CB-NORM ESTIMATES FOR MAPS BETWEEN NONCOMMUTATIVE $L_p$-SPACES AND QUANTUM CHANNEL THEORY

MARIUS JUNGE AND CARLOS PALAZUELOS

Abstract. In the first part of this work we show how certain techniques from quantum information theory can be used in order to obtain very sharp embeddings between noncommutative $L_p$-spaces. Then, we use these estimates to study the classical capacity with restricted assisted entanglement of the quantum erasure channel and the quantum depolarizing channel. In particular, we exactly compute the capacity of the first one and we show that certain nonmultiplicative results hold for the second one.

1. Introduction

Embedding results for $L_p$-spaces have a very long tradition in Banach space theory, see e.g. the handbook [21]. In some sense the starting point are the probabilistic concepts of $p$-stable random variables going back at least as early as [24]. Noncommutative analogues of such embedding results have been established by imitating and modifying the commutative results [14, 18, 19]. The novelty in this paper is to use what should be called “classical ideas” from the emerging new quantum information theory and significantly improve embedding results for (vector-valued) noncommutative $L_p$-spaces, and indicate some applications. On the other hand, operator algebra and functional analysis techniques have been very successfully applied in quantum information theory. For example, operator space techniques have been applied to Bell inequalities ([15], [17], [31]), tools from free probability have been used for the classical capacity of a quantum channel ([1, 8], [19]), and noncommutative versions of Grothendieck theorem were used for efficient approximations for quantum values of quantum games ([10], [36]). There are also some examples using techniques from quantum information to prove new mathematical results. For example Regev and Vidick used the embezzlement state for a simplified proof of the so called Grothendieck theorem for operator spaces ([37]) and Ahlswede/Winter’s application of the Golden-Thompson inequality has found numerous application in compressed sensing (see [35]).

In this paper we will use quantum teleportation, one of the most important quantum information protocols, to provide some very sharp embeddings between noncommutative $L_p$-spaces. Let us recall the definition of the discrete noncommutative vector valued $L_p$-spaces, introduced by Pisier in [33]. For a given natural number $n$ and $1 \leq p \leq \infty$ we will denote by $S^n_p := S_p(\ell^n_2)$ the Schatten $p$-class of operators acting on the $n$-dimensional complex Hilbert space $\ell^n_2$, which can be obtained by interpolation: $S^n_p = [S^n_{\infty}, S^n_1]_p$, where $S^n_{\infty}$ denotes the space of (compact) operators.

The first author is partially supported by NSF DMS-1201886 grant. The second author is partially supported by MINECO (grant MTM2011-26912), the european CHIST-ERA project CQC (funded partially by MINECO grant PRI-PIMCHI-2011-1071) and “Ramón y Cajal” program. Both authors are partially supported by ICMAT Severo Ochoa Grant SEV-2011-0087 (Spain).
acting on $\ell_p^n$ joint with the operator norm and the trace class $S^m_p$ can be seen as the dual space of $S^m_p$ with respect to the dual action $(A, B) = \text{tr}(AB^*)$. In fact, such an interpolation identity can be used to endow the space $S^m_p$ with a natural **operator space structure** ([32], [33]). Note that the diagonal of $S^m_p$ is exactly $\ell^m_p = [\ell^m_\infty, \ell^m_p]_{\frac{1}{2}}$, so one also obtains an operator space structure for these spaces. An **operator space** $E$ is a complex Banach space together with a sequence of **matrix norms** $\alpha_n$ on $M_n[E] = M_n \otimes E$ with $n \geq 1$, satisfying certain “good properties”. Then, given a linear map $T : E \to F$ between operator spaces we say that $T$ is a complete contraction (resp. a complete isomorphism/completely isometry) if the maps $\text{id}_{M_n} \otimes T : M_n[E] \to M_n[E]$ are contractions (resp. isomorphisms/isometries) for every $n$. When working with operator spaces these are precisely the morphisms one has to use in order to preserve the new structure. Finally, given any operator space $E$, we will denote $S^{\infty}_E[E] = S^{\infty}_E \otimes_{\min} E$, where $\min$ denotes the minimal tensor norm in the category of operator spaces. On the other hand, Effros and Ruan introduced the space $S_1[E]$ as the (operator) space $S_1 \hat{\otimes} E$, where $\hat{\otimes}$ denotes the projective operator space tensor norm. Then, using complex interpolation Pisier defined the noncommutative vector valued (operator) space $S_p[E] = [(S^{\infty}_E), S_1[E]]_{\frac{1}{p}}$ for any $1 \leq p \leq \infty$ and he proved that this definition leads to obtain the expected properties of $S_p[E]$, analogous to the commutative setting (see [33] Chapter 3). The first result of this work is the following.

**Theorem 1.1.** Let $1 \leq p, q \leq \infty$. Let $n_1, \cdots, n_k$ be a family of natural numbers and let $d$ be the least common multiplier of $n_1, \cdots, n_k$. There exist a completely positive and completely isometric embedding

$$
\tilde{J}_{p,q} : S^{n_1}_p \oplus_p \cdots \oplus_p S^{n_k}_p \to S^d_p(\ell^m_\infty, \ell^m_p)^{n_1, \cdots, n_k}
$$

and a completely positive and completely contractive map

$$
\tilde{W}_{p,q} : S^d_p(\ell^m_\infty, \ell^m_p)^{n_1, \cdots, n_k} \to S^{n_1}_p \oplus_p \cdots \oplus_p S^{n_k}_p
$$

such that $\tilde{W}_{p,q} \circ \tilde{J}_{p,q} = \text{id}$.

Moreover, the result is also true in the vector valued setting. That is, for any operator space $E$, $S^{n_1}_p[E] \oplus_p \cdots \oplus_p S^{n_k}_p[E]$ is completely isometric to a completely complemented subspace of $S^d_p(e^{m_1}_n, e^{m_k}_n)[E]$.

Finding suitable embeddings of vector valued $L_p$-spaces has a long tradition in Banach space theory, and can be used in noncommutative harmonic analysis, quantum probability theory and operator spaces (see for instance [18], [19], [20] and the references therein). In particular, the type of embeddings given in Theorem 1.1 has been used in order to study notions like type and cotype or $K$-convexity and $B$-convexity in the context of operator spaces. This is the case of the work [18], where the authors, motivated by the study of the previous notions, provided a complete isomorphism from the space $S^m_p$ onto a completely complemented subspace of $S^m_p(e^{m_n}_p)$ with $m \approx n^2$ ([18] Theorem 2]). Moreover, using type/cotype estimates they proved that the order $m \approx n^2$ is optimal. An immediate corollary of Theorem 1.1 is the following result, which significantly improves [18] Theorem 2].

\footnote{Remarkably, this order is different from the well known optimal commutative order $m \approx n$.}
Corollary 1.2. Let $1 \leq p, q \leq \infty$. There exists a complete isometry of $S^n_p$ onto a completely complemented subspace of $S^n_q(S^n_p)$. Moreover, both the isometry and the projection are completely positive maps. The result also holds in the vector valued case.

Hence, while keeping the optimal order $n^2$ in the commutative part ($\ell_p$-space) Corollary 1.2 provides a very tight estimate for the dimension of the noncommutative part ($S_q$-space). Moreover, we have now a complete isometry rather than a complete isomorphism (where a universal constant $C$ appears in the relation of the norms).

Some preliminary calculations show that the techniques developed in this work could be used to define some new embeddings in more general contexts. However, since our main motivation in this work is the use of Theorem 1.1 to study the capacity of certain quantum channels, we postpone this analysis to a future publication.

Finally we will show the following result, which can be understood as a complement of Theorem 1.1. The key point here is to use ideas from the superdense coding, another important protocol of quantum information.

Theorem 1.3. Let $1 \leq p, q \leq \infty$. Then, there exist a completely positive and a completely isometric map

$$H_{p,q} : \ell^n_p \to S^n_q(S^n_p)$$

and a completely positive and completely contractive map

$$Q_{p,q} : S^n_q(S^n_p) \to \ell^n_p$$

such that $Q_{p,q} \circ H_{p,q} = \text{id}$.

Moreover, if $E$ is any operator space, $\ell^n_p[E]$ is completely isometric to a completely complemented subspace of $S^n_q(S^n_p[E])$.

A quantum channel is defined as a completely positive and trace preserving map $\mathcal{N} : M_n \to M_m$. Following [16] we will denote a quantum channel by $\mathcal{N} : S^n_1 \to S^n_m$, where we use $S^n_1$ to denote the trace class of operators acting on $\ell^n_2$. This notation emphasizes the idea that $\mathcal{N}$ must be, in particular, a norm one operator on these spaces. As it was shown in [11] and [16], one can understand some channel capacities as the derivative of certain completely bounded and completely $p$-summing norms. We refer to [16, Section 5] for a brief introduction about channel capacities from a mathematical point of view. In particular, if we denote by $C^{d}_{\text{prod}}(\mathcal{N})$ the product state version of the classical capacity of the quantum channel $\mathcal{N}$ with assisted entanglement restricted to dimension $d$ per channel use, one can see that $C^{d}_{\text{prod}}(\mathcal{N})$ can be written as the derivative (with respect to $p$) of the $\ell_p(S^n_d)$-summing norm of the adjoint map $\mathcal{N}^* : M_m \to M_n$ (see [16, Theorem 1.1] for details). Note that this family of capacities covers, in particular, the well studied classical capacity with non entanglement ($d = 1$) and the classical capacity with unlimited assisted entanglement ($d = n$). Unfortunately, in order to compute the corresponding capacity (rather than its product state version) one has to consider the regularization

$$C^d(\mathcal{N}) := \sup_k \frac{C^{d,k}_{\text{prod}}(\mathcal{N}^{\otimes k})}{k}. \quad (1.1)$$
Since quantum information theory deals with the ways we can send and manipulate the information by using quantum resources, it is not surprising that the study of quantum channel capacities is one of the main topics in the theory and, so, it has captured the attention of many researchers in the area (see for instances [38] and the references therein). Let us consider here the quantum depolarizing channel with parameter $\lambda \in [0,1]$, $D_\lambda : S^n_1 \to S^n_1$, defined by

$$D_\lambda(\rho) = \lambda \rho + (1 - \lambda) \frac{1}{n} \tr(\rho) \mathbb{I}_n$$

for every $\rho \in S^n_1$, and also the quantum erasure channel with parameter $\lambda \in [0,1]$, $E_\lambda : S^n_1 \to S^n_1 \oplus_1 \mathbb{C}$, defined by

$$E_\lambda(\rho) = \lambda \rho \oplus (1 - \lambda) \tr(\rho)$$

for every $\rho \in S^n_1$.

Here $\mathbb{I}_n$ denotes the identity element in $M_n$. The previous two channels are very important in quantum information because, despite its very simple form, they already provide some non-trivial examples. In order to emphasize this idea, let us mention that computing the (non considered in this work) quantum capacity of the depolarizing channel (even in dimension $n = 2$) is an open problem in the area (see [28], [39] for some recent progresses). On the other hand, the classical capacity of the $D_\lambda$ with no assisted entanglement ($C^1(N)$) and with unlimited entanglement ($C^n(N)$) are well understood (see [22] and [5] respectively). The key point here is that both quantities, $C^1_{\text{prod}}$ and $C^n_{\text{prod}}$, are multiplicative when acting on the tensor product of depolarizing channels\footnote{In fact, it was shown in [23] that $C^n_{\text{prod}}$ is multiplicative on every channel so we always have $C^n = C^n_{\text{prod}}$.}, so the regularization (1.1) is not required in this case. On the other hand, a very good property of these two channels is that they are covariant (see definition below) and that allows us to simplify the statement of [16, Theorem 1.1] so that one has to deal with the $d$-norm of the corresponding channel

$$\|N : S^n_1 \to S^n_1\|_d := \|\id \otimes N : M_1(S^n_1) \to M_1(S^n_1)\|,$$

rather than with the $\ell_p(S^n_1)$-summing norm of the adjoint map $N^*$. More precisely, for any covariant quantum channel $N : S^n_1 \to S^n_1$ we have

$$C^n_{\text{prod}}(N) = \ln n + \frac{d}{dp} \|N : S^n_1 \to S^n_1\|_p\big|_{p=1}$$

for every $1 \leq d \leq n$ ([16, Corollary 4.2]). Then, we can use the estimate proved in Theorem 1.1 to obtain the following result.

**Theorem 1.4.** Let $D_\lambda : S^n_1 \to S^n_1$ and $E_\lambda : S^n_1 \to S^n_1 \oplus_1 \mathbb{C}$ be respectively the quantum depolarizing channel and the quantum erasure channel with parameter $\lambda \in [0,1]$ defined as before and let $d$ be a natural number such that $1 \leq d \leq n$. Then,

$$\|D_\lambda : S^n_1 \to S^n_1\|_d = \left(\frac{1}{d} \left(\lambda d + \frac{1 - \lambda}{n}\right)^p + \left(\frac{1 - \lambda}{n}\right)^p (n - \frac{1}{d})\right)^{\frac{1}{p}},$$

which implies

$$C^n_{\text{prod}}(D_\lambda) = \log_2(nd) + \left(\lambda + \frac{1 - \lambda}{nd}\right) \log_2 \left(\lambda + \frac{1 - \lambda}{nd}\right) + (nd - 1) \left(\frac{1 - \lambda}{nd}\right) \log_2 \left(\frac{1 - \lambda}{nd}\right).$$
On the other hand,

\[ \|E_\lambda : S^n_1 \to S^n_p \oplus_p \mathbb{C}\|_d = \left( \lambda p d^{p-1} + (1 - \lambda)^p \right)^{\frac{1}{p}}, \]

so that

\[ C_{prod}^d(E_\lambda) = \lambda \log_2(nd). \]

Both expressions \( C_{prod}^d(D_\lambda) \) and \( C_{prod}^d(E_\lambda) \) extend the previously known expressions for the cases \( d = 1 \) and \( d = n \). This is very surprising in view of the fact that for the depolarizing channel the formula \( C_{prod}^d(D_\lambda) \) is not multiplicative and, hence, \( C^d(D_\lambda) \) does not coincide with \( C_{prod}^d(D_\lambda) \). Indeed, we have the following corollary of the previous theorem.

**Corollary 1.5.** Let us fix \( n = 4, d = 2 \) and \( \lambda \in (0, 1) \). Then,

\[ C_{prod}^d(D_\lambda \otimes D_\lambda) > 2C_{prod}^d(D_\lambda). \]

Hence,

\[ C^d(D_\lambda) > C_{prod}^d(D_\lambda). \]

Interestingly, the quantity \( C^d(D_\lambda) \) has been also studied in some other works by using different techniques (\[13\], \[41\]) and its exact value seems to be unknown. On the other hand, we will show that \( C_{prod}^d \) is multiplicative on the quantum erasure channel \( E_\lambda \) and we will use this estimate to bound the value \( C^d(D_\lambda) - C_{prod}^d(D_\lambda) \). More precisely, we will prove the following result.

**Theorem 1.6.** Let \( D_\lambda : S^n_1 \to S^n_p \) and \( E_\lambda : S^n_1 \to S^n_p \oplus_1 \mathbb{C} \) be respectively the quantum depolarizing channel and the quantum erasure channel with parameter \( \lambda \in [0, 1] \) and let \( d \) be any natural number such that \( 1 \leq d \leq n \). Then,

\[ \lambda \log_2(nd) - H(\mu) \leq C_{prod}^d(D_\lambda) \leq C^d(D_\lambda) \leq \lambda \log_2(nd). \]

Here, \( H(\mu) = -\mu \log_2(\mu) - (1 - \mu) \log_2(1 - \mu) \) is called the Shannon entropy of the probability distribution \( (\mu, 1 - \mu) \), where \( \mu = \lambda + \frac{1 - \lambda}{nd} \). In particular, \( \lambda \log_2(nd) - 1 \leq C_{prod}^d(D_\lambda) \). On the other hand,

\[ C_{prod}^d(E_\lambda \otimes_k) = kC_{prod}^d(E_\lambda) = k\lambda \log_2(nd). \]

Hence,

\[ C^d(E_\lambda) = \lambda \log_2(nd). \]

The paper is organized as follows. In Section 2 we will first introduce some basic notions about operator spaces and noncommutative \( L_p \)-spaces that we will use along the whole paper. Then, we will prove Theorem 1.1 and Theorem 1.3. In Section 3 we will introduce some basic notions about quantum channels and we will explain why computations are easier when we deal with covariant channels. Section 4 is devoted to analyzing the quantum depolarizing channel. There, we will prove those parts of Theorem 1.4 and Theorem 1.6 corresponding to this channel and we will also prove Corollary 1.5. Finally, in Section 5 we will study the quantum erasure channel. In particular, we will show the second part of Theorem 1.4 and Theorem 1.6.
2. Quantum teleportation revised: Some sharp embeddings between noncommutative $L_p$-spaces

2.1. Some basic notions about operator spaces and noncommutative $L_p$-spaces.

In this section we introduce some basic concepts from operator space theory. We focus only on those aspects which are useful for this work and we direct the interested reader to the standard references [12], [32]. Given Hilbert spaces $H$ and $K$, we will denote by $B(H, K)$ the space of bounded operators from $H$ to $K$ endowed with the standard operator norm. When $H = \ell_2^n$ and $K = \ell_2^m$ we will denote $M_{n,m} = B(\ell_2^n, \ell_2^m)$ and in the case where $n = m$ we will just write $M_n$.

