# PROJECTIVE EVOLUTION OF PLANE CURVES

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## 1. Introduction and previous results

The use of partial differential equations in the theory of non-linear geometric scale-spaces has become an important subject in low-level computer vision (see [1], [6], [19], [20], [21], [22]). These methods build the scale-space by means of evolution equations which are invariant under a group of transformations. The first example of such an equation is the Euclidean heat flow. Let us consider a smooth family of closed curves $\hat{\alpha} : [a, b] \times [0, \beta) \to \mathbf{R}^2$. We say that $\hat{\alpha}$ is the Euclidean evolution of the closed curve $\alpha : [a, b] \to \mathbf{R}^2$ if it satisfies

$$\begin{aligned} \frac{\partial \hat{\alpha}}{\partial u}(t, u) &= \kappa \mathbf{n} \\ \hat{\alpha}(t, 0) &= \alpha(t) \end{aligned} \tag{1}$$

Here $t$ parametrizes each curve, $u$ parametrizes the family and $\kappa$ and $\mathbf{n}$ stand for the Euclidean curvature and the Euclidean normal of the curves, respectively. Since $\frac{\partial^2 \hat{\alpha}}{\partial s^2} = \kappa \mathbf{n}$, where $s$ stands for the Euclidean arc length, we see that this is a heat-like equation. There is a large research literature devoted to this equation. Let us mention just a few remarkable properties. The equation is invariant under the Euclidean group. Maybe the more notable property is that any simple closed curve first become convex and then converges to a round point (see [12] and [13]). The equation can also be regarded as the one that shrinks the Euclidean perimeter of the curve as fast as possible (see [13]).

However, the affine and projective groups appears naturally in computer vision, and one might desire an analogous equation invariant under these groups. In the case of the unimodular affine group the answer is very pleasant. If one modifies (1) as follows:

$$\begin{aligned} \frac{\partial \hat{\alpha}}{\partial u}(t, u) &= \kappa^{1/3} \mathbf{n} \\ \hat{\alpha}(t, 0) &= \alpha(t) \end{aligned} \tag{2}$$

then one obtains an equation invariant under the unimodular affine group. Several researches discovered this equation using different approaches (see [1] and [19], [21]). Let us sketch briefly how this equation appears. Given a Lie group of transformations $G$ acting on $\mathbf{R}^2$ there exists a lower order invariant one form $d\sigma$ which is $G$-invariant (this result is true more generally, see [14]). We can then consider the group arc-length $\sigma(t) = \int_{t_0}^{t} d\sigma$. If $\sigma = \sigma(t)$ is actually a reparametrization of the curve we can consider the heat-like equation

$$\begin{aligned} \frac{\partial \hat{\alpha}}{\partial u}(t, u) &= \frac{\partial^2 \hat{\alpha}}{\partial \sigma^2} \\ \hat{\alpha}(t, 0) &= \alpha(t). \end{aligned} \tag{3}$$

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In the case that $G$ is the unimodular affine group one can check (see [20], [21]) that the right hand side of equation (3) can be written as

$$\frac{\partial^2 \hat{\alpha}}{\partial \sigma^2} = \text{some multiple of } \mathbf{t} + \kappa^{1/3}\mathbf{n}, \tag{4}$$

where $\{\mathbf{t}, \mathbf{n}\}$ stands for the usual Euclidean Frenet frame of the curve. But an important fact here is that if we add to the left hand side of an evolution equation any multiple of $\mathbf{t}$ then the solution is the same up to reparametrizations (see [7]), which are immaterial since we are only interested in the traces of the curves. Therefore equation (3) is equivalent to (2) when $G$ is the unimodular affine group.

This fact is of importance for two reasons. First, to compute $\frac{\partial^2 \hat{\alpha}}{\partial \sigma^2}$ we need the knowledge of three derivatives of the curve, nevertheless to set equation (2) we need only two derivatives. The second reason is that equation (3) is singular. This is because the affine arc-length might have singularities. But very fortunately, these singularities appear in the non-written coefficient of $\mathbf{t}$ in formula (4). The singularities arise from the possible inflection points of the curve, so in principle it might appear to be necessary to restrict ourselves to strictly convex curves, but this is not the case if one uses equation (3).

There is a nice theory on the affine evolution of curves, analogous to the Euclidean evolution. We will just mention that when curves evolve according to the affine heat flow, they shrink to a point, as in the Euclidean case (see [2]).

This paper is concerned with possibility of a definition of an analogous projective evolution, i.e, invariant under the full projective group. The classical theory of projective differential geometry of curves dates from the beginning of the twentieth century (see [23]). Later, E. Cartan restated this theory using his powerful method of the moving frame (see [5]), which has been proved to be useful in computer vision (see [8], [15]).

The first problem one finds trying to extent the previous ideas to the projective case is that, although there exists a projective arc length $\sigma = \sigma(t)$, equation (4) makes no sense in the projective plane. The reason is simple: Given two points $\mathbf{p}$ and $\mathbf{q}$ there is a well defined difference vector $\overrightarrow{\mathbf{pq}}$ if we are in the Euclidean or affine settings. However in the projective setting this basic construction makes no sense. From this it follows that there is no canonical notion of parallel transport of tangent vectors, so it makes no sense the notion of acceleration of a curve. However, we will see finally that it is possible to give to this equation an adequate sense if we interpretate it properly.

One can avoid this problem substituting the non-existent acceleration of the right hand side of equation (3) with a vector field along to the curve and attached to it in an invariant way. From classical differential geometry we know that there exists a Frenet frame at regular points of a projective curve. Although this trihedron is formed by points of $\mathbf{R}^3$ and no by tangent vectors to $\mathbf{P}^2$, it is possible to built a frame of vectors $\{V_1, V_2\}$ of the tangent space to $\mathbf{P}^2$ from it, as we will explain in the paper. The first vector will be tangent to the curve, and the frame will be, by its very definition, invariant under projective transformations, and so will be the evolution equations defined by it.

With the aid of this frame we will find all the projective evolution operators. We will see that the lowest order evolution operator is only of order five, although the Frame itself is of order six. This is a similar phenomena to the previously observed in the affine case.

We will study the singularities of these evolution operators, and we will see that they are unavoidable. We will see how singularities arise from what are classically called sextactic points, which are the points of the curve in which the osculating conic has a higher order than expected, thus answering the question possed in [16]. In particular, it is not possible to have an evolution operator well defined for conics.

