Error tolerance of topological codes with independent bit-flip and measurement errors

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Topological quantum error correction codes are currently among the most promising candidates for efficiently dealing with the decoherence effects inherently present in quantum devices. Numerically, their theoretical error threshold can be calculated by mapping the underlying quantum problem to a related classical statistical-mechanical spin system with quenched disorder. Here, we present results for the general fault-tolerant regime, where we consider both qubit and measurement errors. However, unlike in previous studies, here we vary the strength of the different error sources independently. Our results highlight peculiar differences between toric and color codes. This study complements previous results published in New J. Phys. 13, 083006 (2011).

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I. INTRODUCTION

The quest for building a reliable quantum computer involves multiple fields of research, such as several branches of computer science, theoretical and experimental physics, mathematics, and engineering [1,2]. Most notably, disordered spin systems and lattice gauge theories [3–6] in statistical physics have played a pivotal role in the understanding of the theoretical error tolerance of topological quantum computing models in quantum information theory [3,7–11]. The main driver for this fruitful synergy across disciplines was the discovery that methods from statistical physics allow for the numerical study of quantum error correction codes [12–18]. More specifically, error fluctuations in topologically protected codes map [3] directly onto classical spin models with tunable disorder. The level of noise in the quantum code then corresponds to the amount of quenched disorder in a classical spin system. In practice, this means that by carefully analyzing the critical behavior of the classical system, we can learn how resilient a topological code is to a particular source of errors.

The very same approach can also be used to investigate fault-tolerant error correction [19–21], which takes the possibility of faulty measurements during the error-correction procedure into account. This is particularly exciting because topological codes [22] allow for fault-tolerant quantum computation without resorting to code concatenation [3,23–26]. Instead, the new resource is the nontrivial topology of the lattice on which the physical qubits are arranged. The topological quantum code, in turn, is defined by the pattern of the arrangement and the way in which check operators act on these qubits. The key property of these check operators (also known as stabilizers), is that their support is local on the qubits forming the lattice. This locality property is absent in concatenated codes and is beneficial for experimental realizations. Moreover, as long as the external errors act also locally on the code, it is possible to protect the encoded quantum information because the encoded logical qubits are entangled states that spread out globally across the whole system. While implementing such systems might seem to be an insurmountable effort at this time, recently, a complete error-correction code for arbitrary errors using a minimal topological color code has been realized experimentally in a trapped-ion platform [27] that paves the way towards the experimental realization of topological codes, such as the Kitaev code or the color codes in two-dimensional setups [28–32]. As such, gaining a deeper understanding of the interplay between different error sources is of current importance.

Remarkably, assuming that both bit-flip errors and measurement errors occur at the same average rate, previous numerical results [6] suggest that topological color codes [23] exhibit an improved error tolerance over the toric code [22]. While this is potentially only true in the ideal scenario where all physical operations are noise free, it does serve as a guide when comparing the performance of both models on an equal footing. Here, we further investigate this observation by extending the numerical results to qubit and measurement errors of different average strength.

The paper is organized as follows. Section II provides a brief introduction to the toric code and topological color codes in the fault-tolerant regime. Section III summarizes the mappings to classical lattice gauge theories as derived in Ref. [6] for color codes and Ref. [4] for the toric code. In Sec. IV we explain the numerical tools used for our extended analysis, followed by the results in Sec. V, as well as concluding remarks.

II. TOPOLOGICAL STABILIZER CODES

A stabilizer code $C$ of length $n$ is a subspace of the Hilbert space of a set of $n$ physical qubits [16]. The code is defined by means of the stabilizer group $S \subset P_n$ of Pauli operators, which are tensor products of Pauli matrices of length $n$:

$$P_n := \langle 1, X_1, Z_1, \ldots, X_n, Z_n \rangle. \quad (1)$$

The stabilizer group leaves invariant the quantum states belonging to the code:

$$|\psi\rangle \in C \iff \forall O \in S \quad O|\psi\rangle = |\psi\rangle. \quad (2)$$

The Pauli operator $-1$ is excluded from $S$. To fully characterize the code it is sufficient to define the generators of $S$:

$$G := \langle g_1, \ldots, g_t \rangle. \quad (3)$$
The normalizer $N(S)$ of $S$ plays a fundamental role in error correction. It is defined by the operators $O$ satisfying
\[ O \in N(S) \iff OS = SO, \]
which implies that the code space $C$ is left invariant by $N(S)$. When the operators of the normalizer do not belong to the stabilizer itself, then they act in a nontrivial way on the encoded states.