An operator space $E$ is a complex Banach space together with a sequence of matrix norms $\|\cdot\|_k$ on $M_k[E] = M_k \otimes E$ satisfying the following conditions:

- $\|v \oplus w\|_{k+l} = \max\{\|v\|_k, \|w\|_l\}$ and
- $\|\alpha w \beta\|_k \leq \|\alpha\| \|w\|_l \|\beta\|$

for all $v \in M_k[E]$, $w \in M_l[E]$, $\alpha \in M_{k,l}$, and $\beta \in M_{l,k}$. A simple, but important, example of an operator space is $M_n$ with its operator space structure given by the usual sequence of matrix norms $\|\cdot\|_k$ defined by the identification $M_k[M_n] = M_{kn}$.

To understand this theory, one needs to study the morphisms that preserve the operator space structure. In contrast to Banach space theory, where one needs to study the bounded maps between Banach spaces, in the theory of operator spaces we need to study the completely bounded maps. Given operator spaces $E$ and $F$ and a linear map $T : E \to F$, let $T_k : M_k[E] \to M_k[W]$ denote the linear map defined by

$$T_k(v) = (id_k \otimes T)(v) = (T(v_{ij}))_{i,j},$$

The map $T$ is said to be completely bounded if

$$\|T\|_{cb} = \sup_n \|T_k\| < \infty,$$

and this quantity is then called the completely bounded norm of $T$. We will say that $T$ is completely contractive if $\|T\|_{cb} \leq 1$. Moreover, $T$ is said to be a complete isomorphism (resp. complete isometry) if each map $T_k$ is an isomorphism (resp. an isometry).

As in Banach space theory, we can also consider the notion of duality. Given an operator space $E$, we define the dual operator space $E^*$ by means of the acceptable matrix norms $\|\cdot\|_k$ on $M_k[E] = M_k \otimes E$ satisfying the following conditions:

$$\|v \oplus w\|_{k+l} = \max\{\|v\|_k, \|w\|_l\} \quad \text{and} \quad \|\alpha w \beta\|_k \leq \|\alpha\| \|w\|_l \|\beta\|$$

for all $v \in M_k[E]$, $w \in M_l[E]$, $\alpha \in M_{k,l}$, and $\beta \in M_{l,k}$. A simple, but important, example of an operator space is $M_n$ with its operator space structure given by the usual sequence of matrix norms $\|\cdot\|_k$ defined by the identification $M_k[M_n] = M_{kn}$.

To understand this theory, one needs to study the morphisms that preserve the operator space structure. In contrast to Banach space theory, where one needs to study the bounded maps between Banach spaces, in the theory of operator spaces we need to study the completely bounded maps. Given operator spaces $E$ and $F$ and a linear map $T : E \to F$, let $T_k : M_k[E] \to M_k[W]$ denote the linear map defined by

$$T_k(v) = (id_k \otimes T)(v) = (T(v_{ij}))_{i,j},$$

The map $T$ is said to be completely bounded if

$$\|T\|_{cb} = \sup_n \|T_k\| < \infty,$$

and this quantity is then called the completely bounded norm of $T$. We will say that $T$ is completely contractive if $\|T\|_{cb} \leq 1$. Moreover, $T$ is said to be a complete isomorphism (resp. complete isometry) if each map $T_k$ is an isomorphism (resp. an isometry).

As in Banach space theory, we can also consider the notion of duality. Given an operator space $E$, we define the dual operator space $E^*$ by means of the acceptable matrix norms $M_k[E^*] = CB(E, M_k), \ k \geq 1$.

If we denote by $S_1^n$ the space $M_n$ with the trace norm, the duality relation $S_1^n = M_n^*$ allows us to define a natural operator space structure on $S_1^n$. This operator space structure is not given by the linear map identifying matrices in $S_1^n$ with matrices in $M_n$, but the right duality action is the scalar pairing

$$\langle B, C \rangle = \text{tr} (BC^{tr}),$$

which yields completely isometric isomorphisms $M_n^* = S_1^n$ and $(S_1^n)^* = M_n$. It is not difficult to see that $\|T^*\|_{cb} = \|T\|_{cb}$ for every $T : E \to F$, where $T^*$ denotes the adjoint map of $T$. 

There is an equivalent definition of operator spaces, as those closed subspaces of $B(H)$. On the one hand, given a subspace $E \subset B(H)$ it is clear that we have a family of matrix norms, by identifying $M_k[E] \subset M_k(B(H)) = B(\ell^2_k \otimes H)$, which can be shown to be an acceptable sequence of matrix norms. The converse statement is known as Ruan’s Theorem and can be found in [12, Theorem 2.3.5]. This point of view is very suitable to define the minimal tensor product of operator spaces. Given two operator spaces $E \hookrightarrow B(H_E)$ and $F \hookrightarrow B(H_F)$, we have a natural algebraic embedding of $E \otimes F$ in $B(H_E \otimes H_F)$. The minimal operator space tensor product $E \otimes_{\min} F$ is the closure of $E \otimes F$ in $B(H_E \otimes H_F)$. In particular, for every operator space $E$, one has that $M_n[E] = M_n \otimes_{\min} E$! isometrically. One can check that for a couple of linear maps $T_1 : E_1 \to F_1$ and $T_2 : E_2 \to F_2$ one has

$$
\|T_1 \otimes T_2 : E_1 \otimes_{\min} E_2 \to F_1 \otimes_{\min} F_2\|_{cb} = \|T_1 : E_1 \to F_1\|_{cb} \|T_2 : E_2 \to F_2\|_{cb}
$$

and that this tensor norm is commutative and associative (see [32, Chapter 2]). Moreover, if $E$ and $F$ are finite dimensional, one can also check that we have the following completely isometric identification.

$$E \otimes_{\min} F = CB(E^*, F),$$

where here the correspondence is defined by $(\sum_{i=1}^n v_i \otimes w_i)(v^*) = \sum_{i=1}^n (v_i, v^*)w_i$.

The dual tensor norm of the minimal one is the so called projective tensor norm (see [32, Chapter 4]), $E \hat{\otimes} F$, which is defined for a given element $t \in M_k(E \otimes F)$, as

$$\|t\|_{M_k(E \hat{\otimes} F)} = \{\|\alpha\|_{M_{l,n}} \|x\|_{M_n(E)} \|y\|_{M_m(F)} \|\beta\|_{M_{m,n}} : 1 \leq r \leq k, 1 \leq s \leq k\},$$

where the infimum runs over all possible representations $t_{r,s} = \sum_{i,j} \alpha_{r,i,j} (x_{ij} \otimes y_{pq}) \beta_{q,s}$ with $1 \leq r \leq k, 1 \leq s \leq k$. This norm is also commutative and associative and for every finite dimensional operator spaces $E$ and $F$ one has the complete isometric identifications

$$(E \otimes_{\min} F)^* = E^* \hat{\otimes} F^* \quad \text{and} \quad (E \hat{\otimes} F)^* = E^* \otimes_{\min} F^*.$$

In particular, if we denote $S^*_{1} \hat{\otimes} E := S^*_{1}[E]$, one has the completely isometric identification $(M_n[E])^* = S^*_{1}[E^*]$. One can also check that

$$\|T_1 \otimes T_2 : E_1 \otimes_{\min} E_2 \to F_1 \otimes_{\min} F_2\|_{cb} = \|T_1 : E_1 \to F_1\|_{cb} \|T_2 : E_2 \to F_2\|_{cb}
$$

for all linear maps $T_1 : E_1 \to F_1$ and $T_2 : E_2 \to F_2$.

Finally, given two operator spaces $E_0$ and $E_1$ which are compatible interpolation spaces in the sense of [34, Section 2], one can define a natural operator spaces structure on $E_0 = (E_0, E_1)_\theta$ by defining the following family of acceptable norms

$$M_k[E_\theta] = (M_k[E_0], M_k[E_1])_\theta, \quad k \geq 1.$$

As we explained in the introduction, this allows us to define a natural operator space structure on $S_p = (S_{\infty}, S_1)_{\frac{1}{p}}$ (resp. $S^\alpha_p = (M_n, S^\alpha_1)_{\frac{1}{p}}$) and, moreover, on $S_p[E] = (S_{\infty}[E], S_1[E])_{\frac{1}{p}}$ (resp. $S^\alpha_p[E] = (M_n[E], S^\alpha_1[E])_{\frac{1}{p}}$) for every operator space $E$. Here, $S_{\infty}$ denotes the space of compact operators on $\ell^2_k$ with the operator norm. As a particular case, the previous interpolation formula allows us to talk about the $p$-direct sum of operator spaces $\ell^p_k(E_i) := E_1 \oplus_p \cdots \oplus_p E_n$. If
$1 < p < \infty$ and $\theta = \frac{1}{p}$, then for any compatible couple of operator spaces $(E_0, E_1)$ the previous definition yields to the completely isometric identification $S_{p\theta}[E_0] = (S_{p\theta}[E_0], S_{p\theta}[E_1])_\theta$, where $\frac{1}{\theta} = \frac{1-\theta}{p_\theta} + \frac{\theta}{q_\theta}$ and $E_\theta = (E_0, E_1)_\theta$ (see [33, Theorem 1.1]). Moreover, it was shown in [33] that this definition of noncommutative $L_p$-spaces leads to the expected properties analogous to the classical ones. A very useful result, analogous to the classical case, states that given two couples of operator spaces $(E_0, E_1)$ and $(F_0, F_1)$, one has that

$$\|T\|_{CB(E_\theta, F_\theta)} \leq \|T\|_{CB(E_0, F_0)}^{1-\theta}\|T\|_{CB(E_1, F_1)}^\theta.$$  

According to the previous definition of the operator spaces $S_p[E]$ ($1 \leq p < \infty$), it can be seen ([33, Lemma 1.7], [33, Theorem 1.5]) that

$$\|Y\|_{M_d[S_p]} = \sup_{A, B \in B_{S_{pq}^d}} \|Z\|_{B(f_2)\otimes \min E} \|B\|_{S_{p\theta}},$$

and

$$\|X\|_{S_p[E]} = \inf \{ \|A\|_{S_{pq}^d} \|Z\|_{B(f_2)\otimes \min E} \|B\|_{S_{p\theta}} \},$$

where the last infimum runs over all representations of the form $X = (A \otimes \mathbb{I})Z(B \otimes \mathbb{I})$. Here, $B_{S_{p\theta}^d}$ denotes the unit ball $S_{p\theta}^d$ and $\mathbb{I}$ denotes the identity operator in $B(f_2)$. We will usually denote by $\mathbb{I}_n$ the identity matrix in $M_n$ appearing in the corresponding formulae for $\|Y\|_{M_d[S_p]}$ and $\|X\|_{S_p[E]}$.

In the second part of this work, we will mainly deal with the case $E = S_{q\theta}^d$ for some $1 \leq q \leq \infty$. It can be seen that, given $1 \leq p, q < \infty$ and defining $\frac{1}{\theta} = \frac{1}{p} - \frac{1}{q}$, we have:

If $p \leq q$,

$$\|X\|_{S_{p}[S_{q\theta}^d]} = \inf \{ \|A\|_{S_{pq}^d} \|Y\|_{S_{pq}^d} \|B\|_{S_{p\theta}} \},$$

where the infimum runs over all representations $X = (A \otimes \mathbb{I}_d)Y(B \otimes \mathbb{I}_d)$ with $A, B \in M_n$ and $Y \in M_n \otimes M_d$.

If $p \geq q$,

$$\|X\|_{S_{p}[S_{q\theta}^d]} = \sup \{ \|A \otimes \mathbb{I}_d\|_{S_{pq}^d} X(B \otimes \mathbb{I}_d)\|_{S_{pq}^d} : A, B \in B_{S_{p\theta}^d} \}.$$  

As an interesting application of this expression for the norm in $S_{p}[S_{q\theta}]$ in [33, Theorem 1.5 and Lemma 1.7] Pisier showed that for a given linear map between operator spaces $T : E \to F$ we can compute its completely bounded norm as

$$\|T\|_{cb} = \sup_{d \in \mathbb{N}} \|id_d \otimes T : S_{q\theta}^d[E] \to S_{q\theta}^d[F]\|$$

for every $1 \leq t \leq \infty$. That is, we can replace $\infty$ with any $1 \leq t \leq \infty$ in order to compute the cb-norm.

**Remark 2.1.** It is known ([3], [10]) that if $T$ is completely positive we can compute $\|T : S_q \to S_p\|$ by restricting to positive elements $A \in S_q$. Moreover, in this case one can also consider positive elements $X \geq 0$ to compute the cb-norm of $T$ ([11, Section 3]) $\|T\|_{cb} = \|id_{S_q} \otimes T : S_{p}[S_q] \to S_{q}[S_p]\|$. On the other hand, for a positive element $X$, one can consider $A = B > 0$ in the expressions (2.5) and (2.6) for $\|X\|_{S_p[S_{q\theta}^d]}$. According to this, if $X > 0$ and $q = 1$, (2.6)
becomes
\[ \|X\|_{\mathcal{S}_p[S]} = \sup_{A > 0} \frac{\|(A \otimes \mathbb{1}_d)X(A \otimes \mathbb{1}_d)\|_{\mathcal{S}_p}}{\|A\|_{\mathcal{S}_p}^p} = \|(\text{id}_n \otimes \text{tr}_d)(X)\|_p, \]
where \( \frac{1}{p} + \frac{1}{p'} = 1 \). Here and in the rest of the work we use notation \( \text{tr}_n := \text{tr}_{M_n} \).

2.2. General quantum teleportation. We will start this section by introducing a family of unitaries which will be crucial in the rest of the work. For all \( k, l = 1, \cdots, n \) we define the following unitaries on \( \ell_2^n \):
\[ u_k(e_j) = e^{\frac{2\pi i j k}{n}} e_j \text{ and } v_l(e_j) = e_{l+j}, \text{ for every } j = 1, \cdots, n, \]
where \( l + j \) will be always understood mod \( n \). In this sense, we will understand \( u_{-k} = u_{n-k} \) and \( v_{-l} = v_{n-l} \) for any \( k, l = 1, \cdots, n \). We will denote \( \psi_n := \sum_{i=1}^n e_i \otimes e_i \in \ell_2^\otimes \ell_2^n \) and \( \overline{\psi}_n := \frac{1}{\sqrt{n}} \sum_{i=1}^n e_i \otimes e_i \in \ell_2^n \otimes \ell_2^n \). The following properties of the previous unitaries will be very useful in our analysis.

Proposition 2.1.

a) Let \( p \) be any natural number in \( \{1, \cdots, n\} \). Then,
\[ \sum_{k=1}^n e^{\frac{2\pi i k p}{n}} = n\delta_{p,n}. \]

b) For every \( j, k = 1, \cdots, n \), we have
\[ e_j \otimes e_k = \frac{1}{\sqrt{n}} \sum_{s=1}^n e^{-\frac{2\pi isj}{n}} (u_s \otimes v_{k-j})(\overline{\psi}_n). \]

c) Let us define \( \eta_{k,l} = (u_k \otimes v_l)(\overline{\psi}_n) \) for every \( j, k = 1, \cdots, n \). Then, \( (\eta_{k,l})_{k,l=1}^n \) is an orthonormal basis of \( \ell_2^n \otimes \ell_2^n \).

d) Let \( h \in \ell_2^n \). Then,
\[ h \otimes \overline{\psi}_n = \frac{1}{n} \sum_{k,l=1}^n \eta_{k,l} \otimes T_{k,l}(h), \]
where we denote \( T_{k,l} = v_l u_{-k} \) for every \( k, l \). In particular, for every operator \( \rho : \ell_2^n \rightarrow \ell_2^n \) we have
\[ \rho \otimes |\overline{\psi}_n\rangle\langle\overline{\psi}_n| = \frac{1}{n^2} \sum_{k,l=1}^n \sum_{k',l'=1}^n |\eta_{k,l}\rangle\langle \eta_{k',l'}| \otimes T_{k,l} \rho T_{k',l'}^*. \]

Here, given two elements \( \alpha, \beta \) in a Hilbert space \( H \), we denote \( |\alpha\rangle\langle \beta| : H \rightarrow H \) the rank one operator defined by \( |\alpha\rangle\langle \beta| h = \langle \beta|h \rangle \alpha \). In particular, \( |\alpha\rangle\langle \alpha| \) is the rank-one projection on \( \alpha \).

Proof. Part a) is trivial. For the part b), we have
\[ \frac{1}{\sqrt{n}} \sum_{s=1}^n e^{-\frac{2\pi isj}{n}} (u_s \otimes v_{k-j})(\overline{\psi}_n) = \frac{1}{n} \sum_{s,l=1}^n e^{-\frac{2\pi isj}{n}} e^{\frac{2\pi i l k}{n}} e_l \otimes e_{l+k-j}. \]
\[ n \sum_{l=1}^{n} n \delta_{l,j} e_{l} \otimes e_{l+k-j} = e_{j} \otimes e_{k}. \]

In order to show part c) we first note that the fact that \( u_{k} \) and \( v_{l} \) are unitaries on \( \ell_{n}^{2} \) guarantees that \( (u_{k} \otimes v_{l}) \) is a unitary on \( \ell_{n}^{2} \) for every \( k, l \). Hence, since \( \| \psi_{n} \| = 1 \) we conclude that \( \| \eta_{k,l} \| = 1 \) for every \( j, k \). On the other hand, it is very easy to see that these vectors are orthogonal. Indeed, we have that

\[ \langle \eta_{k}, v_{l} \| \eta_{k}, l \rangle = \frac{1}{n} \sum_{s,s'=1}^{n} e^{-2\pi i k s' / n} e^{2\pi i k s} \langle e_{s'} \otimes e_{s'} + v_{l}, e_{s} \otimes e_{s+l} \rangle = \delta_{k,l} \delta_{k',l'}. \]

Finally, in order to show part d), let us consider \( h = \sum_{j=1}^{n} h_{j} e_{j} \). According to part c) above we have

\[ h \otimes \overline{\psi}_{n} = \frac{1}{\sqrt{n}} \sum_{j,l=1}^{n} h_{j} e_{j} \otimes e_{l} \]

\[ = \frac{1}{n} \sum_{j,l=1}^{n} h_{j} \left( \sum_{k=1}^{n} e^{-2\pi i k j / n} (u_{k} \otimes v_{l-j}) (\overline{\psi}_{n}) \right) \otimes e_{l} \]

\[ = \frac{1}{n} \sum_{j,l,k=1}^{n} h_{j} e^{-2\pi i k j / n} \eta_{k,l} \otimes e_{l+j}. \]

On the other hand, note that \( T_{k,l}(h) = \sum_{j=1}^{n} h_{j} e^{-2\pi i k j / n} e_{l+j} \). Therefore,

\[ h \otimes \overline{\psi}_{n} = \frac{1}{n} \sum_{k,l=1}^{n} \eta_{k,l} \otimes v_{l} u_{-k}(h). \]

The second part of the statement can be obtained straightforwardly from the first one just looking at rank one operators \( \rho = |h\rangle \langle k| \).

