POLYNOMIAL SEQUENTIAL CONTINUITY ON $C(K,E)$ SPACES

FERNANDO BOMBAL AND IGNACIO VILLANUEVA

Abstract. We show that, for bounded sequences in $C(K,E)$, the polynomial sequential convergence is not equivalent to the pointwise polynomial sequential convergence. We introduce several conditions on E under which different versions of the result are true when K is a scattered compact space. These conditions are related with some others appeared in the literature and they seem to be of independent interest.

Keywords and phrases: Polynomial sequential continuity, C(K,E) spaces.

1. Introduction.

If K is a Hausdorff compact space and E is a Banach space, the following characterization of weak sequential convergence on $C(K,E)$ is well known (see, f.i., [13, Theorem 9]):

A bounded sequence $(f_n) \subset C(K,E)$ converges weakly to $f \in C(K,E)$ if and only if, for every $t \in K$, the sequence $(f_n(t))$ converges weakly (in E) to $f(t)$. Similarly, a bounded sequence $(f_n) \subset C(K,E)$ is weakly Cauchy if and only if, for every $t \in K$, the sequence $(f_n(t))$ is weakly Cauchy.

In the light of this result, one could ask whether a similar statement would be true when we replace weak convergence in $C(K,E)$ and E by some kind of polynomial convergence. Using a space constructed in [10] as a counterexample to several polynomial conjectures, we show that no polynomial version of the previous result can be true in general, not even for finite K.

Then, we isolate necessary and sufficient conditions on E for several polynomial versions of the result to be true, when K is a scattered compact Hausdorff space. As a corollary we prove, for every such K, that the property *every m-linear form on E is weakly sequentially continuous*, passes to $C(K,E)$. Moreover, if for every such K, every m-homogeneous polynomial on $C(K,E)$ is weakly sequentially continuous, then $C(K,E)$ (and E) verify the above mentioned property about m-linear (not necessarily symmetric) forms (see Theorem 3.7 and Corollary 3.10).

1991 *Mathematics Subject Classification.* 46B25, 46G10.
Partially supported by MCYT grant BFM2001-1284.

We show certain relations between the conditions introduced and provide examples of Banach spaces verifying them, together with some related results.

2. Notation and Preliminaries.

The notation and terminology used in the paper will be the standard in Banach space theory, as for instance in [11]. However, before going any further, we shall recall some terminology: For $k \geq 1$, we shall denote by $\mathcal{L}^k(E;X)$ the space of all continuous k-linear operators from $E^k := E\times \overset{k}{\cdots} \times E$ into X and by $\mathcal{L}^k_s(E;X)$ the subspace of all the *symmetric* k-linear operators. When $X = \mathbb{K}$ or $k = 1$, we will omit them. There is a canonical isomorphism between $\mathcal{L}^k(E;X)$ and $\mathcal{L}(\widehat{\bigotimes}^k_\pi E;X)$, where $\widehat{\bigotimes}^k_\pi E$ denotes the k-fold projective tensor product of E. A map $P : E \to X$ is a k-homogeneous polynomial if it is the restriction to the diagonal of E^k of a continuous k-linear map (unique if we require it to be symmetric; in this case, we shall denote by $\hat{P}$ this map, and we shall call it the *symmetric generator* of P). Both are related by the *polarization formula* (see [18, Theorem 1.10]):

$$\hat{P}(x_1, \cdots, x_N) = \frac{1}{N!2^N} \sum_{\substack{\epsilon_i=\pm 1\\ 1\leq i\leq N}} \epsilon_1 \cdots \epsilon_N P\Big(\sum_{j=1}^N \epsilon_j x_j\Big).$$

$\mathcal{P}^k(E;X)$ will stand for the Banach space of all k-homogeneous polynomials from E into X, with the (usual) sup norm on the unit ball of E.

We shall denote by τ_N (resp., $\tau_{\leq N}$), $1 \leq N \leq \infty$, the weakest topology on E making all $P \in \mathcal{P}^N(E)$ (resp., $P \in \cup_{m=1}^N \mathcal{P}^m(E)$) continuous. It is worth noting that $(x_n) \subset E$ is τ_N-convergent to x (resp., τ_N-Cauchy) if and only if $(x_n\otimes \overset{N}{\cdots} \otimes x_n)$ converges weakly to $x\otimes \overset{N}{\cdots} \otimes x$ (resp., is weakly Cauchy) in $\widehat{\bigotimes}^N_\pi E$; A sequence is $\tau_{\leq N}$ convergent (resp., $\tau_{\leq N}$-Cauchy) if and only if it is τ_k convergent (resp., τ_k-Cauchy), for every $1 \leq k \leq N$.

In analogy to the Schur property, Farmer and Johnson ([15]) call a Banach space $\mathcal{P}^N$-*Schur* if the $\tau_{\leq N}$ convergent sequences are norm convergent (P^∞ Schur spaces where introduced in [9] under the name of Λ-spaces).

We shall use the convention $\overset{[i]}{\ldots}$ to mean that the i-th coordinate is not involved.

Let now Σ_i be σ-algebras (or simply algebras) of subsets of some non void sets Ω_i $(1 \leq i \leq k)$. A function $\gamma : \Sigma_1 \times \cdots \times \Sigma_k \longrightarrow E$ or $\gamma : \Sigma_1 \times \cdots \times \Sigma_k \longrightarrow [0, +\infty]$ is a (countably additive) *k-polymeasure* if it is separately (countably) additive. ([14, Definition 1]) . As in the case $k = 1$ we can define the *semivariation* of a polymeasure $\gamma : \Sigma_1 \times \cdots \times \Sigma_k \longrightarrow E$, as the set function

$$\|\gamma\| : \Sigma_1 \times \cdots \times \Sigma_k \longrightarrow [0, +\infty]$$

given by

$$\|\gamma\|(A_1,\dots,A_k) = \sup\left\{\left\|\sum_{j_1=1}^{n_1}\cdots\sum_{j_k=1}^{n_k} a_1^{j_1}\dots a_k^{j_k}\gamma(A_1^{j_1},\dots,A_k^{j_k})\right\|\right\}$$

where the supremun is taken over all the finite Σ_i-partitions $(A_i^{j_i})_{j_i=1}^{n_i}$ of A_i $(1\le i\le k)$, and all the collections $(a_i^{j_i})_{j_i=1}^{n_i}$ contained in the unit ball of $\mathbb{K}$.

If γ has finite semivariation, an elementary integral $\int(f_1,f_2,\dots,f_k)\,d\gamma$ can be defined, where f_i belongs to the space $B(\Sigma_i,E)$ of all bounded, Σ_i-measurable E-valued functions, by just taking the limit of the integrals of k-tuples of simple functions (with the obvious definition) uniformly converging to the f_i's (see [14]).

