A COMPLEX VERSION OF THE BAER–KRULL THEOREMS

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Abstract

In the setting of the complex spectrum of a field of characteristic zero we give complex analogs of the celebrated (real) Baer–Krull theorems relating the orderings of a field compatible with a given valuation ring with the orderings of the residue class field of that valuation ring.

INTRODUCTION

In the setting of the complex spectrum of a field of characteristic zero we want to give complex analogs of the celebrated (real) Baer–Krull theorems relating the orderings of a field compatible with a given valuation ring with the orderings of the residue class field of that valuation ring.

Recall that a field K is called real if K admits orderings. The set of such orderings is denoted $\mathrm{Spec_r}(K)$ and called the real spectrum of K. Real fields are of zero characteristic.

If B is a local ring, let m_B, U_B and $\overline{K} := B/m_B$ respectively denote the maximal ideal, the group of units and the residue class field of B. If K is a field and $B \subseteq K$ is a valuation ring of K, then let $\lambda_B : K \to \overline{K} \cup \{\infty\}$, $\Gamma_B := K \setminus \{0\}/U_B$ and $v_B : K \to \Gamma_B \cup \{\infty\}$ respectively denote the canonical place, the value group of B (additively written) and the canonical valuation associated to B. The restriction of λ_B to B is the canonical residue class

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morphism. Note that the characteristic of $\overline{K}$ is zero if and only if B is finite over $\mathbb{Q}$, i.e., $\mathbb{Q}$ is a subring of B.

If K is a field and B is a valuation ring of K, recall that
(a) B is called residually real if the residue class field $\overline{K}$ is real i.e., if $\mathrm{Spec}_{\mathrm{r}}(\overline{K}) \neq \emptyset$.
(b) If K is real and β belongs to $\mathrm{Spec}_{\mathrm{r}}(K)$, then B is called β–convex if for all $a \in K$, $b \in B$, the condition $a^2 \leq_\beta b^2$ implies $a \in B$.

In the late 1920's and early 1930's it was proved –using a different language– by Baer and Krull (see [Ba] and [Kr]) that:

(Ba–Kr 1) If B is β–convex then there exists a unique $\overline{\beta}$ in $\mathrm{Spec}_{\mathrm{r}}(\overline{K})$ such that for all $x \in U_B$, $x >_\beta 0$ if and only if $\lambda_B(x) >_{\overline{\beta}} 0$. In particular, B is residually real.

(Ba–Kr 2) If $B \subseteq K$ is residually real, then for every $\gamma \in \mathrm{Spec}_{\mathrm{r}}(\overline{K})$ there exists β in $\mathrm{Spec}_{\mathrm{r}}(K)$ such that for all $a, b \in B$, the condition $a \leq_\beta b$ implies $\lambda_B(a) \leq_\gamma \lambda_B(b)$. Moreover, $\gamma = \overline{\beta}$ and B is β–convex.

Already considered in [Kr], the relationship between γ and all possible β satisfying (Ba–Kr 2) was completely elucidated in [Br]. The notion of real place (the place associated to a residually real valuation ring) was introduced and studied in [L].

Today, we can deduce (Ba–Kr 1) and the second assertion in (Ba–Kr 2) from the well–known fact that the following statements are equivalent:
(i) B is β–convex,
(ii) m_B is β–convex in B,
(iii) there exists a unique $\overline{\beta}$ in $\mathrm{Spec}_{\mathrm{r}}(\overline{K})$ with respect to which $\lambda_B|_B : B \to \overline{K}$ is an order–preserving ring morphism,
(iv) B contains the valuation ring $\{x \in K : x^2 \leq_\beta n^2, \text{some } n \in \mathbb{N}\}$,
(v) there exists γ in $\mathrm{Spec}_{\mathrm{r}}(\overline{K})$ such that for all $a, b \in B$, the condition $a^2 \leq_\beta b^2$ implies $\lambda_B(a^2) \leq_\gamma \lambda_B(b^2)$.

These are assertions about pushing down orderings from K to $\overline{K}$. On the other hand, the first part of (Ba–Kr 2) is a lifting property for orderings from $\overline{K}$ to K, under certain circumstances.

A previous unrefereed version of this paper is [Pu1].

