of the logarithmic derivative of the function $F(p, k)$ as shown by the formula (see chapter 7 of [16])

$$P_k = \frac{1}{2\pi i k} \oint_{\partial D} p^k \frac{\partial}{\partial p} \log F(p, k) \, dp,$$

where $\partial D$ is the boundary of the domain of the $p$ complex plane which contains all the zeros $p_i$ of $F(p, k)$.

### 3 Deformations of algebraic curves

We are looking for evolution equation of the potentials leading to equations for $p$ of the form (2) or, equivalently, in terms of branches, $p_i$ for $i = 1, \ldots, N$,

$$\partial_t p_i = \partial_x Q_i, \quad \text{with} \quad Q_i := \sum_{r=1}^{N} a_r(k, x, t) p_i^{N-r}.$$  \hspace{1cm} (11)

By using the notations

$$p := (p_1, \ldots, p_N)^\top, \quad a := (a_N, \ldots, a_1)^\top,$$

the system (11) can be written as

$$\partial_t p = \partial_x (V a),$$  \hspace{1cm} (12)

where $V$ is the Vandermonde matrix

$$V := \begin{pmatrix}
1 & p_1 & \cdots & p_1^{N-1} \\
& \ddots & \ddots & \ddots \\
& & \ddots & \ddots \\
1 & p_N & \cdots & p_N^{N-1}
\end{pmatrix}.$$  \hspace{1cm} (13)
which in turn is the Jacobian of the transformation from branches to power sum variables \( \mathbf{P} \). Hence, for power sums we have

\[
\partial_t \mathbf{P} = J \mathbf{a}, \quad \mathbf{P} := (\mathcal{P}_1, \ldots, \mathcal{P}_N)^\top, \quad (14)
\]

where \( J \) is the Hamiltonian operator

\[
J := V^\top \partial_x V. \quad (15)
\]

Notice that

\[
J_{11} = N \partial_x, \quad (16)
\]

\[
J_{ij} = (i + j - 2) \mathcal{P}_{i+j-2} \partial_x + (j - 1) \mathcal{P}_{i+j-2,x}, \quad (i, j) \neq (1, 1).
\]

Finally let us consider the evolution law for the potentials. In virtue of the well-known formula \( \frac{\partial S_n(x)}{\partial x_k} = S_{n-k}(x) \), it is straightforward to deduce from \( \mathbf{P} \) that the Jacobian of the transformation from power sums to potentials is given by

\[
T := \begin{pmatrix}
1 & -u_1 & \cdots & -u_{N-1} \\
0 & 1 & \cdots & -u_{N-2} \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & 1 & 0
\end{pmatrix}. \quad (17)
\]

Therefore one has

\[
\partial_t \mathbf{u} = J_0 \mathbf{a}, \quad \mathbf{u} := (u_1, \ldots, u_N)^\top \quad (18)
\]

where \( J_0 \) is the matrix differential operator

\[
J_0 := T^\top J = T^\top V^\top \partial_x V. \quad (19)
\]

The problem now is to determine expressions for \( \mathbf{a} \) depending on \( k \) and \( \mathbf{u} \) such that \( (18) \) is consistent with the polynomial dependence of \( \mathbf{u} \) on the variable \( k \). That is to say, if \( d_n := \text{degree}(u_n) \) are the degrees of the coefficients \( u_n \) as polynomials in \( k \), then \( (18) \) must satisfy

\[
\text{degree}(J_0 \mathbf{a})_n \leq d_n, \quad n = 1, \ldots, N,
\]
where we are taking into account our notation (18) for \( u \). We observe that if (18) is consistent then, as a consequence of (12), the coefficients of the expansions in \( z \) of the branches \( p_i \) are conserved densities.

Our strategy for finding consistent deformations is to use Lenard type relations

\[
J_0 r = 0, \quad r := (r_1, \ldots, r_N)^\top, \quad r_i \in \mathbb{C}((k)),
\]

(20)
to generate systems of the form

\[
u_t = J_0 a, \quad a := r_+.
\]

(21)

Here \((\cdot)_+\) and \((\cdot)_-\) indicate the parts of non-negative and negative powers in \( k \), respectively. The point is that from the identity

\[
J_0 a = J_0 r_+ = -J_0 r_-,
\]

it is clear that a sufficient condition for the consistency of (21) is that

\[
\text{degree}(J_0)^{nm} \leq d_n + 1,
\]

(22)

for all \( n \) and all \( m \) such that \( a_{N-m+1} = (r_+)_m \neq 0 \).

If we impose (22) for all \( 1 \leq n, m \leq N \), we get a sufficient condition for consistency, which only depends on the curve (11) and does not refer to the particular solution of the Lenard relation used.