**Corollary 2.2.** The linear map

\[ \iota: \ell_{n}^{2} \rightarrow S_{p}[S_{p}^{n}], \quad \text{defined by} \quad e_{k,l} \mapsto |\eta_{k,l} \rangle \langle \eta_{k,l}|, \quad k, l = 1, \ldots, n, \]

is completely positive and a complete isometry and the linear map

\[ P: S_{p}[S_{p}^{n}] \rightarrow \ell_{n}^{2} \quad \text{defined by} \quad P(A) = \sum_{k,l=1}^{n} \langle \eta_{k,l} | A | \eta_{k,l} \rangle e_{k,l}, \quad A \in S_{p}[S_{p}^{n}], \]

is completely positive, and it is a completely contractive projection onto the image of \( \iota \).

Moreover, for every operator space \( E \) the map \( \iota \otimes \text{id}_{E} \) defines a complete isometry of \( \ell_{n}^{2}[E] \) onto a subspace of \( S_{p}^{n}[E] \) which is completely complemented via \( P \otimes \text{id}_{E} \).

**Proof.** The proof is immediate from part c) of Proposition 2.1 (see for instance [33, Corollary 1.3]).
For the following lemma we note that $|\psi_n\rangle\langle\psi_n|$ can be seen as an element of $M_n \otimes M_n$ by writing $|\psi_n\rangle\langle\psi_n| = \sum_{i,j=1}^n e_i \otimes e_j$. Moreover, we note that the corresponding map $S^n_1 \to M_n$ is the identity map. Hence, $|\psi_n\rangle\langle\psi_n|$ is an element in the unit ball of $S^n_1 \otimes_{\min} M_n$.

**Lemma 2.3.** Let us define the linear map $\iota : M_n \to M_n \otimes M_n \otimes M_n$ as $\rho \mapsto \rho \otimes |\psi_n\rangle\langle\psi_n|$. Then, for every operator space $E$, $\iota$ verifies that

$$\|\iota \otimes id_E : S^n_1[E] \to S^n_1(S^n_1)[E] \otimes_{\min} M_n\|_{cb} \leq 1.$$  

**Proof.** We must show that

$$\|\iota \otimes id_E \otimes id_k : S^n_1[E] \otimes_{\min} M_k \to S^n_1(S^n_1)[E] \otimes_{\min} M_n \otimes_{\min} M_k\| \leq 1$$

for every $k$. To this end, let us consider an element $x$ in the unit ball of $S^n_1[E] \otimes_{\min} M_k$. Now, it follows from the definition of $\iota$ that

$$(\iota \otimes id_E \otimes id_k)(x) = x \otimes |\psi_n\rangle\langle\psi_n|.$$  

On the other hand, since $|\psi_n\rangle\langle\psi_n|$ is in the unit ball of $S^n_1 \otimes_{\min} M_n$, according to (2.2), we have that $x \otimes |\psi_n\rangle\langle\psi_n|$ is in the unit ball of $S^n_1(S^n_1)[E] \otimes_{\min} M_n \otimes_{\min} M_k$. Here, we have used that $S^n_1[E] \otimes_{\min} S^n_1 = S^n_1(S^n_1)[E]$. This concludes the proof. \qed

**Proposition 2.4.** Let us define the linear map $J : M_n \to \ell^q_2 \otimes M_n$ by

$$J(\rho) = \frac{1}{n} \sum_{k,l=1}^n e_{k,l} \otimes T_{k,l} \rho T_{k,l}^*.$$  

Then, $J$ is completely positive verifying, for every operator space $E$,

$$\|J \otimes id_E : S^n_1[E] \to S^n_q(\ell^2_2[E])\|_{cb} \leq \frac{1}{n^{1-\frac{1}{p}}}$$

for every $1 \leq p \leq q \leq \infty$.  

**Proof.** The fact that $J$ is linear and completely positive is very easy. On the other hand, since it is well known that

$$\|id_n \otimes id_X : M_n[X] \to S^n_q[X]\|_{cb} = n^{\frac{1}{q}}$$

for every operator space $X$, it suffices to show that

$$\|J \otimes id_E : S^n_1[E] \to M_n(\ell^q_2[E])\|_{cb} \leq \frac{1}{n^{1-\frac{1}{p}}}$$

for every $1 \leq p \leq \infty$.

In order to prove the previous estimate for the case $p = 1$,

$$\|J \otimes id_E : S^n_1[E] \to M_n(\ell^2_1[E])\|_{cb} \leq 1,$$  

\footnote{Note that here we are shifting the spaces: $|\psi_n\rangle\langle\psi_n| = \sum_{i,j=1}^n (e_i \otimes e_i) \otimes (e_j \otimes e_j) = \sum_{i,j=1}^n (e_i \otimes e_j) \otimes (e_i \otimes e_j).$}
we invoke part b) in Proposition 2.1 to understand the map $J$ as

$$J = (P \otimes id_n) \circ \iota : S^n_1 \to S^n_1(S^n_1) \otimes_{\min} M_n \to \ell_1^\infty \otimes_{\min} M_n.$$ 

Here, the map $P$ was defined in Corollary 2.2 and the map $\iota$ was defined in Lemma 2.3. Indeed, this identification can be checked by basic calculations

$$((P \otimes id_n) \circ \iota)(\rho) = (P \otimes id_n)((\rho \otimes |\psi_n\rangle\langle\psi_n|)$$

$$= (P \otimes id_n)(\frac{1}{n} \sum_{k,l=1}^{n} \sum_{k',l'}^{n} |\eta_{k,l}\rangle\langle\eta_{k',l'}| \otimes T_{k,l} \rho T_{k,l}^*)$$

$$= \frac{1}{n} \sum_{k,l=1}^{n} |\eta_{k,l}\rangle\langle\eta_{k,l}| \otimes T_{k,l} \rho T_{k,l}^*.$$ 

Hence, the estimate (2.11) follows from Corollary 2.2 and Lemma 2.3.

In order to show the case $p = \infty$, we just note that $J$ is a completely positive map between C*-algebras. Then, it is well known (see for instance [30, Corollary 2.9]) that

$$\|J : M_n \to M_n(\ell_1^\infty)\|_{cb} = \|J(1_n)\|_{M_n(\ell_1^\infty)} = \frac{1}{n}.$$ 

Then, (2.2) immediately implies that

$$\|J \otimes id_E : M_n[E] \to M_n(\ell_1^\infty[E])\|_{cb} = \|J(1_E)\|_{M_n(\ell_1^\infty)} = \frac{1}{n},$$

since $M_n[E] = M_n \otimes_{\min} E$ and $M_n(\ell_1^\infty[E]) = M_n(\ell_1^\infty) \otimes_{\min} E$.

Finally, the case $1 < p < \infty$ follows from (2.11), (2.12) and interpolation (2.4).

$$\|J \otimes id_E : S^n_p[E] \to M_n(\ell_q^p[E])\|_{cb} \leq (\frac{1}{n})^{1-\frac{1}{p}},$$

where $S^n_p[E] = (M_n[E], S^n_1[E])_{\frac{1}{p}}$ and $M_n(\ell_q^p[E]) = (M_n(\ell_1^\infty[E]), M_n(\ell_q^p[E]))_{\frac{1}{p}}$. 

\[ \square \]

**Proposition 2.5.** Let $W : M_n \otimes \ell_1^\infty \to M_n$ be the linear map defined by

$$W \left( \sum_{k,l=1}^{n} A_{k,l} \otimes e_{k,l} \right) = \frac{1}{n} \sum_{k,l=1}^{n} T_{k,l} A_{k,l} T_{k,l}^*$$

for every $(A_{k,l})_{k,l=1}^{n} \subset M_n$. Then, $W$ is completely positive and it verifies, for every operator space $E$,

$$\|W \otimes id_E : S^n_p(E) \to S^n_p(E)\|_{cb} \leq n^{1-\frac{1}{p}-\frac{1}{q}},$$

for every $1 \leq p \leq q \leq \infty$.

**Proof.** The fact that $W$ is a linear map is obvious. Moreover, $W$ is defined as a sum of completely positive maps $A \mapsto T_{k,l}^* A_{k,l} T_{k,l}$, so it is completely positive. On the other hand, since it is well known that

$$\|id_n \otimes id_X : S^n_p[X] \to S^n_p[X]\|_{cb} = n^{\frac{1}{p}-\frac{1}{q}},$$
for every operator space $X$, it suffices to show that
\begin{align}
\|W \otimes \text{id}_E : S^p_n(\ell^2_p[E]) \to S^p_n[E]\|_{cb} &\leq n^{1-\frac{2}{p}}.
\end{align}

for every $1 \leq p \leq \infty$.

Let us first consider the case $p = 1$. The fact that $nW$ is completely positive and trace preserving immediately implies that $nW$ is completely contractive from $S^1_n(\ell^2_1[E])$ to $S^1_n$. Thus, we have
\begin{align}
\|W \otimes \text{id}_E : S^1_n(\ell^2_1[E]) \to S^1_n[E]\|_{cb} &\leq \frac{1}{n},
\end{align}

since $S^1_n(\ell^2_1[E]) = S^1_n(\ell^2_1) \otimes E$ and $S^1_n[E] = S^1_n \otimes E$ \eqref{2.2}. If we consider $p = \infty$, we have a completely positive map between the $C^*$-algebras $\ell^\infty_\infty(M_n)$ and $M_n$. As we have said previously, the completely bounded norm is then attained in the unit. Again, we easily deduce from here that
\begin{align}
\|W \otimes \text{id}_E : M_n(\ell^\infty_\infty[E]) \to M_n[E]\|_{cb} &\leq \left(\frac{1}{n}\right)^{\frac{1}{2}} n^{1-\frac{1}{p}} = n^{1-\frac{2}{p}}.
\end{align}

Equations \eqref{2.14} and \eqref{2.15} allow us to obtain the estimate in \eqref{2.13} for a general case $1 \leq p \leq \infty$ by interpolation \eqref{2.4}. Indeed, we have
\begin{align}
\|W \otimes \text{id}_E : S^p_n(\ell^2_p[E]) \to S^p_n[E]\|_{cb} &\leq \left(\frac{1}{n}\right)^{\frac{1}{2}} n^{1-\frac{1}{p}} = n^{1-\frac{2}{p}},
\end{align}

where we have used that $S^p_n(\ell^2_p[E]) = (M_n(\ell^\infty_\infty[E]), S^p_n(\ell^2_p[E]))_{\frac{1}{p}}$. \hfill \Box

Instead of proving Theorem \ref{1.1} directly we will first show how to obtain Corollary \ref{1.2}. Then, we will explain how to adapt such a proof to obtain Theorem \ref{1.1}.

**Proof of Corollary \ref{1.2}**. It suffices to show the case $1 \leq p \leq q \leq \infty$, since the other case can be obtained by duality.

Let us define the linear maps
\begin{align}
J_{p,q} := n^{1-\frac{1}{p} - \frac{1}{q}} J : M_n \to \ell^\infty_\infty \otimes M_n,
\end{align}

where $J$ was defined in Proposition \ref{2.4} and
\begin{align}
W_{p,q} := n^{\frac{1}{p} + \frac{1}{q} - 1} W : \ell^\infty_\infty \otimes M_n \to M_n,
\end{align}

where $W$ was defined in Proposition \ref{2.13}. According to the previous propositions both maps are completely positive and they verify the estimates
\begin{align}
\|J_{p,q} \otimes \text{id}_E : S^p_n[E] \to S^p_n(\ell^2_p[E])\|_{cb} \leq 1, \quad \text{and} \quad \|W_{p,q} \otimes \text{id}_E : S^p_n(\ell^2_p[E]) \to S^p_n[E]\|_{cb} \leq 1
\end{align}

for every operator space $E$. Therefore, it suffices to show the algebraic identification $W_{p,q} \circ J_{p,q} = \mathbb{I}_n$. This is very easy just noting that for every $\rho \in M_n$, we have that
\begin{align}
W_{p,q}(J_{p,q}(\rho)) = W(J(\rho)) = W\left(\frac{1}{n} \sum_{k,l=1}^n e_{k,l} \otimes T_{k,l} \rho T_{k,l}^*\right) = \frac{1}{n^2} \sum_{k,l=1}^n \rho = \rho.
\end{align}

\hfill \Box
Quantum teleportation is a communication protocol between two people, Alice and Bob, where say Alice can transmit a qubit (basic unit in quantum information theory) to Bob, by just sending two classical bits of information if they are allowed to share a maximally entangled state during the protocol. From a mathematical point of view, this means that there exist a channel (completely positive and trace preserving map) $\mathcal{E} : S^2_1 \otimes S^2_1 \rightarrow \ell^4_1$ (Alice’s encoder from quantum to classical information) and another channel $\mathcal{D} : \ell^4_1 \otimes S^2_1 \rightarrow S^2_1$ (Bob’s decoder from classical to quantum information) so that the following diagram commutes:

$$
\begin{array}{ccc}
\ell^4_1 \otimes S^2_1 & \xrightarrow{id} & \ell^4_1 \otimes S^2_1 \\
\mathcal{E} \otimes id & \ Candidated & \mathcal{D} \\
(S^2_1 \otimes S^2_1) \otimes S^2_1 & \xrightarrow{id} & S^2_1
\end{array}
$$

where here the map $i : S^2_1 \rightarrow S^2_1 \otimes S^2_1$ is defined by $i(\rho) = \rho \otimes |\psi_2\rangle \langle \psi_2|$. A careful study of the channels $\mathcal{E}$, $\mathcal{D}$ in the teleportation protocol (see for instance [27, Section 1.3.7]) should help the reader to identify the maps used in the proof of Corollary 1.2 for the particular case $n = 2$.

The proof of Theorem 1.1 is a generalization of the previous one. However, in this case we need to be more careful since we have to use the same state $|\psi_d\rangle/|\psi_d| \in M_d \otimes M_d$ to define different maps. Let us start by noting that the element $\psi_d$ can be seen as a tensor product element. Indeed,

$$
\psi_d = \sum_{i=1}^n \sum_{j=1}^n (e_i \otimes e_j) \otimes (e_i \otimes e_j) = \sum_{i=1}^n (e_i \otimes e_i) \otimes \sum_{j=1}^n (e_j \otimes e_j).
$$

Therefore,

$$
|\psi_d\rangle \langle \psi_d| = \sum_{i,j=1}^n \sum_{i',j'=1}^n \langle e_i \otimes e_j | (e_i \otimes e_j) \otimes (e_i \otimes e_j) | e_i' \otimes e_j' \rangle
$$

$$
= \sum_{i,j=1}^n |e_i| \langle e_i' | \otimes |e_i| \langle e_i' | \otimes \sum_{j,j'=1} |e_j| \langle e_j' | \otimes |e_j| \langle e_j' |
$$

$$
= |\psi_n\rangle \langle \psi_n| \otimes |\psi_{n1}\rangle \langle \psi_{n1}|.
$$

Similarly, we have that $|\psi_d\rangle \langle \psi_d| = |\psi_{m}\rangle \langle \psi_{m1}| \otimes |\psi_{m1}\rangle \langle \psi_{m1}|$.

We will also need a “more sophisticated” interpolation result here, which allows us to interpolate not just the spaces, but also the operators. We will use the following result, which can be found in [26].

**Theorem 2.6.** Let $\mathcal{S}$ denote the close strip $\{z : 0 \leq \text{Re}(z) \leq 1\}$ in the complex plane and $A(\mathcal{S})$ the algebra of bounded continuous functions on $\mathcal{S}$ that are analytic on the open strip $\mathcal{S}$. Let $(E_0, E_1)$ and $(F_0, F_1)$ be two compatible couples of Banach spaces and $\{T_z\}_{z \in \mathcal{S}}$ be a family of operators on $E_0 \cap E_1$ into $F_0 + F_1$ such that for every $a \in E_0 \cap E_1$ and $b^* \in (F_0 + F_1)^*$, $(b^*, T_z(a)) \in A(\mathcal{S})$, there exist constants $M_0$, $M_1$ so that $\sup_{z \in \mathcal{S}} \|T_z : E_j \rightarrow F_j\| \leq M_j$ for
j = 0, 1, and for every a ∈ E_0 ∩ E_1 we have that \{T_{ia}(a)\}_i lies in a separable subspace of F_0. Then,

\[ ||T_0 : (E_0, E_1)_a \rightarrow (F_0, F_1)_a|| \leq M_0^{1-θ} M_1^θ. \]

To simplify notation, we will show the proof of the main theorem for the case of two spaces and in the scalar case \((E = \mathbb{C})\), \(S^n_p \oplus_p S^m_p\). The reader will see that exactly the same proof applies in the general case.