If K is a compact Hausdorff space, $C(K,E)$ stands for the Banach space of all continuous functions from K into E, with the sup norm. The basic tool we use in our proofs is the representation of multilinear forms on $C(K,E)$ as operator valued polymeasures:

Every $T\in\mathcal{L}^k(C(K,E))$ has a unique *representing polymeasure*

$$\Gamma : \mathrm{Bo}(K)\times\overset{k}{\cdots}\times\mathrm{Bo}(K)\longrightarrow\mathcal{L}^k(E)$$

(where $\mathrm{Bo}(K)$ denotes the Borel σ-algebra of K) with finite semivariation, in such a way that

$$T(f_1,\dots,f_k)=\int(f_1,\dots,f_k)\,d\Gamma \quad\text{for } f_i\in C(K,E), \tag{†}$$

and such that for every choice $A_1,\overset{[i]}{\dots},A_k\in\mathrm{Bo}(K)$ and $x_1,\overset{[i]}{\dots},x_k\in E$, the set function

$$\Gamma_{(A_1,\overset{[i]}{\dots},A_k)(x_1,\overset{[i]}{\dots},x_k)} : \mathrm{Bo}(K)\longrightarrow E^*$$

defined by

$$\Gamma_{(A_1,\overset{[i]}{\dots},A_k)(x_1,\overset{[i]}{\dots},x_k)}(A_i)(x_i) := \Gamma(A_1\dots,A_i,\dots,A_k)(x_1\dots,x_i,\dots,x_k)$$

is a regular, countably additive measure of bounded variation (hence, an element of $C(K,E)^*$, by the well known Dinculeanu-Singer representation theorem; see [12, Th. III.19.9]). As a consequence, T can be extended to a multilinear operator $\overline{T}\in\mathcal{L}^k(B(\mathrm{Bo}(K),E))$ defined by the formula (†) (for $f_i\in B(\mathrm{Bo}(K),E)$) and for every choice of $g_1,\overset{[i]}{\dots},g_k\in B(\mathrm{Bo}(K),E)$ there is a regular, E^*-valued countably additive measure of bounded variation $\Gamma_{(g_1,\overset{[i]}{\dots},g_k)}\in C(K,E)^*$ such that

$$\overline{T}_{(g_1,\overset{[i]}{\dots},g_k)}(g_i) := \overline{T}(g_1\dots,g_i,\dots g_k)=\int(g_1,\dots,g_k)\,d\Gamma=\int g_i\,d\Gamma_{(g_1,\overset{[i]}{\dots},g_k)}$$

for every $g_i\in B(\mathrm{Bo}(K),E)$. (*Cfr.*[20]).

3. The Results.

We shall start with a list of Banach space properties which we will later use. From now on, unless otherwise stated, we consider $N \in \mathbb{N} \cup \{\infty\}$, $M \in \mathbb{N}$

Definition 3.1.

- A Banach space E has *property* $P_{N,M}$ if, for all sequences (x_1^n), …, $(x_M^n) \subset E$ such that (x_i^n) is $\tau_{\leq N}$-convergent to x_i $(1 \leq i \leq M)$ and for every $T \in \mathcal{L}_s^M(E)$, $T(x_1^n, \dots, x_M^n)$ converges to $T(x_1, \dots, x_M)$.
- A Banach space E has *property* $SP_{N,M}$ (S for *strong*) if, for all sequences $(x_1^n), \dots, (x_M^n) \subset E$ such that (x_i^n) is $\tau_{\leq N}$-convergent to x_i $(1 \leq i \leq M)$ and for every $T \in \mathcal{L}^M(E)$, $T(x_1^n, \dots, x_M^n)$ converges to $T(x_1, \dots, x_M)$.
- A Banach space E has *property* $Q_{N,M}$ if, for all $\tau_{\leq N}$-Cauchy sequences $(x_1^n), \dots, (x_M^n) \subset E$ and for every $T \in \mathcal{L}_s^M(E)$, the sequence $T(x_1^n, \dots, x_M^n)$ converges.
- *property* $SQ_{N,M}$ can be defined analogously.

Remark 3.2.

1) When $N = 1$, properties $Q_{1,M}$ and $P_{1,M}$ are equivalent (See (7) below), and they have been widely studied. By the polarization identity, they are equivalent to the fact that every weakly Cauchy (or weakly convergent) sequence in E is τ_M-convergent. In [7], this property is called the *M-Sequential Continuity property.* In [10], a space with property $SP_{1,M}$ is called an $\mathcal{M}_M$-space, and a space with property $P_{1,M}$ is called a $\mathcal{P}_M$-space.

2) For a space E to have any of the properties $SP_{N,M}$ or $P_{N,M}$ it suffices that it verifies the condition of the definition when $x_1 = \cdots = x_M = 0$.

Let us prove this for the $SP_{N,M}$ property, the other case being similar. First we note that, if $(x^n) \subset E$ is $\tau_{\leq N}$-convergent to $x \in E$ then $x^n - x$ is $\tau_{\leq N}$-convergent to 0 ([15, Lemma 1.1]), and now we reason by induction on M. For $M = 1$, the result is clear. We suppose the result true for $M - 1$ and we consider M sequences $(x_i^n)_n \subset E$ such that $(x_i^n)_n$ is $\tau_{\leq N}$ convergent to $x_i \in E$ $(1 \leq i \leq M)$. Then the sequences $(x_i^n - x_i)$ are $\tau_{\leq N}$ convergent to 0, and therefore our hypothesis on E tells us that, for every $T \in \mathcal{L}^M(E)$,

$$T(x_1^n - x_1, \dots, x_M^n - x_M) \to 0. \tag{1}$$

Next we note that

$$T(x_1^n - x_1, \dots, x_M^n - x_M) = T(x_1^n, \dots, x_M^n) - T(x_1^n, \dots, x_{M-1}^n, x_M) -$$

$$- \sum_{k=1}^{M-1} T(x_1^n, \dots, x_{k-1}^n, x_k, x_{k+1}^n - x_{k+1}, \dots, x_M^n - x_M).$$

Taking limits on the above expression and applying the induction hypothesis and (1) we get that

$$0 = \lim_{n\to\infty} T(x_1^n,\dots,x_M^n) - T(x_1,\dots,x_M) + \sum_{k=1}^{M-1} T(x_1,\dots,x_{k-1},x_k,0,\dots,0)$$

and our result follows.