NOTATIONS

$\star$ An involution of a ring A is an element τ of the automorphism group $\mathrm{Aut}\,(A)$ such that $\tau^2 = \mathrm{id}_A$. In this paper we will assume, in addition, that *involutions are not trivial,* i.e., $\tau \neq \mathrm{id}_A$. If τ is an involution of A, then A^τ denotes the subring of A consisting of the fixed points of A; it is called the fixed ring of A.

$\star$ If A_1, A_2 are subfields of a field A_3, then $A_1 \cdot A_2$ denotes the smallest subfield of A_3 containing $A_1 \cup A_2$.

$\star$ Let E be an algebraically closed field. Then $\pm i_E$ denote the solutions of the equation $X^2 + 1 = 0$ in E. When E is understood, we will simply write $\pm i$. In this paper, ϵ is always either i or $-i$.

$\star$ E^a denotes the algebraic closure of a field E.

$\star$ We fix a field extension $R \subseteq K$ with R real. Then K has zero characteristic, obviously. Thus, there exist field morphisms $j : K \to C$ with C some algebraically closed field of zero characteristic and such a field C possesses involutions τ that fix $j(R)$ pointwise (see [S] to learn about the abundance of involutions in zero characteristic algebraically closed fields). Let $\epsilon \in \{\pm i_C\}$ and take triplets (j, τ, ϵ) such that C is algebraic over $j(R) \cdot \tau j(R)$. Say two triplets $(j_s, \tau_s, \epsilon_s)$ with $s = 1, 2$ are equivalent if there exists a field isomorphism $f : C_1 \to C_2$ such that $f(\epsilon_1) = \epsilon_2$, $j_2 = f j_1$ and $f|_{C_1^{\tau_1}} : C_1^{\tau_1} \to C_2^{\tau_2}$ is a field isomorphism (order-preserving, a fortiori). The points of the complex spectrum of K over R, $\mathrm{Spec}_{\mathrm{c}}(K/R)$, are equivalence classes $[j, \tau, \epsilon]$ of such triplets. We have just explained that $\mathrm{Spec}_{\mathrm{c}}(K/R)$ is non-empty, provided that R is real. The complex spectrum has been studied by the author in [Pu3] and earlier in [Pu2]. If K is also real then $\mathrm{Spec}_{\mathrm{c}}(K/R)$ extends $\mathrm{Spec}_{\mathrm{r}}(K)$ in the following sense. For each ordering β of K consider the real closure F of (K, β). Then F^a has a unique involution τ fixing F and having $\tau(i) = -i$. Thus $\beta \mapsto [j, \tau, i]$ defines a (canonical) mapping $\Upsilon : \mathrm{Spec}_{\mathrm{r}}(K) \to \mathrm{Spec}_{\mathrm{c}}(K/R)$, where $j : K \to F^a$ is the field embedding considered above. (In this paper we only use the existence of Υ. However, as one would expect, Υ is continuous and a homeomorphism onto the image, provided that the spectra are endowed with their usual topologies; see 2.5, 2.6 and 2.7 [Pu3]).

Let $R \subseteq K$ be a field extension with R real. In this paper we will deal with just a point in $\mathrm{Spec}_c(K/R)$ at a time, so that the notation $z = [j, \tau, \epsilon]$ can and will often be simplified to $z = [K \subseteq C, \tau, \epsilon]$; this conveys that j is the identity on K and each element of K is identified with its image in C. We have then $R \subseteq C^\tau$ (although the notation does not make this explicit).

First we observe that $z = [K \subseteq C, \tau, \epsilon]$ induces a point in $\mathrm{Spec}_r(K \cap C^\tau)$, namely the ordering induced by the real closed field C^τ on $K \cap C^\tau$. Let us denote it by $\mathrm{re}(z)$. Note that $K \subseteq C^\tau$ iff $K = K \cap C^\tau$ iff $\Upsilon\, \mathrm{re}(z) = z$, with Υ as in the notations. We may say that the point z is *real* if any of the equivalent conditions above hold. Note that $\mathrm{Spec}_c(K/R)$ contains real points iff K is real.

