I. INTRODUCTION

Long ago, Jaffe [1] identified the distinct nature of mesons: those built simply of a quark and an antiquark, and those with additional $\bar{q}q$ pairs. Of course, even well established $\bar{q}q$ resonances, like the $\rho$ and $\omega$, spend part of their time in four and six quark configurations as this is how they decay to $\pi\pi$ and $3\pi$, respectively. However, the $1/N_c$ expansion [2] provides a method of clarifying such differences. If we could tune $N_c$ up from 3, we would see that an intrinsically $\bar{q}q$ state would become narrower and narrower. As $N_c$ increases, the underlying pole, which defines the resonant state, moves along the unphysical sheet(s) towards the real axis. In contrast, a tetraquark state would become wider and wider and its pole would effectively disappear from “physical” effect: if only we could tune $N_c$.

A long recognized feature of the world with $N_c = 3$ is that of “local duality”[3–5]. In a scattering process, as the energy increases from threshold, distinct resonant structures give way to a smooth Regge behavior. At low energy the scattering amplitude is well represented by a sum of resonances (with a background), but as the energy increases the resonances (having more phase space for decay) become wider and increasingly overlap. This overlap generates a smooth behavior of the cross section most readily described not by a sum of a large number of resonances in the direct channel, but the contribution of a small number of crossed channel Regge exchanges. Indeed, detailed studies [4,6] of meson-baryon scattering processes show that the sum of resonance contributions at all energies “averages” (in a well-defined sense to be recalled below) the higher energy Regge behavior. Indeed, these early studies [3,4] revealed how this property starts right from the $\pi N$ threshold, so that this “local duality” holds across the whole energy regime. Thus, resonances in the $s$ channel know about Regge exchanges in the $t$ channel. Indeed, these resonance and Regge components are not to be added like Feynman diagram contributions, but are “dual” to each other: one uses one or the other. Indeed, the wonderful formula discovered by Veneziano [7] is an explicit realization of this remarkable property. This has allowed the idea of “duality” first found in meson-nucleon reactions to be extended to baryon-antibaryon reactions, as well as to the simpler meson-meson scattering channel we consider here [8].

Unlike the idealized Veneziano model with its exact local duality, the real world, with finite width resonances, has a “semilocal duality” quantified by averaging over the typical spacing of resonance towers defined by the inverse of the slope of relevant Regge trajectory.

Regge exchanges too are built from $\bar{q}q$ and multiquark contributions. In a channel like that with isospin 2 in $\pi\pi$ scattering, or isospin 3/2 in $K\pi$ scattering, there are no $\bar{q}q$ resonances, and so the Regge exchanges with these quantum numbers must involve multiquark components. Data teach us that even at $N_c = 3$ these components are suppressed compared to the dominant $\bar{q}q$ exchanges. Semilocal duality means that in $\pi^+\pi^- \rightarrow \pi^-\pi^+$ scattering, the low energy resonances must have contributions to the cross section that “on the average” cancel, since this process is purely isospin 2 in the $t$ channel. The meaning of semilocal duality is that this cancellation happens right from the $\pi\pi$ threshold.

Now in $\pi\pi$ scattering below 900 MeV, there are just two low energy resonances: the $\rho$ with $I = J = 1$ and the $\sigma$ with $I = J = 0$. In the model of Veneziano, where resonances contribute as delta functions, exact local duality is achieved by the $\sigma$ and $\rho$ having exactly the same mass, and the coupling squared of the $\sigma$ is 9/2 times that of the $\rho$. Of course, the Veneziano amplitude is too simplistic and does not respect two body unitarity. Yet nevertheless, in the real world with $N_c = 3$ with finite width resonances “semilocal” duality is at play right from threshold. There
is a cancellation between the \( \rho \) with a width of 150 MeV, which is believed to be predominantly a \( \bar{q}q \) state, and the \( \sigma \), which is very broad, at least 500 MeV wide, with a shape that is not Breit-Wigner like, and might well be a tetraquark, molecular [9] or gluonic state [10,11], or possibly a mixture of all of these. Its short-lived nature certainly means it spends most of its existence in a di-pion configuration. The contribution of these two resonances to the \( \pi^+\pi^- \) cross section do indeed “on average” cancel in keeping with \( I = 2 \) exchange in the \( t \) channel. However, such a distinct nature for the \( \rho \) and \( \sigma \) would prove a difficulty if we could increase \( N_c \) such a distinct nature for the \( \sigma \) and } 

keeping with [12], six more at two loops [21], etc. These have to be } 
pion-pion scattering amplitudes: four at one loop order } 
orders new low energy constants (LECs) enter in the } 
parameter, the pion decay constant. However, at higher } 
resonance). In contrast, at least for not too large \( N_c \), the \( \sigma \) pole became wider and moved away from the 400 to 600 MeV region of the real energy axis, as anticipated by a largely \( \bar{q}q \bar{q}q \) nature. As we shall discuss, and as already } } 
introduced, this means that for the central values and most } 
parameter space, the semilocal duality implicit in finite } 
energy sum rules (FESRs) is not satisfied as \( N_c \) increases. 

Subsequently, one of us (J. R. P.) together with Rios showed [24] that the \( N_c \) dependence of the parameters is determined. Every \( \chi \) PT, starting with \( f_\pi \) has a well-defined leading \( N_c \) behavior [12,22], for } 
instance, \( f_\pi \sim \sqrt{N_c} \). At one-loop order with central values for } 
the LECs, one of us (J. R. P.) has studied unitarized low } 
energy \( \pi\pi \) scattering as \( N_c \) increases [23], showing how the } 
\( \rho \) does indeed become narrower (as expected of a \( \bar{q}q \) resonance). In contrast, at least for not too large \( N_c \), the \( \sigma \) pole became wider and moved away from the 400 to 600 MeV region of the real energy axis, as anticipated by a largely \( \bar{q}q \bar{q} \bar{q} \) nature. As we shall discuss, and as already } 
introduced, this means that for the central values and most } 
parameter space, the semilocal duality implicit in finite } 
energy sum rules (FESRs) is not satisfied as \( N_c \) increases. 

Subsequently, one of us (J. R. P.) together with Rios showed [24] that the \( N_c \) behavior becomes more subtle when two-loop \( \chi \) PT effects are included. In particular, for } 
the best fits of the unitarized two-loop \( \chi \) PT, there is a \( \bar{q}q \) component of the \( \sigma \), which while subordinate at \( N_c = 3 \), becomes increasingly important as \( N_c \) increases. The \( \sigma \) pole still moves away from the 400–600 MeV region of the } 
real axis, but the pole trajectory turns around moving back } 
towards the real axis above 1 GeV as \( N_c \) becomes larger than 10 or so. This occurs rather naturally in the two-loop results but was only hinted in some part of the one-loop } 
parameter space. Such a behavior would indicate that, } 
while the \( \sigma \) is predominantly non-\( \bar{q}q \) at \( N_c = 3 \), it does } 
have a \( \bar{q}q \) component. As we show here, it is this component } 
that ensures FESRs are satisfied. Regge expectations then } 
hold at all \( N_c \). Indeed, imposing this as a physical } 
requirement places a constraint on the second order LECs: } 
a constraint readily satisfied with LECs in fair agreement } 
with current crude estimates. 