3) It is obvious that, for every $N \geq 1$ and $M \geq 2$, property $SP_{N,M}$ implies properties $SP_{N+1,M}$ and $SP_{N,M-1}$ Similarly, property $SQ_{N,M}$ implies properties $SQ_{N+1,M}$ and $SQ_{N,M-1}$

4) The same relations hold for properties $P_{N,M}$ and $Q_{N,M}$. It is clear that $P_{N,M}$ implies $P_{N+1,M}$ and $Q_{N,M}$ implies $Q_{N+1,M}$. To see that $P_{N,M}$ implies $P_{N,M-1}$, let $(x_1^n),\dots,(x_{M-1}^n) \subset E$ be $\tau_{\leq N}$ null sequences and let $x_M^n = x_M \neq 0$ be a constant sequence. Consider $T \in \mathcal{L}_s^{M-1}(E)$ and $x^* \in E^*$ such that $x^*(x_M) = 1$. Let $\tilde{T} \in \mathcal{L}_s^M(E)$ be given by

$$\tilde{T}(x_1,\dots,x_M) = \sum_{i=1}^{M} x^*(x_i) T(x_1,\overset{[i]}{\dots},x_M).$$

Then, $\tilde{T}(x_1^n,\dots,x_M^n)$ converges to 0. Since $x^*(x_i^n)$ converges to 0 for every $1 \leq i \leq M-1$, we get that $x^*(x_M)T(x_1^n,\dots,x_{M-1}^n)$ converges to 0, which proves what we wanted.

Let us now see that $Q_{N,M}$ implies $Q_{N,M-1}$. We will prove, by induction on k, that, for all $1 \leq k \leq M-1$, $Q_{N,M}$ implies $Q_{N,k}$. For $k=1$, it is trivial. Let us consider $k=2$: let $(x_1^n),(x_2^n) \subset E$ be $\tau_{\leq N}$ Cauchy sequences and let $x_3^n = \cdots = x_M^n = x \neq 0$ be constant sequences. Consider $T \in \mathcal{L}_s^2(E)$ and $x^* \in E^*$ such that $x^*(x) = 1$. Let $\tilde{T} \in \mathcal{L}_s^M(E)$ be the symmetric operator associated to the operator $T' \in \mathcal{L}^M(E)$ given by

$$T'(x_1,\dots,x_M) = x^*(x_3)\cdots x^*(x_M)T(x_1,x_2),$$

that is

$$\tilde{T}(x_1,\dots,x_M) = \frac{1}{M!}\sum_{\sigma\in\Xi_M} T'(x_{\sigma(1)},\dots,x_{\sigma(M)}),$$

where Ξ_M stands for all the permutations of the set $\{1,2,\dots,M\}$. Then,

$$\begin{aligned}\tilde{T}(x_1^n,x_2^n,x,\dots,x) &= 2\frac{(M-2)!}{M!}x^*(x)^{M-2}T(x_1^n,x_2^n)\\ &+\frac{1}{M!}\sum_{\sigma\in\Xi'} x^*(x_{\sigma(3)})\cdots x^*(x_{\sigma(M)})T(x_{\sigma(1)},x_{\sigma(2)}),\end{aligned}$$

where

$$\Xi' = \{\sigma \in \Xi_M : \sigma(1) \in \{3,\dots,M\} \text{ or } \sigma(2) \in \{3,\dots,M\}\},$$

and $x_{\sigma(j)} = x$ when $\sigma(j) \in \{3,\dots,M\}$ and $x_{\sigma(j)} = x_{\sigma(j)}^n$ otherwise. Every term in the sum on the right hand side is a product of $M-1$ linear forms acting on weak Cauchy sequences (since at least one of the entries in $T(x_{\sigma(1)},x_{\sigma(2)})$ is equal to x). Hence, using the hypothesis and the fact that

(x_i^n) is $\tau_{\leq N}$-Cauchy, we get that $x^*(x)^{M-2}T(x_1^n, x_2^n)$ is a Cauchy sequence, which proves the case $k = 2$. Using this, we can now prove the case $k = 3$ by a similar argument, and then we can continue to finish the proof.

5) For every N, M, property $SQ_{N,M}$ implies property $SP_{N,M}$: In fact, suppose E does not have property $SP_{N,M}$. Then, for $1 \leq i \leq M$ there exist sequences $(x_i^n) \subset E$ $\tau_{\leq N}$ converging to 0 and a multilinear form $T \in \mathcal{L}^M(E)$ such that $T(x_1^n, \dots, x_M^n)$ does not converge to 0. If the sequence $T(x_1^n, \dots, x_M^n)$ is not Cauchy, then E does not have property $SQ_{N,M}$ and we are finished. If it is a Cauchy sequence, then it converges to $\lambda \neq 0$. In this case the sequence $(-1)^n T(x_1^n, \dots, x_M^n) = T((-1)^n x_1^n, \dots, x_M^n)$ is not Cauchy, which proves that the space does not have property $SQ_{N,M}$, since $((-1)^n x_1^n)$ is $\tau_{\leq N}$ convergent to 0.

6) Similarly, property $Q_{N,M}$ implies property $P_{N,M}$.

7) By using similar reasoning to that of [5, Theorem 2.3 and Lemma 2.4] it is possible to prove that, for every $M \in \mathbb{N}$, property $SP_{1,M}$ implies property $SQ_{1,M}$ and property $P_{1,M}$ implies property $Q_{1,M}$. In general we have not been able to prove that property $SP_{N,M}$ implies property $SQ_{N,M}$. The main problem we face is that, if E has $SP_{N,M}$, $(x_n) \subset E$ is a $\tau_{\leq N}$ Cauchy sequence and $(p(n))_n$, $(q(n))_n$ are increasing sequences of indices, we do not know whether the sequence $(x_{p(n)} - x_{q(n)})_n \subset E$ is $\tau_{\leq N}$ convergent to 0, which seems to be a question of independent interest. If we knew this to be true, then we could again mimic the proof of [5, Theorem 2.3 and Lemma 2.4] to prove that $SP_{N,M}$ implies $SQ_{N,M}$.

8) Using the polarization formula it is not hard to see that, for any N, M, property $P_{N,M}$ is equivalent to the following: for all sequences $(x_1^n), \dots, (x_M^n) \subset E$ such that $(x_i^n) \stackrel{\tau_{\leq N}}{\to} x_i$ $(1 \leq i \leq M)$, the sequence $(x_1^n + \cdots + x_M^n)$ converges $\tau_{\leq M}$ to $x_1 + \cdots + x_M$. This property has been studied in [6]. Something similar can be said of properties $Q_{N,M}$.

9) It is clear that SP properties imply the respective P properties, and similarly with SQ and Q. The converse is an open question already asked in [10] for the case of properties $P_{1,M}$. In Corollary 3.9 we give a partial answer showing that this converse is true for *stable* Banach spaces.