Given $z = [K \subseteq C, \tau, \epsilon]$ and $a \in K$ set $\mathrm{N}_\tau(a) := a\tau(a) \in C^\tau$ and call it the *τ-norm of a*. In fact, $\mathrm{N}_\tau(a)$ is non-negative, being a sum of squares in C^τ. Indeed $\mathrm{N}_\tau(a) = (\frac{a+\tau(a)}{2})^2 + (\frac{a-\tau(a)}{2i})^2$. In general, $\mathrm{N}_\tau(a) \notin K$, since in general $\tau(K) \nsubseteq K$.

1. Definitions. *Let $R \subseteq K$ be a field extension with R real, B a valuation ring of K and $S \subseteq K$ any subset. Let $z = [K \subseteq C, \tau, \epsilon]$ belong to $\mathrm{Spec}_c(K/R)$.*
(a) B is called z-convex (in K) if for all $a \in K$, $b \in B$, the condition $\mathrm{N}_\tau(a) \le \mathrm{N}_\tau(b)$ implies $a \in B$.
(b) S is called τ-invariant if $\tau(S) = S$, (equivalently, if $\tau(S) \subseteq S$).

Though (a) has been defined using a specific representative of z, it is not hard to show that this definition is independent of the representative chosen.

Note that $\mathrm{N}_\tau(a) = a^2$, for all $a \in K$, when z is a real point. Thus, both definitions (real and complex) of convexity agree in this case.

2. Remarks. (a) Assume that B is τ-invariant. Then all m_B, U_B and K are τ-invariant. Call τ' the restriction of τ to K. Then $\tau' = \mathrm{id}_K$ iff $K \subseteq C^\tau$ iff z is real.
(b) Assume that K is algebraically closed, τ is an involution of K and B is τ-invariant. Then the mapping defined by $\overline{\tau}(x + m_B) = \tau(x) + m_B$, for all $x \in B$, is an involution of $\overline{K}$ (note that $i \in U_B$, since B is integrally closed, whence $\overline{\tau}(i + m_B) = -i + m_B \ne i + m_B$, so that $\overline{\tau} \ne \mathrm{id}_{\overline{K}}$). In particular, $\lambda_B \tau = \overline{\tau}\lambda_B$. Conversely, if σ is any involution of $\overline{K}$ such that $\lambda_B\tau = \sigma\lambda_B$, then B is τ-invariant and $\sigma = \overline{\tau}$.
(c) The condition $\lambda_B\tau = \sigma\lambda_B$ obviously implies $\lambda_B \mathrm{N}_\tau = \mathrm{N}_\sigma \lambda_B$.

3. Proposition. *Let $R \subseteq K$ be a field extension with R real, $z = [K \subseteq C, \tau, \epsilon]$ a point in $\mathrm{Spec}_c(K/R)$ and B a z–convex valuation ring of K. Then the following hold true:*
(i) B is finite over $\mathbb{Q}$ and $\mathrm{char}\, \overline{K} = 0$,
(ii) $B \cap C^{\tau}$ is a $\mathrm{re}(z)$–convex valuation ring of $K \cap C^{\tau}$,
(iii) if K is τ–invariant then B is τ–invariant.

Proof. (i) We have $n^{-2} = \mathrm{N}_\tau(n^{-1}) < \mathrm{N}_\tau(1) = 1$, hence $n^{-1} \in B$, for all $1 < n \in \mathbb{N}$, whence $\mathbb{Q}$ is a subring of B and of $\overline{K}$.
(ii) Immediate.
(iii) $\tau(K) = K$ implies $\tau(B) = B$ simply because $\tau(a) \in K$, B is z–convex and $\mathrm{N}_\tau(\tau(a)) = a\tau(a) = \mathrm{N}_\tau(a)$, for all $a \in B$. ∎

4. Proposition. *Let $R \subseteq K$ be a field extension with R real, $z = [K \subseteq C, \tau, \epsilon]$ a point in $\mathrm{Spec}_c(K/R)$ and B a valuation ring of K. If B is τ–invariant and $B \cap C^{\tau}$ is $\mathrm{re}(z)$–convex, then B is z–convex.*