Active error correction is necessary to protect the error-prone logical state: we need to measure a set of generators of $S$. The result of these measurements is called the syndrome—the signature of which error has occurred. Errors can be corrected as long as the syndromes allow us to discriminate among possible errors. As correctable errors always form a vector space, it is enough to consider Pauli operators, which form a basis. A Pauli error $e$ is said to be undetectable if it belongs to the set $N(S) - S$. In this case, the syndrome provides no information:
\[ \forall s \in S \quad s e(\psi) = e s(\psi) = |\psi\rangle. \] (5)
A set of Pauli errors $E$ is said to be correctable if, and only if,
\[ E^\dagger E \cap N(S) \in S. \] (6)

Topological stabilizer codes are peculiar instances of stabilizer codes employing a regular arrangement of qubits on a topologically nontrivial surface [33] and local stabilizer operators. Because both codes are Calderbank-Shor-Steane codes [14,15], bit-flip and phase errors can be phased independently and analogously. Here we focus only on bit-flip errors occurring at a rate $p$.

**Toric code.** The physical qubits are arranged on the edges of a two-dimensional lattice, with stabilizers at each vertex being the tensor product of $\hat{Z}$ operators for adjacent qubits [22]. Thus flipping qubit $Q_\ell$ changes the sign of the measured eigenvalue for the check operators at either end of the edge $\ell$. The first example of such a topological code was the Kitaev toric code defined with a square-lattice arrangement. In this case, each stabilizer operator $\hat{Z}_{\ell}^{04}$ is the tensor product of exactly four $\hat{Z}$ operators.

**Topological color codes.** Initially conceived to expand the computational capabilities of topological codes by increasing the set of topological gates that can be applied [23,24], here we consider a hexagonal arrangement of the physical qubits, with stabilizers $\hat{Z}_{\ell}^{06}$ on each plaquette acting on the adjacent qubits.

**Error correction.** For all check operators, encoded states satisfy $\hat{Z}_4 |\psi\rangle = |\psi\rangle$. Such states exist because the group generated by check operators, called the stabilizer group, does not contain $-1$, so that in particular check operators commute with each other. The dimension of the encoded subspace depends only on the topology of the surface where the code lives. For example, a regular lattice with periodic boundary conditions has the topology of a torus and encodes two logical qubits [23].

**Fault-tolerant regime.** With measurements being faulty at a rate $q$, new errors are introduced involuntarily during the error-correction procedure. To detect local inconsistencies with the code, check operators need to be measured repeatedly over time and error correction amounts to guessing which error happened with the highest probability $P(\hat{E})$. Therefore, error correction is highly successful when for typical errors there is a class that dominates over the others.

### III. Mapping to Lattice Gauge Theories

For both types of topological stabilizer codes introduced in the previous section, the mapping of the setup to a classical spin system produces a lattice gauge theory with disorder [3,6]. For a side-by-side comparison, it is instructive to describe both in terms of their local equivalences. The results of the respective mappings (with some minor adjustments to match the notations) can be summarized as follows.

**Toric codes ($\mathbb{Z}_2$ lattice gauge theory).** The toric code for bit-flip errors occurs on the edges of a square lattice with stabilizer operators at each vertex acting on the four adjacent qubits (see Fig. 1). Fault tolerance is added by stacking multiple of these lattices and connecting the vertices vertically to a three-dimensional cubic grid. We can then interpret the vertical axis as time with each vertical edge representing a measurement of the stabilizer operator it connects.

In addition to the regular local equivalence of flipping four qubits around a horizontal plaquette, this stacked model also has a second type that consists of flipping the same qubit in adjacent layers along with the two measurements connecting them. This equivalence is a vertical plaquette in the three-dimensional lattice and represents the scenario of two consecutive bit-flip errors which go unnoticed because of two concurrent measurement errors.