Example 3.3. a) Every Banach space E with the Dunford-Pettis property (i.e., weakly compact operators on E transform weakly convergent sequences into norm convergent ones) has the $SP_{1,M}$ (equivalently, $SQ_{1,M}$, see below) property, and consequently the $SP_{N,M}$ and $SQ_{N,M}$ properties for every $N, M \in \mathbb{N}$. This is a very well known result which goes back to Pełczynski ([19, Proposition 5])

b) Tsirelson's original space T^* is a reflexive space with the $SP_{1,M}$ property for every M([1]). However, Tsirelson's dual space T does not have the $P_{1,2}$ property ([1]). In the same way, on the quasi-reflexive James space J there is a homogeneous polynomial of degree 2 which is not weakly sequentially continuous ([4]). Hence, J does not have property $P_{1,2}$

c) Every $\mathcal{P}^N$-Schur space has trivially properties $SP_{N,M}$, for every M. Examples of $\mathcal{P}^N$-Schur spaces are ℓ_p and L_p for $N \geq p$. In general, if E^* has type $q > 1$, then E is $\mathcal{P}^N$-Schur for $N > p$, where $\frac{1}{p} + \frac{1}{q} = 1$ ([15, th. 3.5]). Also James space J and Tsirelson's dual space T are $\mathcal{P}^N$-Schur for $N \geq 2$ by [6, Proposition 3.5], since their duals have property S_2 and do not contain copies of ℓ_1 ([16])

d) For $1 < p < \infty$, ℓ_p has the $SP_{1,M}$ for every $M < p$ (see, e.g., [2]) and, as mentioned before, ℓ_p is $\mathcal{P}^N$-Schur if $N \geq p$. Therefore, ℓ_p has properties $SP_{N,M}$ for every $N \in \mathbb{N} \cup \{\infty\}$ if $M < p$, and properties $SP_{N,M}$ for every $M \in \mathbb{N}$ if $N \geq p$. Since $SP_{1,M}$ implies $SQ_{1,M}$, we get that ℓ_p has properties $SQ_{N,M}$ for every $N \in \mathbb{N}\cup\{\infty\}$ if $M < p$. We claim that it has also properties $SQ_{N.M}$ for every $M \in \mathbb{N}$ if $N \geq p$: In fact, reasoning as in [3, Theorem 1.6] (note that ℓ_p^* is in W_p), we can prove that, if $(x_n), (y_n) \subset \ell_p$ are two sequences such that $P(x_n) - P(y_n)$ converges to 0 for every homogeneous polynomial of degree less than or equal to N, then $\|x_n - y_n\|$ converges to 0. It follows that, if $(x_n) \subset \ell_p$ is a $\tau_{\leq N}$-Cauchy sequence, and $p(n), q(n)$ are increasing sequences of indices, then $x_{p(n)} - x_{q(n)}$ is norm null. From this, the claim follows easily.

On the other hand, if $N < p$ and $M \geq p$, then ℓ_p does not have property $P_{N,M}$, since the canonical basis (e_n) is $\tau_{\leq N}$ null, but not $\tau_{\leq M}$ null.

e) For $N \in \mathbb{N}$, a space is said to have the $\mathcal{P}^{\leq N}$ Dunford-Pettis property ($\mathcal{P}^{\leq N}$ DP) if, for each $\tau_{\leq N}$ null sequence $(x_n) \subset X$ and every weakly null sequence $(P_n) \subset \mathcal{P}(^N X)$, we have that $P_n(x_n)$ converges to 0. This property was defined in [15] and further studied in [6] (see also [7] for a related notion). A Banach space is said to have the $\mathcal{P}$-Dunford-Pettis property ($\mathcal{P}$-DP) if, for each $m \in \mathbb{N}$, for every weakly null sequence $(P_n) \subset \mathcal{P}(^m X)$ and for every $\tau_{\leq\infty}$ null sequence $(x_n) \subset X$, we have that $P_n(x_n)$ converges to 0. This property was introduced in [6] where it is shown, among other things, that DP=$\mathcal{P}^{\leq 1}$-DP $\Rightarrow \mathcal{P}^{\leq 2}$-DP $\Rightarrow \cdots \Rightarrow \mathcal{P}^{\leq N}$-DP $\Rightarrow \cdots \Rightarrow \mathcal{P}$-DP.

Let us see that, if E has the $\mathcal{P}^{\leq N}$-DP, then E has property $SP_{N,M}$ for every M. Similarly, if E has the $\mathcal{P}$-DP, the E has property $SP_{\infty,M}$ for every $M \in \mathbb{N}$. We show this for the $\mathcal{P}^{\leq N}$-DP property, the other case being similar. We reason by induction on M. For $M = 1$, the statement is trivial. We suppose now the result true for $M-1$ and consider $(x_1^n), \ldots, (x_M^n) \subset E$ to be M $\tau_{\leq N}$ null sequences. Let $T \in \mathcal{L}^M(E)$. By the induction hypothesis, $(x_1^n \otimes \cdots \otimes x_{M-1}^n) \subset \widehat{\bigotimes}_\pi^{M-1} E$ converges weakly to 0, so $(T_{x_1^n \otimes \cdots \otimes x_{M-1}^n}) := (T(x_1^n, \ldots, x_{M-1}^n, \cdot)) \subset E^*$ is a weakly null sequence. Again we can apply [6, Theorem 2.3] to conclude that $T(x_1^n, \ldots, x_M^n) = T_{x_1^n \otimes \cdots \otimes x_{M-1}^n}(x_M^n)$ converges to 0.

So, for instance $E := \ell_3 \oplus c_0$ is $\mathcal{P}^{\leq 3}$-DP (but not $\mathcal{P}^N$-Schur for any N; see [15]). It follows that it has properties $SP_{N,M}$ for all $N \geq 3, M \in \mathbb{N}$. From (d), it also follows that E does not have properties $P_{2,M}$ for $M \geq 3$.

The converse to the previous relation is not true: Tsirelson' original space does not have the $\mathcal{P}$-DP property ([6]), yet, as stated before, it has properties $SP_{N,M}$ for all N, M.

f) In [10, Theorem 5.5], the authors exhibit a remarkable example of a space $d_*(\omega) \times d(\omega; 1)$, where $d(\omega; 1)$ is a certain Lorentz sequence space and $d_*(\omega)$ is its predual, with the following property: if (e_n) and (e_n^*) are the canonical bases of $d(\omega; 1)$ and $d_*(\omega)$, then $(e_n^*, 0)$ and $(0, e_n)$ are $\tau_{\leq\infty}$ null in $d_*(\omega) \times d(\omega; 1)$, but $T((e_n^*, 0), (0, e_n)) = \langle e_n, e_n^* \rangle = 1$, where $T \in \mathcal{L}_s^2(d_*(\omega) \times d(\omega; 1))$ is the bilinear form given by $T((x_*, x), (y_*, y)) = \frac{1}{2}(\langle x, y_* \rangle + \langle y, x_* \rangle)$. This proves that $d_*(\omega) \times d(\omega; 1)$ has none of the properties $P_{N,M}$, $P_{\infty,M}$, for $N, M \in \mathbb{N}$, $M \geq 2$. Considering the sequences $((e_n^*, 0))_n$ and $((0, (-1)^n e_n))_n$, we see that it does not have properties $Q_{N,M}$, $Q_{\infty,M}$, for $N, M \in \mathbb{N}$, $M \geq 2$.

As can be seen, it is easy to find examples of spaces without the $P_{N,M}$ properties when $M > N$, but we only know of one space (Example (f) above) which does not have some of the properties $P_{N,N}$, and it has none of them.

