Basis for paraxial surface-plasmon-polariton packets

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We present a theoretical framework for the study of surface-plasmon polariton (SPP) packets propagating along a lossy metal-dielectric interface within the paraxial approximation. Using a rigorous formulation based on the plane-wave spectrum formalism, we introduce a set of modes that constitute a complete basis set for the solutions of Maxwell’s equations for a metal-dielectric interface in the paraxial approximation. The use of this set of modes allows us to fully analyze the evolution of the transversal structure of SPP packets beyond the single plane-wave approximation. As a paradigmatic example, we analyze the case of a Gaussian SPP mode, for which, exploiting the analogy with paraxial optical beams, we introduce a set of parameters that characterize its propagation.

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I. INTRODUCTION

Surface-plasmon polaritons (SPPs) emerge from the mixing of electromagnetic fields and collective oscillations of conduction electrons near metal surfaces [1,2]. These excitations are capable of propagating along hundreds of microns at visible and near-infrared frequencies being, at the same time, confined in small transversal regions near the metallic surface [3–6]. Thanks to these properties, SPPs have become a versatile tool to guide and manipulate electromagnetic fields at length scales well below the diffraction limit of light [7–9], thus enabling applications as diverse as photonic interconnects [6,10–13] and ultrasensitive biosensors [14–16], to cite a few.

Although the fundamental properties of SPPs have been known for decades, there remain certain issues that can benefit from further analysis. Frequently, SPPs propagating at metal-dielectric interfaces are modeled using a single plane wave [7,8,10]. Even if this level of description may be sufficient in many situations, it ignores any possible transversal structure, thus missing, for instance, interesting cases such as nondiffractive solutions [17–20]. Therefore, a complete characterization of the evolution and the confinement properties of SPPs requires a description beyond the single plane-wave description.

II. GENERAL FORMULATION

Our analysis starts by considering a flat interface perpendicular to the z axis, which separates two homogeneous nonmagnetic media as shown in Fig. 1. The region below the interface (i.e., z < 0) corresponds to a metallic medium characterized by a complex dielectric function $\varepsilon_c$ with Re$\varepsilon_c < 0$ and Im$\varepsilon_c > 0$. On the other hand, the region z > 0 is filled with a dielectric medium whose dielectric function $\varepsilon_d$ is real and positive and satisfies $\varepsilon_d + \text{Re}\varepsilon_c < 0$. An electromagnetic wave of frequency $\omega$ propagates along the interface with an electric field in the dielectric medium given by

$$E_d(x,y,z) = E_{d0}(x,y) e^{i k_{sp} z}.$$ 

Here, $k_{sp}^2 = k_{sp}^2 (\varepsilon_d / \varepsilon_c)$, and $k_{sp}^2 = (\omega/c)^2 \varepsilon_c \varepsilon_d / (\varepsilon_d + \varepsilon_c)$ is the SPP wave vector, which is a complex quantity for lossy metals. Interestingly, $k_{sp}$ also varies in the absence of losses, which ensures that the SPP is confined in the direction perpendicular to the interface. Since $E_d$ satisfies Maxwell’s equations and the corresponding boundary conditions, $E_{d0}$ must verify the two-dimensional Helmholtz equation,

$$\frac{\partial^2 E_{d0}}{\partial x^2} + \frac{\partial^2 E_{d0}}{\partial y^2} + k_{sp}^2 E_{d0} = 0,$$ 

along with the transversality condition,

$$\left( \frac{\partial}{\partial x} i k_{sp}, \frac{\partial}{\partial y} i k_{sp} \right) \cdot E_{d0} = 0.$$ 

Using the angular plane-wave spectrum formalism of optical fields [29,30] as done in Ref. [23], it can be proven that, for $x > 0$, a solution of Eqs. (1) and (2) can be written as follows:

$$E_{d0}(x,y) = \int_{-\infty}^{\infty} du \, \tilde{E}(u) e^{ik_{sp}(u)x} e^{i\gamma u |y|},$$ 

where $\tilde{E}(u)$ is the angular plane-wave spectrum of the electric field $E_{d0}(x,y)$, $\gamma$ is the Gouy phase, and $\eta$ is the Rayleigh range, whose analysis serves to complete the characterization of the SPP packet evolution.
So that the components of the surface-plasmon field can be understood as a superposition of inhomogeneous two-dimensional fields with weights given by $\psi_{\text{sp}}(x,y)$, which decay at different rates along the $x$ axis. Furthermore from Eq. (5) we see that the problem of SPP propagation at a metal-dielectric interface is polarization dependent and, therefore, strictly vectorial. However, we can always choose one of the components of the field and use it as a scalar potential from which all the other field components can be deduced. Therefore, without loss of generality, the problem can effectively be formulated in terms of the component of the electric field normal to the interface $f(x,y)$, which obeys the scalar Helmholtz equation,