**Proof of Theorem 1.1.** Again, it suffices to show the result for the case \(1 \leq p \leq q \leq \infty\), since the general case can be then obtained by duality. In order to prove the first part of the theorem, let \(d\) be the least common multiplier of \(m\) and \(n\) so that \(d = m n_1 = n m_1\) for certain natural numbers \(n_1\) and \(m_1\). Let us denote by \(P^k\), \(J^k\) and \(W^k\) the linear maps introduced in Corollary 2.2 Proposition 2.4 and Proposition 2.5 respectively, when they are defined in dimension \(k\) equal \(n\) or \(m\).

Motivated by (2.16), we consider the projection

\[ \tilde{P}^n : M_n(M_d) \rightarrow \ell^n_{∞} \]

defined as

\[ \tilde{P}^n(\rho) = P^n\left((id_n \otimes tr_{n_1}) (\rho)\right) \quad \text{for every} \quad \rho \in M_n(M_d) = M_n(M_n \otimes M_{n_1}). \]

We define \(\tilde{P}^m : M_m(M_d) \rightarrow \ell^m_{∞}\) analogously. Moreover, for every \(1 \leq p \leq q \leq \infty\) we consider the linear map

\[ \tilde{J}_{p,q} : M_n \oplus M_m \rightarrow \ell^n_{∞}(M_d) \oplus \ell^m_{∞}(M_d) = (\ell^n_{∞} \oplus \ell^m_{∞})(M_d). \]

defined by

\[ \tilde{J}_{p,q}(\rho_1 \oplus \rho_2) = d^{-\frac{1}{2}} \left[ (n^{\frac{1}{p}} \tilde{P}^n \otimes id_d) (\rho_1 \otimes |\psi_d\rangle \langle \psi_d|) \oplus (m^{\frac{1}{p}} \tilde{P}^m \otimes id_d) (\rho_2 \otimes |\psi_d\rangle \langle \psi_d|) \right]. \]

According to (2.16) we have

\[ \tilde{J}_{p,q}(\rho_1 \oplus \rho_2) = d^{-\frac{1}{2}} \left[ \frac{1}{n^{\frac{1}{p}}} \sum_{j=1}^{n_1} \sum_{k,l=1}^{n} e_{k,l} \otimes T_{k,l} p_1 T_{k,l}^* \otimes |e_j\rangle \langle e_j| \right. \]

\[ \left. \oplus \frac{1}{m^{\frac{1}{p}}} \sum_{i=1}^{m_1} \sum_{k',l'=1}^{m} e_{k',l'} \otimes T_{k',l'} p_2 T_{k',l'}^* \otimes |e_i\rangle \langle e_i| \right]. \]

\(\tilde{J}_{p,q}\) is a direct sum of two completely positive maps. Thus, it is completely positive. We claim that

(2.17) \[ \|\tilde{J}_{p,q} : S^n_p \oplus_p S^m_p \rightarrow S^n_{\ell^n_{∞}} \oplus_p S^m_{\ell^m_{∞}}\| \leq 1. \]

As we explained before, it suffices to show that

(2.18) \[ \left\| d^{\frac{1}{2}} \tilde{J}_{p,q} : S^n_p \oplus_p S^m_p \rightarrow M_d \oplus_{\min} (\ell^n_{\ell^n_{∞}} \oplus_p \ell^m_{\ell^m_{∞}})\right\| \leq 1. \]
Since $d^2 J_{p,q}$ does not depend on $q$, let us just denote $J_p$ this map. Indeed, for the case $p = 1$ we invoke the same argument as in the proof of Proposition 2.4 to state that the map

$$\tilde{i} : S_{1}^{n} \oplus S_{1}^{m} \to (S_{1}^{n} \oplus S_{1}^{m})(S_{1}^{d}) \otimes_{\min} M_{d} = (S_{1}^{nd} \oplus S_{1}^{md}) \otimes_{\min} M_{d},$$

defined by

$$\tilde{i}(p_1 \oplus p_2) = (p_1 \oplus p_2) \otimes |\psi_d \rangle \langle \psi_d|,$$

is completely contractive. On the other hand, since $\tilde{P}^n : S_{1}^{nd} \to \ell_{1}^{n^2}$ and $\tilde{P}^m : S_{1}^{md} \to \ell_{1}^{m^2}$ are completely contractive maps, we conclude (see for instance [33, Chapter 2]) that

$$(\tilde{P}^n \oplus \tilde{P}^m) \otimes id_{d} : (S_{1}^{nd} \oplus S_{1}^{md}) \otimes_{\min} M_{d} \to (\ell_{1}^{n^2} \oplus \ell_{1}^{m^2}) \otimes_{\min} M_{d}$$

is a complete contraction. Since $\tilde{J}_1 = ((\tilde{P}^n \oplus \tilde{P}^m) \otimes id_{d}) \circ \tilde{i}$, we obtain that

$$\|\tilde{J}_1 : S_{1}^{n} \oplus S_{1}^{m} \to M_{d} \otimes_{\min} (\ell_{1}^{n^2} \oplus \ell_{1}^{m^2})\|_{cb} \leq 1. \tag{2.19}$$

For the case $p = \infty$ we can proceed as in some previous proofs (just by evaluating the norm of $\tilde{J}_{\infty}(1_{n} \oplus 1_{m})$) or we can realized that, since $\tilde{J}_{\infty} : M_{n} \otimes_{\infty} M_{m} \to M_{d} \otimes_{\min} (\ell_{1}^{n^2} \otimes \ell_{1}^{m^2})$ is defined as a direct sum of two maps, it suffices to see that each of these maps $\tilde{J}_{\infty}^{1} : M_{n} \to M_{d}(\ell_{1}^{n^2})$ and $\tilde{J}_{\infty}^{2} : M_{m} \to M_{d}(\ell_{1}^{m^2})$ are completely contractive respectively. This is trivial since both of them are completely positive and unital. Therefore,

$$\|\tilde{J}_{\infty} : M_{n} \otimes_{\infty} M_{m} \to M_{d} \otimes_{\min} (\ell_{1}^{n^2} \otimes \ell_{1}^{m^2})\|_{cb} = 1. \tag{2.20}$$

The general case (2.18) for $1 < p < \infty$ follows now by interpolation. However, in this case we need to use a more general result, since we must also interpolate the operators $\tilde{J}_{p}$. To this end, we can apply Theorem 2.6 with

$$\tilde{J}_{z} := (n^{1-z} \tilde{P}^n \otimes 1_{d})(1_{1} \otimes |\psi_d \rangle \langle \psi_d|) \oplus (m^{1-z} \tilde{P}^m \otimes 1_{d})(1_{1} \otimes |\psi_d \rangle \langle \psi_d|).$$

In fact, since the theorem is stated for the norm of operators, in order to obtain our estimate for the completely bounded norm, we must consider the family of operators $T_{z} = id_{M_{k}} \otimes \tilde{J}_{z}$ for an arbitrary but fixed $k$. Then, we must understand (2.19) and (2.20) as estimates about the norm of $id_{M_{k}} \otimes \tilde{J}_{1}$ and $id_{M_{k}} \otimes \tilde{J}_{\infty}$ respectively. On the one hand, according to our explanation in Section 2.6, we can indeed obtain the spaces $M_{k}(S_{p}^{n} \oplus p S_{p}^{m})$ and $M_{k}(M_{d} \otimes_{\min} (p^{n^2} \oplus p^{m^2})$) by interpolating the spaces involved in the estimates (2.19) and (2.20) when they are tensored with $M_{k}$. On the other hand, since all the spaces are finite dimensional and the dependence of $T_{z}$ with respect to $z$ is so simple, all regularity conditions of Theorem 2.6 are trivially verified and we just need to see that $sup_{t} \|T_{z+it}\| \leq 1$ for $j = 0, 1$. Let us recall that $\tilde{J}_{z}$ is a direct sum of two maps $\tilde{J}_{1}^{z}$ and $\tilde{J}_{2}^{z}$. Then, we see that $T_{z} = id_{M_{k}} \otimes (n^{-it} \tilde{J}_{\infty}^{z} \oplus m^{-it} \tilde{J}_{\infty}^{z})$ and similarly $T_{1+it} = id_{M_{k}} \otimes (n^{-it} \tilde{J}_{1}^{z} \oplus m^{-it} \tilde{J}_{1}^{z})$. However, it is very easy to see that the arguments in (2.19) and (2.20) are not affected if we multiply $\tilde{J}_{1}^{z}$ and $\tilde{J}_{2}^{z}$ by a number of modulus one. Therefore, the same estimates hold in this new case. Hence, we obtain (2.18).

---

4This second proof, although more stilted, will make the interpolation argument below easier.
Let us consider now the linear map $\Gamma_{p,q} : S^d_q(\ell^2_p \oplus \ell^{m^2}_p) \to M_n \oplus M_m$ defined by

$$
\Gamma_{p,q} = \frac{1}{d^{1-\frac{1}{q}}} \left( \frac{1}{n^p} \bar{W}^n \oplus \frac{1}{m^p} \bar{W}^m \right),
$$

where $\bar{W}^n : M_d \otimes \ell^2_\infty \to M_n$ is defined by

$$
\bar{W}^n \left( \sum_{k,l=1}^n A_{k,l} \otimes e_{k,l} \right) = \sum_{k,l=1}^n T_{k,l} ((id_n \otimes tr_{n_i})(A_{k,l})) T_{k,l},
$$

and $\bar{W}^m : M_d \otimes \ell^m_\infty \to M_m$ is defined analogously. It is clear that $\Gamma_{p,q}$ is completely positive. We claim that

$$
\|\bar{W}^n : \ell^2_\infty(M_d) \to M_n\|_{cb} \leq (dn)^\frac{1}{d} \quad \text{and} \quad \|\bar{W}^m : \ell^m_\infty(M_d) \to M_m\|_{cb} \leq (dm)^\frac{1}{m}
$$

for ever $1 \leq p \leq \infty$. We show the estimate for $\bar{W}^n$ since the second one is completely analogous.

Let us first consider $p = 1$. Then, $\|\bar{W}^n : \ell^1_\infty(S^d_q) \to S^n\|_{cb} \leq 1$ follows from the fact that $\bar{W}^n$ is completely positive and trace preserving. On the other hand, the case $p = \infty$ follows from the estimate

$$
\|\bar{W} : \ell^\infty_\infty(M_d) \to M_n\|_{cb} = \|\bar{W} \left( \sum_{k,l=1}^n e_{k,l} \otimes I_d \right)\|_{M_n} = dn.
$$

The general estimate (2.21) can be obtained now by interpolation.

With (2.21) at hand, one can show that

$$
\|\Gamma_{p,q} : S^d_q(\ell^2_\infty \oplus \ell^{m^2}_p) \to S^n \oplus_p S^m\|_{cb} \leq 1.
$$

To this end, we use once more that

$$
\|\Gamma_{p,q} : S^d_q(\ell^2_\infty \oplus \ell^{m^2}_p) \to S^n \oplus_p S^m\|_{cb} \leq d^{\frac{p-1}{2}} \|\Gamma_{p,q} : S^d_q(\ell^2_\infty \oplus \ell^{m^2}_p) \to S^n \oplus_p S^m\|_{cb}.
$$

Therefore, we need to show that

$$
\|d^{\frac{p-1}{2}} \Gamma_{p,q} : S^d_q(\ell^2_\infty \oplus \ell^{m^2}_p) \to S^n \oplus_p S^m\|_{cb} \leq 1.
$$

Since $d^{\frac{p-1}{2}} \Gamma_{p,q}$ does not depend on $q$, let us denote it by $\Gamma_p$. Now, noting that

$$
\Gamma_p = \frac{1}{(nd)^{\frac{1}{q}}} \bar{W}^n \oplus \frac{1}{(md)^{\frac{1}{q}}} \bar{W}^m,
$$

the previous estimate is a direct consequence of (2.21).

Therefore, we conclude our proof if we show that

$$
\Gamma_{p,q} \circ \tilde{J}_{p,q} = id_{S^p_n \oplus_p S^m_q}.
$$

Indeed, given $\rho_1 \oplus \rho_2 \in S^n \oplus_p S^m$, we have that

$$
\Gamma_p(\tilde{J}_{p}(\rho_1 \oplus \rho_2)) = d^{\frac{1}{p}} \Gamma_p \left( \frac{1}{n^p} \sum_{j=1}^n \sum_{k,l=1}^n e_{k,l} \otimes T_{k,l} \rho_1 T_{k,l}^* \otimes |e_j\rangle \langle e_j| \right).
$$
Proposition 2.7. Let us define the linear map $H : \ell_p^2 \to M_n \otimes M_n$ by

$$H(\epsilon_k, l) = n|\eta_{k,l}\rangle\langle \eta_{k,l}|$$

for every $k, l$. Then, $H$ is completely positive and it verifies, for every operator space $E$,

$$\|H \otimes id_E : \ell_p^2[E] \to S^n(S_p^n[E])\|_{cb} \leq n^{1 + \frac{1}{p} - \frac{1}{2}}$$

for every $1 \leq p \leq q \leq \infty$.

Proof. Since the domain space is a commutative C*-algebra, completely positivity is equivalent to positivity. Hence, the fact that $n|\eta_{k,l}\rangle\langle \eta_{k,l}|$ is a positive element for every $k, l$ assures that $H$ is indeed completely positive. On the other hand, we have already explained that

$$\|H \otimes id_E : \ell_p^2[E] \to S^n(S_p^n[E])\|_{cb} \leq n^{1 + \frac{1}{p} - \frac{1}{2}}$$

so we must show the estimate

$$(2.23) \quad \|H \otimes id_E : \ell_p^2[E] \to M_n(S_p^n[E])\|_{cb} \leq n^{1 - \frac{1}{p}}$$

for every operator space $E$. In order to show this estimate let us start with the case $p = 1$,

$$(2.24) \quad \|H \otimes id_E : \ell_1^2[E] \to M_n(S_1^n[E])\|_{cb} \leq 1.$$ 

Since $\ell_1^2$ is a maximal operator space (see [32, Chapter 3]), we have that

$$\|H : \ell_1^2 \to M_n(S_1^n)\|_{cb} = \|H : \ell_1^2 \to M_n(S_1^n)\|.$$ 

Furthermore, by a convexity argument one can easily deduce that $\|H\| = \sup_{k, l} \|H(\epsilon_k, l)\|_{M_n(S_1^n)}$. Now, by noting that

$$H(\epsilon_k, l) = n|\eta_{k,l}\rangle\langle \eta_{k,l}|$$

and recalling that $\sum_{i,j=1}^n \epsilon_{i,j} \otimes \epsilon_{i,j} \|M_n(S_1^n)\| = 1$ (see the proof of Lemma 2.3), it is very easy to conclude that $\|H(\epsilon_k, l)\|_{M_n(S_1^n)} = 1$ for every $k, l$. On the other hand, according to (2.3) the previous estimate implies that

$$\|H \otimes id_E : \ell_1^2[E] \to M_n(S_1^n)\otimes E\|_{cb} \leq 1.$$
Hence, (2.24) follows from the fact that \( \| \text{id} : M_n(S^n_p) \otimes E \to M_n(S^n_p[E]) \|_{cb} \leq 1 \), which can be obtained from the definition of the projective tensor norm.

In order to prove the estimate for \( p = \infty \) we just note that
\[
\| H : \ell^\infty_p \to M_{n^2} \|_{cb} = \| H(1) \|_{M_{n^2}} = \| n \mathbb{I}_{n^2} \|_{M_{n^2}} = n.
\]

According to (2.2), this implies that
\[
(2.25) \quad \| H \circ id_E : \ell^\infty_p[E] \to M_n[M_n[E]] \|_{cb} = n.
\]

The estimate (2.23) for the general case \( 1 < p < \infty \) can be now deduced from (2.24), (2.25) and a standard interpolation argument (2.4).

\[ \square \]

**Proposition 2.8.** Let \( Q : M_n \otimes M_n \to \ell^\infty_p \) be the linear map defined by
\[
Q(\rho) = \frac{1}{n} \sum_{k,l=1}^n \langle \nu_{k,l}, [\rho] \nu_{k,l} \rangle e_{k,l}, \quad \rho \in M_{n^2}.
\]

Then, \( Q \) is completely positive and it verifies, for every operator space \( E \),
\[
(2.26) \quad \| Q \circ id_E : S^n_q(S^n_p[E]) \to \ell^\infty_p(E) \|_{cb} \leq \| P \circ id_E : S^n_q(S^n_p[E]) \to \ell^\infty_p(E) \|_{cb} = \| \mathbb{I}_{n^2} \|_{cb} = \| \mathbb{I}_{n^2} \|_{cb} = n^\frac{1}{p} - \frac{1}{q} - 1.
\]

for every \( 1 \leq p \leq q \leq \infty \).