Thus, chiral dynamics already contains the resolution of the } 
paradox that was the motivation for this study: namely, how } 
does the suppression of \( I = 2 \) Regge exchanges happen if } 
resonances like the \( \rho \) and \( \sigma \) are intrinsically } 
different. We will see that the \( \sigma \) may naturally } 
contain a small but all important \( \bar{q}q \) component. At large } 
\( N_c \) this would be the seed of this state. As \( N_c \) is lowered this } 
state will have an increased coupling to pions, and it is } 
these that dominate its existence when \( N_c = 3 \). We will, } 
of course, discuss the range of \( N_c \) for which the IAM applies } 
and where replacing the LECs (at \( N_c = 3 \)) with their } 
leading \( N_c \) form is appropriate.
II. SEMILocal DUALITY AND FINITE ENERGY SUM RULES

A. Regge theory and semilocal duality

Regge considerations lead us to study $s$-channel $\pi\pi$ scattering amplitudes with definite isospin in the $t$ channel, labeled $A^d(s, t)$. These can, of course, be written in terms of amplitudes with definite isospin in the direct channel, $A^d(s, t)$, using the well-known crossing relationships, so that

$$
A^0(s, t) = \frac{1}{3} A^0(s, t) + A^1(s, t) + \frac{5}{3} A^2(s, t)
$$

$$
A^1(s, t) = \frac{1}{3} A^0(s, t) + \frac{1}{2} A^1(s, t) - \frac{2}{5} A^2(s, t)
$$

$$
A^2(s, t) = \frac{1}{3} A^0(s, t) - \frac{1}{2} A^1(s, t) + \frac{2}{5} A^2(s, t).
$$

It is convenient to denote the common channel threshold by $s_{th} \equiv t_{th} = 4m^2_f$. The amplitudes of Eq. (1) have definite symmetry under $s \leftrightarrow t$ and this will be reflected in writing them as functions of $\nu = (s - t)/2$, a variable for which $\nu = s = -t$ along the line $t = t_{th}$. To check semilocal duality, we need to continue the well-known Regge asymptotics at fixed $t$ down to threshold. To do this we follow [25] with

$$
\text{Im}^\prime A^d_{\text{Regge}}(\nu, t) = \sum_R \beta_R(\nu) \Theta(\nu) \left[ \alpha^2(\nu^2 - \nu_{th}^2) \right]^{\alpha(t)/2},
$$

where as usual the $\alpha_R(t)$ denote the Regge trajectories with the appropriate $t$-channel quantum numbers, $\beta_R(t)$ their Regge couplings, and $\alpha'$ is the universal slope of the $q\bar{q}$ meson trajectories ($\sim 0.9 \text{ GeV}^{-2}$). The crossing function $\Theta(\nu) = [1 - 2\nu/\nu^2]^{1+\gamma}$ having $\gamma = 0$ for $s - u$-even amplitudes, and $\gamma = 1/2$ if they are crossing odd, ensures the imaginary parts of the amplitudes vanish at threshold, while being unity when $\nu$ is large. $\nu_{th}$ is the value of $\nu$ at threshold, viz. $\nu_{th} = (s_{th} + t_{th})/2$. For the amplitude with $I = 1$ in the $t$ channel, for which $\gamma = 1$, the sum in Eq. (2) will be dominated by $\rho$ exchange with a trajectory $\alpha(t) = \alpha_0 + \alpha' t$ that has the value 1 at $t = m^2_\rho$ and 3 at $t = m^2_{\rho'}$ [26], i.e. $\alpha_0 = 0.467$ and $\alpha' = 0.889 \text{ GeV}^{-2}$. For isoscalar exchange the dominant trajectories are the Pomeron with $\alpha_p(t) = 1.083 + 0.25t$ (with $t$ in GeV$^2$ units) [27] and the $f_2$ trajectory which is almost degenerate with that of the $\rho$. For the exotic $I = 2$ channel with its leading Regge exchange being a $\rho - \rho$ cut, we expect $\alpha(0) \ll \alpha_p(0)$, and its couplings to be correspondingly smaller.

Semilocal duality between Regge and resonance contributions teaches us that

$$
\int_{\nu_{th}}^{\nu_{max}} d\nu \nu^{-\alpha} \text{Im} A^d_{\text{Regge}}(\nu, t) \approx \int_{\nu_{th}}^{\nu_{max}} d\nu \nu^{-\alpha} \text{Im} A^d_{\text{Regge}}(\nu, t).
$$

the “averaging” should take place over at least one resonance tower. Thus, the integration region $\nu_{max} - \nu_{th}$ should be a multiple of $1/\alpha'$, typically 1 GeV$^2$. We will consider two ranges from threshold to 1 GeV$^2$ and up to 2 GeV$^2$.

This duality should hold for values of $t$ close to the forward scattering direction, and so we consider both $t = 0$ and $t = t_{th}$. The difference in results between these two gives us a measure of the accuracy of semilocal duality, as expressed in Eq. (3). Since we are interested in the resonance integrals being saturated by the lightest states, we consider values of $n = 0$ to $n = 3$. We will find that with $n = 1, 2, 3$ the low mass resonances do indeed control these finite energy sum rules.

B. Finite energy sum rules from data (i.e. $N_c = 3$)

Let us first look at $\pi\pi$ scattering data and see how well it approximates this relationship, before we consider the various resonances contributions that make up the “data” and in turn how these might change with $N_c$. To do this it is useful to define the following ratio:

$$
R_n = \frac{\int_{\nu_{th}}^{\nu_{max}} d\nu \nu^{-\alpha} \text{Im} A^d_{\text{Regge}}(\nu, t)}{\int_{\nu_{th}}^{\nu_{max}} d\nu \nu^{-\alpha} \text{Im} A^d_{\text{Regge}}(\nu, t)}.
$$

The behavior of such a ratio tests the way the low energy amplitudes average the expected leading Regge energy dependence of Eq. (2)—the leading Regge behavior because only then does the Regge coupling $\beta_R(t)$ cancel out in the ratio. We will consider these ratios with $\nu_1$ at its threshold value, $\nu_2 = 1 \text{ GeV}^2$ and $\nu_3 = 2 \text{ GeV}^2$. In evaluating the amplitudes in Eq. (1), we represent them by a sum of $s$-channel partial waves, so that

$$
\text{Im} A^d(s, t) = \sum_{\ell} (2\ell + 1) \text{Im} A^d_{\ell}(s) P_\ell(z_s),
$$

where the sum involves only even $\ell$ for $I = 0$, 2 and odd $\ell$ for $I = 1$. $P_\ell(z_s)$ are the usual Legendre polynomials, with $z_s$ the cosine of the $s$ channel c.m. scattering angle related to the Mandelstam variables by $z_s = 1 + 2t/(s - s_{th})$. It is useful to note that the partial wave amplitudes behave towards threshold like $A^d_\ell \sim (s - s_{th})^{2\ell+1}$, so that the imaginary parts that appear in Eqs. (3)–(5) behave like $(s - s_{th})^{2\ell+1}$ from unitarity.