## 2. Differential projective geometry

The projective plane is an abstract differentiable manifold, in the sense that it most natural definition does not give it as imbedded in any Euclidean space:

**Definition 2.1.** The real projective plane is the quotient $\mathbf{P}^2 = (\mathbf{R}^3 - \{0\})/\mathbf{R}^\star$, so a point $\alpha_0 \in \mathbf{P}^2$ is an equivalence class $\alpha_0 = [A_0] = \{\lambda \cdot A_0 : \lambda \in \mathbf{R}^\star\}$ for some $A_0 \in \mathbf{R}^3 - \{0\}$.

It can be endowed with a structure of differentiable manifold using just three charts, as follows. Let $U_3 \subset \mathbf{P}^2$ be given by all the points $\alpha_0 = [A_0]$ such that the third coordinate $A_{03}$ of $A_0$ does not vanish. We define a bijection $\varphi_3 : U_3 \to \mathbf{R}^2$ by the rule $\varphi_3(A_0) = (A_{01}/A_{03}, A_{02}/A_{03})$. We define analogously the subsets $U_i$ and bijections $\varphi_i$ for $i = 1, 2$. These three charts cover the whole $\mathbf{P}^2$ and define a differentiable structure in the projective plane. So we can speak about differentiable projective curves. A curve $\alpha : I \to \mathbf{P}^2$, $I$ being an open interval of the real line $\mathbf{R}$ is said to be differentiable if the compositions $\varphi_i \circ \alpha$ are differentiable whenever they are well defined.

We now propose to use the following alternative model of the tangent space to $\mathbf{P}^2$ which is better suited for our purpouses that the usual one.

Let $\pi : \mathbf{R}^3 - \{0\} \to \mathbf{P}^2$ be the natural projection. We define a lift of a differentiable curve $\alpha : I \to \mathbf{P}^2$ as a differentiable mapping $A : I \to \mathbf{R}^3 - \{0\}$ such that $\alpha = \pi \circ A$. Since $I$ is contractible, any differentiable curve in $\mathbf{P}^2$ can be lifted to $\mathbf{R}^3 - \{0\}$.

Two lifts $A$ and $\bar{A}$ differ by a non-vanishing differentiable function $\lambda = \lambda(t)$ such that $\bar{A} = \lambda \cdot A$. Therefore, the tangent vectors $(A(t), A'(t)) \in T_{A(t)}\left(\mathbf{R}^3 - \{0\}\right)$ and $(\bar{A}(t), \bar{A}'(t)) = (\lambda(t)A(t), \lambda'(t)A(t) + \lambda(t)A'(t))$ project onto the same tangent vector in $T_{[A(t)]}\mathbf{P}^2$. This leads us to define the equivalence relation $\sim$ in $T(\mathbf{R}^3 - \{0\})$ : we say that $(A, v)$ and $(A', v') \in T(\mathbf{R}^3 - \{0\})$ are $\sim$-related if there exists real numbers $\lambda \neq 0$ and $\lambda'$ such that $(A', v') = (\lambda \cdot A, \lambda' \cdot A + \lambda \cdot v)$. Then we have the following Definition

**Definition 2.2.** We define the tangent space to the projective plane as

$$T\mathbf{P}^2 = T(\mathbf{R}^3 - \{0\})/\sim .$$

It is straightforward to check that this definition is equivalent to the usual one.

Now we introduce some of the elements we need of the projective differential geometry of curves. We follow [5]. Given the lift $A = A(t)$ of the projective curve $\alpha = \alpha(t)$, let us denote

$$\begin{aligned} p &= -\frac{\det(A''', A', A)}{\det(A'', A', A)} \\ q &= \frac{\det(A''', A'', A)}{\det(A'', A', A)} \\ r &= -\frac{\det(A''', A'', A')}{\det(A'', A', A)}. \end{aligned}$$

Here we are denoting by

$$\det(A'', A', A) = \begin{vmatrix} x''(t) & x'(t) & x(t) \\ y''(t) & y'(t) & y(t) \\ z''(t) & z'(t) & z(t) \end{vmatrix}$$

and so on, where $A(t) = (x(t), y(t), z(t))^T$. Then the curve $A$ satisfies the following tautological differential equation

$$A''' + pA'' + qA' + rA = 0. \tag{5}$$

Of course it is necessary to assume that $\det(A'', A', A) \neq 0$, which is equivalent to assume that the curve has no inflection points. The interest of this equation is that it is invariant under the action of homographies. In fact, if $h : \mathbf{P}^2 \to \mathbf{P}^2$ is a homography given by the matrix (defined up to a proportionality factor) $M = (m_{ij})_{i,j=1,2,3}$ then the curve $M \cdot A : I \to \mathbf{R}^3 - \{0\}$ is a lift of the projective curve $h \circ \alpha$, and it is immediate to check that it satisfies the same differential equation.

However, equation (5) is not geometric, in the sense that it is not invariant under reparametrizations of the curve nor under the change of the representative $A$ by some $\lambda \cdot A$ where $\lambda = \lambda(t)$ is a function.

The search for geometric invariants of the curve leads (see [5, p. 50]) to the consideration of the function $H = r - \dfrac{1}{3}pq + \dfrac{2}{27}p^3 + \dfrac{1}{3}pp' - \dfrac{1}{2}q'' + \dfrac{1}{6}p''$. The function $H$ is invariant under homographies, since $p$, $q$ and $r$ are invariants themselves. The interest of the function $H$ lies in that it is independent on the particular lift of the curve chosen and that it changes in a very simple way under reparametrizations of the curve: If $\tau = \tau(t)$ is a reparametrization then $H(\tau) = \left(\frac{dt}{d\tau}\right)^3 H(t)$. Therefore we have a well-defined differential 1-form $d\sigma = H(t)^{1/3}dt$ which turns out to be invariant under reparametrizations and also under the action of the projective group. We can now define the projective arc length as the function $\sigma = \sigma(t)$ given by $\sigma(t) = \int_{t_0}^{t} d\sigma$.

The projective arc length is defined up to an additive constant, and it is also invariant under the projective group and under reparametrizations of the curve.

Given a point of the curve, the osculating conic at the point is the conic characterized by having the highest contact with the curve at the point. Since a conic is determined by five points, generically we can only expect that the curve and the conic have a contact of order four. Points of the curve with higher order of contact are called sextactic points. An important property of the function $H$ is that a point $\alpha(t)$ is sextactic if and only if $H(t) = 0$ (see [5, p. 56]. As a corollary of this property, we have that the function $H$ vanishes along the curve if and only if the curve is a conic.