The probability of an error arbitrary history $E$, consisting of $h$ bit-flip errors and $v$ faulty measurements, can be written as
\[ P(E) = (1 - p)^{H-h} p^h (1 - q)^v q^v \]
\[ \propto \left( \frac{p}{1 - p} \right)^h \left( \frac{q}{1 - q} \right)^v. \] (7)

**FIG. 1.** Stacked layers representing the mapped model for the fault-tolerant toric code. Qubits reside on the edges and stabilizer operators $\hat{Z}_{\ell}^{04}$ act on the qubits surrounding each vertex. (a) Horizontal loops correspond to the usual local equivalence of the toric code: flipping the four qubits around a plaquette leaves the error syndrome invariant. (b) The second type of local equivalence involves measurement errors which are represented by vertical links connecting stabilizer operators. (c) The resulting model consists of spatial and timelike links forming a three-dimensional cubic lattice.
where $p$ is the bit-flip rate and $q$ the measurement error rate, while both $H$ (total number of qubits) and $V$ (total number of measurements) are constants of the cubic lattice.

A specific error history $E$ can be represented by a set of variables $\tau_\ell \in [\pm 1]$, each indicating whether the qubit or measurement corresponding to edge $\ell$ is faulty. Furthermore, we can enumerate all histories in the error class of $E$ (i.e., those that differ only by local equivalences) by attaching a binary variable $\sigma_{h,v} \in \pm 1$ to each equivalence. To numerically sample from these, one then constructs a classical Hamiltonian which has Boltzmann weights proportional to Eq. (7):

$$H_E = -J \sum_{j \in Q} \tau_j \sigma_h^{\otimes 2} \sigma_v^{\otimes 2} - K \sum_{k \in M} \tau_k \sigma_v^{\otimes 4}. \quad (8)$$

Note that the first sum (which iterates over all qubits $Q$, i.e., horizontal links) essentially counts the number of flipped qubits. By definition, a qubit is flipped if the product of $\tau_j$ and all the equivalences it is affected by (two horizontal and two vertical ones) is negative. Similarly, the second sum iterates over all measurements $M$ and adds up the number of faulty ones. Therefore, we can see that the correct Boltzmann weights are produced with

$$e^{-2\beta J} = p/(1-p), \quad e^{-2\beta K} = q/(1-q). \quad (9)$$

which is called the Nishimori condition [34]. The Hamiltonian in Eq. (8) is equivalent to the one given by Dennis et al. [3] however, with separated terms for qubit and measurement errors.

**Color codes (tricolored lattice gauge theory).** For topological color codes, consider a three-dimensional lattice consisting of stacked triangular and hexagonal layers, with qubits residing on intermediate hexagonal layers. There is a stabilizer operator $Z^{\otimes 6}$ for each of the hexagonal tiles, acting on the six qubits surrounding the plaquette.

As for the toric code, there are again two distinct types of elementary equivalences. The first is a horizontal loop consisting of the six qubits around a plaquette, while the second consists of adjacent qubits in two layers, connected by three measurement errors (see Fig. 2). This represents again the scenario of two subsequent qubit flips on the same qubit, which remain unnoticed because of three concurrent measurement errors. The resulting Hamiltonian takes the form

$$H_E = -J \sum_{j \in Q} \tau_j \sigma_h^{\otimes 3} \sigma_v^{\otimes 2} - K \sum_{k \in M} \tau_k \sigma_v^{\otimes 6}. \quad (10)$$

with identical requirements for the constants $J$ and $K$. This corresponds to the Hamiltonian calculated by Andrist et al. (see Refs. [6,35] for details).

### IV. Numerical Methods

Based on the Hamiltonians in Eq. (8) for the toric code and Eq. (10) for color codes, the error threshold for a particular code is given by the largest error rates for which the model remains in an ordered state at the temperature $T$ specified by the Nishimori condition. Dennis et al. [3] have demonstrated that this property is found at the multicritical point of the $p$-$T$ ($p$ the bit-flip error rate) phase diagram where the Nishimori line intersects the phase boundary. For independent qubit and measurement error rates, the Nishimori condition translates to a Nishimori sheet in the three-dimensional parameter space spanned by $p,q$ and the model’s temperature $T = 1/\beta$. Note that the purpose of this “virtual” temperature is merely to achieve the desired Boltzmann statistics via Eq. (9), while any physical temperature effects in the quantum device are implicitly captured by the error rates $p$ and $q$.