We use the partial wave parametrization from Kamiński, Peláez, and Yndurain (KPY) [29] to represent the data. The partial wave sum is performed in two ways: first including partial waves up to and including $\ell = 2$, and second with just the $S$ and $P$ waves. We compare each of these in Table I with the evaluation of the ratios in Eq. (4) using the leading Regge pole contribution. This serves as a guide as to

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1Because of the rapid convergence of the sum rules we consider, the fact the Pomeron form used violates the Froissart bound is of no consequence. This has been explicitly checked by also using the parametrization of Cadell et al. [28].

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TABLE I. \( R_n^i \) ratios defined in Eq. (4) evaluated using the Regge model of Eq. (2) and the KPY \( \pi \pi \) parametrization [29] with and without \( D \) waves.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( t = t_{\text{th}} ) ( I_t = 0 )</th>
<th>( t = 0 )</th>
<th>( t = t_{\text{th}} )</th>
<th>( t = 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>REGGE</td>
<td>0</td>
<td>0.225 0.325</td>
<td>0.342 0.452</td>
<td>0.347 0.578</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>0.425 0.578</td>
<td>0.582 0.757</td>
<td>0.725 0.914</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.705 0.839</td>
<td>0.815 0.947</td>
<td>0.894 0.992</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0.916 0.966</td>
<td>0.953 0.992</td>
<td>0.971 0.992</td>
</tr>
</tbody>
</table>

| KPY S, P, D | 0 | 0.337 ± 0.093 | 0.342 ± 0.083 | 0.479 ± 0.213 | 0.492 ± 0.191 |
| KPY S, P   | 0 | 0.615 ± 0.169 | 0.572 ± 0.133 | 0.743 ± 0.187 | 0.709 ± 0.103 |

(i) how well semilocal duality of Eq. (3) works from experimental data in the world of \( N_s = 3 \) by comparing the Regge “prediction” with the KPY representation of experiment, and

(ii) by comparing how well the integrals are dominated by just the lowest partial waves \( \ell = 1 \) with \( \ell = 2 \).

This will be needed to address how the duality relation of Eq. (3) puts constraints on the nature of the \( \rho \) and \( \sigma \) resonances. We present these results in Table I. The \( n = 1 \) integral would with \( t = 0 \) be closest to averaging the total cross section. The table shows that the data follow the expectations of semilocal duality from the dominant Pomeron and \( \rho \) Regge exchange immediately above threshold to 1 and 2 GeV\(^2\). As expected this works best for \( n \geq 1 \) when the low energy regime dominates. We see that including just \( S \) and \( P \) waves is not sufficient for this agreement. For the \( n = 0 \) sum rule even higher waves than \( D \) are crucial in integrating up to 2 GeV\(^2\). In contrast for \( n = 3 \) of course just \( S \) and \( P \) are naturally sufficient. Higher values of \( n \) would weight the near threshold behavior of all waves even more and this region is less directly controlled by resonance contributions alone but their tails down to threshold, where Regge averaging is less likely to be valid. Thus, we restrict attention to our finite energy sum rules with \( n = 1–3 \). It is important to note that all we require is the fact that the \( I_t = 2 \) exchange is lower lying than those with \( I_t = 0, 1 \). That the continuation of Regge behavior for the absorptive parts of the amplitude actually does average resonance-dominated low energy data even with sum rules with \( n = 2, 3 \) is proved by considering the \( P \) and \( D \)-wave scattering lengths. With scattering lengths defined by being the limit of the real part of the appropriate partial waves, Eq. (5), as the momentum tends to zero:

\[
a^i_\ell = \lim_{p \to 0} \mathcal{A}^i_\ell(s)/(p/m_\pi)^{2\ell},
\]

where \( p = \frac{1}{2} \sqrt{s - s_{\text{th}}} \). Then by using the Froissart-Gribov representation for the partial wave amplitudes, we have

\[
a^1_\ell = \frac{4}{3\pi} \int_{s_{\text{th}}}^{\infty} ds \frac{s}{s^2} \text{Im} A^i_\ell(s, t_{\text{th}}),
\]

\[
a^0_2 = \frac{16}{15\pi} \int_{s_{\text{th}}}^{\infty} ds \frac{s}{s^2} \text{Im} A^{i0}_\ell(s, t_{\text{th}}).
\]

If we evaluate these integrals using just the Regge representation from threshold up, we find the following result:

\[
m^2_\pi a^1_\ell = \frac{1}{12\pi} \beta^\rho(t_{\text{th}})(\alpha(s_{\text{th}})\alpha_\rho)^{1/2} \Gamma\left(1 + \frac{\alpha_p}{2}\right) \Gamma\left(1 - \frac{\alpha_p}{2}\right). \quad (7)
\]

\[
m^2_\pi a^0_2 = \frac{1}{120\pi} \sum_{R = \rho, f_2} \beta^\rho(t_{\text{th}})(\alpha(s_{\text{th}})\alpha_\rho)^{1/2} \Gamma\left(1 + \frac{\alpha_R}{2}\right)
\times \Gamma\left(1 - \frac{\alpha_R}{2}\right). \quad (10)
\]

where each \( \alpha_R \) is to be evaluated at \( t = t_{\text{th}} \). Analysis of high energy \( NN \) and \( \pi N \) scattering [30,31] determines the couplings \( \beta^\rho \) of the contributing Regge poles to \( \pi \pi \) scattering through factorization [25]. In the case of the \( \rho \), the value of the residue is known to be almost proportional to \( \alpha_p(t) \) putting a zero close to \( t \approx -0.5 \) (GeV\(^2\)) and reproducing the correct \( \rho \pi \pi \) coupling at \( t = m_\rho^2 \). This is more like the shape shown in Ref. [32] than that proposed earlier by Harita et al. [30,31]. This fixes \( \beta^\rho(t = t_{\text{th}}) = 0.84 \pm 0.13 \) from the “best value” of the analysis of Ref. [31].

The suppression of \( I = 2 \) \( s \)-channel amplitudes that is basic to our assumptions here requires an exchange degeneracy between the \( \rho \) and \( f_2 \) trajectories, so that

\[ \text{Note that the amplitudes defined in [31] are } \pi/4 \text{ times those used here.} \]
\( \beta_f = 3 \beta_p / 2 \), as in the “best value” fit of Ref. [31]. With the Pomeron contribution proportional to a \( \pi \pi \) cross section of \( 16 \pm 2 \) mb for \( s \approx 5-8 \) GeV\(^2\). This gives

\[
\begin{align*}
    m^2_{a_1} &= (3.4 \pm 0.5) \times 10^{-2}, \\
    m^2_{a_2} &= (1.67 \pm 0.19) \times 10^{-3}.
\end{align*}
\]