**Proof.** Note that \( Q = \frac{1}{n} P \), where \( P \) was introduced in Corollary 2.2. Therefore, the statement of the proposition is clear just noting that
\[
\| Q \circ id_E : S^n_q(S^n_p[E]) \to \ell^\infty_p(E) \|_{cb} \leq \| P \circ id_E : S^n_q(S^n_p[E]) \to \ell^\infty_p(E) \|_{cb} = \| \mathbb{I}_{n^2} \|_{cb} = n^\frac{1}{p} - \frac{1}{q} - 1.
\]

\[ \square \]

**Proof of Theorem 1.3.** Again, by duality it suffices to consider the case \( 1 \leq p \leq q \leq \infty \).

Let us define the linear maps
\[
H_{p,q} := n^{\frac{1}{p} - \frac{1}{q}} \circ H : \ell^\infty_p \to M_n \otimes M_n,
\]
where \( H \) was defined in Proposition 2.7 and
\[
Q_{p,q} := n^{\frac{1}{p} + \frac{1}{q}} \circ Q : M_n \otimes M_n \otimes M_n \to \ell^\infty_p,
\]
where \( Q \) was defined in Proposition 2.8. According to Proposition 2.7 and Proposition 2.8 both maps are completely positive and they verify the following estimates:
\[
\| H_{p,q} \circ id_E : S^n_q(S^n_p[E]) \|_{cb} \leq 1, \quad \text{and} \quad \| Q_{p,q} \circ id_E : S^n_q(S^n_p[E]) \|_{cb} \leq 1.
\]

Therefore, it suffices to show the algebraic identification \( Q_{p,q} \circ H_{p,q} = id_{\ell^\infty_p} \). This is very easy by noting that for every \( e_{k,l} \in \ell^\infty_p \)
\[
Q_{p,q}(H_{p,q}(e_{k,l})) = Q(H(e_{k,l})) = e_{k,l}.
\]

\[ \square \]
3. Some results about covariant channels

In this section we will introduce a nice family of channels and we will explain why computing some capacities of these channels is easier than in the general case. First, let us recall that a state (or density operator) $\rho$ is a positive operator (acting on Hilbert spaces) with trace equal one. In fact, in this work we will restrict to finite dimensional Hilbert spaces, so a state (or density matrix) is a semidefinite positive matrix $\rho \in M_n$ such that $\text{tr}(\rho) = 1$. We will write $\rho \in S^n_1$ to denote a general state. In fact, very often we will consider bipartite states, which means that $\rho$ is a state acting on the tensor product of two Hilbert spaces, say $\ell_2^d \otimes \ell_2^n$. In this case, we will denote $\rho \in S^d_1 \otimes S^n_1 = S^{dn}$. We will say that $\rho$ is a pure state if it is a rank one projection $\rho = |\psi\rangle\langle\psi|$ onto a unit vector $|\psi\rangle \in \ell_2^n$. To be consistent with the standard notation in quantum information, we will write $|\psi\rangle \in \mathbb{C}^n$ to denote one of these unit vectors.

Then, a general pure bipartite state will be described by $\rho = |\psi\rangle\langle\psi|$ with $|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^n = \mathbb{C}^{dn}$. We will also make use of a very important quantity in quantum information called von Neumann entropy. Given a state $\rho$, its von Neumann entropy is defined as

$$S(\rho) = -\text{tr}(\rho \log_2 \rho).$$

This is a generalization of the Shannon entropy of a probability distribution already introduced in Theorem 1.6. We start this section by recalling the following well known result, which can be found in [1].

**Lemma 3.1.** The function $F(\rho, p) = \frac{1 - \|\rho\|_p}{p - 1}$ is well defined for $p$ positive with $p \neq 1$ and $\rho$ a density matrix. It can be extended by continuity to $p \in (0, \infty)$ and this extension verifies

$$F(\rho, 1) = \left. \frac{d}{dp} \|\rho\|_p\right|_{p=1} = S(\rho).$$

Moreover, the convergence at $p = 1$ is uniform in the states $\rho$.

In particular, for every net $(\rho_p)_p$ of states such that $\lim_{p \to 1} \rho_p = \rho$ in the trace class norm, we have that $\lim_{p \to 1} F(\rho_p, p) = S(\rho)$.

Indeed, although the first part of the result was proved in [1] for the function $\frac{1 - \|\rho\|_p}{p - 1}$, it is very easy to conclude that, then, the same result must hold for the function $F(\rho, p)$. On the other hand, the second part of the statement is a direct consequence of the uniform convergence and the continuity of the von Neumann entropy (see for instance [2]):

**Theorem 3.2.** For all $n$-dimensional states $\rho$, $\sigma$ we have

$$|S(\rho) - S(\sigma)| \leq T \log(n - 1) + H((T, 1 - T)),$$

where $T = \frac{\|\rho - \sigma\|_1}{2}$ and $H$ denotes the Shannon entropy.

Lemma 3.1 has motivated the study of channel capacities by means of the derivative of certain $p$-norms defined on these channels (see for instance [1] and [11]). More precisely, since a quantum channel $\mathcal{N}$ is nothing else than a completely positive and trace preserving map from $M_n$ to $M_m$ (we will denote it by $\mathcal{N} : S^n_1 \to S^m_1$) one can consider (and differentiate) de function

\footnote{Ket-notation $|\psi\rangle$ denotes a general unit element in a Hilbert space, while bra-notation $\langle\psi|$ is used to denote it as a dual element.}
The quantity $cb-min$ corresponds to $S_{\psi}^{\dagger}$. Indeed, the quantity $\frac{d}{dp}(f(p))|_{p=1}$ has been shown to be related to the (product state) classical capacity, also called Holevo capacity, of the quantum channel $\mathcal{N}$. However, in the recent paper [16] the authors showed that, in order to exactly describe the (product state) classical capacity of a quantum channel with $d$-assisted entanglement, $C_{prod}^{d}(\mathcal{N})$, as a derivative of a function, one has to consider the completely $\ell_{q}(S_{q}^{d})$-summing norm of the channel. Formally, one has! the following result.

**Theorem 3.3.** Given a quantum channel $\mathcal{N} : S_{1}^{n} \to S_{1}^{n}$ and a natural number $d$ verifying $1 \leq d \leq n$, we find

$$C_{prod}^{d}(\mathcal{N}) = \frac{d}{dp}[\pi_{q,d}(\mathcal{N}^{*})]|_{p=1},$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Here, $\pi_{q,d}(\mathcal{N}^{*})$ denotes the $\ell_{q}(S_{q}^{d})$-summing norm of $\mathcal{N}^{*} : M_{m} \to M_{n}$.

**Remark 3.1.** Actually, to have the equality in the previous theorem we must define $C_{prod}^{d}(\mathcal{N})$ ([16] Equation (1.3)) by using the ln-entropy, $S(\rho) := -tr(\rho \ln \rho)$, instead of using $\log_{2}$ as it is usually done in quantum information. Since both definitions are the same up to a multiplicative factor, we can use the standard entropy $S$ and we must then write the previous expression as $C_{prod}^{d}(\mathcal{N}) = \frac{1}{\ln 2} \frac{d}{dp}[\pi_{q,d}(\mathcal{N}^{*})]|_{p=1}$. In order to avoid the $\ln 2$ term in all our statements, we will still consider here the definition of $C_{prod}^{d}(\mathcal{N})$ as in the previous work [16]. However, in order to state our results in Theorem 1.4 and Theorem 1.6 (where we want to consider the standard definitions in quantum information theory) we will need to multiply our results by $\frac{1}{\ln 2}$. As the reader will see, this will be only reflected in replacing $\ln$ by $\log_{2}$ and ln-entropies by log$_{2}$-entropies, since these are the only terms appearing in our main statements.

In many cases, the factorization associated to the $\ell_{q}(S_{q}^{d})$-summing norm of $\mathcal{N}^{*}$ has a particularly nice form. This is the case of covariant channels where one can show that

$$C_{prod}^{d}(\mathcal{N}) = \ln n + \frac{d}{dp}[\|\mathcal{N} : S_{1}^{n} \to S_{p}^{m}\|_{d}]|_{p=1},$$

where here $\|\mathcal{N} : S_{1}^{n} \to S_{p}^{m}\|_{d}$ denotes the $d$-norm: $||id_{d} \otimes \mathcal{N} : M_{d}(S_{p}^{m}) \to M_{d}(S_{p}^{m})||$.

In this work we will mainly deal with covariant channels. The next result shows that one can restrict to pure states in the computation of this quantity.

**Theorem 3.4.** Given a quantum channel $\mathcal{N} : S_{1}^{n} \to S_{1}^{n}$ and $1 \leq d \leq n$, let us define the quantity

$$S_{d}(\mathcal{N}) := \frac{d}{dp}[\|\mathcal{N} : S_{1}^{n} \to S_{p}^{m}\|_{d}]|_{p=1}.$$

Then,

$$S_{d}(\mathcal{N}) = \sup \left\{ S(id_{d} \otimes tr_{n})(|\psi\rangle\langle\psi|) - S(id_{d} \otimes \mathcal{N})(|\psi\rangle\langle\psi|) \right\}$$

where the supremum is taking over all unit vectors $|\psi\rangle \in \mathbb{C}^{d} \otimes \mathbb{C}^{n}$.

The quantity $S_{d}(\mathcal{N})$ is a generalization of the cb-min entropy introduced in [11]. In particular, the quantity cb-min corresponds to $S_{n}(\mathcal{N})$. 
Proof. According to (2.7) we have
\[
\frac{d}{dp} \left\| N : S_1^n \to S_p^m \right\|_{d_{p=1}} = \frac{d}{dp} \left\| \iota d \otimes N : S_p^d(S_1^n) \to S_p^d(S_p^m) \right\|_{p=1}
\geq \lim_{p \to 1} \sup_{\rho \in S_1^n} \frac{1}{\|\rho\|_{S_1^n}} \left\| (\iota d \otimes N)(\rho) \right\|_{S_p^m} - \left\| \rho \right\|_{S_p^d(S_1^n)}
= \lim_{p \to 1} \sup_{\rho \in S_1^n} \frac{1}{\|\rho\|_{S_1^n}} \left( \left\| (\iota d \otimes N)(\rho) \right\|_{S_p^m} - 1 \right) + \frac{1}{p} \left( 1 - \left\| \rho \right\|_{S_p^d(S_1^n)} \right)
= \sup_{\rho \in S_1^n} \left\{ S(\iota d \otimes tr_n)(\rho)) - S(\iota d \otimes N)(\rho)) \right\}.
\]
Here, the first inequality is due to the fact that we are restricting the computation of the norm to states \( \rho \in S_1^n \) rather than to general matrices \( \rho \in M_{dn} \). We have also used that, by Lemma 3.1 and the fact that \( \left\| \rho \right\|_{S_p^d(S_1^n)} = \left\| (\iota d \otimes tr_n)(\rho) \right\|_{S_p^d} \) for positive elements (see Remark 2.1), we have that
\[
\lim_{p \to 1} \frac{\left\| (\iota d \otimes N)(\rho) \right\|_{S_p^m} - 1}{p - 1} = -S(\iota d \otimes N)(\rho))
\]
and
\[
\lim_{p \to 1} \frac{1}{p - 1} = 1 - \frac{\left\| (\iota d \otimes tr_n)(\rho) \right\|_{S_p^d}}{p - 1} = S((\iota d \otimes tr_n)(\rho))
\]
uniformly. Therefore, we can iterate the limite and the supremum.

On the other hand, according to (2.7) we also have
\[
\left\| N : S_1^n \to S_p^m \right\|_d = \left\| \iota d \otimes N : S_1^n \to S_p^d(S_p^m) \right\| = \sup_{|\psi\rangle \in \mathbb{C}^m} \left\| (\iota d \otimes N)(|\psi\rangle\langle\psi|) \right\|_{S_p^d(S_p^m)}.
\]
Here, we have used that, since \( N \) is completely positive, we can compute its completely bounded norm by restricting to positive elements (Remark 2.1). Then, by normalizing we can restrict to states. Furthermore, since pure states are exactly the extreme points of the set of states, we have the last equality. Then,
\[
\frac{d}{dp} \left\| N : S_1^n \to S_p^m \right\|_{d_{p=1}} = \lim_{p \to 1} \sup_{|\psi\rangle \in \mathbb{C}^m} \frac{\left\| (\iota d \otimes N)(|\psi\rangle\langle\psi|) \right\|_{S_p^d(S_p^m)} - 1}{p - 1}
\leq \lim_{p \to 1} \sup_{|\psi\rangle \in \mathbb{C}^m} \frac{\text{tr}_d \left( (\iota d \otimes tr_m)(|\psi\rangle\langle\psi|) \right)^{\frac{1}{p}} - 1}{p - 1},
\]
where here we have used that for ever positive element \( x \in M_d \otimes M_m \) we have (see [23])
\[
\left\| x \right\|_{S_p^d(S_p^m)} \leq \left\| \left( (\iota d \otimes tr_m)(x^p) \right)^{\frac{1}{p}} \right\|_{S_1^n}.
\]
Let us call for a fixed \( |\psi\rangle \in \mathbb{C}^m \), \( (\iota d \otimes N)(|\psi\rangle\langle\psi|) \) and note that
\[
\frac{\text{tr}_d \left( (\iota d \otimes tr_m)(\rho^p_\psi) \right)^{\frac{1}{p}} - 1}{p - 1} = \frac{\text{tr}_d((\iota d \otimes tr_m)(\rho^p_\psi))^{\frac{1}{p}} - (\text{tr}_d \otimes tr_m)(\rho^p_\psi) + (\text{tr}_d \otimes tr_m)(\rho^p_\psi) - 1}{p - 1}.
\]
covariant is the function

Here we have used functional calculus and Remark 3.2 in [16]. Now, it is not difficult to see that the second term in (3.2) is a direct consequence of Lemma 3.1. On the other hand, the uniform convergence of the first term in (3.2) can be easily obtained from Theorem 3.2.

Hence, we can finish our proof by using (3.2) and noting that

\[
\lim_{p \to 1} G(p, \rho) = S((id_d \otimes tr_m)(\rho)) - S(\rho)
\]

and that this convergence is uniform in the states \( \rho \in S_1^{dm} \). Indeed, the uniform convergence for the second term in (3.2) is a direct consequence of Lemma 3.1. On the other hand, the uniform convergence of the first term in (3.2) can be easily obtained from Theorem 3.2.

In this work, we are interested in dealing with quantum channels of the form

\[
N : S_1^n \to S_1^n \oplus \cdots \oplus S_1^n
\]

such that

\[
\lambda_j^{\mathcal{N}}(\rho) = \mu_1\mathcal{N}_1(\rho) \oplus \cdots \oplus \mu_m\mathcal{N}_m(\rho),
\]

where \((\mu_j)_{j=1}^m \) is a probability distribution and \( \mathcal{N}_j : S_1^n \to S_1^{n_j} \) is a quantum channel for every \( j \).

**Definition 3.1.** Let \( G \) be a compact group and let us consider unitary representations \( \pi : G \to U(n) \) and \( \sigma_j : G \to U(n_j) \) for every \( j = 1, \cdots, m \). We say that a quantum channel \( \mathcal{N} \) of the form (3.3) is **covariant** (with respect to \((G, \pi, \sigma_1, \cdots, \sigma_m)\)) if

1. \( \int_{G} \sigma_j(g)^* \rho \sigma_j(g) dg = tr(\rho) \mathbb{1}_{n_j} \) for every \( \rho \in S_1^n \) and for every \( j \). Here, \( U(n_j) \) represents the unitary group in dimension \( n_j \) and the integral is with respect to the Haar measure of \( G \).
2. \( \mathcal{N}_j(\pi(g)^* \rho \pi(g)) = \sigma_j(g)^* \mathcal{N}_j(\rho) \sigma_j(g) \) for every \( g \in G \) and every \( \rho \in S_1^n \).

**Proposition 3.5.** Given a quantum channel \( \mathcal{N} : S_1^n \to S_1^{n_1} \oplus \cdots \oplus S_1^{n_m} \) as in (3.3), we have

\[
C_{\text{prod}}^d(\mathcal{N}) = \sup \sum_{j=1}^m \mu_j \left\{ S \left( \sum_{i=1}^N \lambda_i \mathcal{N}_j \left( (tr_d \otimes id_n)(\rho_i) \right) \right) \right\}
\]

\[
+ \sum_{i=1}^N \lambda_i \left\{ S \left( (id_d \otimes tr_n)(\rho_i) \right) - S \left( (id_d \otimes \mathcal{N}_j)(\rho_i) \right) \right\}.
\]
Here, the supremum runs over all $N \in \mathbb{N}$, all probability distributions $(\lambda_i)_{i=1}^N$ and all families $(\rho_i)_{i=1}^N$, where $\rho_i \in S_d^d$ is a state for every $i = 1, \ldots, N$.

**Proof.** According to [16, Proposition 5.5], for a channel $N : S_1^n \to S_1^n$, we have that

$$C^d_{\text{prod}}(N) = \sup \left\{ S \left( \sum_{i=1}^{N} \lambda_i N \left( (tr_d \otimes id_n)(\rho_i) \right) \right) + \sum_{i=1}^{N} \lambda_i \left[ S \left( (id_d \otimes tr_n)(\rho_i) \right) - S \left( (id_d \otimes N)(\rho_i) \right) \right] \right\},$$

where the supremum runs over all $N \in \mathbb{N}$, all probability distributions $(\lambda_i)_{i=1}^N$, and all families $(\rho_i)_{i=1}^N$, where $\rho_i \in S_d^d$ is a state for every $i = 1, \ldots, N$.

However, it is very easy to check that if $N : S_1^n \to S_1^n \oplus_1 \cdots \oplus_1 S_1^{m} \subset S_1^{n_1 + \cdots + n_m}$ as in (3.3) we have

$$S \left( \sum_{i=1}^{N} \lambda_i N \left( (tr_d \otimes id_n)(\rho_i) \right) \right) = H \left( (\mu_j)_{j=1}^m \right) + \sum_{j=1}^{m} \mu_j S \left( (id_d \otimes N_j)(\rho_i) \right),$$

$$S \left( (id_d \otimes N)(\rho_i) \right) = H \left( (\mu_j)_{j=1}^m \right) + \sum_{j=1}^{m} \mu_j S \left( (id_d \otimes N_j)(\rho_i) \right).$$

Here, $H \left( (\mu_j)_{j=1}^m \right)$ is the Shannon entropy of the probability distribution $(\mu_j)_{j=1}^m$ already introduced in Theorem 1.6. Then, the result follows. \hfill \Box

Let us now define, for a channel $N : S_1^n \to S_1^n \oplus_1 \cdots \oplus_1 S_1^n$ as in (3.3), the quantity

$$V_d(N) = \sup \left\{ \sum_{j=1}^{m} \mu_j \left[ S \left( (id_d \otimes tr_n)(|\psi\rangle\langle\psi|) \right) - S \left( (id_d \otimes N_j)(|\psi\rangle\langle\psi|) \right) \right] \right\},$$

where the supremum is taking over all pure states $|\psi\rangle \in C^d \otimes \mathbb{C}^n$.