We relate now properties SP and SQ of the space E with the same properties in $C(K, X)$ when K is scattered.

Theorem 3.4. *Let E be a Banach space, and let $N, M \in \mathbb{N}$. Then the following assertions are equivalent:*

(a) *E has the $SQ_{N,M}$ (resp. the $SP_{N,M}$) property.*
(b) *For every scattered compact Hausdorff space K, $C(K, E)$ has the $SQ_{N,M}$ (resp. the $SP_{N,M}$) property.*
(c) *There is some scattered compact Hausdorff space K such that $C(K, E)$ has the $SQ_{N,M}$ (resp., $SP_{N,M}$) property.*

A similar assertion can be made about the $SQ_{\infty,M}$ and $SP_{\infty,M}$ properties.

Proof. (b) $\Rightarrow$ (c) is obvious and (c) $\Rightarrow$ (a) follows immediately from the fact that E is isomorphic to a complement subspace of $C(K, E)$. For the proof of (a) $\Rightarrow$ (b) we will apply induction on M. If $M = 1$ there is nothing to prove. Let us now suppose the result true for $M-1$ and let $(f_1^n), \ldots, (f_M^n) \subset C(K, E)$ be $\tau_{\leq N}$ Cauchy (resp. sequences such that (f_i^n) is $\tau_{\leq N}$ convergent to f_i, $1 \leq i \leq M$).

There exists C such that $\|f_i^n\| \leq C$ for every $n \in \mathbb{N}$, $1 \leq i \leq M$. Let $T \in \mathcal{L}^M(C(K, E))$ and let Γ be the polymeasure associated to T. Then

$$T(f^1, \ldots, f^M) = \int (f^1, \ldots, f^M) d\Gamma = \int f^j d\Gamma_{(f^1, \overset{[j]}{\ldots}, f^M)}$$

where $\Gamma_{(f^1, \overset{[j]}{\ldots}, f^M)}$ is the measure associated to the functional

$$\begin{array}{rcl} C(K, E) & \longrightarrow & \mathbb{K} \\ h & \mapsto & \overline{T}_j(h) = T(f^1, \ldots, f^{j-1}, h, f^{j+1}, \ldots, f^M) \end{array}$$

We note here that the same formula is valid for functions in $B(\mathrm{Bo}(K), E)$ (see Section 2). It follows from the induction hypothesis (recall that $SQ_{N,M}$

implies $SQ_{N,M-1}$) that, for any $1 \leq j \leq M$, the sequence $(f_1^n \otimes \overset{[j]}{\cdots} \otimes f_M^n)$ is weakly Cauchy in $\widehat{\bigotimes}_\pi^{M-1} C(K,E)$. Consider the mappings

$$\begin{array}{ccc} C(K,E)\times \overset{(M-1)}{\cdots} \times C(K,E) & \xrightarrow{T_j} & C(K,E)^* \\ (f^1, \overset{[j]}{\ldots}, f^M) & \mapsto & T_j(f^1, \overset{[j]}{\ldots}, f^M) \end{array}$$

where

$$T_j(f^1, \overset{[j]}{\ldots}, f^M)(h) = T(f^1, \ldots, f^{j-1}, h, f^{j+1}, \ldots, f^M).$$

All the T_j's are (M-1)-linear and continuous, and therefore their linearizations are continuous. It follows that the set $\{\Gamma_{(f^1, \overset{[j]}{\ldots}, f^M)} : n \in \mathbb{N}\} \subset C(K,E)^*$ is weakly conditionally compact (or Rosenthal). Therefore, the scalar measures $\{v(\Gamma_{(f^1, \overset{[j]}{\ldots}, f^M)}) : n \in \mathbb{N}\}$ (where v stands for the variation) have a control measure λ_j (see [8, Proposition 3.1]. Note that the authors use the name *conditionally weakly compact* to mean *relatively weakly compact*, but the proof of (ii) implies (i) works also for Rosenthal's sets. Note also that for measures with values in a dual space, their variation and semivariation coincides). Let us consider λ_1. Since K is scattered, λ_1 is concentrated on a countable set of atoms $K_1 = \{t_n^{(1)} : n \in \mathbb{N}\} \subset K$ (see, f.i. [17, Theorem 2.10]). Therefore, since $\bigcap_{r=1}^{\infty} \{t_n^{(1)} : n > r\} = \emptyset$, given $\epsilon > 0$ there exists $B_1 = \{t_n^{(1)} : n \leq r_1\}$ such that

$$\sup_{n \in \mathbb{N}} v(\Gamma_{(f_2^n, \ldots, f_M^n)})(K \setminus B_1) \leq \epsilon.$$

Therefore, if $n, m \in \mathbb{N}$,

$$|T(f_1^n, \ldots, f_M^n) - T(f_1^m, \ldots, f_M^m)| \leq$$

$$\leq \left| \int_{B_1} f_1^n d\Gamma_{(f_2^n, \ldots, f_M^n)} - \int_{B_1} f_1^m d\Gamma_{(f_2^m, \ldots, f_M^m)} \right| + 2\epsilon C.$$

For the case of the $SP_{N,M}$ property we note that

$$|T(f_1^n, \ldots, f_M^n) - T(f_1, \ldots, f_M)| \leq$$

$$\leq \left| \int_{B_1} f_1^n d\Gamma_{(f_2^n, \ldots, f_M^n)} - \int_{B_1} f_1 d\Gamma_{(f_2, \ldots, f_M)} \right| + 2\epsilon C.$$

But, in any case,

$$\int_{B_1} f_1^n d\Gamma_{(f_2^n, \ldots, f_M^n)} = \int (\chi_{B_1} f_1^n, f_2^n, \ldots, f_M^n) d\Gamma = \overline{T}(\chi_{B_1} f_1^n, f_2^n, \ldots, f_M^n) =$$

$$= \int f_2^n d\Gamma_{(\chi_{B_1} f_1^n, f_3^n, \ldots, f_M^n)}$$

with the notation of Section 2.