¿From now on, we must suppose that the curves we are considering have no sextactic points. With this regularity assumption, the projective arc length turns out to be a reparametrization of the curve. The reparametrized lift of the curve $A = A(\sigma)$ will satisfy its own tautological differential equation. It is possible to find a function $\lambda : I \to \mathbf{R}^\star$ such that the lift $\bar{A} = \lambda A$ has a third-order differential equation with no second-order term. In fact, we will find $\lambda$ as a function of $t$, in order to avoid to compute projective arc length:

**Proposition 2.1.** *The function $\lambda$ is given by $\lambda = C \cdot H^{1/3} \exp\left(\frac{1}{3}\int p(t)dt\right)$ where $C$ is a constant which can be fixed by the condition* $\det\left(\bar{A}, \bar{A}', \bar{A}''\right) = 1$.

We will postpone the proof of this proposition to an appendix.

Note that the function $\lambda$ depends on the lift of the projective curve. In fact, if $\hat{A}(t) = w(t)A(t)$ is another lift, then it is immediate to check that

$$\hat{p}(t) = -\frac{\det(\hat{A}''', \hat{A}', \hat{A})}{\det(\hat{A}'', \hat{A}', \hat{A})} = -\frac{3}{w(t)}\frac{dw}{dt} + p(t),$$

and so we have that $\hat{\lambda}(t) = w(t)^{-1}\lambda(t)$ and therefore $\lambda(t)A(t) = \hat{\lambda}(t)\hat{A}(t)$ is well defined.

¿From now on, we will denote $\bar{A} = \lambda A$ again by $A$. The tautological differential equation is now reduced to

$$\frac{d^3A}{d\sigma^3} + 2\,k(\sigma)\frac{dA}{d\sigma} + h(\sigma)A(\sigma) = 0.$$

¿From the relation $H(t)^{1/3} = d\sigma/dt$ we conclude that if $\sigma = t$ then we must have $H(\sigma) = 1$. If we write this relation using the definition of $H$ we obtain that $h = dk/d\sigma + 1$, and so, there is really only one independent function $k = k(\sigma)$. This function is called projective curvature. So the lift $A = A(\sigma)$ satisfies the following differential equation:

$$\frac{d^3A}{d\sigma^3} + 2k(\sigma)\frac{dA}{d\sigma} + (k'(\sigma) + 1)A(\sigma) = 0. \tag{6}$$

The Frenet frame of the curve is the moving frame of $\mathbf{R}^3$ denoted by $A, A^{(1)}, A^{(2)}$ defined by the following formulas

$$A^{(1)} = \frac{dA}{d\sigma}$$
$$A^{(2)} = \frac{d^2A}{d\sigma^2} + kA.$$

It gives a moving frame of $\mathbf{R}^3$. Note that thanks to Lemma 3.3 we have that $\det(A, A^{(1)}, A^{(2)}) = 1$. The following Frenet formulas are an immediate consequence of the definition of the Frenet frame and the third order differential equation (6):

$$\begin{cases} \dfrac{dA}{d\sigma} & = & & A^{(1)} & \\ \dfrac{dA^{(1)}}{d\sigma} & = & -kA & & +A^{(2)} \\ \dfrac{dA^{(2)}}{d\sigma} & = & -A & -kA^{(1)} & \end{cases}$$

Let us give now some formulas to explicitly compute the Frenet frame from an arbitrary representative of the projective curve. The projective curvature is given by

$$k = H^{-2/3}\left(-\frac{1}{2}p' - \frac{1}{6}p^2 + \frac{1}{2}q - \frac{1}{3}\frac{H''}{H} + \frac{7}{18}\frac{H'^2}{H^2}\right).$$

Now the Frenet frame is given by

$$A^{(1)}(t) = \frac{dA}{d\sigma} = \frac{dA}{dt}\frac{dt}{d\sigma} = H^{-1/3}\frac{dA}{dt},$$
$$A^{(2)}(t) = \frac{dA^{(1)}}{d\sigma} + kA = \frac{dA^{(1)}}{dt}\frac{dt}{d\sigma} + kA = H^{-1/3}\frac{dA^{(1)}}{dt} + kA.$$

It is worth to remark that, although the projective curvature depends on the seventh derivative of the curve, the Frenet frame depends only on the sixth derivative.

## 3. Evolution operators

First we introduce the notion of $r$-jet of a curve. Given two projective curves $\alpha$ and $\beta$ defined around some $t \in \mathbf{R}$, we say that they produce the same $r$-jet if $\alpha(t) = \beta(t)$ and for some coordinate chart $(U, \varphi)$ with $\alpha(t) \in U$ we have that

$$\frac{d^j(\varphi\circ\alpha)}{dt^j}(t) = \frac{d^j(\varphi\circ\beta)}{dt^j}(t), \text{for } j = 1, \ldots, r.$$

It is easy to see that this notion is independent of the coordinate chart used. Let us denote by $j_t^r\alpha$ the equivalence class given by all the curves defined around $t$ such that they have at $t$ the same $r$-jet as the curve $\alpha$. Now, we can define $J^r(\mathbf{R}, \mathbf{P}^2)$ as the set of all such $r$-jets. It is not hard to see that this space is in fact a differentiable manifold and that the mapping

$$j_t^r\alpha \mapsto \left(t, (\varphi\circ\alpha)(t), \frac{d(\varphi\circ\alpha)}{dt}(t), \dots, \frac{d^r(\varphi\circ\alpha)}{dt^r}(t)\right)$$

gives a coordinate chart on $J^r(\mathbf{R},\mathbf{P}^2)$. Note that since each $\varphi\circ\alpha$ has two coordinates, this makes $J^r(\mathbf{R},\mathbf{P}^2)$ a manifold of dimension $2r+3$.

We will denote by $J^r_{\mathrm{reg}}(\mathbf{R},\mathbf{P}^2)$ the open subset of $J^r(\mathbf{R},\mathbf{P}^2)$ given by those jets of curves with non-vanishing tangent vector.