**Lemma 3.6.** Given a channel $N : S_1^n \to S_1^n \oplus_1 \cdots \oplus_1 S_1^n$ as in (3.3), we have

$$S_d(N) = V_d(N) - H \left( (\mu_j)_{j=1}^m \right).$$

Furthermore,

$$V_d(N) = \sup \left\{ \sum_{j=1}^{m} \mu_j \left[ S \left( (id_d \otimes tr_n)(\rho) \right) - S \left( (id_d \otimes N_j)(\rho) \right) \right] \right\},$$

where the supremum is taking over all states $\rho \in S_d^d \oplus S_d^d$.

**Proof.** According to Theorem 3.4 we have

$$S_d(N) = \sup \left\{ S \left( (id_d \otimes tr_n)(|\psi\rangle\langle\psi|) \right) - S \left( (id_d \otimes N)(|\psi\rangle\langle\psi|) \right) \right\},$$

where the supremum is taking over all pure states $|\psi\rangle \in C^d \otimes \mathbb{C}^n$. On the other hand, it is very easy to see that for every state (pure or not)

$$S \left( (id_d \otimes N)(\rho) \right) = H \left( (\mu_j)_{j=1}^m \right) + \sum_{j=1}^{m} \mu_j S \left( (id_d \otimes N_j)(\rho) \right).$$
Therefore, the first statement follows.

The second part of the statement follows from the fact that the definition of $S_d(N)$ doesn’t change if we take the supremum over all states (see Theorem 3.4). □

In the following proposition we give a nice formula to compute $C^d_{\text{prod}}(N)$ for covariant channels.

**Proposition 3.7.** Let $N : S_1^n \rightarrow S_1^{n_1} \oplus_1 \cdots \oplus_1 S_1^{n_m}$ be a quantum channel as in (3.3) which is covariant. Then,

$$C^d_{\text{prod}}(N) = \sum_{j=1}^m \mu_j \ln n_j + V_d(N).$$

**Proof.** Since $S\left( \sum_{i=1}^N \lambda_i N_j ((tr_d \otimes id_n)(\rho_i)) \right) \leq \ln n_j$ for every $N \in \mathbb{N}$, all probability distributions $(\lambda_i)_i^N$, and all families $(\rho_i)_i^N$ of states $\rho_i \in S_1^d \otimes S_1^n$, Proposition 3.5 guarantees that

$$C^d_{\text{prod}}(N) \leq \sum_{j=1}^m \mu_j \ln n_j + \sup \left\{ \sum_{j=1}^m \mu_j \sum_{i=1}^N \lambda_i \left[ S\left( (id_d \otimes tr_n)(\rho_i) \right) - S\left( (id_d \otimes N_j)(\rho_i) \right) \right] \right\}$$

$$= \sum_{j=1}^m \mu_j \ln n_j + \sup \left\{ \sum_{i=1}^N \lambda_i \left[ S\left( (id_d \otimes tr_n)(\rho_i) \right) - \sum_{j=1}^m \mu_j S\left( (id_d \otimes N_j)(\rho_i) \right) \right] \right\},$$

where the supremum runs over all $N \in \mathbb{N}$, all probability distributions $(\lambda_i)_i^N$, and all families $(\rho_i)_i^N$ of states $\rho_i \in S_1^d \otimes S_1^n$. Now, by convexity it is clear that this is the same as

$$C^d_{\text{prod}}(N) \leq \sum_{j=1}^m \mu_j \ln n_j + \sup \left\{ \sum_{j=1}^m \mu_j S\left( (id_d \otimes tr_n)(\rho) \right) - S\left( (id_d \otimes N_j)(\rho) \right) \right\}$$

$$= \sum_{j=1}^m \mu_j \ln n_j + \sup \left\{ \sum_{j=1}^m \mu_j \left[ S\left( (id_d \otimes tr_n)(\rho) \right) - S\left( (id_d \otimes N_j)(\rho) \right) \right] \right\},$$

where the supremum runs over all states $\rho \in S_1^d \otimes S_1^n$. Then, we conclude that

$$C^d_{\text{prod}}(N) \leq \sum_{j=1}^m \mu_j \ln n_j + V_d(N).$$

Let us now consider a general state $\rho \in S_1^d \otimes S_1^n$ (in particular, any pure state). For every $g \in G$ we denote $\rho_g := (id_d \otimes \pi(g)) \rho (id_d \otimes \pi(g))$ and we consider the ensemble $\{dg, (\rho_g)_g\}_G$. Then, according to Proposition 3.5 we have

$$C^d_{\text{prod}}(N) \geq \sum_{j=1}^m \mu_j \left\{ S \left( \int_G N_j ((tr_d \otimes id_n)(\rho_g)) \, dg \right) \right. \right.$$  

$$\left. + \int_G \left[ S\left( (id_d \otimes tr_n)(\rho_g) \right) - S\left( (id_d \otimes N_j)(\rho_g) \right) \right] \, dg \right\}.  
$$

\[\text{[6]Although we usually consider finite ensembles } \{(\lambda_i)_i^N, (\rho_i)_i^N\} \text{ one can also work with infinite ones and obtain the corresponding result by approximation.}\]
Now, for every $j$ we have that
\begin{equation}
S\left(\int_G N_j((tr_d \otimes id_n)(\rho))dg\right) = S\left(\int_G \sigma_j(g)^* N_j((tr_d \otimes id_n)(\rho))\sigma_j(g)dg\right) = S\left(\frac{\mathbb{1}}{n_j}\right) = \ln n_j,
\end{equation}
where in the second equality we have used the covariant properties of our channel.

On the other hand, for every $j$ we also have
\begin{equation}
\int_G S\left((id_d \otimes tr_n)(\rho)\right)dg = \int_G S\left((id_d \otimes tr_n)((id_d \otimes \pi(g))\rho(id_d \otimes \pi(g)))\right)dg = \int_G S((id_d \otimes tr_n)(\rho))dg = S((id_d \otimes tr_n)(\rho)),
\end{equation}
and
\begin{equation}
S\left((id_d \otimes N_j)(\rho)\right) = S\left((id_d \otimes N_j)((id_d \otimes \pi(g)^*)\rho(id_d \otimes \pi(g)))\right) = S\left((\mathbb{1} \otimes \sigma_j(g)^*) (id_d \otimes N_j)(\rho) (\mathbb{1} \otimes \sigma_j(g))\right) = S\left((id_d \otimes N_j)(\rho)\right).
\end{equation}

Here in the last equality we have used that the von Neumann entropy is invariant under unitaries.

Equations (3.7), (3.8) and (3.9) imply that
\begin{equation}
C^d_{\text{prod}}(N) \geq \sum_{j=1}^{m} \mu_j \ln n_j + \sum_{j=1}^{m} \mu_j \left[S\left((id_d \otimes tr_n)(\rho)\right) - S\left((id_d \otimes N_j)(\rho)\right)\right].
\end{equation}

Since this happens for every state $\rho \in S^d_1 \otimes S^n_1$, we conclude that
\begin{equation}
C^d_{\text{prod}}(N) \geq \sum_{j=1}^{m} \mu_j \ln n_j + V_d(N).
\end{equation}

\[\square\]

4. \textit{d}-restricted capacity of the quantum depolarizing channel

In this section we will prove the part of Theorem 1.4 corresponding to the depolarizing channel (Equation (1.2)) and also Corollary 1.5. Finally, we will see how to obtain the first part of Theorem 1.6 (Equation (1.4)) by assuming (1.5), which will be proved in the next section.

It is very easy to see that $D_\lambda$ is a covariant channel with respect to $(U(n), id_{U(n)}, id_{U(n)})$. Therefore, according to Proposition 3.7 and Lemma 3.6, the expression for $C^d_{\text{prod}}(D_\lambda)$ in Theorem 1.4 can be obtained from Equation (1.2) by differentiation (and adding a $\ln n$ term). Indeed, if we differentiate in Equation (1.2) we obtain
\begin{equation}
\frac{d}{dp} \|D_\lambda : S^d_1 \rightarrow S^n_1\|_{d=1} = \left(\frac{1}{nd}\right) \ln \left(\frac{1}{nd}\right) + (nd - 1) \left(\frac{1}{nd}\right) \ln \left(\frac{1}{nd}\right) + \ln d.
\end{equation}
Adding a \( \ln n \) term we obtain desired equation \(^7\).

In order to prove \(^{1,2}\) we will start by defining the following family of linear maps\(^8\): \( \theta_{\lambda}^{d,p} : S_i^d \to S_i^d \otimes_1 S_i^d \subseteq S_i^{2d} \) for every \( p \geq 1 \), define by

\[
\theta_{\lambda}^{d,p}(\rho) = \left( \lambda \rho + \frac{1 - \lambda}{n} Tr(\rho) \mathbb{I}_d \right) \otimes \frac{1 - \lambda}{n} Tr(\rho) \left( \frac{n - d}{d} \right)^{\frac{p}{2}} \mathbb{I}_d
\]

for every \( \rho \in S_i^d \).

**Proposition 4.1.** Let \( D_\lambda : S_p^n \to S_p^n \) be the quantum depolarizing channel with parameter \( \lambda \) and \( \theta_{\lambda}^{d,p} \) defined as above. Then,

\[
\| D_\lambda : S_p^n \to S_p^n \|_{cb} \leq \| \theta_{\lambda}^{d,p} : S_p^d \to S_p^d \otimes_p S_p^d \|_{cb}.
\]

Before proving the proposition, we will show the following easy lemma.

**Lemma 4.2.** Given \( 1 \leq d \leq n \), let us define the linear map \( V : S_p^d \to S_p^{n-d} \) by

\[
V(\rho) = \frac{Tr(\rho)}{(n - d)^{\frac{p}{2}} d^{\frac{p}{2}}} \mathbb{I}_{n-d}, \quad \rho \in S_p^d
\]

Then, \( \| V \|_{cb} = 1 \). Moreover,

\[
\| \text{id} \otimes V : S_p^d \otimes_p S_p^d \to S_p^d \otimes_p S_p^{n-d} \|_{cb} = 1.
\]

**Proof.** Since \( V \) has rank one, we know that \( \| V \|_{cb} = \| V \| \). Let us then consider an element \( \rho \) in the unit ball of \( S_p^d \). We have that

\[
\| V(\rho) \|_{S_p^{n-d}} = \frac{|Tr(\rho)|}{(n - d)^{\frac{p}{2}} d^{\frac{p}{2}}} \| \mathbb{I}_{n-d} \|_{S_p^{n-d}} \leq \frac{d^p}{(n - d)^{\frac{p}{2}} d^{\frac{p}{2}}} (n - d)^{\frac{p}{2}} = 1.
\]

The second statement follows straightforward from the first one. \( \square \)

We prove now Proposition \(^{4,1}\).

**Proof.** According to \(^2,7\), it suffices to show that

\[
\| \text{id} \otimes D_\lambda : S_i^d (S_i^n) \to S_i^d (S_i^n) \| \leq \| \text{id} \otimes \theta_{\lambda}^{d,p} : S_i^d (S_i^n) \to S_i^d (S_i^d \otimes_p S_i^d) \|.
\]

In fact, since \( D_\lambda \) is completely positive we can restrict the computation of the first norm to positive elements (see Remark \(^2,1\)) so, by normalization, to states \( \rho \in S_i^{dn} \). Moreover, since pure states are exactly the extreme points of general states, by convexity we can restrict to pure states \( \xi = |\eta\rangle \langle \eta| \in S_i^{dn} \), where \( |\eta\rangle \) is a unit vector in \( \mathbb{C}^d \). Now, according to the Hilbert-Schmidt decomposition we can assume that \( |\eta\rangle = \sum_{i=1}^d \lambda_i |f_i\rangle \otimes |g_i\rangle \) for certain orthonormal systems \( (|f_i\rangle) \subseteq \mathbb{C}^d \), \( (|g_i\rangle) \subseteq \mathbb{C}^n \) respectively and \( \sum_{i=1}^d |\lambda_i|^2 = 1 \). Moreover, by the unitary invariance of our channel \( D_\lambda \), we can assume that \( |\eta\rangle = \sum_{i=1}^d \lambda_i c_i \otimes c_i \in \mathbb{C}^{d} \otimes \mathbb{C}^{d} \subset \mathbb{C}^{d} \otimes \mathbb{C}^{n} \). Indeed, this is because we have

\[
\| (\text{id} \otimes D_\lambda)(\xi) \|_{S_i^d (S_i^n)} = \| (U \otimes V)( (\text{id} \otimes D_\lambda)(\xi) )(U^* \otimes V^*) \|_{S_i^d (S_i^n)}
\]

\(^7\)Recall that, according to Remark \(^4,1\), we must replace our \( \ln \) terms by \( \log_2 \) terms in order to consider the right capacity.

\(^8\)It is very easy to see that \( \theta_{\lambda}^{d,p}(\rho) \) is a quantum channel. However, we will consider the whole family \( \{ \theta_{\lambda}^{d,p}(\rho) \} \)

in order to compute the \((1,p)\)-norm of our channel.
defines a complete isometry of $S$ and completely positive map. Moreover, defines a complete isometry of $S \otimes S$. It is trivial to check that

$$(\text{id}_d \otimes D_\lambda)(\xi) = \lambda \xi + \frac{1 - \lambda}{n} \sum_{i=1}^d |\lambda_i|^2 |i\rangle \langle i| \otimes \mathbb{1}_n.$$ 

Now, we can see that $\mathbb{1}_n = \mathbb{1}_d \oplus \mathbb{1}_{n-d}$ and since $\xi \in S_d \otimes S_1 \subset S_d \otimes S_1$, we have

$$(\text{id}_d \otimes D_\lambda)(\xi) = \left( \lambda \xi + \frac{1 - \lambda}{n} \sum_{i=1}^d |\lambda_i|^2 |i\rangle \langle i| \otimes \mathbb{1}_d \right) \oplus \left( \frac{1 - \lambda}{n} \sum_{i=1}^d |\lambda_i|^2 |i\rangle \langle i| \otimes \mathbb{1}_{n-d} \right).$$

Let us now consider

$$(\mathbb{1}_d \otimes \theta_\lambda^{d,p})(\xi) = \left( \lambda \xi + \frac{1 - \lambda}{n} \sum_{i=1}^d |\lambda_i|^2 |i\rangle \langle i| \otimes \mathbb{1}_d \right) \oplus \left( \frac{1 - \lambda}{n} \left( \frac{n - d}{d} \right)^{\frac{1}{2}} \sum_{i=1}^d |\lambda_i|^2 |i\rangle \langle i| \otimes \mathbb{1}_d \right).$$

Since $\mathbb{1}_{n-d} = V \left( \frac{n-d}{d} \right)^{\frac{1}{2}} \mathbb{1}_d$, the result follows from Lemma 4.2. \hfill \Box

In order to find an upper bound for the quantity $\|\theta_\lambda^{d,p}: S_d^p \rightarrow S_d^p \oplus_p S_p^d\|_{cb}$ we will use Theorem [1.1]. In the particular case we need, the theorem states that the map

$$j_p(\rho) = \frac{1}{d^p} \sum_{k,l=1}^d T_{k,l}^* \rho T_{k,l} \otimes e_{k,l}$$

defines a complete isometry of $S_d^p$ in $M_d(\ell_p^d)$, which is complemented by a completely contractive and completely positive map. Moreover, 

$$\tilde{J}_p(\rho_1 \oplus \rho_2) = \frac{1}{d^p} \left( \sum_{k,l=1}^d T_{k,l} \rho_1 T_{k,l}^* \otimes e_{k,l,1} \oplus \sum_{k,l=1}^d T_{k,l} \rho_2 T_{k,l}^* \otimes e_{k,l,2} \right)$$

defines a complete isometry of $S_d^p \oplus_p S_d^p$ in $M_d(\ell_p^d \oplus_p \ell_p^d)$ which is complemented by a completely contractive and completely positive map. Here, we denote by $e_{k,l,1}$ the elements of the canonical basis of the first $\ell_p^d$ space and by $e_{k,l,2}$ the elements of canonical basis of the second $\ell_p^d$ space.

**Lemma 4.3.** Let us consider the linear map $\Psi_{\alpha,\beta,\gamma}: \ell_{p_1}^d \rightarrow \ell_{p_2}^p \oplus_p \ell_{p_2}^d$ defined by

$$\Psi_{\alpha,\beta,\gamma} \left( \sum_{i,j=1}^d a_{i,j} e_{i,j} \right) = \alpha \sum_{i,j=1}^d a_{i,j} e_{i,j,1} + \beta \left( \sum_{i,j=1}^d a_{i,j} \right) \left( \sum_{i,j=1}^d e_{i,j,1} \right) + \delta \left( \sum_{i,j=1}^d a_{i,j} \right) \left( \sum_{i,j=1}^d e_{i,j,2} \right).$$

Then,

$$\|\Psi_{\alpha,\beta,\gamma}\|_{cb} = \|\Psi_{\alpha,\beta,\gamma}\| = \left( |\alpha + \beta|^p + (d^2 - 1)|\beta|^p + d^2|\delta|^p \right)^{\frac{1}{p}}.$$ 

**Proof.** The equality $\|\Psi_{\alpha,\beta,\gamma}\|_{cb} = \|\Psi_{\alpha,\beta,\gamma}\|$ follows from the fact that we consider the natural operator space structure on $\ell_{p_2}^d$, which is the maximal one (see [32, Chapter 3]). On the other hand, in order to estimate $\|\Psi_{\alpha,\beta,\gamma}\|$ it suffices to check the elements of the canonical basis $e_{i,j}$. 

