EULER CHARACTERISTIC OF THE MILNOR FIBRE
OF PLANE SINGULARITIES

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Abstract. We give a formula for the Euler characteristic of the Milnor fibre
of any analytic function $f$ of two variables. This formula depends on the
intersection multiplicities, the Milnor numbers and the powers of the branches
of the germ of the curve defined by $f$. The goal of the formula is that it use
neither the resolution nor the deformations of $f$. Moreover, it can be use for
giving an algorithm to compute it.

1. Introduction

In this note we deal with germs of analytic functions $f$ of two complex variables
with $f(0) = 0$ and its factorization $f = f_1^{q_1} \cdots f_r^{q_r}$ into irreducible factors, such
that $f_i/f_j$, $1 \leq i, j \leq r$, are as power series not units. Let $(C, 0)$ be the germ of the
plane curve defined by the local equation $f = 0$ and let $(C_i, 0)$, $i = 1, \ldots, r$, be its
reduced branches defined by $f_i = 0$.

The local curve $C$ defines a link with multiplicities $L := C \cap S^3_\varepsilon$, in the sphere
of radius $\varepsilon > 0$ around $0 \in \mathbb{C}^2$, which does not depend on $\varepsilon$ provided $\varepsilon$ is small
enough. The link $L$ consists of the components $C_i \cap S^3_\varepsilon$, with multiplicities $q_i$ and
determines the topological type of the germ $C$. Moreover, Milnor proved that the
map $f: S^3_\varepsilon \setminus L \to S^1$ is a $C^\infty$-locally trivial fibration, the Milnor fibration. Any
fibre $F$ of this fibration is called the Milnor fibre of $f$ (see [M, Theorem 4.8]).
A’Campo [A] and Eisenbud-Neumann [EN], using different methods, calculated
many topological invariants of the fibration $f$ from the resolution graph or the
splicing diagrams. We are only interested in the Euler characteristic $\chi(F)$ of the
surface $F$. If $f$ is reduced, i.e. every power $q_i$ is equal to one, the Euler characteristic
of $F$ is $1 - \mu(C, 0)$, where $\mu(C, 0)$ is the Milnor number of the isolated singularity
of $C$. Moreover the Euler characteristic of $F$ is related to topological and geometric
invariants of its branches by the well-known formula:

$$\chi(F) = -2 \sum_{1 \leq i < j \leq r} (C_i, C_j)_0 + \sum_{i=1}^{r} (1 - \mu(C_i)),$$

where $(C_i, C_j)_0$ is the intersection multiplicity of $C_i$ and $C_j$ at the origin and $\mu(C_i)$
is the Milnor number of $C_i$ at the origin (e.g. see [BK]).

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On the other hand, when \( f \) is non-reduced Schrauwen [S] expressed the Euler characteristic of \( F \) in terms of special points of suitable deformations of \( f \). For calculating \( \chi(F) \) in this case one can use the methods of A’Campo or Eisenbud-Neumann and construct the resolution graph or the splicing diagram.

The aim of this note is to give a closed formula for the Euler characteristic of \( F \) without the construction of these graphs.

For every \( q \in \mathbb{N}^r \) set
\[
F^q := \{ z \in S_e : \prod_{1 \leq i \leq r, q_i \neq 0} \left( \frac{f_i}{|f_i|} \right)^{q_i}(z) = 1 \text{ and } f_i(z) \neq 0 \ \forall i = 1, \ldots, r \}.
\]

Note that, for \( \epsilon \) small, the surface \( F^q \) is the Milnor fibre of the local curve \( C^q := \{ f_1^{q_1} \cdots f_r^{q_r} = 0 \} \) if and only if all \( q_i \neq 0 \). If some \( q_i \) are zero, but \( q \neq 0 \), then \( F^q \) is the Milnor fibre of \( \prod_{1 \leq i \leq r, q_i \neq 0} f_i^{q_i} \) with punctures, where the number of punctures equals \( \sum_{1 \leq i < j \leq r, q_i \neq 0, q_j = 0} (C_i, C_j)_0(q_i) \). For \( q = 0 \) the space \( F^q \) is just the complement of the link of the curve \( C \).