We propose the following definition of evolution operator:

**Definition 3.1.** Let $U$ be an open subset of $J^r_{\mathrm{reg}}(\mathbf{R},\mathbf{P}^2)$. An evolution operator is a mapping $V$ which assigns to each $r$-jet of curve $j_t^r\alpha\in U$ a vector of the one-dimensional quotient space $V(j_t^r\alpha)\in T_{\alpha(t)}\mathbf{P}^2/\langle\alpha'(t)\rangle$, i.e, $V(j_t^r\alpha)$ is a tangent vector defined up to multiples of $\alpha'(t)$, and such that it is invariant under the natural action of homographies and under reparametrizations of the curve.

Given an evolution operator $V$, a representative $\hat V$ of $V$ is a mapping $\hat V: U\to T\mathbf{P}^2$ (or maybe defined in some smaller open subset $U'\subset U$) such that $\hat V(j_t^r\alpha)\in T_{\alpha(t)}\mathbf{P}^2$ is a representative of the equivalence class $V(j_t^r\alpha)\in T_{\alpha(t)}\mathbf{P}^2/\langle\alpha'(t)\rangle$. A simple way to obtain a representative is the following: Let us suppose that $\alpha(t)\subset U_3$, and let us identify $U_3$ with $\mathbf{R}^2$ as usual using the chart $\varphi_3: U_3\to\mathbf{R}^2$. We can consider the standard metric on $\mathbf{R}^2$ and the Euclidean Frenet frame $\{\mathbf{t},\mathbf{n}\}$, which depends only on $j_t^1\alpha$. Let $\hat V(j_t^r\alpha)=F(j_t^r\alpha)\mathbf{n}(j_t^1\alpha)$ be the only multiple of $\mathbf{n}(j_t^1\alpha)$ which is a representative of $V(j_t^r\alpha)$. Note that $F(j_t^r\alpha)\mathbf{n}(j_t^1\alpha)$ is not invariant itself: If $j_{\bar t}^r\bar\alpha$ is obtained reparametrizing $\alpha$ and moving it by an homography $h$ (such that $h(\alpha(t))\in U_3$) then we should have that

$$h_\star\left(F(j_t^r\alpha)\mathbf{n}(j_t^1\alpha)\right) - F(j_{\bar t}^r\bar\alpha)\mathbf{n}(j_{\bar t}^1\bar\alpha)\in\langle\bar\alpha'(\bar t)\rangle\,, \tag{7}$$

where in this formula we are denoting by $h_\star: T\mathbf{P}^2\to T\mathbf{P}^2$ the tangent mapping to $h$. Of course there are many other possible choices of such a representative $\hat V$.

Given a curve $\alpha: I\to\mathbf{P}^2$ we define the evolution equation of $\alpha$ associated to the evolution operator $V$ as the partial differential equation

$$\begin{aligned}\frac{\partial\hat\alpha}{\partial u}(t,u) &= \hat V(j_t^r\hat\alpha^u)\\ \hat\alpha(t,0) &= \alpha(t)\end{aligned} \tag{8}$$

where $\hat\alpha^u=\hat\alpha(\cdot,u)$.

The important point is that if we consider the same equation but with a different representative of $V$ then we will obtain the same solutions *up to a reparametrization.*

Let $r\ge 5$ and let $\mathfrak{O}^r\subset J^r_{\mathrm{reg}}(\mathbf{R},\mathbf{P}^2)$ be the open dense subset given by the conditions $\det(A'',A',A)\neq 0$ and $H\neq 0$. Note that although these functions are not well defined on $J^r_{\mathrm{reg}}(\mathbf{R},\mathbf{P}^2)$, the set $\mathfrak{O}^r$ is well defined. Geometrically, $\mathfrak{O}^r$ is the set of jets of curves at points which are not inflexion nor sextactic points. Since the projective Frenet frame depends on derivatives of the curve up to order six we have a well defined mapping $\mathfrak{O}^r\ni j_t^r\alpha\mapsto\left(A,A^{(1)},A^{(2)}\right)$. We can use this frame to define two evolution operators on $\mathfrak{O}^r$ for $r\ge 6$ given by $j_t^r\alpha\mapsto V_i(j_t^r\alpha)=[A,A^{(i)}]\in T_{\alpha(t)}\mathbf{P}^2,\quad i=1,2$. Of course the evolution given by $V_1$ is trivial, since it is tangent to the curve. But since $\{[A,A^{(1)}],[A,A^{(2)}]\}$ is a basis of $T_{\alpha(t)}\mathbf{P}^2$ we can write a representative of any evolution operator $V$ restricted to $\mathfrak{O}^r$ as $\hat V(j_t^r\alpha)=f(j_t^r\alpha)V_2(j_t^r\alpha)$.

**Proposition 3.1.** *Given an evolution operator $V: U\subset J^r_{reg}(\mathbf{R},\mathbf{P}^2)\to T\mathbf{P}^2$, let $f=f(j_t^r\alpha)$ be such that $\hat V(j_t^r\alpha)=f(j_t^r\alpha)V_2(j_t^r\alpha)$ is a representative of $V$ at $U\cap\mathfrak{O}^r$.*

*Then $f(j_t^r\alpha)$ is a differential invariant, i.e., it is invariant under homographies and reparametrizations.*

*Proof.* We have to check that $f(j_t^r\alpha)$ is invariant under homographies and reparametrizations. Let $h$ be an homography. By the naturality of the moving frame $\{V_1, V_2\}$ we have that $h_\star(V_i(j_t^r\alpha)) = V_i(j_t^r(h\circ\alpha))$ for $i = 1, 2$. Since we must have that

(9)
$$h_\star(f(j_t^r\alpha)V_2(j_t^r\alpha)) - f(j_t^r(h\circ\alpha))V_2(j_t^r(h\circ\alpha)) \in \langle (h\circ\alpha)'(t)\rangle = \langle V_1(j_t^r(h\circ\alpha))\rangle$$

we conclude that $h_\star(f(j_t^r\alpha) = f(j_t^r(h\circ\alpha))$ and thus $f(j_t^r\alpha)$ is invariant under homographies. The proof that it is invariant under reparametrizations is entirely analogous. □

It is known that any projective differential invariant must be a function of the projective curvature and its derivatives with respect the projective arc-length (see [14]), i.e,

$$f = f(k, dk/d\sigma, \dots, d^pk/d\sigma^p)$$

where $p = r - 7$ since $k$ itself is an invariant of order 7.

Note that since we are restricting ourselves to the open dense subset $\mathfrak{O}^r$ the projective curvature and the projective arc-length are well defined and without singularities. Therefore, given any function $f$ the corresponding evolution operator is well defined and without singularities.