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + k_{\text{sp}}^2 f = 0.$$  

If we assume that the transverse profile of the SPP packet is slowly varying with respect to the propagation coordinate $x$, the function $F(u)$ has to be peaked around $u = 0$, and therefore we can approximate $k_s(u)$ in Eq. (6) to obtain a simpler expression (see Appendix A),

$$f(x,y) = e^{ik_{\text{sp}} f_0(x,y)}.$$  

where

$$f_0(x,y) = \int_{-\infty}^{\infty} du F(u) \exp \left( -i x \frac{u^2 k_{\text{sp}}^*}{2} \right) e^{i y k_{\text{sp}} u}.$$  

Here, the asterisk stands for complex conjugation. It is straightforward to show that $f_0$ satisfies the so-called paraxial equation,

$$\frac{\partial^2 f_0}{\partial y^2} + 2 i k_{\text{sp}} \frac{\partial f_0}{\partial x} = 0.$$  

It is important to notice that in this case $k_{\text{sp}}$ is a complex quantity which leads, as will be seen below, to a number of important differences with respect to the well-known paraxial beam propagation in dielectric media.

### III. A BASIS FOR PARAXIAL SPP PACKETS

Hermite-Gauss (HG) modes constitute a complete set of solutions of the paraxial wave equation for free space propagation [32]. The functions describing the transversal profile of HG modes are expressed, apart from normalization factors, as the product of a Hermite polynomial and a Gaussian function. Here, we exploit the analogy between the paraxial equation of optical beams and Eq. (8) to construct a basis set that is well suited to analyze the propagation for $x \geq 0$ of a SPP packet defined at $x = 0$. Let us introduce the following set of functions:

$$u_n(x,y) = A_n \sqrt{\frac{q_0}{q_{\text{sp}}(x)}} \frac{q_0 - x}{q_0 + x}^{n/2} H_n \left( \frac{y}{w_0^2(x)} \right) \exp \left( i y^2 k_{\text{sp}} \right).$$  

where $A_n$ is a normalization constant, $q_{\text{sp}}(x) = q_0 + x$, $p^2(x) = i(q_0^2 - x^2)/(k_{\text{sp}} q_0)$, and $q_0 = -i k_{\text{sp}} w_0^2$. Here, $w_0^2$ is a real quantity that represents the Gaussian SPP width at $x = 0$. When $k_{\text{sp}}$ becomes real this set of functions reduces to the usual Hermite-Gauss modes. Furthermore, it can be proven that the functions $u_n(x,y)$ satisfy Eq. (8) and constitute, at $x = 0$, a complete set of orthonormal functions characterized by a single complex parameter $q_0$. Therefore, they can be used as a basis set to expand any arbitrary $z$ component of the field of a SPP packet. In fact, given a certain $f(x,y)$ at $x = 0$, we have $f(0,y) = f_0(0,y)$, and since $u_n(0,y)$ is a complete basis, we can write

$$f_0(0,y) = \sum_{n=0}^{\infty} f_{0n} u_n(0,y),$$  

where

$$f_{0n} = \int_{-\infty}^{\infty} dy f_0(0,y) u_n^*(0,y).$$  

Then, using Eq. (7), we have that for any $x \geq 0$,

$$f(x,y) = e^{i k_{\text{sp}} x} \sum_{n=0}^{\infty} f_{0n} u_n(x,y).$$  

This allows us to write the complete expression for the propagation of the SPP packet as

$$E_{\text{sp}}(x,y) = e^{i k_{\text{sp}} x} \sum_{n=0}^{\infty} f_{0n} \left( -i \frac{\partial u_n}{\partial x} + k_{\text{sp}} u_n, -i \frac{\partial u_n}{\partial y}, -\frac{\epsilon_e k_{\text{sp}}}{\epsilon_d} u_n \right).$$  