Our generalized and closed formula is:
\[
\chi(F^q) = - \sum_{1 \leq i < j \leq r} (C_i, C_j)_0(q_i + q_j) + \sum_{i=1}^r q_i (1 - \mu(C_i)).
\]

I am indebted to the referee for suggesting how to improve the presentation of the proof of the formula.

2. Proof of the formula

The formula follows from the two following lemmas.

**Lemma 1.** The function \( q \in \mathbb{N}^r \to \chi(F^q) \) is additive.

**Proof.** Let \( \pi : X \to \mathbb{C}^2 \) be a proper modification of \( \mathbb{C}^2 \) above the origin such that, for every point on the divisor \( E := \pi^{-1}(0) \), the total transform of the \( \bigcup_{1 \leq i \leq r} C_i \) has normal crossing singularities. Let \( C_i \) be the strict transform of \( C_i \) by \( \pi \) and \( E_\alpha, \alpha \in A \), the components of \( E \).

First assume \( q \neq 0 \). Put \( f^q = \prod_{1 \leq i \leq r, q_i \neq 0} f_i^{q_i} \). Observe that \( F^q \) retracts on \( E \setminus \left( \bigcup_{1 \leq i \leq r, q_i = 0} \overline{C_i} \right) \). With the formula of A’Campo we get:
\[
\chi(F^q) = \sum_{\alpha \in A} m(f^q, E_\alpha) \chi(\tilde{E}_\alpha),
\]
where \( \tilde{E}_\alpha := E_\alpha \setminus \left( \bigcup_{\beta \neq \alpha} E_\beta \cup \bigcup_{1 \leq i \leq r} C_i \right) \). Then
\[
\chi(F^q) = \sum_{\alpha \in A} \sum_{i=1}^r q_i m(f_i, E_\alpha) \chi(\tilde{E}_\alpha),
\]
since \( m(f^q, E_\alpha) = \sum_{1 \leq i \leq r} q_i m(f_i, E_\alpha) \).

To prove the additivity it remains to observe that \( \chi(F^0) = 0 \).

Put \( \epsilon_i = (0, \ldots, 1, \ldots, 0) \). From the additivity we get:
\[
\chi(F^q) = \sum_{i=1}^r q_i \chi(F^{\epsilon_i}).
\]
Lemma 2.
\[ \chi(F^{e_i}) = - \sum_{\substack{j=1,\ldots,r \setminus i \neq j}} (C_i, C_j)_0 + (1 - \mu(C_i)). \]

Proof. Remember that \( F^{e_i} \) is the Milnor fibre \( F_i \) with \( \sum_{1 \leq j \leq r, j \neq i} (C_i, C_j)_0 \) punctures. \[ \square \]

Remark that Lemma 1 holds for the case where the germs of the curves \( C_i \) are reduced and have no branch in common. Thus, if we assume
1. each \( f_i \) has no multiple components (i.e. \( f_i \) is squarefree) and
2. for \( i, j \in \{1, \ldots, r\}, i \neq j \), the germ \( f_if_j \) has no multiple components,
then we finally get for the Euler characteristic of the Milnor fibre of \( F \) of \( f = f_1^{q_1} \cdots f_s^{q_s}, q_i > 0 \), the formula:
\[ \chi(F) = - \sum_{1 \leq i < j \leq s} (C_i, C_j)_0 (q_i + q_j) + \sum_{i=1}^s q_i (1 - \mu(C_i, 0)). \]

To have this formula for squarefree factorization is particularly useful for inductive calculations. If \( R \) is a computable ring with \( \text{char}(R) = 0 \) and \( f \) is a polynomial in \( R[x, y] \), then there exists an algorithm that computes a squarefree decomposition of \( f \) in \( R[x, y] \) (see [BWK, Proposition 2.86, Corollary 2.92]). This is also a squarefree decomposition in \( R\{x, y\} \) and one may then compute the intersection multiplicities and the Milnor numbers. I would like to thank Bernd Martin for showing me the implementation of this algorithm using the computer algebra system SINGULAR, [GPS].

References

[GPS] G.M. Greuel, G. Pfister, H. Schoenemann, SINGULAR. A computer algebra system for singularity theory and algebraic geometry, It is available via anonymous ftp from helios.mathematik.uni-kl.de.

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