These values are to be compared with the precise values found by Colangelo, Gasser, and Leutwyler [33] from a dispersive analysis of \( \pi \pi \) amplitudes combining Roy equations and \( \chi \)PT predictions

\[
\begin{align*}
    m^2_{a_1} &= (3.79 \pm 0.06) \times 10^{-2}, \\
    m^2_{a_2} &= (1.75 \pm 0.03) \times 10^{-3},
\end{align*}
\]

or the recent dispersive analysis by two of us and other collaborators in [34], which includes the latest NA48/2 \( K_e4 \) decay results [35] and no \( \chi \)PT,

\[
\begin{align*}
    m^2_{a_1} &= (3.81 \pm 0.09) \times 10^{-2}, \\
    m^2_{a_2} &= (1.78 \pm 0.03) \times 10^{-3}.
\end{align*}
\]

We see that the presumption that Regge parametrization averages the low energy scattering in terms of sum rules with \( n = 2, 3 \) is borne out with remarkable accuracy: far greater accuracy than underlies our fundamental assumption that \( I = 2 \) \( s \)-channel resonances and \( t \)-channel exchanges are suppressed relative to those with \( I = 0 \) and 1. This is further supported by the fact that the \( I = 2 \) \( D \)-wave scattering length as determined in [33,34] is indeed a factor of 10 smaller than that for \( I = 0 \). The required cancellation between the \( \rho \) and the \( \sigma \) contributions that is the subject of this paper requires a less stringent relation than nature imposes at \( N_c = 3 \).
imposing the leading Black lines correspond to the fit described in the text [36].

For different values of \(N_c\) at \(N_c/3\) and \(N_c/6\), since it is so very close to the central line.

To quantify the \(N_c\) dependence at different orders in \(\chi PT\) and with different choices of LECs, we calculate the estimated uncertainty from the choice of \(\mu\). This range is not plotted for the \(\rho\), since it is so very close to the central line.

FIG. 2 (color online). Position of the \(\rho\) and \(\sigma\) poles in the complex energy plane as a function of \(N_c\) in one-loop \(\chi PT\). Black lines correspond to the fit described in the text [36] imposing the leading \(1/N_c^0\) behavior of the LECs at the usual renormalization scale \(\mu = 770\) MeV. Note the different vertical scales for the \(\rho\) and \(\sigma\) poles. The lighter points delineate the estimated uncertainty from the choice of \(\mu\). This range is not plotted for the \(\rho\), since it is so very close to the central line.

at \(N_c = 3\) the positive \(\sigma\) and negative \(\rho\) components cancel. This is not the case as \(N_c\) increases to 12.

To quantify the \(N_c\) dependence at different orders in \(\chi PT\) and with different choices of LECs, we calculate the value of finite energy sum rules (FESR) ratios:

\[
F_{n}^{\mu}(t) = \frac{\int_{T_0}^{T_{\text{max}}} d\nu \text{Im} A_{\mu}^{\mu}(\nu, t, N_c) / \nu^n}{\int_{T_0}^{T_{\text{max}}} d\nu \text{Im} A_{\mu}^{\mu}(\nu, t, N_c) / \nu^n},
\]

for different values of \(n = 0\) to 3, and \(N_c\), \(t\), \(T_{\text{max}}\), and isospin \(I, I'\) channels. The ratio \(F^{\mu}\) compares the amplitude given by \(\rho\) Regge exchange with that controlled by the Pomeron, while the ratio \(F^{21}\) compares the “exotic” four quark exchange with \(\bar{q}q\) \(\rho\) exchange.

We show the results in Table III and plot the data in Fig. 4. If Regge expectations were working at one-loop order, we would expect \(F^{10}\) to tend to 0.66 and for \(F^{21}\) to be very small in magnitude, just as they are at \(N_c = 3\), particularly for a cutoff of 2 GeV\(^2\), the results for which are shown as the bolder lines. However, as \(N_c\) increases we find that the ratio \(F^{10}\) tends to 0.5, while that for \(F^{21}\) tends to −1. This is in accord with the \(n = 1, 2\) sum rules becoming increasingly dominated by the \(\rho\) with very little scalar contribution. This difference is a consequence of the seeming largely non-\(\bar{q}q\) nature of the \(\sigma\) being incompatible with Regge expectations. All these results use values for the one-loop LECs that accurately fit the low energy \(\pi\pi\) phase shifts up to 1 GeV.

Finally, let us recall that the LECs carry a dependence on the regularization scale \(\mu\) that cancels with those of the loop functions to give a finite result order by order. As a consequence, when rescaling the LECs with \(N_c\), a specific choice of \(\mu\) has to be made. In other words, despite the \(\chi PT\) and IAM amplitudes being scale independent, the \(N_c\) evolution is not. Intuitively, \(\mu\) is related to a heavier scale, which has been integrated out in \(\chi PT\) and it is customary to take \(\mu\) between 0.5 and 1 GeV [23,37]. This range is confirmed by the fact that at these scales the measured LECs satisfy their leading \(1/N_c\) relations fairly well [37]. All the previous considerations about the one-loop IAM have been made with an \(N_c\) scaling at the usual choice of renormalization scale \(\mu = 770\) MeV \(\approx M_{\rho}\), which is the most natural choice given the fact that the values of the LECs are mainly saturated by the first octet of vector resonances, with additional contributions from scalars above 1 GeV [37].

Thus, in Fig. 2 we have also illustrated the uncertainties in the pole movements for the \(\rho(770)\) and \(f_0(600)\) due to the choice of \(\mu\). Note that the \(\rho(770)\bar{q}q\) behavior is rather stable, since for the LECs in Table II the variation is negligible. Other sets of LECs [23,38], which also provide a relatively good description of the \(\rho(770)\), show a bigger variation with \(\mu\), but they always lead to the expected \(\bar{q}q\) behavior. In contrast, we observe that the only robust

<table>
<thead>
<tr>
<th>LECs((\times10^3))</th>
<th>One-loop IAM</th>
</tr>
</thead>
<tbody>
<tr>
<td>(L_1)</td>
<td>0.60 ± 0.09</td>
</tr>
<tr>
<td>(L_2)</td>
<td>1.22 ± 0.08</td>
</tr>
<tr>
<td>(L_3)</td>
<td>−3.02 ± 0.06</td>
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<tr>
<td>(L_4)</td>
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<tr>
<td>(L_5)</td>
<td>1.90 ± 0.03</td>
</tr>
<tr>
<td>(L_6)</td>
<td>−0.07 ± 0.20</td>
</tr>
<tr>
<td>(L_7)</td>
<td>−0.25 ± 0.18</td>
</tr>
<tr>
<td>(L_8)</td>
<td>0.84 ± 0.23</td>
</tr>
</tbody>
</table>

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feature of the $f_0(600)$ is that it does not behave predominantly as a $\bar{q}q$. Unfortunately, its detailed pole behavior is not well determined except for the fact that it moves away from the 400 to 600 MeV region of the real axis and that at $N_c$ below 15 its width always increases. However, for $N_c$ around 20 or more and for the higher values of the $\mu$ range, the width may start decreasing again and the pole would start behaving as a $\bar{q}q$.