On the other hand, the mapping

$$\begin{array}{ccc} C(K,E)\times \overset{(M-1)}{\cdots} \times C(K,E) & \longrightarrow & \widehat{\bigotimes}_\pi^{M-1} B(\mathrm{Bo}(K),E) \\ (f^1,f^3,\dots,f^M) & \mapsto & (\chi_{B_1}f^1\otimes f^3\otimes\cdots\otimes f^M) \end{array}$$

is multilinear and continuous, so the set $\{\chi_{B_1}f_1^n\otimes f_3^n\otimes\cdots\otimes f_M^n : n\in\mathbb{N}\}\subset \widehat{\bigotimes}_\pi^{M-1} B(\mathrm{Bo}(K),E)$ is a Rosenthal set. Hence, the set $\{\Gamma_{(\chi_{B_1}f_1^n,f_3^n,\cdots,f_M^n)} : n\in\mathbb{N}\}\subset C(K,E)^*$ is Rosenthal. Therefore, there exists a control measure λ_2 for the set $\{v(\Gamma_{(\chi_{B_1}f_1^n,f_3^n,\cdots,f_M^n)}) : n\in\mathbb{N}\}$. Reasoning as before, there exists a finite set $B_2\subset K$ such that

$$\left|\int f_2^n d\Gamma_{(\chi_{B_1}f_1^n,f_3^n,\dots,f_M^n)} - \int f_2^m d\Gamma_{(\chi_{B_1}f_1^m,f_3^m,\dots,f_M^m)}\right| \leq$$

$$\leq 2\epsilon C + \left|\int_{B_2} f_2^n d\Gamma_{(\chi_{B_1}f_1^n,f_3^n,\dots,f_M^n)} - \int_{B_2} f_2^m d\Gamma_{(\chi_{B_1}f_1^m,f_3^m,\dots,f_M^m)}\right| =$$

$$= 2\epsilon C + |\overline{T}(\chi_{B_1}f_1^n,\chi_{B_2}f_2^n,f_3^n,\dots,f_M^n) - \overline{T}(\chi_{B_1}f_1^m,\chi_{B_2}f_2^m,f_3^m,\dots,f_M^m)|.$$

Again, in the case of the $P_{N,M}$ property we get that

$$\left|\int f_2^n d\Gamma_{(\chi_{B_1}f_1^n,f_3^n,\dots,f_M^n)} - \int f_2 d\Gamma_{(\chi_{B_1}f_1,f_3,\dots,f_M)}\right| \leq$$

$$\leq 2\epsilon C + \left|\int_{B_2} f_2^n d\Gamma_{(\chi_{B_1}f_1^n,f_3^n,\dots,f_M^n)} - \int_{B_2} f_2 d\Gamma_{(\chi_{B_1}f_1,f_3,\dots,f_M)}\right| =$$

$$= 2\epsilon C + |\overline{T}(\chi_{B_1}f_1^n,\chi_{B_2}f_2^n,f_3^n,\dots,f_M^n) - \overline{T}(\chi_{B_1}f_1,\chi_{B_2}f_2,f_3,\dots,f_M)|.$$

Repeating the process M times, we get finite sets $B_1,B_2,\dots,B_M\subset K$ such that

$$|T(f_1^n,\dots,f_M^n) - T(f_1^m,\dots,f_M^m)| \leq$$

$$\leq 2M\epsilon C + \left|\int_{B_1\times\cdots\times B_M}(f_1^n,\dots,f_M^n)d\Gamma - \int_{B_1\times\cdots\times B_M N}(f_1^m,\dots,f_M^m)d\Gamma\right| =$$

$$= 2M\epsilon C + \Bigg|\sum_{t_{i_j}^{(j)}\in B_j} \Gamma(\{t_{i_1}^{(1)}\},\dots,\{t_{i_M}^{(M)}\})(f_1^n(t_{i_1}^{(1)}),\dots,f_M^n(t_{i_M}^{(M)}))-$$

$$-\Gamma(\{t_{i_1}^{(1)}\},\dots,\{t_{i_M}^{(M)}\})(f_1^m(t_{i_1}^{(1)}),\dots,f_M^m(t_{i_M}^{(M)}))|$$

but, for every $t_{i_j}^{(j)}\in B_j$, $(1\leq j\leq M)$ the sequences $(f_1^n(t_{i_1}^{(1)}))_n$, $\dots$, $(f_M^n(t_{i_M}^{(M)}))_n\subset E$ are $\tau_{\leq N}$ Cauchy, so, our hypothesis on E implies the existence of an $n_0\in\mathbb{N}$ such that, for every $n,m\geq n_0$,

$$|T(f_1^n,\dots,f_M^n) - T(f_1^m,\dots,f_M^m)| \leq 2M\epsilon C + \epsilon.$$

In the case of the $SP_{N,M}$ property, we get that

$$|T(f_1^n,\dots,f_M^n) - T(f_1,\dots,f_M)| \leq$$

$$\leq 2M\epsilon C + \left| \sum_{t_{i_j}^{(j)} \in B_j} \Gamma(\{t_{i_1}^{(1)}\}, \ldots, \{t_{i_M}^{(M)}\})(f_1^n(t_{i_1}^{(1)}), \ldots, f_M^n(t_{i_M}^{(M)})) - \right.$$

$$\left. -\Gamma(\{t_{i_1}^{(1)}\}, \ldots, \{t_{i_M}^{(M)}\})(f_1(t_{i_1}^{(1)}), \ldots, f_M(t_{i_M}^{(M)})) \right|$$

but, for every $t_{i_j}^{(j)} \in B_j$, $(1 \leq j \leq M)$ the sequences $(f_j^n(t_{i_j}^{(j)}))_n \subset E$ is $\tau_{\leq N}$-convergent to $f_j(t_{i_j}^{(j)})$, so our hypothesis on E implies again the existence of an $n_0 \in \mathbb{N}$ such that, for every $n \geq n_0$,

$$|T(f_1^n, \ldots, f_M^n) - T(f_1, \ldots, f_M)| \leq 2M\epsilon C + \epsilon$$

which finishes the proof. The cases of the $SP_{\infty,M}$ and $SQ_{\infty,M}$ properties are entirely similar. □

We remark that the $P_{N,M}$ property of the space does not imply, in general, the $P_{N,M}$ property of $C(K,E)$ (see Theorems 3.6 and 3.7).

Embedded in the proof of the previous result is the following lemma.

Lemma 3.5. *Let K be a scattered compact Hausdorff space and E a Banach space with the $SQ_{N,M}$ property. If $(f_1^n), \ldots, (f_M^n) \subset C(K,E)$ are sequences such that*

1. $(f_1^n \otimes \overset{[i]}{\cdots} \otimes f_M^n) \subset \widehat{\bigotimes}_\pi^{M-1} C(K,E)$ *are weakly Cauchy* $(1 \leq i \leq M)$
2. *For every $t \in K$, $(f_i^n(t))_n \subset E$ is $\tau_{\leq N}$-Cauchy,* $(1 \leq i \leq M)$

Then $(f_1^n \otimes \cdots \otimes f_M^n) \subset \widehat{\bigotimes}_\pi^M C(K,E)$ is weakly Cauchy.

Moreover, if E has the $SP_{N,M}$ property and $(f_1^n), \ldots, (f_M^n) \subset C(K,E)$ are sequences such that

1. $(f_1^n \otimes \overset{[i]}{\cdots} \otimes f_M^n) \subset \widehat{\bigotimes}_\pi^{M-1} C(K,E)$ *converges weakly to* $(f_1 \otimes \overset{[i]}{\cdots} \otimes f_M)$ $(1 \leq i \leq M)$
2. *For every $t \in K$, $(f_i^n(t))_n \subset E$ is $\tau_{\leq N}$-convergent to $f_i(t)$* $(1 \leq i \leq M)$

Then $(f_1^n \otimes \cdots \otimes f_M^n) \subset \widehat{\bigotimes}_\pi^M C(K,E)$ converges weakly to $(f_1 \otimes \cdots \otimes f_M)$.