Proof. Suppose that $a \in K$, $b \in B$ and $\mathrm{N}_\tau(a) \leq \mathrm{N}_\tau(b)$. Since $\tau(B) = B$, then $\mathrm{N}_\tau(b) = b\tau(b)$ lies in $B \cap C^{\tau}$, whence $\mathrm{N}_\tau(a) = a\tau(a)$ also lies in $B \cap C^{\tau}$, by $\mathrm{re}(z)$–convexity. Moreover if $a \notin B$ then $a^{-1} \in m_B$, whence $\tau(a) = a^{-1}a\tau(a) \in m_B \subseteq B = \tau(B)$. Applying τ we get $a \in B$, a contradiction. Therefore, $a \in B$ holds, as was to be shown. ∎

Since we will only be interested in z–convex valuation rings, then we will restrict ourselves to valuation rings finite over $\mathbb{Q}$, by Proposition 3 (i).

Let $z = [K \subseteq C, \tau, \epsilon]$ be a point in $\mathrm{Spec}_c(K/R)$ and set $\mathcal{H} := \{x \in K : \mathrm{N}_\tau(x) \leq n^2$, some $n \in \mathbb{N}\}$. Then *$\mathcal{H}$ is a z–convex valuation ring in K.* Indeed, it is easy to verify that $\mathcal{H}$ is closed under products and sums, using that $\mathrm{N}_\tau(x + y) \leq 4\max\{\mathrm{N}_\tau(x), \mathrm{N}_\tau(y), \mathrm{N}_\tau(x\tau(y)), 1\}$, for all $x, y \in K$. It is also clear that $x \in K \setminus \mathcal{H}$ implies $x^{-1} \in \mathcal{H}$.

5. Proposition. *Let $R \subseteq K$ be a field extension with R real, $z = [K \subseteq C, \tau, \epsilon]$ a point in $\mathrm{Spec}_c(K/R)$. If B is any valuation ring of K finite over $\mathbb{Q}$, then the following are equivalent:*
(a) B is z–convex in K,
(b) m_B is z–convex in B,
(c) $\mathcal{H} \subseteq B$.

Proof. (a) $\Rightarrow$ (b) Let $a \in B$, $b \in m_B$ and suppose $\mathrm{N}_\tau(a) \leq \mathrm{N}_\tau(b)$. If $b = 0$ then $0 = \mathrm{N}_\tau(a) = a = \tau(a)$, whence a belongs to m_B. If $a \neq 0$ then

$b \neq 0$, $\mathrm{N}_\tau(b^{-1}) \leq \mathrm{N}_\tau(a^{-1})$ and $b^{-1} \notin B$, whence $a^{-1} \notin B$, by z–convexity of B. Thus, a belongs to m_B.

(b) $\Rightarrow$ (c) Let $x \in K$ be such that $\mathrm{N}_\tau(x) \leq n^2$, for some $n \in \mathbb{N}$. Suppose $x \notin B$. Then $n \neq 0$, $x^{-1} \in m_B$ and $n^{-2} \leq \mathrm{N}_\tau(x^{-1})$ whence n^{-1} belongs to m_B, by z–convexity of m_B and finiteness of B over $\mathbb{Q}$. It follows that 1 belongs to m_B, which is absurd.

(c) $\Rightarrow$ (a) Let $a \in K$, $b \in B$ and suppose $\mathrm{N}_\tau(a) \leq \mathrm{N}_\tau(b)$. If $b = 0$ then $0 = \mathrm{N}_\tau(a) = a = \tau(a)$. If $b \neq 0$ then $\mathrm{N}_\tau(ab^{-1}) \leq 1$ whence $ab^{-1} \in \mathcal{H} \subseteq B$ and so $a = (ab^{-1})b \in B$. ∎

6. Definitions. *Let $R \subseteq K$ be a field extension with R real. Consider $z = [K \subseteq C, \tau, \epsilon] \in \mathrm{Spec}_c(K/R)$. Let E, F be intermediate fields with $R \subseteq E \subseteq K \subseteq F \subseteq C$.*

(a) C is algebraic over $K \cdot \tau(K)$ whence also algebraic over $F \cdot \tau(F)$. We define the extension of z to F to be the point $[F \subseteq C, \tau, \epsilon] \in \mathrm{Spec}_c(F/R)$ and denote it by $\mathrm{ext}(z, F)$.