In Fig. 5 we show how the IAM uncertainty translates into our calculations of the $F_n$ ratio for the most interesting cases $n = 2, 3$. The thick continuous line stands for the central values we have been discussing so far, which at larger $N_c$ tend to grow in absolute value and, as already commented, spoil semilocal duality. The situation is even worse when the $N_c$ scaling of our LECs is performed at $\mu = 500$ MeV. This is due to the fact, seen in Fig. 2, that, with this choice of $\mu$, the $\sigma$ pole moves deeper and deeper into the complex plane and its mass even decreases. Let us note that this behavior—compatible with our IAM results when the uncertainty in $\mu$ is taken into account—is also found when studying the leading $N_c$ behavior within other unitarization schemes, or for certain values of the LECs within the one-loop IAM [39,40]. We would therefore also expect that in these treatments semilocal duality would deteriorate very rapidly. In [39], there is the $f_0(980)$, as well as other scalar states above 1300 MeV, but all of them seem insufficient to compensate for the disappearance of the $\sigma$ pole. As we will discuss in Sec. V, this is because the contributions of the $f_0(980)$ resonance and the region above 1300 MeV to our $F_n$ ratios are rather small, and in [39] they seem to become even smaller, since all those resonances become narrower as $N_c$ increases. Of course, as pointed out in [39] this deserves a detailed calculation within their approach.


In Fig. 5 we also find that the $F_n^{21}$ are much smaller and may even seem to stabilize if we apply the $N_c$ scaling of the LECs at $\mu = 1000$ MeV. In such a case, the $\sigma$ pole, after moving away from the real axis, returns back at higher masses, above roughly 1 GeV. For simplicity we only show $F_n^{21}$ for the $t = 0$ case, but a similar pattern is found at $t = 4M_0^2$; the turning back of the $\sigma$ pole at higher masses helps to keep the $F_n^{21}$ ratios smaller. This behavior follows from the existence of a subdominant $qq$ component within the $f_0(600)$ with a mass which is at least twice that of the original $f_0(600)$ pole. However, at one-loop order such behavior only occurs at one extreme of the $\mu$ range. In contrast, as we will see next, it appears in a rather natural way in the two-loop analysis.

### IV. $N_c$ Dependence of $\pi\pi$ Scattering

To Two-Loop UChPT: Subdominant $qq$ Component of the $\sigma$

Now let us move to two-loop order in $\chi$PT [21] and see if this situation changes. The IAM to two loops for pion-pion scattering was first formulated in [15], and first analyzed in [19]. With a larger number of LECs appearing, we clearly have more freedom. In studying the $1/N_c$
FIG. 4. Ratios $F_{n}^{10}$ of Eq. (13) with $n = 0–3$. The top four graphs are for $F^{10}$, and the lower four for $F^{21}$. One-loop χPT IAM parameters are from a coupled channel $SU(3)$ fit with $N_c = 3$ to data.
behavior, Peláez and Rios [24] consider three alternatives within single channel $SU(2)$ chiral theory for fixing these, which we follow here too. These three cases involve combining agreement with experiment with different underlying structures for the $\rho$ and $\sigma$. Agreement with experiment for the $I=0$ and 2 $S$ waves and the $I=1$ $P$ wave is imposed by minimizing a suitable $\chi^2_{\text{data}}$. Our whole approach is one of considering the $1/N_c$ corrections to the physical $N_c=3$ results. Consequently, to impose an underlying structure for the resonances, we note that if a resonance is predominantly a $\bar{q}q$ meson, then as a function of $N_c$, its mass $M \sim O(1)$ and width $\Gamma \sim O(1/N_c)$. Taking into account the subleading orders in $1/N_c$, it is sufficient to consider a resonance a $\bar{q}q$ state, if

$$M_{N_c}^{\bar{q}q} = M_0 \left(1 + \frac{\epsilon_M}{N_c}\right), \quad \Gamma_{N_c}^{\bar{q}q} = \frac{\Gamma_0}{N_c} \left(1 + \frac{\epsilon_\Gamma}{N_c}\right), \quad (15)$$

where $M_0$ and $\Gamma_0$ are unknown but $N_c$ independent, with $\epsilon_M$ and $\epsilon_\Gamma$ naturally taken to be one. Thus, for a $\bar{q}q$ state the expected $M_{N_c}$ and $\Gamma_{N_c}$ can be obtained from those generated by the IAM,

$$M_{N_c}^{\bar{q}q} = M_{N_c} - \frac{1}{N_c} = M_{N_c} + \Delta M_{N_c}^{\bar{q}q},$$

$$\Gamma_{N_c}^{\bar{q}q} = \frac{\Gamma_{N_c}}{N_c} = \frac{\Gamma_{N_c} + \Delta \Gamma_{N_c}^{\bar{q}q}}{N_c}. \quad (16)$$

We therefore define an averaged $\chi^2_{\bar{q}q}$ to measure how close a resonance is to a $\bar{q}q$ behavior, using as uncertainty the $\Delta M_{N_c}^{\bar{q}q}$ and $\Delta \Gamma_{N_c}^{\bar{q}q}$:

$$\chi^2_{\bar{q}q} = \frac{1}{2n} \sum_{N_c=3}^{n} \left[ (M_{N_c}^{\bar{q}q} - M^{\bar{q}q})^2 + \frac{(\Gamma_{N_c}^{\bar{q}q} - \Gamma^{\bar{q}q})^2}{(\Delta M_{N_c}^{\bar{q}q})^2 + (\Delta \Gamma_{N_c}^{\bar{q}q})^2} \right]. \quad (17)$$

This $\chi^2$ is added to $\chi^2_{\text{data}}$ and the sum is minimized. Case A is where the data are fitted assuming that the $\rho$ is a $\bar{q}q$ meson, while case B assumes that both the $\sigma$ and the $\rho$ are $\bar{q}q$ states. Last, case C is where we minimize $\chi^2_{\text{data}}$ and just $\chi^2_{\bar{q}q}$, for the $\sigma$.

We show in Table IV the values of the $\chi^2$ contributions for each case, where we sum over $N_c$ from 3 to 12. The two-loop LECs [24] for each case are shown in Table V. We see from Table IV that constraining the $\rho$ to be a $\bar{q}q$ state by imposing Eq. (17) is completely compatible with data at $N_c=3$. In contrast, imposing a $\bar{q}q$ configuration for the $\sigma$ gives much poorer agreement with data and can distort the

TABLE IV. Values of the $\chi^2$ for the different $SU(2)$ fits.

<table>
<thead>
<tr>
<th>IAM Fit</th>
<th>$\chi^2_{\text{data}}$</th>
<th>$\chi^2_{\rho,\bar{q}q}$</th>
<th>$\chi^2_{\sigma,\bar{q}q}$</th>
<th>$\chi^2_{\sigma,\bar{q}q,N_c=9}$</th>
<th>$\chi^2_{\sigma,\bar{q}q,N_c=12}$</th>
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</thead>
<tbody>
<tr>
<td>Case A: $\rho$ as $\bar{q}q$</td>
<td>1.1</td>
<td>0.9</td>
<td>15.0</td>
<td>4.8</td>
<td>3.4</td>
</tr>
<tr>
<td>Case B: $\rho$ and $\sigma$ as $\bar{q}q$</td>
<td>1.5</td>
<td>1.3</td>
<td>4.0</td>
<td>0.8</td>
<td>0.5</td>
</tr>
<tr>
<td>Case C: $\sigma$ as $\bar{q}q$</td>
<td>1.4</td>
<td>2.0</td>
<td>3.5</td>
<td>0.6</td>
<td>0.5</td>
</tr>
</tbody>
</table>

TABLE V. Two-loop IAM LECs for the different cases we have used [24].