Here, the asterisk stands for complex conjugation. It is straightforward to show that $f_0$ satisfies the so-called paraxial equation,

$$\frac{\partial^2 f_0}{\partial y^2} + 2 i k_{\text{sp}} \frac{\partial f_0}{\partial x} = 0.$$  

It is important to notice that in this case $k_{\text{sp}}$ is a complex quantity which leads, as will be seen below, to a number of important differences with respect to the well-known paraxial beam propagation in dielectric media.
The coefficients $f_{on}$ depend on the arbitrary choice of $q_0$ at $x = 0$. Therefore, in general, there is not a unique way of choosing the waist size for the basis set to expand a given SPP packet, similar to what happens in the case of free space propagating paraxial beams. We may attempt to choose this parameter in such a way that it results in the expansion that best fits the actual SPP field with the smaller number of terms. Figure 2 shows the square modulus of $u_0(x,y)$ (left column), $u_2(x,y)$ (center column), and $u_4(x,y)$ (right column) plotted for different values of $w_0$: $2\lambda$, $5\lambda$, and $8\lambda$, with $\lambda$ being the wavelength in vacuum, which we take equal to 633 nm. We consider the case of a planar interface separating two half spaces made of silver and vacuum (i.e., $\varepsilon_d = 1$). The dielectric function of the metal is obtained from the data tabulated in Ref. [31]. As expected, larger values of $n$ result in a more complex structure. Furthermore, the intensity profile spreads faster for smaller values of $w_0$, which is a consequence of the well-known trade-off between beam size and beam divergence for the paraxial propagation of beams.

As a direct consequence of the losses in the metal, $\text{Im}[k_{sp}] \neq 0$. This has two important implications: (i) It produces a global exponential decay for $u_0(x,y)$ along the $x$ axis, characterized by a decay length $d_z = 1/(2\text{Im}[k_{sp}])$ [cf. Eqs. (5) and (7)], (ii) it makes the transversal intensity $I(x)$, defined as $I(x) = \int_{-\infty}^{\infty} |f_0(x,y)|^2 \, dy$, dependent on $x$. Indeed, it can be proven that, within the paraxial approximation, the variation of $I(x)$ is given by (see Appendix B)

$$dI(x) \frac{dx}{dx} = - \frac{\text{Im}[k_{sp}]}{|k_{sp}|^2} \int_{-\infty}^{\infty} \left| \frac{\partial f_0(x,y)}{\partial y} \right|^2 \, dy,$$

and therefore, for finite $\text{Im}[k_{sp}]$, $I(x)$ decreases with $x$. This decay, which does not exist in the single-plane-wave approximation, enhances the one caused by $e^{-x^2/(2d_z)}$. Furthermore, as a consequence of the introduction of a complex argument in the Hermite polynomials, the propagation features of SPP packets differ substantially from those of the standard paraxial optical beams. In fact, the transversal intensity profile of the standard Hermite modes changes only by a scaling factor on propagation, whereas the transversal profile of modes given in Eq. (9) experience rather complicated changes.

**IV. GAUSSIAN SPP MODE**

In order to get a deeper insight on the physical properties of the introduced basis we can consider the lowest-order mode from Eq. (9) (i.e., $n = 0$), which corresponds to a Gaussian SPP mode,

$$u_0(x,y) = A_0 \sqrt{\frac{q_0}{q_{sp}(x)}} \exp\left(\frac{i y^2 k_{sp}}{2q_{sp}(x)}\right).$$

By analogy with the well-known $q$ parameter of paraxial beams [30,32], we can introduce the real parameters,

$$R_{sp}(x) = \frac{|2x - ik_{sp} u_0^2|}{4x},$$

and

$$w_{sp}^2(x) = \frac{|2x - ik_{sp} u_0^2|}{2x \text{Im}[k_{sp}'] + |k_{sp}'|^2 u_0^2}.$$

According to these definitions the parameter $q_{sp}(x)$ can be written as

$$\frac{1}{q_{sp}(x)} = \frac{\text{Re}[k_{sp}]}{k_{sp}R_{sp}(x)} + \frac{2i}{k_{sp}' w_{sp}^2(x)}.$$