(b) Let C' be the algebraic closure of $E \cdot \tau(E)$ inside C. Clearly ϵ belongs to C'. Now, for every $a \in C'$, $\tau(a)$ is a root of the image by τ of the minimal polynomial of a over $E \cdot \tau(E)$, whence $\tau(a)$ belongs to C'. It follows that the restriction τ' of τ to C' is an involution (if $\tau' = \mathrm{id}_{C'}$ then C' would embed into a real closed field, contradicting that C' is algebraically closed). The point $[E \subseteq C', \tau', \epsilon]$ belongs to $\mathrm{Spec}_c(E/R)$ and is called the restriction of z to E and denoted $\mathrm{res}(z, E)$.

The extensions of z to all intermediate fields F are dominated by the extension to C, in the sense that $\mathrm{ext}(z, F) = \mathrm{res}(\mathrm{ext}(z, C), F)$. For short, write $z^\dagger = \mathrm{ext}(z, C)$.

The proof of the following result is immediate.

7. Proposition. *Let $R \subseteq E \subseteq K$ be a field extension with R real. Let $z = [K \subseteq C, \tau, \epsilon]$ be a point in $\mathrm{Spec}_c(K/R)$ and $B \subseteq K$ a valuation ring. Consider $w = \mathrm{res}(z, E) \in \mathrm{Spec}_c(E/R)$. If B is z–convex then $B \cap E$ is w–convex.* ∎

8. Definition and Proposition. *Let $R \subseteq K \subseteq C$ be a field extension with R real and C algebraically closed. Let B be a subring of K and τ an involution of C. The set*

$$\mathcal{O}_\tau(B) := \{x \in C : \mathrm{N}_\tau(x) \leq \mathrm{N}_\tau(b), \text{ some } b \in B\}$$

is called the τ–convex hull of B. It is a w–convex, τ–invariant valuation ring of C, where $w = [C \subseteq C, \tau, \epsilon] \in \mathrm{Spec}_c(C/R)$. In addition, if B is a valuation ring of K then B is z–convex if and only if $\mathcal{O}_\tau(B)$ is an extension of B, i.e., $\mathcal{O}_\tau(B) \cap K = B$.

Proof. It is easy to verify that $\mathcal{O}_\tau(B)$ is closed under products and sums, using that $\mathrm{N}_\tau(x+y) \leq 4\max\{\mathrm{N}_\tau(x), \mathrm{N}_\tau(y), \mathrm{N}_\tau(x\tau(y)), 1\}$, for all $x, y \in C$. It is straightforward to show that $\mathcal{O}_\tau(B)$ is a w–convex valuation ring. It is τ–invariant just because $\mathrm{N}_\tau(x) = \mathrm{N}_\tau(\tau(x))$. Finally, it is immediate to realize that $\mathcal{O}_\tau(B) \cap K = B$ if and only if B is z–convex. ∎

From here on, we set $f(\infty) = \infty$, for any map f.

9. Construction: Complex (Ba–Kr 1). Let $B \subseteq K$ be a valuation ring finite over $\mathbb{Q}$ and $z = [K \subseteq C, \tau, \epsilon] \in \mathrm{Spec}_c(K/R)$. Write $\overline{R}$ for the residue class field of the valuation ring $B \cap R$. Suppose that B is z–convex. Then $B \cap R$ is convex in R with respect to the ordering induced by C^τ.

Easy case: Suppose $K = C$. Then K is algebraically closed and τ–invariant. By Proposition 3 (iii), B is τ–invariant. By Remarks 2 (b), τ induces an involution $\overline{\tau}$ on $\overline{K}$ such that $\lambda_B\tau = \overline{\tau}\lambda_B$. Moreover $\overline{K}$ is algebraically closed, by [PC] lemma 17, page 47. The fixed field $\overline{K}^{\overline{\tau}}$ contains $\overline{R}$, since $R \subseteq K^\tau$. In this situation, we say that *z and B induce the point* $z_B := [\overline{K} \subseteq \overline{K}, \overline{\tau}, \lambda_B(\epsilon)] \in \mathrm{Spec}_c(\overline{K}/\overline{R})$. Such a point z_B is unique in $\mathrm{Spec}_c(\overline{K}/\overline{R})$ with the conditions $\lambda_B\tau = \overline{\tau}\lambda_B$ and $\lambda_B(\epsilon)$ appearing in the last entry (clearly $\lambda_B(\epsilon) \in \{\pm i\}$).