<table>
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<tr>
<th>LECs</th>
<th>Case A</th>
<th>Case B</th>
<th>Case C</th>
</tr>
</thead>
<tbody>
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<td>$-5.7$</td>
<td>$-5.7$</td>
</tr>
<tr>
<td>$f_0(10^3)$</td>
<td>$1.8$</td>
<td>$2.5$</td>
<td>$2.6$</td>
</tr>
<tr>
<td>$f_0(10^3)$</td>
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<td>$0.39$</td>
<td>$-1.7$</td>
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<td>$3.5$</td>
<td>$1.7$</td>
</tr>
<tr>
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<td>$-0.58$</td>
<td>$-0.6$</td>
</tr>
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<td>$1.5$</td>
<td>$1.3$</td>
</tr>
<tr>
<td>$r_3(10^3)$</td>
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<td>$-3.2$</td>
<td>$-4.4$</td>
</tr>
<tr>
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<td>$-0.49$</td>
<td>$-0.03$</td>
</tr>
<tr>
<td>$r_6(10^3)$</td>
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<td>$2.7$</td>
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<td>$-0.62$</td>
<td>$-0.7$</td>
</tr>
</tbody>
</table>
simple structure for the \(\rho\). It is interesting to point out that,
the lower energy at which such a sigma’s \(\bar{q}q\) behavior
emerges, the higher energy at which the \(\rho\) pole moves
with \(N_c\). Therefore, as much as we try to force the \(\sigma\) to
behave as a \(\bar{q}q\) meson, less the \(\rho\) meson does. However,
requiring a \(\bar{q}q\) composition for the \(\sigma\) for larger \(N_c\)
causes no such distortion.

In all parameter sets at two loops, including case A,
which fits the data best and in which the \(\rho\) has a clear \(\bar{q}q\)
structure, we do see a subleading \(\bar{q}q\) behavior for the \(\sigma\)
meson emerge between 1 and 1.5 GeV\(^2\). This is evident
from Fig. 6 where the imaginary part of the \(I = J = 0\)
amplitude is plotted. We see a clear enhancement above
1 GeV emerge as \(N_c\) increases. That this enhancement is
related to the \(\sigma\) at larger \(N_c\) can be seen by tracking the
movement of the \(\rho\) and \(\sigma\) poles at two loops, and compar-
ing this with the one-loop trajectories in Fig. 2.

We see clearly how the \(\sigma\) pole moves away as \(N_c\)
increases above 3, just as in the one-loop case, but then
subleading terms take over as \(N_c\) increases above 6 and the
\(\sigma\) pole moves back to the real axis close to 1.2 GeV. This
clearly indicates dominance of a \(\bar{q}q\) component in its Fock
space, which may well be related to the existence of a scalar
\(\bar{q}q\) nonet above 1 GeV, as suggested in [17,41–44].

This is directly correlated with the enhancement seen in
Fig. 6 (the pole movement shown in Fig. 7) and of course

![Fig. 6. Absorptive parts of the \(I = J = 0\) partial wave amplitude, \(\text{Im} T_{00}^\rho(s)\), at one loop with the parameters of an \(SU(3)\) fit (cf. the corresponding coupled channel fit in Fig. 1) and at two loops an \(SU(2)\) fit with \(N_c = 3\) to data below 0.9 GeV. These both involve only the \(\pi\pi\) channel and so the strong inelastic effects from \(KK\) threshold are not included, in contrast to Fig. 1.](image)

![Fig. 7. Position of the \(\rho\) and \(\sigma\) poles in the complex energy plane as a function of \(N_c\) in two-loop \(\chi PT\) with parameters from the \(SU(2)\) fit A of Table IV. This is to be compared with the one-loop trajectories of Fig. 2. Note the different vertical scales for the \(\rho\) and \(\sigma\) plots.](image)
this enhancement makes its presence felt in the amplitudes with definite \( t \)-channel isospin. Indeed, with \( I = 2 \) we see the growth of a positive contribution to the imaginary part that might cancel the negative \( \rho \) component as \( N_c \) increases; see Fig. 8 and compare with the one-loop forms in Fig. 3.

In addition, and though these ratios have only been evaluated at one-loop order, as shown in Fig. 1, to go further one would need to extend this analysis to two or more loops. Notwithstanding this caveat, we now compute the finite energy sum rule ratios, \( F(t) I^0 \) of Eq. (6) with these same two-loop parameters. These ratios are set out in Table VI.

We should be just a little cautious in recognizing the limitations of the single channel approach we use here at two loop in \( \chiPT \). Despite the unitarization, we are restricted to a region below 1 GeV, where strong coupling inelastic channels are not important. We see in Fig. 7 (and Fig. 6) that the subdominant \( \bar{q}q \) components move above 1 GeV as \( N_c \) increases beyond 10 or 12. Consequently, if we take \( N_c \) much beyond 15 without including coupled channels, we do not expect to have a detailed description of the resonances up to 2 GeV\(^2\). However, in the scenario where the sigma has a subdominant \( \bar{q}q \), it should be interpreted as a Fock space state that is mixed in all the \( f_0 \) resonant structures in that region [45], which survives as \( N_c \) increases. Then it is easy to see that its contribution would be dominant in our ratios, and still provide a large cancellation with the \( \rho \) contribution. The reason is that, when this subdominant \( \bar{q}q \) component approaches the real axis above 1 GeV, it has a much larger width than any other \( f_0 \) resonant state in that region. For instance, we see in Fig. 7 that for \( N_c = 12 \), the width of the \( \bar{q}q \) subcomponent in the sigma is roughly 450 MeV, whereas the width of any other \( \bar{q}q \) component that may exist in that region would have already decreased by \( 3/12 = 1/4 \). Since the other

FIG. 8. Absorptive parts of amplitudes with definite \( t \)-channel isospin, \( \text{Im} A^{11}(t_{th})/\nu^0 \), using the parameters of the two-loop \( SU(2) \) fit case A. The top pair of graphs has \( I = 1 \) and the lower with \( I = 2 \), on the left hand \( n = 0 \) and right hand \( n = 2 \). One sees from the lower pair how integrating the curves the positive and negative contributions cancel for all \( N_c \).
components would be heavier and much narrower, their contributions would be much smaller than that of the $\bar{q}q$ state subdominant in the $\sigma$. Note that it is also likely that some of the $f_0$’s may have large glueball components (see, for instance, [44]), which also survive as $N_c$ increase, but then their widths would decrease even faster—like $1/N_c^2$, and our argument would apply even better. For the scenario when we do not see the sigma subdominant component (as in Fig. 4), we still expect that the other resonances by themselves will not be able to cancel the $\rho$ contribution, so that the IAM would still provide a qualitatively good picture of this “noncancellation.” For this reason, although the IAM much beyond $N_c = 15$ may not necessarily yield a detailed description of the resonance structure, we expect the $N_c$ behavior of the ratios to be qualitatively correct for both scenarios even at larger $N_c$.