**Examples**

For \( N = 2 \) the conditions (22) reduce to

\[
d_1 \leq d_2 + 1.
\]

(23)

This indicates that one can take \( u_2(k) \) to be a polynomial of arbitrary degree, which implies that there exist integrable deformations for hyperelliptic curves with arbitrary genus.

For \( N = 3 \) the consistency conditions (22) become

\[
d_1 \leq 1, \quad d_2 \leq d_1 + 1, \\
d_3 \leq d_2 + 1, \quad d_2 \leq d_3 + 1,
\]

(24)
which lead to the following twelve nontrivial choices for \((d_1, d_2, d_3)\)

\[
(0, 0, 1), (0, 1, 0), (0, 1, 1), (0, 1, 2), \\
(1, 0, 0), (1, 0, 1), (1, 1, 0), (1, 1, 1) \\
(1, 1, 2), (1, 2, 1), (1, 2, 2), (1, 2, 3).
\]

This implies that our deformations allows one to have only trigonal curves 
with genus less than or equal to one. This subject will be discussed elsewhere.

There is a natural class of solutions of the Lenard relations (15). Indeed, 
from the expression (19) of \(J_0\) it is obvious that for any constant vector 
\(c \in \mathbb{C}^N\)

\[
r := V^{-1} c,
\]

is a solution of (15). Hence by taking into account that \(V\) and \(T\) are the 
Jacobians for the transformation from branches to power sums and power 
sums to potentials, respectively, we have that

\[
r_i := \nabla_p p_i = T \nabla_u p_i, \quad i = 1, \ldots, N, 
\]

\[
\nabla_p p_i := \left( \frac{\partial p_i}{\partial P_1} \ldots \frac{\partial p_i}{\partial P_N} \right)^\top, \\
\nabla_u p_i := \left( \frac{\partial p_i}{\partial u_1} \ldots \frac{\partial p_i}{\partial u_N} \right)^\top,
\]

are solutions of the Lenard relations.

In summary, given an algebraic curve (1) satisfying (22) then evolution equations of the form

\[
\partial_t u = J_0 \left( T \nabla_u R \right), \\
R(z, p) = \sum_i f_i(z) p_i,
\]

where \(f_i \in \mathbb{C}[z]\) define a deformation of the curve provided \(R \in \mathbb{C}[k]\).

We conclude with the observation that the nonlinear equations discussed 
above have a very simple form in terms of branches. Indeed by applying 
Cramer rule to find the columns of \(V^{-1}\) it follows that

\[
\frac{\partial p_i}{\partial P_j} = \frac{|1 \ p \ldots e_i \ldots p^{N-1}|}{|1 \ p \ldots e_j \ldots p^{N-1}|},
\]

where \(|1 \ p \ldots e_i \ldots p^{N-1}|\) denotes the determinant of Vandermonde matrix 
\(V\) and \(|1 \ p \ldots e_i \ldots p^{N-1}|\) stands for the determinant of the matrix resulting
from substituting in $V$ the $j$-th column by the vector $(e_i)_j = \delta_{ij}$. Thus from (12) we find
\[
\partial_t p = \partial_x \left( V (\nabla_P R)_+ \right)
\]
\begin{equation}
(\nabla_P R)_i = \frac{|1 p \ldots R_i \ldots p^{N-1}|}{|1 p \ldots p^{i-1} \ldots p^{N-1}|}, \quad i = 1, \ldots, N,
\end{equation}
where $|1 p \ldots R_i \ldots p^{N-1}|$ is the determinant of the matrix resulting from substituting in $V$ the $i$-th column by the vector $(R_i)_j = f_j(z)$.

Finally, several remarks are in order:

1. If (1) is an irreducible curve the condition $R \in \mathbb{C}[k]$ is verified by $z^i \mathcal{L}_i$ where $\mathcal{L}_i$ are the so-called Lagrange resolvents [13]
\[
\mathcal{L}_i = \sum_j \epsilon^{j i} p_j(z), \quad \epsilon := \exp \frac{2\pi i}{N},
\]
where $i = 1, \ldots, N - 1$.

2. The equations (26) admit a simpler expression when they are formulated in terms of power sums. Indeed, they reduce to
\[
\partial_t p = J \left( \nabla_P R \right)_+.
\]
This form should be particularly convenient for analyzing their Hamiltonian content as $J$ defined in (15) is a Hamiltonian operator.

3. The main problem which remains in our analysis is the classification of deformations of algebraic curves (1) for general potentials of fixed degrees in $k$. In [9] a complete description of the deformations of hyperelliptic curves ($N = 2$) was given. Since for $N = 3$ only twelve cases of consistent sets of general potential degrees arise, one is lead to think that for higher $N$ admissible cases different from the Gelfand-Dikii choice ($d_1 = \ldots = d_{N-1} = 0, d_N = 1$) should be very special. This of course must be expected for $N \geq 5$ because of Abel theorem.

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