We use large-scale Monte Carlo simulations to analyze the phase diagram along different projections, namely $p = 2q$ and $p = q/2$. In both cases we expect to find the system in an ordered Higgs phase for weak disorder and low temperatures $T$. This indicates that error histories observed at these error rates typically exhibit only small fluctuations. Once the phase boundary is crossed, the system enters the disordered confinement phase, indicating that the topologically encoded information is vulnerable to failures.

The crossing point is determined as follows. For increasingly larger error rates $p$ and $q$, we use the peak position in the measured system’s susceptibility as done in Ref. [4], as well as the skewness of the Wilson look distribution [6] to locate the phase transition temperature $T_c(p,q)$. As long as this transition occurs at a higher temperature than the one specified by the Nishimori condition, we know that the system still exhibits an ordered state. Because the error rates $p$ and $q$ are merely parameters to generate the quenched random interactions, these calculations need to be repeated for many independent disorder realizations to obtain the desired statistical-mechanical average. This and the fact that disordered lattice gauge theories are inherently hard to simulate necessitates considerable numerical efforts for every single point generated in the phase diagram in Fig. 3. To mitigate this challenge, we use the parallel tempering Monte Carlo technique [36], with the detailed simulation parameters listed in Table I. Equilibration for each sample is tested by a logarithmic binning of the data. Once the last three bins agree within statistical error bars, the system is deemed to be in thermal equilibrium.
FIG. 3. Summary of the numerically calculated error thresholds in context of the previous results. The plot indicates the phase boundaries of the ordered phase for both types of codes, projected onto the Nishimori surface where the mapping to the quantum setup is valid. The estimates indicate that the difference in resilience remains even for nonmatching error rates. Simulations for $p \to 0$ are difficult and we have no estimates for this regime.

TABLE I. Simulation parameters: $L$ is the layer size, $M$ is the number of layers, $N_{sa}$ is the number of disorder samples, $t_{eq} = 2^p$ is the number of equilibration sweeps, $T_{\text{min}}$ ($T_{\text{max}}$) is the lowest (highest) temperature, and $N_T$ is the number of temperatures used for a given error rate $p$. The corresponding values of $q$ are given by the simulation paths chosen, namely $q = 2p$, $q = p$, and $q = p/2$.

\begin{tabular}{c|cccccc}
$p$ & $L$ & $M$ & $N_{sa}$ & $b$ & $T_{\text{min}}$ & $T_{\text{max}}$ & $N_T$ \\
\hline
0.00 & 6.9 & 6.8 & 1600 & 15 & 1.20 & 2.00 & 64 \\
0.00 & 12 & 12 & 800 & 15 & 1.20 & 2.00 & 64 \\
0.02 & 6.9 & 6.8 & 1600 & 16 & 0.90 & 1.80 & 52 \\
0.02 & 12 & 12 & 800 & 17 & 0.90 & 1.80 & 52 \\
0.03-0.039 & 6.9 & 6.8 & 1600 & 17 & 0.70 & 1.40 & 52 \\
0.03-0.039 & 12 & 12 & 800 & 19 & 0.70 & 1.40 & 52 \\
0.04-0.060 & 6.9 & 6.8 & 1600 & 18 & 0.50 & 1.20 & 52 \\
0.04-0.060 & 12 & 12 & 800 & 20 & 0.50 & 1.20 & 52 \\
\end{tabular}

\section{VI. CONCLUSIONS}

Our numerical results indicate that the difference in error resilience between the toric code and topological color codes persists when bit-flip and measurement errors occur at different rates. Only in the limit of a vanishing measurement error rate the two lines in Fig. 3 converge to a common point, i.e., both toric and color codes have the same error threshold to bit-flip errors. Exploring the regime where measurement errors are far more common than bit-flip errors is extremely difficult numerically because of the anisotropy of the resulting lattice gauge theory. However, for a large portion of the phase diagram in the $p$-$q$ plane both topological schemes show different error tolerance. Gaining a complete understanding of the underlying cause for the differences between the two types of topological error-correction codes in the fault-tolerant regime will require new analysis approaches by improving the error model with more realistic features like taking into account the unavoidable noise introduced by real physical operations during the correction protocol. Whether the differences between both codes vanish under external noise remains an open problem, and this may also require more detailed studies of lattice gauge theories with quenched bond disorder. This, however, is a numerically and analytically extremely challenging problem.

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