We can now write the evolution operator $V_2$ in terms of the Euclidean Frenet Frame $\{\mathbf{t}, \mathbf{n}\}$. The vector $V_2$ can be written in terms of this basis as

$$V_2 = C_1\mathbf{t} + C_2\mathbf{n}.$$

Let us compute these coefficients. In order to do that, we must make explicit how the identification of $\mathbf{R}^2$ with the open subset $U_3 \subset \mathbf{P}^2$ leads to an identification between the corresponding tangent spaces. So let us consider a tangent vector $[A, v]$ given by the equivalence class of a vector $(A, v) \in T_A\left(\mathbf{R}^3 - \{0\}\right)$. Let us take a curve $A(t)$ such that $A(0) = A$ and $A'(0) = v$. We can consider the representative of the same projective curve given by $a(t) = A(t)/A_3(t)$, which is contained in the plane of $\mathbf{R}^3$ given by $\{(x_1, x_2, 1) : (x_1, x_2) \in \mathbf{R}^2\}$, which in turn is identified with $\mathbf{R}^2$. The vector $[A, v] \in T_{[A]}\mathbf{P}^2$ is identified with $a'(0) \in T_p\mathbf{R}^2$, $p = a(0)$ just by disregarding the third component $a'(0) = (a_1'(0), a_2'(0), 0)$. Explicitely, $[A, v]$ is identified with $(p, \bar{v})$ given by

$$\begin{aligned} p &= \frac{1}{A_3}A \\ \bar{v} &= \frac{1}{A_3}v - \frac{v_3}{(A_3)^2}A. \end{aligned}$$

Since $V_2$ is invariant under homographies (of which Euclidean transformations are particular instances) and it is independent of the parametrization of the curve, we conclude that the coefficients $C_1$ and $C_2$ must be Euclidean invariants. So in order to compute them we can use any Euclidean reference, in particular we can use the Euclidean reference with origin at $\alpha(s)$ and vectors $\{\mathbf{t}(s), \mathbf{n}(s)\}$, where we are denoting by $s = s(t)$ the Euclidean arc-length of the curve $\alpha$. This is particularly convenient, since in the associate coordinates $(x, y)$ the curve $\alpha(s) = (x(s), y(s))$ parametrized by arc length in such a way that $t$ corresponds with $s = 0$, will satisfy that $x(0) = 0$, $y(0) = 0$, $x'(0) = 1$, $y'(0) = 0$. From this and from the Euclidean Frenet formulas

$$\begin{aligned} x'' &= -\kappa y' \\ y'' &= \kappa x' \end{aligned}$$

and the derivatives of these formulas we conclude that any expression in $x$, $y$ and its derivatives can be written in terms of the Euclidean curvature and its derivatives. We obtain after some computations (made with the aid of MAPLE) that

$$\begin{aligned} C_1 &= 3\cdot 2^{2/3}\kappa\Big(-36\kappa^5\tfrac{d^2\kappa}{ds^2}+45\tfrac{d^2\kappa}{ds^2}^2\kappa^2+200\tfrac{d\kappa}{ds}^4-300\tfrac{d^2\kappa}{ds^2}\tfrac{d\kappa}{ds}^2\kappa \\ &\qquad +72\tfrac{d\kappa}{ds}\tfrac{d^3\kappa}{ds^3}\kappa^2-9\tfrac{d^4\kappa}{ds^4}\kappa^3+36\kappa^4\tfrac{d\kappa}{ds}^2\Big) \\ &\qquad \Big/\Big(-40\tfrac{d\kappa}{ds}^3-9\kappa^2\tfrac{d^3\kappa}{ds^3}+45\kappa\tfrac{d\kappa}{ds}\tfrac{d^2\kappa}{ds^2}-36\kappa^4\tfrac{d\kappa}{ds}\Big)^{5/3} \\ C_2 &= 9\cdot 2^{2/3}\kappa^3\Big/\Big(-40\tfrac{d\kappa}{ds}^3-9\kappa^2\tfrac{d^3\kappa}{ds^3}+45\kappa\tfrac{d\kappa}{ds}\tfrac{d^2\kappa}{ds^2}-36\kappa^4\tfrac{d\kappa}{ds}\Big)^{2/3}, \end{aligned} \tag{10}$$

ans in the same way we obtain that

$$H=\frac{1}{54\kappa^3}\left(-40\left(\frac{d\kappa}{ds}\right)^3-36\kappa^4\frac{d\kappa}{ds}+45\frac{d\kappa}{ds}\frac{d^2\kappa}{ds^2}-9\kappa^2\frac{d^3\kappa}{ds^3}\right) \tag{11}$$

where the Euclidean curvature is denoted by $\kappa$. Since $\kappa$ depends on $j^2_t\alpha$ we see that $C_1$ depends on $j^6_t\alpha$ and that $C_2$ depends on $j^5_t\alpha$. So a representative of the evolution $V_2$ is given by $C_2\mathbf{n}$, which is of order 5 and we see that it is not possible to obtain a lower order representative modifing it with any multiple of $\mathbf{t}$. Therefore any evolution operator can we written as $f(k, dk/d\sigma, \dots, d^pk/d\sigma^p)\, C_2\,\mathbf{n}$. Since $k$ is an invariant of order 7, the lowest order is obtained when $f$ is a constant function. We now include all these information in the next Theorem, together with the main result: It is not possible to define an evolution operators without singularities on the whole jet space $J^r_{\mathrm{reg}}(\mathbf{R},\mathbf{P}^2)$.