General case: If K is not necessarily equal to C, then extend z to $z^\dagger \in \mathrm{Spec}_c(C/R)$ and consider $H := \mathcal{O}_\tau(B)$, which is a $z^\dagger$–convex valuation ring of C extending B, using Definitions 6 and 8. To $z^\dagger$ and H we apply the construction of the previous easy case, thus obtaining an induced point $(z^\dagger)_H$. It lies in $\mathrm{Spec}_c(\overline{C}/\overline{R})$, being $\overline{C}$ (respectively, $\overline{R}$) the residue class field of H (respectively, of $H \cap R = B \cap R$).

The condition $H \cap K = B$ conveys inclusions $B \subseteq H$ and $m_B \subseteq m_H$. They induce a field embedding $l : \overline{K} \to \overline{C}$. In particular, the place $\lambda_H : C \to \overline{C} \cup \{\infty\}$ satisfies $l\lambda_B = \lambda_H|_B$. If we identify each element of $\overline{K}$ with its image by l then λ_H is an extension of λ_B to C and we will simply write λ for either λ_B or λ_H. Now, the field extension $\overline{K} \cdot \overline{\tau}(\overline{K}) \subseteq \overline{C}$ is algebraic, since $K \cdot \tau(K) \subseteq C$ is algebraic. We have a point $\mathrm{res}((z^\dagger)_H, \overline{K})$ that will be denoted z_B and called *point induced by z and B in* $\mathrm{Spec}_c(\overline{K}/\overline{R})$, namely, $z_B = [\overline{K} \subseteq \overline{C}, \overline{\tau}, \lambda_B(\epsilon)]$. ■

Note that H is τ–invariant. Thus the equality $\lambda\tau = \overline{\tau}\lambda$ holds and it follows that $\lambda \mathrm{N}_\tau = \mathrm{N}_{\overline{\tau}}\lambda$.

How does this fit with the real (Ba–Kr1)? $\mathrm{re}(z^\dagger)$ is nothing but the unique ordering of C^τ and $\mathrm{re}((z^\dagger)_H)$ is the unique ordering of $\overline{C}^{\overline{\tau}}$. Thus $\mathrm{re}(z_B)$ is the ordering induced on $\overline{K} \cap \overline{C}^{\overline{\tau}}$. Then *the ordering* $\overline{\mathrm{re}(z)}$ *given by the real (Ba–Kr 1) coincides with* $\mathrm{re}(z_B)$. Indeed, given $x \in U_B \cap C^\tau$, we have $\lambda(x) >_{\overline{\mathrm{re}(z)}} 0$ iff $x >_{\mathrm{re}(z)} 0$ iff $x = a^2$, for some $a \in C^\tau \setminus \{0\}$. This implies that $0 \neq \lambda(x) = \lambda(a)^2$ and this holds iff $\lambda(x) >_{\mathrm{re}(z_B)} 0$.

10. Definition. *Let $R \subseteq K$ be a field extension with R real, B a valuation ring of K. Suppose that the residue class field $\overline{R}$ of $B \cap R$ is a real subfield of $\overline{K}$. If $w = [\overline{K} \subseteq D, \sigma, \epsilon]$ belongs to* $\mathrm{Spec}_c(\overline{K}/\overline{R})$, *then a lifting of w via λ_B is a point z in* $\mathrm{Spec}_c(K/R)$, *such that B is z–convex and $z_B = w$.*

The generalization of (Ba-Kr 2) to the complex spectrum claims that liftings exist. It will follow easily from the next result, the proof of which is due to A. Prestel.