Moreover, by the symmetry of the problem suffices to check $e_{1,1}$. Then,

$$\|\Psi_{\alpha,\beta,\gamma}\|_p = \|\Psi_{\alpha,\beta,\gamma}(e_{1,1})\|_{p^2} = \|\alpha e_{1,1} + \beta \sum_{i,j=1}^d e_{i,j;1} + \gamma \sum_{i,j=1}^d e_{i,j;2}\|_{p^2} = (|\alpha| + \beta|\sum_{i,j=1}^d (d^2 - 1)|\beta| + d^2|\delta|)^{\frac{1}{2}}.
$$

The key result in our analysis is the following factorization.

**Proposition 4.4.** Let us fix $\alpha = \lambda d^{\frac{1}{p}}$, $\beta = \frac{1 - \lambda}{d^{\frac{1}{p}}}$ and $\delta = \frac{1 - \lambda}{d^{\frac{1}{p}}} (\frac{n - d}{d})^{\frac{1}{2}}$. Then, we have

$$(id_d \otimes \Psi_{\alpha,\beta,\gamma}) \circ j_1 = J_p \otimes \theta_\lambda^{d,p}.
$$

**Proof.** Let consider an element $\rho \in S_k$. Then we have

$$(id_d \otimes \Psi_{\alpha,\beta,\gamma}) \circ j_1(\rho) = (id_d \otimes \Psi_{\alpha,\beta,\gamma})(\frac{1}{d} \sum_{k,l=1}^d T_{k,l} \rho T_{k,l}^* \otimes e_{k,l})$$

$$= \frac{1}{d} \sum_{k,l=1}^d T_{k,l} \rho T_{k,l}^* \otimes \Psi_{\alpha,\beta,\gamma}(e_{k,l})$$

$$= \frac{1}{d} \left( \sum_{k,l=1}^d T_{k,l} \rho T_{k,l}^* \otimes (\alpha e_{k,l;1} + \beta \sum_{i,j=1}^d e_{i,j;1} + \gamma \sum_{i,j=1}^d e_{i,j;2}) \right)$$

$$= \left( \frac{1}{d} \sum_{k,l=1}^d T_{k,l} \rho T_{k,l}^* \otimes e_{k,l;1} + \beta tr(\rho) \mathbb{I}_d \otimes \sum_{i,j=1}^d e_{i,j;1} \right) \otimes \left( \gamma tr(\rho) \mathbb{I}_d \otimes \sum_{i,j=1}^d e_{i,j;2} \right),$$

where in the last step we have used that $\sum_{k,l=1}^d T_{k,l} \rho T_{k,l}^* = dtr(\rho) \mathbb{I}_d$. Indeed, this can be easily checked by noting that

$$\sum_{k,l=1}^d T_{k,l} \rho |q| T_{k,l}^* = \delta_{p,q} d \mathbb{I}_d$$

for every $p, q = 1, \cdots, d$.

If we consider the specific values for $\alpha$, $\beta$ and $\gamma$ stated in the proposition, we obtain

$$\left( \frac{\lambda}{d^{\frac{1}{p}}} \sum_{k,l=1}^d T_{k,l} \rho T_{k,l}^* \otimes e_{k,l;1} + \frac{1 - \lambda}{d^{\frac{1}{p}}} tr(\rho) \mathbb{I}_d \otimes \sum_{i,j=1}^d e_{i,j;1} \right) \otimes \left( \frac{1 - \lambda}{d^{\frac{1}{p}}} \left( \frac{n - d}{d} \right)^{\frac{1}{2}} tr(\rho) \mathbb{I}_d \otimes \sum_{i,j=1}^d e_{i,j;2} \right).$$

On the other hand,

$$(J_p \circ \theta_\lambda^{d,p})(\rho) = J_p \left( (\lambda \rho + \frac{1 - \lambda}{n} tr(\rho) \mathbb{I}_d) \otimes \frac{1 - \lambda}{n} tr(\rho) \left( \frac{n - d}{d} \right)^{\frac{1}{2}} \mathbb{I}_d \right)$$

$$= \frac{1}{d^{\frac{1}{p}}} \left( \sum_{k,l=1}^d T_{k,l} \left( \lambda \rho + \frac{1 - \lambda}{n} tr(\rho) \mathbb{I}_d \right) T_{k,l}^* \otimes e_{k,l;1} \otimes \sum_{k,l=1}^d T_{k,l} \frac{1 - \lambda}{n} tr(\rho) \left( \frac{n - d}{d} \right)^{\frac{1}{2}} \mathbb{I}_d T_{k,l}^* \otimes e_{k,l;2} \right),$$

which is equal to

$$\left( \frac{\lambda}{d^{\frac{1}{p}}} \sum_{k,l=1}^d T_{k,l} \rho T_{k,l}^* \otimes e_{k,l;1} + \frac{1 - \lambda}{d^{\frac{1}{p}}} tr(\rho) \mathbb{I}_d \otimes \sum_{k,l=1}^d e_{k,l;1} \right) \otimes \left( \frac{1 - \lambda}{d^{\frac{1}{p}}} \left( \frac{n - d}{d} \right)^{\frac{1}{2}} tr(\rho) \mathbb{I}_d \otimes \sum_{k,l=1}^d e_{k,l;2} \right).$$

This concludes the proof. □
Corollary 4.5. Let $\theta^{d,p}_{\lambda}$ be the linear map defined in (4.1). Then,
\[ \|\theta^{d,p}_{\lambda} : S^d_1 \to S^d_p \otimes_p S^d_p\|_{cb} \leq \left( \frac{1}{d} \left( \lambda d + \frac{1 - \lambda}{n} \right)^p + \left( \frac{1 - \lambda}{n} \right)^p \left( n - \frac{1}{d} \right) \right)^{1/p}. \]

Proof. By Proposition 4.4 and the fact that $j_p$ and $\tilde{j}_p$ are complete isometries it suffices to show that
\[ \|\psi_{\alpha,\beta,\gamma} : M_d(\ell^d_p) \to M_d(\ell^d_p \otimes_p \ell^d_p)\|_{cb} \leq \left( \frac{1}{d} \left( \lambda d + \frac{1 - \lambda}{n} \right)^p + \left( \frac{1 - \lambda}{n} \right)^p \left( n - \frac{1}{d} \right) \right)^{1/p}. \]

Now, it follows from the definition of the completely bounded norm that
\[ \|\psi_{\alpha,\beta,\gamma} : M_d(\ell^d_p) \to M_d(\ell^d_p \otimes_p \ell^d_p)\|_{cb} = \|\psi_{\alpha,\beta,\gamma} : \ell^d_p \to \ell^d_p \otimes_p \ell^d_p\|_{cb} \]
\[ = \left( |\alpha + \beta|^p + (d^2 - 1)|\beta|^p + d^2|\delta|^p \right)^{1/p}, \]
where the last equality follows from Lemma 4.3. By considering the values for $\alpha$, $\beta$ and $\gamma$ stated in Proposition 4.4 we obtain
\[ \|\psi_{\alpha,\beta,\gamma} : \ell^d_p \to \ell^d_p \otimes_p \ell^d_p\|_{cb} = \lambda d \left( \frac{1 - \lambda}{d} \right)^p \left( \frac{1 - \lambda}{n} \right)^p + \frac{d(1 - \lambda)^p}{n^p} - \frac{(1 - \lambda)^p}{n^p} + \frac{d(1 - \lambda)^p}{n^p} \left( n - \frac{1}{d} \right). \]
We are now ready to prove (1.2).

Proof of Equation (1.2) in Theorem 1.4. The upper bound in Equation (1.2) follows from Proposition 4.1 and Corollary 4.5. Thus, we must only show the lower bound.

Let us consider the particular element $\xi = \frac{1}{d} \sum_{i,j=1}^{d} |i\rangle \langle j| \otimes |i\rangle \langle j| \in M_d(M_n)$. We have already mentioned that for a positive element $\xi$ in $M_d(M_n)$ one has
\[ \|\xi\|_{S^d_p(S^d_p)} = \|\psi_{\alpha,\beta,\gamma} : \ell^d_p \to \ell^d_p \otimes_p \ell^d_p\|_{cb} = \frac{d^2}{d} = \frac{1}{d^{p/2}}. \]
On the other hand,
\[ \|\psi_{\alpha,\beta,\gamma} : \ell^d_p \to \ell^d_p \otimes_p \ell^d_p\|_{cb} = \|\psi_{\alpha,\beta,\gamma} : \ell^d_p \to \ell^d_p \otimes_p \ell^d_p\|_{cb} \]
\[ = \left( \lambda + \frac{1 - \lambda}{nd} \right)^p + \left( nd - 1 \right) \left( \frac{1 - \lambda}{n} \right)^p \left( n - \frac{1}{d} \right)^{1/p}. \]
Then, using that $\xi = |\eta\rangle \langle \eta|$ is a pure state with $\eta = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle$, the element $\lambda \xi + (1 - \lambda) \frac{1}{nd}$ can be seen as a matrix in $M_{nd}$ with all eigenvalues equal $\frac{1 - \lambda}{nd}$ up to one which is $\lambda + \frac{1 - \lambda}{nd}$. Hence,
\[ \|\psi_{\alpha,\beta,\gamma} : \ell^d_p \to \ell^d_p \otimes_p \ell^d_p\|_{cb} = \left( \lambda + \frac{1 - \lambda}{nd} \right)^p + \left( nd - 1 \right) \left( \frac{1 - \lambda}{n} \right)^p \left( n - \frac{1}{d} \right)^{1/p}. \]
We immediately conclude that
\[ \|\psi_{\alpha,\beta,\gamma} : \ell^d_p \to \ell^d_p \otimes_p \ell^d_p\|_{cb} = \frac{d^2}{d} \left( \lambda + \frac{1 - \lambda}{nd} \right)^p + \left( nd - 1 \right) \left( \frac{1 - \lambda}{n} \right)^p \left( n - \frac{1}{d} \right)^{1/p}. \]
According to (4.2), we want to show that
\[f(n, d, \lambda) = (\lambda + \frac{1 - \lambda}{nd^2}) \ln (\lambda + \frac{1 - \lambda}{nd^2}) + (nd^2 - 1) \frac{1 - \lambda}{nd^2} \ln \frac{1 - \lambda}{nd^2} \]
\[+ (\lambda + \frac{1 - \lambda}{n}) \ln (\lambda + \frac{1 - \lambda}{n}) + (n - 1) \frac{1 - \lambda}{n} \ln \frac{1 - \lambda}{n} \]
\[- 2(\lambda + \frac{1 - \lambda}{nd}) \ln (\lambda + \frac{1 - \lambda}{nd}) - 2(nd - 1) \frac{1 - \lambda}{nd} \ln \frac{1 - \lambda}{nd}.
\]

Now, it is very easy to see that this is exactly the same expression as the one in Equation (1.2). Indeed,
\[\frac{d}{p} \left(\left(\lambda + \frac{1 - \lambda}{nd}\right)^p + (nd - 1)\left(\frac{1 - \lambda}{nd}\right)^p\right)^\frac{1}{p} \]
\[= \frac{d}{p - 1} \left[\frac{1}{dp} \left(\lambda d + \frac{1 - \lambda}{n}\right)^p + \frac{1}{dp} (nd - 1) \left(\frac{1 - \lambda}{n}\right)^p\right] \]
\[= \left(\frac{1}{d}\left(\lambda d + \frac{1 - \lambda}{n}\right)^p + (nd - 1) \left(\frac{1 - \lambda}{n}\right)^p\right) \]
\[= \left(\frac{1}{d}\left(\lambda d + \frac{1 - \lambda}{n}\right)^p + (n - 1) \left(\frac{1 - \lambda}{n}\right)^p\right).
\]
Therefore, the result follows. \(\square\)

4.1. Non additivity of \(C^d_{\text{prod}}\) for the depolarizing channel. As we said in the previous section the quantity \(C^d_{\text{prod}}(D_\lambda)\) in Theorem 1.4 extends the corresponding results for the product state classical capacity of the quantum depolarizing channel (with no assisted entanglement), so \(d = 1\), and for the product state (unlimited) assisted entanglement classical capacity, \(d = n\). In fact, it is known that in both cases the quantity \(C^d_{\text{prod}}(D_\lambda)\) coincides with the capacity \(C^d(D_\lambda)\).

Somehow surprisingly, this is no longer true if \(1 < d < n\) as we stated in Corollary 1.5.

First of all, note that it is very easy to see that
\[(4.2) \quad C^d_{\text{prod}}(D_\lambda \otimes D_\lambda) \geq C^d_{\text{prod}}(D_\lambda) + C^1_{\text{prod}}(D_\lambda).
\]

Indeed, from a physical point of view this means that a particular strategy for Alice and Bob with a \(d^2\)-dimensional entangled state consists of using all the entanglement in one of the channel and using the other channel without assisted entanglement. From a mathematical point of view, this can be deduced from the fact that
\[\|D_\lambda \otimes D_\lambda : S^n_1 \otimes S^n_1 \rightarrow S^n_p \otimes S^n_p\|_{\mathcal{D}_2} \geq \|D_\lambda : S^n_1 \rightarrow S^n_p\|_{\mathcal{D}_2} + \|D_\lambda : S^n_1 \rightarrow S^n_p\|,
\]
which is obvious by restricting to elements of the form \(x = y \otimes z\), with \(y \in S^n_p(S^n_1)\) and \(z \in S^n_1\) in the computation of the norm. The fact that we have a complete description of \(C^d_{\text{prod}}(D_\lambda)\) for every \(n, d\) and \(\lambda\) allows us to exactly compute the quantity
\[(4.3) \quad f(n, d, \lambda) = C^d_{\text{prod}}(D_\lambda) + C^1_{\text{prod}}(D_\lambda) - 2C^d_{\text{prod}}(D_\lambda).
\]

According to (4.2), we want to show that \(f(n, d, \lambda)\) is strictly positive for some values of \(n, d\) and \(\lambda\). Now,
The most basic example can be found for \( n = 4 \) and \( d = 2 \). The function \( h(\lambda) = f(4, 2, \lambda) \) is represented below. Recall that, according to Remark 3.1 in order to compute the real quantity \( C_{\text{prod}}^d(D_\lambda) \) we must multiply by \( \frac{1}{\ln 2} \). We can see that the “amount of violation” \( h(\lambda) \) is very small. Some other examples can be found where the amount of violation is arbitrary large. Indeed, it was shown in [16, Theorem 1.2] that for every natural number \( n \), one can find a quantum channel \( N : S_1^{2n} \to S_1^{2n} \) such that

\[
C_{\text{prod}}^n(N \otimes N) - 2C_{\text{prod}}^{\sqrt{n}}(N) \geq \frac{1}{3} \log_2 n,
\]

where we use the symbol \( \succeq \) to denote inequality up to universal (additive) constants which do not depend on \( n \). One could wonder whether we can have a similar result for the quantum depolarizing channel so that the reason for our small value in the violation is that we are considering parameters \( n \) and \( d \) very small. In fact, our Theorem 1.6 (Equation (1.4)) shows that for the quantum depolarizing channel the amount of violation is bounded by \( \ln 2 \) independently of \( n \) and \( d \) (and the number of uses of the channel). To finish this section we will prove (1.4) by assuming Equation (1.5), which will be proved in the next section.

**Proof of Equation (1.4) in Theorem 1.6** Equation (1.5) states that \( C^d(\mathcal{E}_\lambda) = \lambda \ln(nd) \), where \( \mathcal{E}_\lambda : S_1^n \to S_1^n \oplus_1 \mathbb{C} \) denotes the quantum erasure channel with parameter \( \lambda \), defined by

\[
\mathcal{E}_\lambda(\rho) = \lambda \rho \oplus (1 - \lambda)tr(\rho) \quad \text{for every} \quad \rho \in S_1^n.
\]

Since it is very easy to see that \( C^d(D_\lambda) \leq C^d(\mathcal{E}_\lambda) \), the last inequality in (1.4) follows. On the other hand, we know that the inequality \( C_{\text{prod}}^d(D_\lambda) \leq C^d(D_\lambda) \) holds for every channel. Therefore, we just need to show the first inequality in (1.4). To this end, note that

\[
C_{\text{prod}}^d(D_\lambda) = \ln(nd) + \mu \ln \mu + \left( \frac{nd - 1}{nd} \right) (1 - \lambda) \ln \left( \frac{1 - \lambda}{nd} \right)
\]

\[
= \ln(nd) + \mu \ln \mu + \left( \frac{nd - 1}{nd} \right) (1 - \lambda) \left[ \ln \left( \frac{1 - \lambda}{nd} \right) + \ln(nd - 1) - \ln(nd - 1) \right]
\]

\[
= \ln(nd) - H(\mu, 1 - \mu) - \left( \frac{nd - 1}{nd} \right) (1 - \lambda) \ln(nd - 1)
\]

\[\text{It can be shown that for } n = 3, \ C_{\text{prod}}^3(D_\lambda) + C_{\text{prod}}^1(D_\lambda) - 2C_{\text{prod}}^2(D_\lambda) < 0 \ \text{for every } \lambda \in (0, 1).\]
Let us show the case $E$ is covariant. In fact, one can also easily check that the channel of the quantum depolarizing channel, it is very easy to see that the quantum erasure channel $\mathcal{E}_d$ is a stronger result than computing its capacity. This point will be particularly important in this work we are interested in computing the $d$-norm of the channels, so we will show here Equation (1.3) from which the previous quantity can be obtained by differentiating (and adding an extra in-term). It is interesting to remark here that, computing the $d$-norm of a channel is a stronger result than computing its capacity. This point will be particularly important in

\[ C_{\text{prod}}(\mathcal{E}_}\mathcal{E}_d) = \lambda \ln n + V_d(\mathcal{E}_d) \]
\[ = \lambda \ln n + \sup \left\{ S\left((id_d \otimes tr_n)(|\psi \rangle \langle \psi|)\right) - \lambda S\left((id_d \otimes id_n)(|\psi \rangle \langle \psi|)\right) \right\} \]
\[ = \lambda \ln n + \lambda \ln d \leq \lambda \ln n + \lambda \ln d = \lambda \ln (nd). \]

Here, the supremum runs over all pure states $|\psi \rangle \in \mathbb{C}^d \otimes \mathbb{C}^n$ and we have used that $S(\rho) = 0$ for every pure state $\rho$ and also that $S(\eta) \leq \ln d$ for every $d$-dimensional state $\eta$.