**Theorem 3.2.** *Any projective evolution operator $V : U \subset J^r_{reg}(\mathbf{R},\mathbf{P}^2) \to T\mathbf{P}^2$ restricted to the open dense subset $U\cap\mathfrak{O}^r$ admits a representative of the form*

$$\hat V = f(k, dk/d\sigma,\dots, d^pk/d\sigma^p)\frac{1}{H^{2/3}}\kappa\mathbf{n}, \tag{12}$$

*The lowest order evolution operator is obtained when $f$ is constant and is of order five. Moreover, there exists no evolution operator defined over the conics (and in particular over the whole $J^r_{reg}(\mathbf{R},\mathbf{P}^2)$) without singularities*

*Proof.* Since it is immediate that $\dfrac{1}{H^{2/3}}\kappa = C_2$, it only remains to prove the statement on the impossibility to define an evolution operator without singularities on the whole $J^r_{\mathrm{reg}}(\mathbf{R},\mathbf{P}^2)$. In fact, we will see that there exists no evolution operator defined over conics. This depends on the following technical Lemma:

**Lemma 3.3.** *For each $(\Lambda_0,\dots,\Lambda_p)\in\mathbf{R}^p$ there exists a curve of jets $\epsilon\mapsto j^r_0\alpha^\epsilon$ such that when $\epsilon\to 0$*

1. *The jets $j^r_t\alpha^\epsilon$ converge to $j^r_t\alpha^0$, where $\alpha^0$ is a circle.*
2. *The derivatives $\dfrac{d^jk^\epsilon}{d\sigma^j}(0)$ converge to $\Lambda_j$, $j=0,\dots,p$, where $k^\epsilon$ stands for the projective curvature of $\alpha^\epsilon$ and $\sigma$ for its projective arc length.*
3. *The values $C_2(j^r_0\alpha^\epsilon)$ goes to $\infty$.*

We will postpone the proof of this Lemma to the Appendix. ¿From it we see that the only way $fC_2\mathbf{n}$ can be non-singular at $j^r_0\alpha^0$ is that $f(\Lambda_0,\dots,\Lambda_p)=0$. Since $(\Lambda_0,\dots,\Lambda_p)\in\mathbf{R}^p$ is arbitrary we see that $f$ is identically 0 and therefore the evolution must be trivial.

Finally, since all (non-degenerate) conics are projectively equivalent, it follows that any evolution operator must fail to be well defined over them. ∎

*Remark* 3.1. It is possible to arrive to equation 12 as follows. Let $\bar{A}$ be any lift of the curve $\alpha$ and let $A = \lambda\bar{A}$ be its canonical representative. Then we have:

$$\begin{aligned}
V_2 &= [A, A^{(2)}] = [A, \frac{d^2A}{d\sigma^2} + kA] \\
&= [A, \frac{d^2A}{d\sigma^2}] \\
&= [\lambda\bar{A}, \frac{d^2(\lambda\bar{A})}{d\sigma^2}] = [\lambda\bar{A}, \lambda\frac{d^2\bar{A}}{d\sigma^2}] + [\lambda\bar{A}, 2\frac{d\lambda}{d\sigma}\frac{d\bar{A}}{d\sigma}] + [\lambda\bar{A}, 2\frac{d^2\lambda}{d\sigma^2}\bar{A}] \\
&= [\lambda\bar{A}, \lambda\frac{d^2\bar{A}}{d\sigma^2}] \mod \alpha'(t) = [\bar{A}, \frac{d^2\bar{A}}{d\sigma^2}] \\
&= [\bar{A}, \frac{d}{d\sigma}\left(\frac{d\bar{A}}{ds}\frac{ds}{d\sigma}\right)] = [\bar{A}, \frac{d^2\bar{A}}{ds^2}\left(\frac{ds}{d\sigma}\right)^2 + \frac{d\bar{A}}{ds}\frac{d^2s}{d\sigma^2}] \\
&= [\bar{A}, \frac{d^2\bar{A}}{ds^2}\left(\frac{ds}{d\sigma}\right)^2] \mod \alpha'(t) = [\bar{A}, \frac{1}{H^{2/3}}\frac{d^2\bar{A}}{ds^2}].
\end{aligned} \tag{13}$$

If the curve $\alpha$ lies in $U_3$ we can take the lift $\bar{A}$ of the form $\bar{A} = (\bar{A}_1, \bar{A}_2, 1)$ so $\varphi_3 \circ \bar{A} = (\bar{A}_1, \bar{A}_2)$ can be identified with $\bar{A}$ and $[\bar{A}, \frac{1}{H^{2/3}}\frac{d^2\bar{A}}{ds^2}]$ with $\frac{1}{H^{2/3}}\frac{d^2\bar{A}}{ds^2}$. Therefore, in terms of the affine coordinates given by $\varphi_3$ the evolution given by $V_2$ is equivalent to

$$\hat{V}_2 = \frac{1}{H^{2/3}}\frac{d^2\bar{A}}{ds^2} = \frac{1}{H^{2/3}}\kappa\mathbf{n}$$

and we recover in this way (12). We see that although (3) makes no sense in the projective case, the same kind of equation in an affine coordinate chart of the projective plane produces an evolution which gives traces of curves which are projectively invariant, although the curves themselves are not (cf. [16] and [17])

## 4. Appendix

*Proof of Proposition 2.1.* Let us define $\mu(t) = \lambda^{-1}(t)$. Then we can write $A(t) = \mu(t)\bar{A}(\sigma(t))$. So we have that

$$\begin{aligned}
0 &= \frac{d^3A}{dt^3} + p(t)\frac{d^2A}{dt^2} + q(t)\frac{dA}{dt} + r(t)A(t) \\
&= \frac{d^3}{dt^3}\left(\mu(t)\bar{A}(\sigma(t))\right) + p(t)\frac{d^2}{dt^2}\left(\mu(t)\bar{A}(\sigma(t))\right) + q(t)\frac{d}{dt}\left(\mu(t)\bar{A}(\sigma(t))\right) \\
&+ r(t)\mu(t)\bar{A}(\sigma(t)) \\
&= \mu(t)\frac{d^3\bar{A}}{d\sigma^3}(\sigma(t))\frac{d\sigma}{dt}^3 + \left(3\mu(t)\frac{d\sigma}{dt}\frac{d^2\sigma}{dt^2} + 3\frac{d\mu}{dt}\frac{d\sigma}{dt}^2 + p(t)\mu(t)\frac{d\sigma}{dt}^2\right)\frac{d^2\bar{A}}{d\sigma^2}(\sigma(t)) \\
&+ \left(\mu(t)\frac{d^3\sigma}{dt^3} + 2p(t)\frac{d\mu}{dt}\frac{d\sigma}{dt} + 3\frac{d^2\mu}{dt^2}\frac{d\sigma}{dt} + p(t)\mu(t)\frac{d^2\sigma}{dt^2} + q(t)\mu(t)\frac{d\sigma}{dt}\right. \\
&\left. +3\frac{d\mu}{dt}\frac{d^2\sigma}{dt^2}\right)\frac{d\bar{A}}{d\sigma}(\sigma(t)) + \left(\frac{d^3\mu}{dt^3} + r(t)\mu(t) + p(t)\frac{d^2\mu}{dt^2} + q(t)\frac{d\mu}{dt}\right)\bar{A}(\sigma(t)).
\end{aligned}$$