Additional arguments to consider the IAM only as a qualitative description beyond $N_c = 15$ or 30 have been given in [38] since the error made in approximating the left cut, as well as the effect of the $\eta'$ may start to become numerically relevant around those $N_c$ values.

Remarkably we see with two-loop ChPT that the unitarized amplitudes do reflect semilocal duality with $I = 2$ in the $t$ channel suppressed. This is most readily seen from the plots of the ratios $F_\Pi^{21}$ for the two-loop amplitudes shown in Fig. 9 (to be compared with the one-loop ratios of Fig. 4). For $F_\Pi^{21}$, it is clear that, if only considering the integrals up to 1 GeV$^2$, the ratios are still not small in

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<th>$n$</th>
<th>$N_c$</th>
<th>$\nu_{\text{max}} = 1 \text{ GeV}^2$</th>
<th>$\nu_{\text{max}} = 2 \text{ GeV}^2$</th>
<th>$\nu_{\text{max}} = 1 \text{ GeV}^2$</th>
<th>$\nu_{\text{max}} = 2 \text{ GeV}^2$</th>
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<td>12</td>
<td>-0.332</td>
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FIG. 9. Ratios $F_n^{\mu}$ of Eq. (6) with $n = 0–3$. The top four graphs are for $F_{10}$, and the lower four for $F_{21}$. Two-loop $\chi$PT IAM parameters are from the SU(2) fit with $N_c = 3$ to data: case A.
magnitude. Indeed, their absolute value increases with $N_c$. However, integrating up to 2 GeV$^2$ takes into account the subdominant component, and then the ratios stabilize at much smaller values for all $N_c$, consistent with expectations from semilocal duality.

V. THE EFFECT OF HEAVIER RESONANCES

So far we have restricted the analysis of the $N_c$ behavior to the $\rho$ and $\sigma$ resonances. Of course, one may wonder what is the effect of heavier resonances on our analysis and conclusions. In particular, since the subdominant $\bar{q}q$ component of the $\sigma$ emerges between 1 and 1.5 GeV$^2$, one might worry about the $f_0(980)$ and even the $f_0(1370)$ resonance, since the latter has a width of several hundred MeV and may overlap with the region of interest. [The $f_0(1500)$ and $f_0(1710)$ lie beyond that energy range and are therefore suppressed by the $1/s^a$ in the denominator.] In addition, we might worry about resonances in higher waves; in this case the $f_2(1270)$ in the $D$ wave would yield the largest contribution.

Actually, Fig. 1 has been calculated in an $SU(3)$ coupled channel formalism and includes the $f_0(980)$ as a very sharp drop in $\text{Im} T_{00}^f$, which disappears as $N_c$ increases. By comparing with Fig. 6, with no $f_0(980)$ present, it is clear that, by removing the $f_0(980)$ the variation in the $\text{Im} T_{00}^f$ integrals, and therefore in the $F_{n}^f$ of Eq. (6), is small compared to the systematic uncertainty that we have estimated as the difference between the $t = 0$ and $t = t_{th}$ calculations. Actually, if the $f_0(980)$ is included in a coupled channel IAM calculation, as in Fig. 1, the new $F_{n}^{21}$ values would all lie between our $t = 0$ and $t = t_{th}$ results listed in Table III without the $f_0(980)$. The error we make by ignoring the $f_0(980)$ is, at most, 30$\%$ of the estimated systematic uncertainty. For sure the $f_0(980)$ will not be able to compensate the $\rho$ contribution. Still, one might wonder whether this is also the case at two loops if the $f_0(980)$ or $f_0(1370)$ have a $\bar{q}q$ component around 1 to 1.5 GeV$^2$ that survives when $N_c$ increases. However, at least the lightest such component would be precisely the same $\bar{q}q$ state that we already see in the $f_0(600)$. Actually the interpretation of the IAM results is that all these scalars are a combination of all possible states from Fock space [45], namely, $\bar{q}q$, tetraquarks, molecules, glueballs, etc., . . . , but as $N_c$ grows only the $\bar{q}q$ survives between 1 and 1.5 GeV$^2$, whereas the other components are either more massive or disappear in the deep complex plane. It is precisely that component, which we already have in our calculation, the one compensating the $\rho$ contributions, as we have just seen above.

In the very preliminary interpretation of [45], the $\bar{q}q$ subdominant component of the $f_0(600)$ within the IAM naturally accounts for 20$\%$–30$\%$ of its total composition. This is in fairly good agreement with the 40$\%$ estimated in [46]. Indeed, given the two caveats raised by the authors of [46], their 40$\%$ may be considered an upper bound. First, this 40$\%$ refers to the “tree level masses” of the scalar states. These mesons, of course, only acquire their physical mass and width after unitarization, which is essentially generated by $\pi\pi$ final state interactions. Intuitively we would expect these to enhance the non-$\bar{q}q$ component, and so bring the $\bar{q}q$ fraction below the “bare” 40$\%$. Second, in [46] the authors also suggest that “a possible glueball state is another relevant effect” not included in their analysis. In [45], the glueball component is of the order of 10$\%$. Consequently, the results of [46], those presented here and in [45], are all quite consistent.

Finally, we will show that the contribution of the $f_2(1270)$ to the FESR cancellation, even assuming it follows exactly a $\bar{q}q$ leading $N_c$ behavior, is rather small and does not alter our conclusions. All other resonances coupling to $\pi\pi$ are more massive and therefore less relevant.

In order to describe the $I = 0$ $J = 2$ channel we will again use the parametrization of KPY in terms of the corresponding phase shift $\delta_2^0$, namely,

$$A_2^0 = \frac{1}{\sigma(s)} \frac{1}{\cot \delta_2^0 - i},$$

(19)

where $\cot \delta_2^0$, which is proportional to $s - M_{f_2}^2$, is given in detail in the Appendix of KPY [29]. Now, by replacing

$$\cot \delta_2^0 \rightarrow N_c \cot \delta_2^0,$$

(20)

we ensure that the amplitude itself scales as $1/N_c$. This also ensures that the resonance mass $M_{f_2}$ is constant, and its width scales as $1/N_c$. We require the $f_2(1270)$ to behave as a perfect $\bar{q}q$ at leading order in $1/N_c$, while reproducing the KPY fit to the $D$ wave at $N_c = 3$. As can be noticed in Fig. 10, for $F_{21}^0$ and $F_{31}^0$, which are the most relevant ratios for our arguments, the difference between adding this $D$-wave contribution to our previous results is smaller than the effect of the sigma $\bar{q}q$ component around 1 to 1.5 GeV$^2$. For the ratio $F_{31}^0$, the effect of the $D$-wave contribution is larger, but it is the effect of the sigma $\bar{q}q$ subcomponent the one that makes the curves flatter and bounded between $-0.2$ and $0.2$, whereas the slope is clearly negative without such a contribution and the absolute value of the ratio can be as large as 0.5 and still growing. Note that in Fig. 10 we compare our previous one- and two-loop $F_{n}^{21}$ calculations (bolder line) to those which include the $f_2(1270)$ resonance as a pure $\bar{q}q$ (thin lines). Therefore the effect of including the $f_2(1270)$ does not modify our conclusions. The main FESR cancellation at $N_c$ larger than 3 is between the $\rho(770)$ and the subdominant $\bar{q}q$ component of the $f_0(600)$ resonance, which appears around 1 to 1.5 GeV$^2$.