On the other hand, if we consider the $d$-maximally entangled state $|\psi_d\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |e_i \otimes e_i \rangle \in \mathbb{C}^d \otimes \mathbb{C}^n$ we can check that

\[ C_{\text{prod}}(\mathcal{E}_d) \geq \lambda \ln n + \lambda S\left((id_d \otimes tr_n)(|\psi_d \rangle \langle \psi_d|)\right) = \lambda \ln n + \lambda \ln d = \lambda \ln (nd). \]

Therefore, the previous argument already gives us the right expression for $C_{\text{prod}}(\mathcal{E}_d)$. However, in this work we are interested in computing the $d$-norms of the channels, so we will show here Equation (1.3) from which the previous quantity can be obtained by differentiating (and adding an extra in-term). It is interesting to remark here that, computing the $d$-norm of a channel is a stronger result than computing its capacity. This point will be particularly important in
the study of $E_k^C$ below, since we couldn’t find a good expression for its $d^k$-norm and we directly computed $C_{\prod d}(E_k^\otimes s)$.

**Proof of Equation (1.3) in Theorem 1.4.** Let us first note that

\[
\|\mathcal{E}_\Lambda : S_1^n \to S_p^d(S_1^n) \|_{id_d} = \|\text{id}_d \otimes \mathcal{E}_\Lambda : S_p^d(S_1^n) \to S_p^d(S_1^n) \|.
\]

In order to compute this norm, let us consider an element $\rho \in M_d \otimes M_n$ with $\|\rho\|_{S_p^d(S_1^n)} = 1$. It is very easy that this implies, in particular, that $\|(\text{id}_d \otimes \text{tr}_n)(\rho)\|_{S_p^d} \leq 1$. Indeed, this is a trivial consequence of the fact that $\text{tr} : S_1^n \to \mathbb{C}$ is a (complete) contraction. On the other hand,

\[
(id_d \otimes \mathcal{E}_\Lambda)(\rho) = \lambda \rho \oplus (1 - \lambda)(id_d \otimes \text{tr}_n)(\rho).
\]

Thus,

\[
\|(id_d \otimes \mathcal{E}_\Lambda)(\rho)\|_{S_p^d(S_1^n) \oplus_p S_p^d} = \left(\lambda^p \|\rho\|_{S_p^d}^p + (1 - \lambda)^p \|(\text{id}_d \otimes \text{tr}_n)(\rho)\|_{S_p^d}^p\right)^{\frac{1}{p}} \\
\leq \left(\lambda^p \|\rho\|_{S_p^d}^p + (1 - \lambda)^p\right)^{\frac{1}{p}} \\
\leq \left(\lambda^p d^{p-1} + (1 - \lambda)^p\right)^{\frac{1}{p}}.
\]

Here, in the last inequality we have used that

\[
\|\text{id}_n : S_1^n \to S_p^n\|_d = \|(\text{id}_d \otimes \text{id}_n) : S_p^d(S_1^n) \to S_p^d(S_1^n)\| = d^{1 - \frac{1}{p}}.
\]

On the other hand, one can see that

\[
\|\text{id}_d \otimes \mathcal{E}_\Lambda : S_p^d(S_1^n) \to S_p^d(S_1^n) \| \geq \left(\lambda^p d^{p-1} + (1 - \lambda)^p\right)^{\frac{1}{p}},
\]

by testing this norm at the $d$-maximally entangled state $\rho = |\psi_d\rangle \langle \psi_d|$. \hfill \Box

In order to show Equation (1.5) in Theorem 1.6 we must deal with an arbitrary number of tensor products of the channel $\mathcal{E}_\Lambda$. To this end, we need to introduce some notation. Let us fix $k \in \mathbb{N}$ and consider a natural number $s$ with $0 \leq s \leq k$. We note that there are $\binom{k}{s}$ subsets $A$ of $\{1, \ldots, k\}$ with cardinal $|A| = s$. For each of these sets we will denote

\[
\mathcal{N}_A : S_1^n \to S_1^n|A|,
\]

defined by

\[
\mathcal{N}_A(\rho) = (\text{id}_A \otimes \text{tr}_{A^c})(\rho) \quad \text{for every} \quad \rho \in S_1^n,
\]

where $(\text{id}_A \otimes \text{tr}_{A^c})(\rho) \in S_1^n|A|$ denotes the state $\rho$ after tracing out all the systems $j \in A^c$. Then, it is clear that

\[
\mathcal{E}_\Lambda^\otimes_k : S_1^n \to \bigoplus_{s=0}^k \bigoplus_{A \subseteq \{1 \ldots k\} \atop |A| = s} S_1^n|A|.
\]
is given by
\[
\mathcal{E}^{\otimes k}_\chi(\rho) = \bigoplus_{s=0}^{k} \bigoplus_{A \subseteq \{1, \ldots, k\}} \lambda^s(1 - \lambda)^{k-s} \mathcal{N}_A(\rho) \quad \text{for every } \rho \in S_1^{n^k}.
\]

**Lemma 5.1.** For every \(k \in \mathbb{N}\) we have
\[
C^d_{\text{prod}}(\mathcal{E}^{\otimes k}_\chi) \leq \sum_{s=0}^{k} \binom{k}{s} \lambda^s(1 - \lambda)^{k-s} \ln n^s + \sum_{s=0}^{k} \binom{k}{s} \lambda^s(1 - \lambda)^{k-s} V_d(\mathcal{N}_s).
\]
Here,
\[
\mathcal{N}_s : S_1^{n^k} \to \bigoplus_{A \subseteq \{1, \ldots, k\}} S_1^{n^{\lvert A \rvert}}
\]
is defined by
\[
\mathcal{N}_s(\rho) = \frac{1}{\binom{k}{s}} \bigoplus_{A \subseteq \{1, \ldots, k\}} \mathcal{N}_A(\rho) \quad \text{for every } \rho \in S_1^{n^k}.
\]

**Proof.** According to Proposition 3.7 and the covariant property of \(\mathcal{E}^{\otimes k}_\chi\) we have that
\[
C^d_{\text{prod}}(\mathcal{E}^{\otimes k}_\chi) = \sum_{s=0}^{k} \binom{k}{s} \lambda^s(1 - \lambda)^{k-s} \ln n^s + V_d(\mathcal{E}^{\otimes k}_\chi).
\]
On the other hand, by definition, \(V_d(\mathcal{E}^{\otimes k}_\chi)\) is equal to
\[
\sup \left\{ S \left( (id_d \otimes tr_{n^k})(\langle \psi | \psi \rangle) \right) \right\} = \sum_{s=0}^{k} \binom{k}{s} \lambda^s(1 - \lambda)^{k-s} \ln n^s + V_d(\mathcal{E}^{\otimes k}_\chi).
\]

where the supremum is taking over all pure states \(\ket{\psi} \in \mathbb{C}^d \otimes \mathbb{C}^{n^k}\). Here, we have used the identity
\[
(5.1) \quad 1 = (\lambda + (1 - \lambda))^k = \sum_{s=0}^{k} \binom{k}{s} \lambda^s(1 - \lambda)^{k-s}.
\]
It follows now easily that the previous quantity is lower than or equal to
\[
\sum_{s=0}^{k} \binom{k}{s} \lambda^s(1 - \lambda)^{k-s} \sup \left\{ S \left( (id_d \otimes tr_{n^k})(\langle \psi | \psi \rangle) \right) - \frac{1}{\binom{k}{s}} \sum_{A \subseteq \{1, \ldots, k\}} \mathcal{N}_A(\rho) \right\} = \sum_{s=0}^{k} \binom{k}{s} \lambda^s(1 - \lambda)^{k-s} V_d(\mathcal{N}_s),
\]
where all the supremums are taking over all pure states \(\ket{\psi} \in \mathbb{C}^d \otimes \mathbb{C}^{n^k}\).
The statement of the lemma follows.

**Lemma 5.2.** Let \( k \) and \( s \) be two natural numbers such that \( 0 \leq s \leq k \). Then,

\[
V_d(N_s) \leq \frac{s}{k} \ln d.
\]

Before proving this lemma, we will show how to deduce the main result of this section from Lemma 5.1 and Lemma 5.2.

**Proof of Equation (1.5) in Theorem 1.6.** The inequality \( C^d_{\text{prod}}(E^{\otimes k}) \geq kC^d_{\text{prod}}(E) \) holds for every channel (since one could use each copy of the channel independently). According to Theorem 1.4 this implies that \( C^d_{\text{prod}}(E^{\otimes k}) \geq k\lambda \ln(nd) \). On the other hand, according to Lemmas 5.1 and Lemma 5.2 we have

\[
C^d_{\text{prod}}(E^{\otimes k}) \leq \sum_{s=1}^{k} \binom{k}{s} \lambda^s(1-\lambda)^{k-s} \ln n^s + \sum_{s=0}^{k} \binom{k}{s} \lambda^s(1-\lambda)^{k-s} V_d(N_s)
\]

\[
\leq \sum_{s=1}^{k} \binom{k}{s} \lambda^s(1-\lambda)^{k-s} \ln n^s + \sum_{s=0}^{k} \binom{k}{s} \lambda^s(1-\lambda)^{k-s} \frac{s}{k} \ln d^k
\]

\[
= \sum_{s=1}^{k} \binom{k}{s} \lambda^s(1-\lambda)^{k-s}s \ln(\ln(nd))
\]

\[
= k\lambda \ln(nd).
\]

Here, we have used that

\[
\sum_{s=1}^{k} \binom{k}{s} \lambda^s(1-\lambda)^{k-s}s = k\lambda.
\]

In order to see this, let us proceed by induction.

For \( k = 2 \) we have \( \sum_{s=1}^{2} \binom{2}{s} \lambda^s(1-\lambda)^{2-s} = \binom{2}{1}(1-\lambda) + \binom{2}{2} \lambda^2 = 2\lambda \). Let us now assume the result for \( k \). Then,

\[
\sum_{s=1}^{k+1} \binom{k+1}{s} \lambda^s(1-\lambda)^{k+1-s} = \lambda(k+1) \sum_{s=1}^{k} \binom{k}{s-1} \lambda^{s-1}(1-\lambda)^{k-(s-1)}
\]

\[
= \lambda(k+1) \sum_{s=0}^{k} \binom{k}{s} \lambda^s(1-\lambda)^{k-s} = \lambda(k+1),
\]

where in the last equality we have used again the identity (5.1).

This finishes the proof.

**Lemma 5.2** can be obtained as a simple consequence of the following deep and extremely useful result in information theory.

**Theorem 5.3** (Strong subadditivity inequality, [25]). For every tripartite state \( \rho \in S_1 \otimes S_1^n \otimes S_1^n \) the following inequality holds.

\[
S(\rho) + S\left(tr_n \otimes id_n \otimes tr_n\right)(\rho) \leq S\left(id_n \otimes id_n \otimes tr_n\right)(\rho) + S\left(tr_n \otimes id_n \otimes id_n\right)(\rho).
\]
In general, if we call the respective systems \(A, B,\) and \(C,\) the strong subadditivity inequality can be written by
\[
S(ABC) + S(B) \leq S(AB) + S(BC).
\]
Of course, the system \(B\) can be replaced by system \(A\) and \(C\) and the analogous inequality holds. It is also interesting to mention that the strong subadditivity inequality can be obtained by differentiating the norm \(\|\rho\|_{S_1(S_\rho)}\) and using a Minkowski-type inequalities (see [11, Section 6]).

We thank Andreas Winter for the explanation of the following proof which simplified very much a previous proof by the authors (not using the strong subadditivity inequality).

**Proof of Lemma 5.2.** According to our definition (3.6), \(V_d(\mathcal{N}_s)\) can be trivially written as
\[
\sup \left\{ \frac{s}{k} S((id_d \otimes tr_{n^s})(\rho)) + \left( \frac{k-s}{k} \right) S((id_d \otimes tr_{n^s})(\rho)) - \frac{1}{(\frac{k}{s})} \sum_{A \subseteq \{1, \ldots, k\}} S((id_d \otimes N_A)(\rho)) \right\},
\]
where here the supremum is taken over all pure states \(\rho \in S^d_1(S_1^n).\) Since, we clearly have \(S((id_d \otimes tr_{n^s})(\rho)) \leq \ln d\) for every state \(\rho,\) it suffices to show that for every pure state \(\rho \in S^d_1(S_1^n)\) we have
\[
\left( \frac{k-s}{k} \right) S((id_d \otimes tr_{n^s})(\rho)) \leq \frac{1}{(\frac{k}{s})} \sum_{A \subseteq \{1, \ldots, k\}} S((id_d \otimes N_A)(\rho)).
\]

Now, since we are assuming that \(\rho\) is pure, the previous inequality is the same as
\[
\left( \frac{k-s}{k} \right) S((tr_d \otimes id_{n^s})(\rho)) \leq \frac{1}{(\frac{k}{s})} \sum_{|A|=s} S((tr_d \otimes N_A^s)(\rho)).
\]

Since we must prove the result for every \(0 \leq s \leq k,\) by replacing \(s\) with \(k-s,\) we see that it suffices to show that for every not necessarily pure state \(\rho \in S_1^n\) and for every \(0 \leq s \leq k\) one has
\[
\frac{s}{k} S(\rho) \leq \frac{1}{(\frac{k}{s})} \sum_{A \subseteq \{1, \ldots, k\}} S(N_A(\rho)).
\]
Let us simplify the notation of the previous inequality by writing it as
(5.2)
\[
\frac{s}{k} S(A_1 \cdots A_k) \leq \frac{1}{(\frac{k}{s})} \sum_{|A|=s} S(A_A),
\]
with the obvious interpretation. We will first prove this inequality for the particular case \(s = k-1\) and we will obtain the general case by induction. In this case, we must show

(5.3) 
\[
(k-1)S(A_1 \cdots A_k) \leq \sum_{i=1}^{k} S(A_{[k]-\{i\}}),
\]
Let us consider a purification $\{W A_1 \cdots A_k\}$ of the system $A_1 \cdots A_k$ (that is, the state $\rho \in S_{1^n}$) so that we can write the previous expression as

$$ (k - 1)S(W) \leq \sum_{i=1}^{k} S(W A_i). \tag{5.4} $$

Here, we are using that for every multipartite pure state the von Neumann entropy of any subsystem is the same as the von Neumann entropy of the complement subsystem, which is a direct consequence of the Hilbert Schmidt decomposition. Now, a direct application of Theorem 5.3 implies that for every $0 \leq s \leq k - 1$,

$$ S(W) + S(W A_1 A_2 \ldots A_{k-1} \{0, \ldots, s\}) \leq S(W A_s) + S(W A_{k-1} \{0, \ldots, s+1\}). $$

Then, we can obtain Equation (5.4) by applying this inequality $k-1$ times iteratively. With Equation (5.3) at hand, we can finish our proof by using induction. Checking that (5.2) holds for $k = 2$ ($s = 0, 1, 2$) is very easy by just using the subadditivity of the von Neumann entropy $\dagger$. On the other hand, let us assume that (5.2) holds for every state $\rho \in S_{n^{k-1}}$ (so for every systems $A_1, \ldots, A_{k-1}$) and every $0 \leq s \leq k - 1$ and we will show that, then, it must also hold for $k$. First of all, note that the case $s = k$ is completely trivial, so it suffices to consider $0 \leq s \leq k - 1$. Then, we can write

$$ \frac{s}{k} S(A_1 \cdots A_k) \leq \frac{s}{k(k - 1)} \sum_{i=1}^{k} S(A_{k-1} - \{i\}) \leq \frac{s}{k(k - 1)} \sum_{i=1}^{k} \frac{1}{s} \left( \frac{k-1}{s} \right) \sum_{\delta | s} S(A_{k-1} - \{i\}; \delta) = \frac{1}{s} \sum_{\delta | s} S(A_{\delta}). $$

Here, the first inequality follows from Equation (5.3) and the second inequality follows from the induction hypothesis. The last equality is straightforward.

**Acknowledgments**

We thank Andreas Winter and Toby S. Cubitt for helpful conversations. A part of this work was done at the Isaac Newton Institute (Cambridge, U.K.), during the programme on Mathematical Challenges in Quantum Information in Fall 2013.

**References**


\dagger Given any state $\rho \in S_{n^n}$, we can always find a unit vector $|\psi\rangle \in \mathbb{C}_M \otimes \mathbb{C}_N$ so that $(tr_M \otimes id_N) (|\psi\rangle\langle\psi|) = \rho$.

\dagger This result is a trivial consequence of Theorem 5.3.


[38] G. Smith, *Quantum channel capacities*, Information Theory Workshop (ITW), 2010 IEEE, pp. 1-5.

Marius Junge  
Department of Mathematics  
University of Illinois at Urbana-Champaign  
1409 W. Green St. Urbana, IL 61801. USA  
junge@math.uiuc.edu

Carlos Palazuelos  
Instituto de Ciencias Matemáticas, ICMAT  
Facultad de Ciencias Matemáticas  
Universidad Complutense de Madrid  
Plaza de Ciencias s/n. 28040, Madrid. Spain  
carlospalazuelos@ucm.es