Therefore the tautological differential equation for $\bar{A}(\sigma)$ has no second-order term if and only if $\mu$ satisfies the following differential equation:

$$3\mu(t)\frac{d\sigma}{dt}\frac{d^2\sigma}{dt^2} + 3\frac{d\mu}{dt}\frac{d\sigma}{dt}^2 + p(t)\mu(t)\frac{d\sigma}{dt}^2 = 0$$

and so $\mu$ is given by

$$\mu(t) = c \cdot \exp\left(-\int\left(\frac{\frac{d^2\sigma}{dt^2}}{\frac{d\sigma}{dt}} + \frac{1}{3}p(t)\right)dt\right),$$

being $c \neq 0$ a constant. By the very definition of the projective arc we obtain

$$\frac{\frac{d^2\sigma}{dt^2}}{\frac{d\sigma}{dt}} = \frac{1}{3H}\frac{dH}{du}$$

and so we have that

$$\mu(t) = c \cdot H(t)^{-1/3} \exp\left(\frac{-1}{3}\int p(t)dt\right),$$

and therefore

$$\lambda(t) = C \cdot H(t)^{1/3} \exp\left(\frac{1}{3}\int p(t)dt\right).$$

□

*Proof of Lemma 3.3.* Without lost of generality, we will work in the chart $U_3 \simeq \mathbf{R}^2$. Given an evolution operator $f(k, dk/d\sigma, ..., d^pk/d\sigma^p)C_2\mathbf{n}$, for defining the desired parametrized family of jets $j_0^r\alpha^\varepsilon$, $r = p+7$, we are going to give the $(p+5)$-jet of the Euclidean curvature, that is, $j_0^{r-2}\kappa^\varepsilon \in J^{r-2}(\mathbf{R},\mathbf{R})$, provided the curves are parametrized with the Euclidean arc in $U_3$. The fundamental theorem of differential geometry of plane curves gives the equivalence, up to an isometry, between the $r$-jet of a curve and the $(r-2)$-jet of its Euclidean curvature.

One can check, again with the aid of MAPLE, that the expression of the projective curvature in terms of the Euclidean curvature is the following

$$k = \frac{A\left(\frac{d^5\kappa}{ds^5}\right) + P}{B} \tag{14}$$

where

$$A = -\frac{3}{2}2^{\frac{2}{3}}\left(720\kappa^4\left(\frac{d\kappa}{ds}\right)^3 + 162\kappa^6\frac{d^3\kappa}{ds^3} + 648\kappa^8\frac{d\kappa}{ds} - 810\kappa^5\frac{d\kappa}{ds}\frac{d^2\kappa}{ds^2}\right),$$

$$B = \left(9\kappa^2\frac{d^3\kappa}{ds^3} + 36\kappa^4\frac{d\kappa}{ds} + 40\left(\frac{d\kappa}{ds}\right) - 45\frac{d\kappa}{ds}\frac{d^2\kappa}{ds^2}\right)^{\frac{8}{3}},$$

and

$$
\begin{aligned}
P = -\frac{3}{2}2^{2/3}\Bigg( & -3024\kappa^{10}\ {\frac{d^2\kappa}{ds^2}}^2 + 405\kappa^8 {\frac{d^3\kappa}{ds^3}}^2 - 189\kappa^6 {\frac{d^4\kappa}{ds^4}}^2 \\
& - 3888\kappa^{12} {\frac{d\kappa}{ds}}^2 + 7560\kappa^7 {\frac{d^2\kappa}{ds^2}}^3 + 61440 {\frac{d\kappa}{ds}}^6 \kappa^4 - 6480\ {\frac{d\kappa}{ds}}^4 \kappa^8 \\
& - 4725\kappa^4 {\frac{d^2\kappa}{ds^2}}^4 - 7875 {\frac{d^2\kappa}{ds^2}}^3 \kappa^3 {\frac{d\kappa}{ds}}^2 - 97200 \frac{d^2\kappa}{ds^2}\kappa^5 {\frac{d\kappa}{ds}}^4 \\
& - 33600 \frac{d^2\kappa}{ds^2}\kappa {\frac{d\kappa}{ds}}^6 + 2925 {\frac{d^2\kappa}{ds^2}}^2 \kappa^6 {\frac{d\kappa}{ds}}^2 + 7560 \frac{d^2\kappa}{ds^2}\kappa^9 {\frac{d\kappa}{ds}}^2 - 756 {\frac{d\kappa}{ds}}^2 \kappa^4 {\frac{d^3\kappa}{ds^3}}^2 \\
& - 6048\kappa^7 {\frac{d\kappa}{ds}}^2 \frac{d^4\kappa}{ds^4} + 648\kappa^{10}\frac{d\kappa}{ds}\frac{d^3\kappa}{ds^3} + 1890\kappa^5\ {\frac{d^2\kappa}{ds^2}}^2 \frac{d^4\kappa}{ds^4} - 1512\kappa^8 \frac{d^2\kappa}{ds^2}\frac{d^4\kappa}{ds^4} \\
& - 2835 \frac{d^2\kappa}{ds^2}\kappa^5 {\frac{d^3\kappa}{ds^3}}^2 + 31500\ {\frac{d^2\kappa}{ds^2}}^2 \kappa^2 {\frac{d\kappa}{ds}}^4 + 6720 {\frac{d\kappa}{ds}}^5 \kappa^2 \frac{d^3\kappa}{ds^3} + 35280 {\frac{d\kappa}{ds}}^3 \kappa^6 \frac{d^3\kappa}{ds^3} \\
& - 12600\ \frac{d^2\kappa}{ds^2}\kappa^3 {\frac{d\kappa}{ds}}^3 \frac{d^3\kappa}{ds^3} - 3150\kappa^4 {\frac{d\kappa}{ds}}^2 \frac{d^4\kappa}{ds^4}\frac{d^2\kappa}{ds^2} + 1134\kappa^5 \frac{d\kappa}{ds}\frac{d^4\kappa}{ds^4}\frac{d^3\kappa}{ds^3} \\
& -2106 \frac{d^2\kappa}{ds^2}\kappa^7\ \frac{d\kappa}{ds}\frac{d^3\kappa}{ds^3} + 13230\ {\frac{d^2\kappa}{ds^2}}^2 \kappa^4 \frac{d^3\kappa}{ds^3}\frac{d\kappa}{ds} + 11200\ {\frac{d\kappa}{ds}}^8 \Bigg).
\end{aligned}
$$