This is even more evident if we extrapolate our results to even higher $N_c$, as in Fig. 11, where all curves include the effect of the $f_2(1275)$. As already explained above, for such high $N_c$ the IAM cannot be trusted as a precise description, but just as a qualitative model of the effect of a $\bar{q}q$ state around 1 to 1.5 GeV$^2$, which has a width
FIG. 10. Results for $F_{21}^{n}$ with and without the $f_2(1270)$ resonance scaled as a pure $\bar{q}q$ (thin and bolder lines, respectively). The left panels are for one-loop IAM results, and the right ones for the two-loop results. The latter contain a subdominant $\bar{q}q$ component of the $f_0(600)$ around 1 to 1.5 GeV² whose effect is relevant for the cancellation of $F_{21}^{n}$. 

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much larger than the states seen there at $N_c = 3$ and will dominate the integrals in $F_{n}^{21}$. It is clearly seen that the effect of such a state will compensate the $\rho(770)$ contribution and preserve semilocal duality. Other states that survive the $N_c$ limit in that region—which would be heavier and much narrower—would only provide smaller corrections to this qualitative picture. Nevertheless, it would be desirable to extend this study to a more ambitious treatment of the higher mass states in future work.

### VI. DISCUSSION

It is a remarkable fact that hadronic scattering amplitudes from threshold upwards build their high energy Regge behavior. This was learnt from detailed studies of meson-nucleon interactions more than 40 years ago. This property is embodied in semilocal duality, expressed through finite energy sum rules. Perhaps just as remarkably we have shown here that the Regge parameters fixed from high energy $NN$ and $\pi N$ scattering yield the correct $\pi \pi P$ and $D$-wave scattering lengths, cf. Eqs. (11) and (12). Indeed, there is probably no closer link between amplitudes with definite $t$-channel quantum numbers and their low energy behavior in the $s$-channel physics region than that shown here. What is more, such a relationship should hold at all values of $N_c$. At low energy the scattering amplitudes of pseudo-Goldstone bosons are known to be well described by their chiral dynamics, and their contribution to finite energy sum rules is dominated by the $\rho(770)$ and $f_0(600)$ contributions. However, there are many proposals in the...
literature, including the $N_c$ dependence of the unitarized chiral amplitudes, suggesting that the $f_0(600)$, contrary to the $\rho(770)$, may not be an ordinary $q\bar{q}$ meson. This is a potential problem for the concept of semilocal duality between resonances and Regge exchanges. The reason is that for $I = 2$-channel exchange it requires a cancellation between the $\rho(770)$ and $f_0(600)$ resonances, which may no longer occur if the $f_0(600)$ contribution becomes comparatively smaller and smaller as $N_c$ increases.

This conflict actually occurs for the most part of one-loop unitarized chiral perturbation theory parameter space. In contrast, for a small part of the one-loop parameter space and in a very natural way at higher order in the chiral expansion, the $\sigma$ may have a $q\bar{q}$ component in its Fock space, which though subdominant at $N_c = 3$, becomes increasingly important as $N_c$ increases. This is critical, as we have shown here, in ensuring semilocal duality for $I = 2$ exchanges is fulfilled as $N_c$ increases. As we show in Fig. 11 this better fulfillment of semilocal duality keeps improving even at much larger $N_c$, where the IAM can only be interpreted as a very qualitative average description.

Thus, the chiral expansion contains the solution to the seeming paradox of how a distinctive nature for the $\rho$, $\sigma$ at $N_c = 3$ is reconciled with semilocal duality at larger values of $N_c$. Indeed, despite the additional freedom brought about by the extra low energy constants at two-loop order, fixing these from experiment at $N_c = 3$ automatically brings this compatibility with semilocal duality as $N_c$ increases. This is a most satisfying result.

The $P$ and $D$-wave scattering lengths evaluated using Eqs. (7) and (8) that agree so well with local duality at $N_c = 3$ can, of course, be computed at larger $N_c$ by inputting chiral amplitudes on each side of the defining equations. The scattering lengths themselves involve only the real parts, while the Froissart-Gribov integrals require the imaginary parts that are determined by the unitarization procedure. Explicit calculation shows that these agree as $N_c$ increases. While the agreement at one-loop order is straightforward, at two loops there is a subtle interplay of dominant and subdominant terms placing constraints on the precise values of the LECs. As this takes us beyond the scope of the present work, we leave this for a separate study.

Though beyond the scope of this work, we can then ask what does this tell us about the nature of the enigmatic scalars $[9]$? At $N_c = 3$, the behavior of the $\sigma$ is controlled by its coupling to $\pi\pi$. Its Fock space is dominated by this non-$q\bar{q}$ component $[42,43,45]$. In dynamical calculations of resonances and their propagators, like that of van Beveren, Rupp, and their collaborators $[41]$ and of Tornqvist $[47]$, the seeds for the lightest scalars are an ideally mixed $q\bar{q}$ multiplet of higher mass. These seeds may leave a conventional $q\bar{q}$ nonet near 1.4 GeV $[17,41,43,44]$, while the dressing by hadron loops dynamically generates a second set of states, whose decay channels dominate their behavior at $N_c = 3$ and pull their masses close to the threshold of their major decay: the $\sigma$ down towards $\pi\pi$ threshold, and the $f_0(980)$ and $a_0(980)$ to $K\bar{K}$ threshold. The leading order in the $1/N_c$ expansion discussed here may be regarded $a$ posteriori as providing a quantitative basis for this. The scalars are at $N_c$, larger than 3 controlled by $q\bar{q}$ seeds of mass well above 1 GeV (1.2 GeV for the intrinsically nonstrange scalar). Switching on decay channels, as one does as $N_c$ decreases, changes their nature dramatically, inevitably producing non-$q\bar{q}$ or di-meson components in their Fock space at $N_c = 3$ $[9]$. We see here that the $\sigma$ having a subdominant $q\bar{q}$ component with a mass above 1 GeV is essential for semilocal duality, that suppresses $I = 2$ amplitudes, to hold.

**ACKNOWLEDGMENTS**

M. R. P is grateful to Bob Jaffe for discussions about the issues raised at the start of this study. The authors (J. R. de E, M. R. P and D. J. W) acknowledge partial support of the EU-RTN Programme, Contract No. MRTN–CT-2006-035482, “Flavianet” for this work, while at the IPPP in Durham. D. J. W is grateful to the UK STFC for the award of a postgraduate studentship and to Jefferson Laboratory for hospitality while this work was completed. This work was supported in part by DOE Contract No. DE-AC05-06OR23177, under which Jefferson Science Associates, LLC, operates Jefferson Laboratory.