We now study the relation between the polynomial convergence of bounded sequences in $C(K,E)$ and the pointwise polynomial convergence of these same sequences. The next two results are both of similar nature. We just prove the first of them.

Theorem 3.6. *Let E be a Banach space. Then the following are equivalent:*

1. *E has the $SQ_{N,M}$ property.*
2. *For every scattered compact Hausdorff space K, $C(K,E)$ has the $SQ_{N,M}$ property.*
3. *For every scattered compact Hausdorff space K, $C(K,E)$ has the $Q_{N,M}$ property.*

(4) *If* $K = \{1, 2, \dots, M\}$ *endowed with the discrete topology, then* $C(K, E)$ *has the* $Q_{N,M}$ *property*

(5) *For every scattered compact Hausdorff space* K*, a bounded sequence* $(f_n) \subset C(K, E)$ *is* $\tau_{\leq M}$ *Cauchy if for every* $t \in K$*,* $(f_n(t))_n$ *is* $\tau_{\leq N}$ *Cauchy.*

Proof. (1) and (2) are equivalent by Theorem 3.4, and trivially (2) implies (3) and (3) implies (4). Let us see that (4) implies (1): let $(x_1^n), \dots, (x_M^n) \subset E$ be $\tau_{\leq N}$ Cauchy sequences. Consider the sequences $(f_i^n)_n \subset C(K, E)$ defined by $f_i^n(t) = \chi_{\{i\}}(t) x_i^n$, $(1 \leq i \leq M)$. Since the linear maps $E \ni x \mapsto \varphi_i(x) := \chi_{\{i\}} \cdot x \in C(K, E)$ are continuous, the sequences (f_i^n) $(1 \leq i \leq M)$ are also $\tau_{\leq N}$ Cauchy. Let $T \in \mathcal{L}^M(E)$ and consider the mapping

$$\tilde{T} = T \circ (\delta_1 \times \cdots \times \delta_M) \in \mathcal{L}^M(C(K, E)),$$

where $\delta_j(f) := f(j)$ for $f \in C(K, E)$ $(1 \leq j \leq M)$. Let $S \in \mathcal{L}_s^M(C(K, E))$ be the symmetric operator associated to $\tilde{T}$. Then

$$\frac{1}{M!} T(x_1^n, \dots, x_M^n) = S(f_1^n, \dots, f_M^n)$$

and our hypothesis guarantees that this sequence is Cauchy.

Let us now see that (1) implies (5): We reason by induction on M (recall that property $SQ_{N,M}$ implies property $SQ_{N,M-1}$). For $M = 1$ the result is well known. Let us suppose it true for $M - 1$ and let $(f_n) \subset C(K, E)$ be a sequence such that, for every $t \in K$, $(f_n(t))$ is $\tau_{\leq N}$ Cauchy. In that case, our induction hypothesis tells us that (f_n) is $\tau_{\leq M-1}$ Cauchy. So, Lemma 3.5 says that $(f_n \otimes \overset{(M)}{\cdots} \otimes f_n) \subset \widehat{\bigotimes}_\pi^M C(K, E)$ is weakly Cauchy, i.e., (f_n) is τ_M Cauchy. Since it is also $\tau_{\leq M-1}$ Cauchy, we get that it is $\tau_{\leq M}$ Cauchy.

To see that (5) implies (1), let $(x_1^n), \dots, (x_M^n) \subset E$ be $\tau_{\leq N}$-Cauchy sequences and let $T \in \mathcal{L}^M(E)$. Let $K = \{1, 2, \dots, M\}$ with the discrete topology. Let us define $f_n \in C(K, E)$ by

$$f_n := \sum_{i=1}^M \chi_{\{i\}} x_i^n.$$

So defined, (f_n) is a bounded sequence and $(f_n(i)) = (x_i^n)$ is a $\tau_{\leq N}$ Cauchy sequence $(1 \leq i \leq M)$. Therefore, our hypothesis says that (f_n) is $\tau_{\leq M}$ Cauchy. Let us again consider the mapping

$$\tilde{T} = T \circ (\delta_1 \times \cdots \times \delta_M) \in \mathcal{L}^M(C(K, E)).$$

Then

$$T(x_1^n, \dots, x_M^n) = \tilde{T}(f_n, \dots, f_n)$$

and our hypothesis says that this is a Cauchy sequence. □

In a similar way, we get:

Theorem 3.7. *Let* E *be a Banach space. Then the following are equivalent:*

(1) E *has the* $SP_{N,M}$ *property.*

(2) *For every scattered compact Hausdorff space K, $C(K,E)$ has the $SP_{N,M}$ property.*
(3) *For every scattered compact Hausdorff space K, $C(K,E)$ has the $P_{N,M}$ property.*
(4) *If $K = \{1, 2, \dots, M\}$ endowed with the discrete topology, then $C(K,E)$ has the $P_{N,M}$ property.*
(5) *For every scattered compact Hausdorff space K, a bounded sequence $(f_n) \subset C(K,E)$ is $\tau_{\le M}$ convergent to $f \in C(K,E)$ if for every $t \in K$, $(f_n(t))_n$ is $\tau_{\le N}$ convergent to $f(t)$.*

When $N = M$, we obtain the announced characterization of polynomial convergence in $C(K,E)$:

Corollary 3.8. *Let E be a Banach space. The following assertions are equivalent:*
(a) *E has the $SQ_{N,N}$ (resp., $SP_{N,N}$) property.*
(b) *For every scattered compact Hausdorff space K, a bounded sequence $(f_n) \subset C(K,E)$ is $\tau_{\le N}$-Cauchy (resp., is $\tau_{\le N}$-convergent to $f \in C(K,E)$) if and only if , for every $t \in K$, $(f_n(t))_n$ is $\tau_{\le N}$-Cauchy (resp., $\tau_{\le N}$ convergent to $f(t)$.)*

As promised in Section 1, we have now

Corollary 3.9. *If E is a stable Banach space (i.e., isomorphic to its cartesian product $E \times E$), then it has the $SP_{N,M}$ property if and only it has the $P_{N,M}$ property. The same holds for properties $SQ_{N,M}$ and $Q_{N,M}$.*

Proof. If we take $K := \{1, 2\}$ with the discrete topology, we have $C(K,E) \approx E$. The result follows from Theorems 3.6 and 3.7. □

As the referee kindly pointed out to us, the next corollary, which is nothing but Theorem 3.7 in case $N = 1$, is probably worth mentioning.