11. Proposition. *Let $R \subseteq K$ be a field extension with R real, $B \subseteq K$ a valuation ring, D an algebraically closed extension of $\overline{K}$ with* $\mathrm{char}\, D = 0$, *σ an involution of D. Suppose that the residue class field $\overline{R}$ of $B \cap R$ satisfies $\overline{R} \subseteq D^\sigma$. Then there exist an algebraically closed extension C^* of K and an involution τ^* of C^* such that for some extension H^* of B to C^* we have*
(a) H^* is τ^*-invariant (and thus τ^* induces an involution $\overline{\tau^*}$ of $\overline{C^*}$),
(b) $D \subseteq \overline{C^*}$ and $\overline{\tau^*}$ extends σ,
(c) $R \subseteq (C^*)^{\tau^*}$.

Proof. It is well known that there exists an extension v^a of v_B to the algebraic closure K^a and that the value group of v^a is the divisible hull Δ of Γ_B. There exists an embedding of R into $D^\sigma((\Delta))$ as valued fields (see [Ka], page 318). Since $D^\sigma((\Delta))$ is real closed then some real closure E of R inside K^a embeds into $D^\sigma((\Delta))$ as ordered compatibly–valued fields, by [PC] page 118, theorem 27. Write $C = D((\Delta))$, denote by u the canonical valuation of C and by τ the canonical involution in C extending σ (so that $\overline{\tau} = \sigma$ holds and $C^\tau = D^\sigma((\Delta))$). The embedding $(E, v^a|_E) \hookrightarrow (C^\tau, u\check{C}^\tau)$ has a canonical extension to some embedding of $E(i)$ into C as valued fields. So we have $(E(i), v^a\check{E}(i)) \xhookrightarrow{\varphi} (C, u)$, for some embedding φ. We would like to extend φ to some embedding of (K^a, v^a) into some extension (C^*, v^*) of (C, v).

By general model theory (see [Pr1] page 125 or [Pr2] page 47), there exists a card$(K)^+$-saturated elementary extension (C^*, u^*, τ^*) of the structure (C, u, τ). Now, all the conditions in [K–P], claim in page 192, are satisfied for $K_1 = E(i)$, $K_2 = K^a$ and $K^* = C^*$. Indeed, $E(i)$ and C^* are obviously henselian valued fields. Their residue class fields satisfy $\overline{E(i)} \subseteq \overline{K^a} \subseteq \overline{C^*}$ and $\overline{E(i)}$ is algebraically closed (in $\overline{K^a}$). And their value groups satisfy $v^a(E(i)) \subseteq \Delta \subseteq u^*(C^*)$ and the residue class group $\Delta / v^a(E(i))$ is torsion–free, since $v^a(E(i))$ is divisible.. Thus, by [K–P], φ extends to $(K^a, v^a) \overset{\varphi}{\hookrightarrow} (C^*, u^*)$.

Clearly, τ^* is an involution extending τ and thus $\varphi(R) \subseteq (C^*)^{\tau^*}$. If we now identify each element of K^a with its image by φ then (c) holds.

The definition of τ implies that $\rho\tau = \sigma\rho$, where ρ denotes the canonical place of C. Therefore the associated valuation ring $H \subseteq C$ is τ–invariant, by Remarks 2 (b). This means that $u(x) \geq 0$ implies $u\tau(x) \geq 0$, for $x \in C$. If $H^* \subseteq C^*$ denotes the valuation ring associated to u^* then (a) follows, since the extension is elementary.

The condition $H^* \cap C = H$ conveys inclusions $H \subseteq H^*$ and $m_H \subseteq m_{H^*}$. They induce a field embedding $l : D \to \overline{C^*}$. If we identify each element of D with its image by l then $\overline{\tau^*}|_D = \overline{\tau} = \sigma$ and (b) holds. ∎

Note that for a power series to be saturated, the value group must be saturated. For instance, $C = \mathbb{C}((\mathbb{Q}))$ is not saturated.

12. Corollary: Complex (Ba–Kr 2). *Let $R \subseteq K$ be a field extension with R real, B a valuation ring of K. Suppose that the residue class field $\overline{R}$ of $B \cap R$ is a real subfield of $\overline{K}$. Then for each $w = [\overline{K} \subseteq D, \sigma, \epsilon] \in \mathrm{Spec_c}(\overline{K}/\overline{R})$ there exists a lifting of w via λ_B.*

Proof. We apply Proposition 11 and obtain C^*, τ^* and H^*, with the notations of the previous proof. Take $\delta \in \rho^{-1}(\epsilon) \cap \{\pm i\}$ and consider the points $z^* = [C^* \subseteq C^*, \tau^*, \delta] \in \mathrm{Spec_c}(C^*/R)$ and $z = \mathrm{res}(z^*, K) \in \mathrm{Spec_c}(K/R)$. We have $z = [K \subseteq C', \tau', \delta]$ where C' is the algebraic closure of $K \cdot \tau^*(K)$ in C^* and τ' is the restriction of τ^* to C'. We must verify that B is z–convex and $z_B = w$.