Notice that for a real $k_{sp}$ this expression reduces to the usual $q$ parameter of paraxial beams. Taking into account the previous definitions, the lowest-order mode can be expressed as

$$u_0(x,y) = A_0 \left(\frac{|k_{sp}' u_0^2}{2x - ik_{sp} u_0^2} \exp\left[-i \phi_0(x)\right]\right)^{1/2} \times \exp\left(\frac{i y^2 \text{Re}[k_{sp}]}{2R_{sp}(x)}\right) \exp\left(-\frac{y^2}{w_{sp}^2(x)}\right),$$

where

$$\tan[\phi_0(x)] = \frac{2x \text{Re}[k_{sp}]}{2x \text{Im}[k_{sp}'] + |k_{sp}'|^2 u_0^2}.$$
Examining the three panels of Fig. 3, we observe that the intensity profile size increases faster for smaller values of \( \omega_0 \), which is consistent with the discussion of Fig. 2. Similarly, larger values of \( \varepsilon_d \) produce more confined SPP packets. On the other hand, the Gouy phase changes from 0 at \( x = 0 \) to \( \pi/2 \) as the SPP packet propagates. Interestingly, this process happens faster for smaller values of \( \omega_0 \) and \( \varepsilon_d \). For higher-order modes the expression of \( q_{sp} \) given in Eq. (12) remains useful, but for \( n \geq 2 \) there are significant differences with the Gaussian SPP mode. The complex argument in the Hermite polynomials gives an additional phase variation that modifies the spherical phase variation given by \( R_{sp}(x) \).

V. CONCLUDING REMARKS

To summarize, we have presented a theoretical framework to analyze the propagation of SPP packets along a lossy metal-dielectric interface within the paraxial approximation. Starting from the plane-wave spectrum, we have introduced a set of modes that is exceptionally well suited to describe the evolution of the transversal structure of SPP packets as they travel along the interface. Furthermore, by exploiting the analogy with the paraxial optical beams, we have adapted several parameters frequently used in that field to characterize SPP packets, namely, the \( q_{sp} \) parameter, the radius of curvature \( R_{sp} \), the transversal size \( w_{sp}^2 \), the Gouy phase \( \phi_0 \), and the Rayleigh range \( x_0 \). Incidentally, due to the complex character of \( k_{sp} \), the evolution of these parameters with the propagation differs strongly from the usual case of paraxial optical beams. The work presented here brings a different point of view to the characterization of the properties of SPP packets beyond the single plane-wave approximation and, therefore, is expected to trigger the development of new applications exploiting the exceptional properties of these excitations.

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APPENDIX A: \( k_s(u) \) IN THE PARAXIAL APPROXIMATION

In the limit of \( u \ll 1 \), the real and imaginary parts of \( k_s(u) \) given in Eq. (4) can be approximated by the following Taylor expansions:

\[
\text{Re}[k_s(u)] \approx \text{Re}[k_{sp}] \left[ 1 - \frac{u^2}{2} \right],
\]

\[
\text{Im}[k_s(u)] \approx \text{Im}[k_{sp}] \left[ 1 + \frac{u^2}{2} \right],
\]

which combined produce \( k_s(u) \approx k_{sp} - k_{sp}^* \frac{u^2}{\pi} \).
APPENDIX B: DERIVATION OF EQ. (11)

Starting from Eq. (8) and multiplying by $f_0^* k_{sp}^*$, we obtain

$$f_0^* k_{sp}^* \frac{\partial^2 f_0}{\partial y^2} + 2i |k_{sp}| f_0^* \frac{\partial f_0}{\partial x} = 0, \quad (B1)$$

whose complex conjugate is

$$f_0 k_{sp} \frac{\partial^2 f_0^*}{\partial y^2} - 2i |k_{sp}| f_0 \frac{\partial f_0^*}{\partial x} = 0. \quad (B2)$$

Now subtracting Eq. (B2) from Eq. (B1), we have

$$2i |k_{sp}|^2 \frac{\partial |f_0|^2}{\partial x} + f_0^* k_{sp} \frac{\partial^2 f_0}{\partial y^2} - f_0 k_{sp} \frac{\partial^2 f_0^*}{\partial y^2} = 0.$$  

Finally, integrating over $y$ taking into account that $f_0 \to 0$ for $|y| \to \infty$, we obtain

$$2i |k_{sp}|^2 \frac{d}{dx} \int |f_0|^2 dy + (k_{sp} - k_{sp}^*) \int \left| \frac{\partial f_0}{\partial y} \right|^2 dy = 0.$$  