Corollary 3.10. *Let E be a Banach space and $M \in \mathbb{N}$. Then the following assertions are equivalent:*
(a) *Every M-linear continuous form on E is weakly sequentially continuous.*
(b) *For every scattered compact Hausdorff space K, every M-linear continuous form on $C(K,E)$ is weakly sequentially continuous.*
(c) *For every scattered compact Hausdorff space K, every scalar M-homogeneous polynomial P on $C(K,E)$ is weakly sequentially continuous.*

Moreover, (a) clearly implies
(d) *Every M-homogeneous scalar polynomial on E is weakly sequentially continuous*

but it is unknown whether (d) implies (a).

The same techniques, used in a much simpler way, can be used to prove the following expected result.

Proposition 3.11. *Let K be a scattered compact Hausdorff space and E a Banach space. Then $C(K,E)$ has the $\mathcal{P}^{\leq N}$-DP property if and only if E has the same property. A similar statement is true for the $\mathcal{P}$-DP property.*

Proof. Since E is isomorphic to a complemented subspace of $C(K,E)$, one of the implications is clear, . For the other, we use the characterization (2) in [6, Theorem 2.3] which says that E has the $\mathcal{P}^{\leq N}$-DP property if and only if, for every Banach space Y, every weakly compact operator $T : E \longrightarrow Y$ maps $\tau_{\leq N}$ convergent sequences in E into norm convergent sequences in Y. Actually, an inspection of the proof in [6] shows that if we replace "convergent sequences" with "null sequences", the statement remains true. So, let $(f_n) \subset C(K,E)$ be a $\tau_{\leq N}$ null sequence which we may suppose contained in the unit ball; let $T : C(K,E) \longrightarrow Y$ be a weakly compact operator, and $m : \text{Bo} \longrightarrow \mathcal{L}(E;Y^{**})$ its associated measure. It is known that m actually takes values in the closed subspace of the weakly compact operators between E and Y ([8, Th. 4.1]). Moreover, the semivariation of m, $|m|$, is continuous at $\emptyset$ (or s-bounded), so $|m|$ has a control measure $\lambda : \text{Bo} \longrightarrow [0,+\infty)$ ([13, Lemma 2]). Since K is scattered, λ is concentrated on a countable set of atoms $K' = \{t_n : n \in \mathbb{N}\} \subset K$, and so $|m|(K \setminus K') = 0$. As in the proof of Theorem 3.4, $\bigcap_{r=1}^{\infty}\{t_n : n > r\} = \emptyset$, so, given $\epsilon > 0$ there exists $B = \{t_n : n \leq r\}$ such that $|m|(K \setminus B) \leq \epsilon$. Therefore,

$$\|T(f_n)\| = \left\|\int_K f_n dm\right\| \leq \left\|\int_{K\setminus B} f_n dm\right\| + \left\|\int_B f_n dm\right\| \leq$$

$$\leq \epsilon + \left\|\sum_{j=1}^{r} m(\{t_j\})(f_n(t_j))\right\|.$$

Since (f_n) is $\tau_{\leq N}$ null, for every $t \in K$, $(f_n(t))$ is also $\tau_{\leq N}$ null. The fact that $m(\{t\})$ is a weakly compact operator for every $t \in K$ and our hypothesis on E suffice to finish the proof. ☐

We want to thank sincerely the anonymous referee for her/his thorough examination of our paper and very valuable suggestions, which have certainly improved our paper. In particular, Corollary 3.10 and the final presentation of Theorem 3.4 are due to her/him.

References

[1] R. Alencar, R. Aron and S. Dineen, A reflexive space of holomorphic functions in infinitely many variables, *Proc. Amer. Math. Soc.* **90** (1984), 407–411.

[2] R. Alencar and K. Floret, Weak-strong continuity of multilinear mappings and the Pełczyński-Pitt Theorem, *J. Math. Anal. Appl.* **206** (1997), 532–546.

[3] R. Aron, Y. S. Choi and J. G. Llavona, Estimates by polynomials, *Bull. Austral. Math. Soc.*, **52** (1995), 475-486.

[4] R. Aron, C. Hervés, Weakly sequentially continuous analytic functions on a Banach space, in *Functional Analysis, Holomorphy and Approximation Theory II.* Zapata (ed.) North Holland (1984), 23-38.

[5] R. M. Aron, C. Hervés, and M. Valdivia, Weakly continuous mappings on Banach spaces,*J. Funct. Anal.* **52** (1983), 189–204.

[6] P. Biström, J. A. Jaramillo and M. Lindström, Polynomial compactness in Banach spaces. *Rocky Mountain J.Math.* **28** (4) (1998), 1203-1226.

[7] F. Bombal, On Polynomial properties in Banach spaces. *Atti Sem. Mat. Fis. Univ. Modena,* **XLIV** (1996), 135-146.

[8] J. K. Brooks and P.W. Lewis, Linear operators and vector measures. *Trans. Amer. Math. Soc.* **192** (1974), 139-162.

[9] T. Carne, B. Cole and T. Gamelin, A uniform algebra of analytic functions on a Banach space. *Trans. Amer. Math. Soc.* **314** (1989), 639-659.

[10] J. Castillo, R. García and R. Gonzalo, Banach spaces in which all multilinear forms are weakly sequentially continuous, *Stud. Math.* **136** (2) (1999), 121–145.

[11] J. Diestel, *Sequences and Series in Banach Spaces.* Graduate Texts in Math. **92**, Springer, Berlin 1984.

[12] N. Dinculeanu, *Vector Measures.* Pergamon Press 1967.

[13] I. Dobrakov, On representation of linear operators on $C_0(T,X)$, *Czech. Math. J.* **21** (1971), 13–30.

[14] I. Dobrakov, On integration in Banach spaces, VIII (polymeasures), *Czech. Math. J.* **37 (112)** (1987), 487–506.

[15] J. Farmer and W. B. Johnson, Polynomial Schur and Polynomial Dunford-Pettis properties. *Comtemporary Math.*, **144** (1993), 95–105.

[16] R. Gonzalo, Upper and lower estimates in Banach sequence spaces. *Comment. Math. Univ. Carolinae,* 36 (1995), 641-653.

[17] H.E. Lacey, *The isometric theory of classical Banach spaces.* Springer-Verlag, 1974.

[18] J. Mujica, *Complex Analysis in Banach Spaces.* Math. Studies **120**, North Holland, msterdam, 1986.

[19] A. Pelczynski, On weakly compact polynomial operators on B-spaces with the Dunford-Pettis property. *Bull. Acad. Pol. Sci.* **XI** (1963), 371-378

[20] I. Villanueva, Representation of multilinear operators on $C(K,X)$ spaces, *Rev. Real Acad. Cienc.: serie A Math.,* **96** (2002), 35-43.

E-mail address: Fernando_Bombal@mat.ucm.es, ignaciov@mat.ucm.es

Departamento de Análisis Matemático, Facultad de Matemáticas, Universidad Complutense de Madrid, Madrid 28040, Spain