Using the notations in the proof of Proposition 11, recall that C^τ is an ordered compatibly–valued field, which implies that $H \cap C^\tau$ is a convex valuation ring of C^τ. By Proposition 4, H is t–convex, where $t = [C \subseteq$

$C, \tau, \delta]$. This means that $u(b) \geq 0$, $b\tau(b) - a\tau(a) = c^2$ and $c = \tau(c)$ imply $u(a) \geq 0$, for $a, b, c \in C$. Since the extension is elementary then H^* is z^*-convex. Thus $B = H^* \cap K$ is z-convex, by Proposition 7.

Finally, $z_B = w$ because the extensions of z to C' (respectively, of z_B and w to $\overline{C'}$) are dominated by z^* (respectively, by $(z^*)_{H^*}$). ■

The proof of the existence of liftings of w, in the case $R = \mathbb{Q}$ is simpler than the proof given above. It uses a version of Proposition 11, which does not call for saturated structures. We just embed K into $D((\Delta))$ as valued fields and, of course, this embedding maps $\mathbb{Q}$ into $D^\sigma((\Delta))$.

The question of determining all the liftings of w via λ_B does not seem to follow easily from the above.

It seems possible to derive a complex (Ba–Kr 2) from 2.12 in [H].

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REFERENCES

[Ba] *R. Baer*, Über nicht–archimedisch geordnete Körper (Beiträge zur Algebra 1). Sitz. Ber. der Heidelberger Akademie, 8. Abhandl., (1927), 3–13.

[Br] *R. Brown*, Real places and ordered fields, Rocky Mountain J. Math. **1**, (1971), 633–636.

[H] *R. Huber*, Spezialisierungen im Reellen Spectrum zu einer Komplexen Varietät, Geom. Dedic. **38**, (1991), 309–327.

[Ka] *I. Kaplansky*, Maximal fields with valuations, Duke Math. J. **9**, (1942), 303–321.

[Kr] *W. Krull*, Allgemeine Bewertungstheorie, J. reine angew. Math. **167**, (1932), 160–196.

[K–P] *F.V. Kuhlmann, A. Prestel*, On places of algebraic function fields, J. reine angew. Math. **353**, (1984), 181–195.

[L] *S. Lang*, The theory of real places, Ann. of Math. **57** (1953), 378–391.

[Pr1] *A. Prestel*, Einführung in die Mathematische Logik und Modeltheorie, Vieweg Studium, Braunschweig/Wiesbaden, 1986.

[Pr2] *A. Prestel*, Model theory for the real algebraic geometer, Instituti Editoriali e Poligrafici Internazionali, Pisa–Roma, 1998.

[PC] *S. Priess-Crampe*, Angeordnete Structuren: Gruppen, Körper, projective Ebenen, Ergebnisse der Math. 98, Springer–Verlag, Berlin–Heidelberg–New York, 1983.

[Pu1] *M.J. de la Puente*, A complex version of the Baer–Krull theorems, Prepubications, Equipe de Logique, Séminaire de Structures Algébriques Ordonnées, no. 61, Université Paris VII, (1997).

[Pu2] *M.J. de la Puente*, The complex spectrum of a ring, Prepubications, Equipe de Logique, Séminaire de Structures Algébriques Ordonnées, no. 61, Université Paris VII, (1997).

[Pu3] *M.J. de la Puente*, The complex spectrum of a ring, in Cont. Math. AMS, Proceedings of the RAGOS special semester, C. Delzell and J. Madden eds., (2000), 235–249.

[S] *B. Schnor*, Involutions in the group of automorphisms of an algebraically closed field, J. of Alg. **152**, (1992), 520–524.