Let us denote $T_0 = P/B$. The derivatives of $k$ with respect to Euclidean arc will be always of the form

$$\frac{d^n k}{ds^n} = \frac{A}{B}\frac{d^{n+5}\kappa}{ds^{n+5}} + T_n, \tag{15}$$

where $T_n$ is a rational function of $\kappa$ and its derivatives up to order $n+4$ recursively defined as

$$T_n = \frac{d}{ds}\left(\frac{A}{B}\right)\frac{d^{n+4}\kappa}{ds^{n+4}} + \frac{dT_{n-1}}{ds}. \tag{16}$$

We are going to define $d^i\kappa^\varepsilon/ds^i$, $i = 5, ..., p+5$, inductively. We set from the beginning

$$\kappa^\varepsilon(0) = 1, \frac{d\kappa^\varepsilon}{ds}(0) = \varepsilon, \frac{d^2\kappa^\varepsilon}{ds^2}(0) = 0, \frac{d^3\kappa^\varepsilon}{ds^3}(0) = 0, \frac{d^4\kappa^\varepsilon}{ds^4}(0) = 0.$$

With this choice, from (14), we have for $s = 0$

$$A_{s=0} = -\frac{3}{2}2^{\frac{2}{3}}\left(720\varepsilon^3 + 648\varepsilon\right),$$

$$B_{s=0} = \left(36\varepsilon + 40\varepsilon^2\right)^{\frac{8}{3}},$$

$$\left(\frac{A}{B}\right)_{s=0} = \frac{-\frac{3}{2}2^{\frac{2}{3}}(720\varepsilon^2 + 648)}{(40\varepsilon^2 + 36)^{\frac{8}{3}}}\varepsilon^{-\frac{5}{3}}.$$

It is straightforward to check (by means of the chain rule) the following relation between the derivatives of the projective curvature with respect to the projective arc and the derivatives with respect to the Euclidean arc

(17)

$$\frac{d^n k}{d\sigma^n} = \frac{d^n k}{ds^n}\xi^n + \sum_{i=1}^{n-1}\frac{d^i k}{ds^i}\left(\sum_{\alpha_1+2\alpha_2+\ldots+(n-1)\alpha_{n-1}=n-i} w^i_{\alpha_1,\ldots,\alpha_{n-1}}\left(\frac{1}{\xi}\frac{d\xi}{ds}\right)^{\alpha_1}\cdots\left(\frac{1}{\xi}\frac{d^{n-1}\xi}{ds^{n-1}}\right)^{\alpha_{n-1}}\right)\xi^n,$$

where

$$\xi = \frac{ds}{d\sigma} = \frac{1}{H^{\frac{1}{3}}}$$

and $w^i_{\alpha_1,\ldots,\alpha_{n-1}}$ are certain entire numbers, which are not going to be relevant in the following.

If we set $d^n k/d\sigma^n = \Lambda_n$, from (17) we have

(18)

$$\frac{d^n k}{ds^n} = H^{\frac{n}{3}}\Lambda_n - \sum_{i=1}^{n-1} \frac{d^i k}{ds^i}\left(\sum_{\alpha_1+2\alpha_2+\ldots+(n-1)\alpha_{n-1}=n-i} w^i_{\alpha_1,\ldots,\alpha_{n-1}}\left(\frac{1}{\xi}\frac{d\xi}{ds}\right)^{\alpha_1}\cdots\left(\frac{1}{\xi}\frac{d^{n-1}\xi}{ds^{n-1}}\right)^{\alpha_{n-1}}\right).$$

for the sake of simplicity we are going to denote $X_n$ the left hand side of the previous formula. From (15) we conclude that

$$\frac{d^{n+5}\kappa}{ds^{n+5}} = \frac{B}{A}(X_n - T_n). \tag{19}$$

For the given choice of $\kappa^\varepsilon(0)$ and its first four derivatives, $A$ and $B$ do not vanish. Therefore, we can inductively define $d^{n+5}\kappa^\varepsilon/ds^{n+5}$ as $B$, $A$, $X_n$ and $T_n$ only depend on the derivatives of $\kappa$ up to order $n+4$.

Next we prove that $d^i\kappa^\varepsilon/ds^i \to 0$, $\forall i$, for $\varepsilon \to 0$; that is, $j^r_0 s^\varepsilon$ converges to the $r$-jet of a circle. In fact, we are going to prove by induction that the derivatives of $\kappa$ are of the form

$$\frac{d^i\kappa^\varepsilon}{ds^i} = \varepsilon F_i(\varepsilon), \quad \forall i,$$

for certain functions $F_i$ which in turn are continuous for $\varepsilon = 0$. In general, we will say that a function $h^\varepsilon$ is of "order $\gamma$" if it can be written as $h^\varepsilon = \varepsilon^\gamma F(\varepsilon)$, for $F$ continuous. So we supose that $d^i\kappa^\varepsilon/ds^i$, $i = 1, \ldots, n+4$ are of order one and we are going to prove that $d^{n+5}\kappa^\varepsilon/ds^{n+5}$ is also of order one.

A basic and easy fact is the following: If we have a rational function $Q$ of the euclidean curvature $\kappa^\epsilon$ and its derivatives up to $n+3$ of order $\gamma$, then $dQ/ds$ is of the same order. Hence from the expression (11), we see that $d^i(H^{-\frac{1}{3}})/ds^i$, $i = 0, \ldots, n$, are of order $-1/3$, and then $(1/\xi)d^i\xi/ds^i$ in formula (17) are of order 0, $\xi$ being $H^{-\frac{1}{3}}$. Similarly, from formula (14), we see that $k$ is of order $-2/3$. Then its derivatives up to $n-1$ are of order $-2/3$ as well. Hence, $X_n$ in formula (18) is of order $-2/3$.

In formula (14), we see that $T_0$ is of order $-2/3$ and $A/B$ is of order $-5/3$. Hence, using the induction hyponthesis, one can check from (16) that $T_1$ is of order $-2/3$ as well as $T_2$,...,$T_n$. Taking into account that the order of $B/A$ is $5/3$, we have from formula (19) that the order of $d^{n+5}\kappa^\varepsilon/ds^{n+5}$ is 1, thus concluding the induction. □

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