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Very rapidly varying boundaries in equations with Nonlinear Boundary Conditions. The case of a non uniformly Lipschitz deformation.

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Abstract

We continue the analysis started in [3] and announced in [2], studying the behavior of solutions of nonlinear elliptic equations $\Delta u + f(x,u) = 0$ in Ω_ϵ with nonlinear boundary conditions of type $\frac{\partial u}{\partial n} + g(x,u) = 0$, when the boundary of the domain varies very rapidly. We show that if the oscillations are very rapid, in the sense that, roughly speaking, its period is much smaller than its amplitude and the function g is of a dissipative type, that is, it satisfies $g(x,u)u \geq b|u|^{d+1}$, then the boundary condition in the limit problem is $u = 0$, that is, we obtain a homogeneus Dirichlet boundary condition. We show the convergence of solutions in H^1 and C^0 norms and the convergence of the eigenvalues and eigenfunctions of the linearizations around the solutions. Moreover, if a solution of the limit problem is hyperbolic (non degenerate) and some extra conditions in g are satisfied, then we show that there exists one and only one solution of the perturbed problem nearby.

1 Introduction

In this article we continue the analysis initiated in [3], and announced in [2], on the behavior of the solutions of an elliptic equation with nonlinear boundary conditions of the type

$$\begin{cases} \Delta u + f(x,u) = 0 \text{ in } \Omega_\epsilon \\ \dfrac{\partial u}{\partial n} + g(x,u) = 0 \text{ in } \partial\Omega_\epsilon. \end{cases} \tag{1.1}$$

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when the boundary of the domain presents a highly oscillatory behavior as the parameter $\epsilon \to 0$. The main assumption in [3] was that $\partial\Omega_\epsilon$, the boundary of the domain Ω_ϵ, is expressed in local charts as a Lipschitz deformation of $\partial\Omega_0$ with the Lipschitz constant uniformly bounded in ϵ. Roughly speaking, this kind of perturbation is characterized by the fact that locally around each point $x_0 \in \partial\Omega_0$ and for all $0 < r < 1$ we have $\dfrac{|\partial\Omega_\epsilon \cap B(x_0, r)|}{|\partial\Omega \cap B(x_0, r)|} \leq C$, for some constant C independent of x_0, r and ϵ, where we denote by $|\cdot|$ the $(N-1)$-dimensional measure. For instance, in a two dimensional setting, if $\partial\Omega_0$ is given locally around certain point by the graph of the function $y = \psi_0(x)$, then $\partial\Omega_\epsilon$ is given locally by the graph of a function $y = \psi_\epsilon(x)$ where $\psi_\epsilon \to \psi_0$ and $|\psi'_\epsilon(x)| \leq C$ uniformly in ϵ. This includes the case where $\psi_\epsilon(x) = \psi_0(x) + \epsilon \sin(x/\epsilon^\alpha)$, with $\alpha \leq 1$. In [3], we were able to obtain the correct limit equation of (2.5) when $\epsilon \to 0$. As a matter of fact, for a broad class of nonlinearities f and g, the differential equation is the same: $\Delta u + f(x, u) = 0$ in Ω_0 while the limit boundary condition is $\dfrac{\partial u}{\partial n} + \gamma(x)g(x, u) = 0$ with $\gamma(x) \geq 1$ a factor that depends on the oscillations of the boundary. In certain sense we may say that the oscillations at the boundary amplify the effect of the nonlinearity $g(x, u)$ at the point $x \in \partial\Omega_0$ by a factor $\gamma(x) \geq 1$. Hence, if $g(x, u)$ is a dissipative nonlinearity so that energy is lost through the boundary, then the oscillations increase the energy loss. While if the effect of the nonlinearity is to drive energy into the system through the boundary the oscillations increase the intake of energy.

In the present paper we treat the case where $\partial\Omega_\epsilon$ is not a uniform Lipschitz deformation of $\partial\Omega_0$. This case, which includes the example above with $\alpha > 1$, can be characterized (roughly speaking) by $\dfrac{|\partial\Omega_\epsilon \cap B(x_0, r)|}{|\partial\Omega \cap B(x_0, r)|} \to +\infty$, which is to say that the factor $\gamma(x) = +\infty$. This extremely high oscillating behavior of the boundary interacts in a nontrivial way with the nonlinear boundary condition. With the interpretation described above we expect that if the nonlinearity $g(x, u)$ is strongly dissipative, that is $g(x, u)u \geq b|u|^{d+1}$ for some $d \geq 0$, then the effect of the oscillations is to amplify the dissipativity of the boundary and we expect to converge to the most dissipative boundary condition, which is the boundary condition $u = 0$, that is, the homogeneous Dirichlet boundary condtion. This is what is shown in this paper (see Theorem 2.2). Notice that the case $g \equiv 0$, that is, the case of homogeneous Neumann boundary condition was treated in [5] and it was shown that this condition is also preserved in the limit for many different perturbations of the boundary, including the ones treated in this work (see [5], Section 5.1). On the other hand, if $g(u) \sim -\alpha u$ near $u = 0$, then, as it was announced in [2] we do expect a rather complicated behavior of the set of stationary solutions, with a sequence of bifurcations from the trivial solution $u = 0$, as we make the parameter $\epsilon \to 0^+$.

The behavior of solutions of elliptic partial differential equations in the presence of boundary oscillations is a subject that has been addressed in the literature by different authors. In [13] the authors study the nonlinear problem with linear Robin boundary conditions of the type $\frac{\partial u}{\partial n} + bu = 0$ with $b > 0$ and they show that the limit behaves like in the present paper. We would like to mention [20] for a general reference of homogenization, including boundary homogenization and to [10] for a general reference on reticulated structures, which have some similarities with our problem. Also, the works [9, 14] deal with boundary homogenization with different boundary conditions and the coefficients appearing in the boundary condition depend also on the parameter ϵ. In [18], the authors treat homogeneous Dirichlet boundary conditions for the Poisson problem in the presence of boundary oscillations and they are able to obtain good estimates on the asymptotic expansion of the solution in terms of the parameter ϵ. We also refer to [19, 25] for interesting applications of boundary homogenization to climatization models. For the behavior of the equation of fluids with roughs boundaries we refer to [1, 15, 8] and references therein. Also, we refer to [6, 12] for interesting problems where the amplitudes of the oscillations at the boundary do not go to zero. In [25, 11] they treat an interesting homogenization problem of a reticulated structure with nonlinear boundary conditions depending also on the parameter ϵ and obtain a limit problem where the combined effect of the reticulated structure and the boundary condtions produces an extra term in the equation.

The articles mentioned above, appart from [25, 11], and most of the references in the literature address linear problems. In this respect, our work is different since we study the nonlinear problem, even nonlinear boundary condtions, we show the convergence of solutions of the nonlinear problem to the limiting problem in strong norms, that is H^1 and L^∞, and even show that the stability properties of the solutions are preserved in the limit. Moroever, we provide an interesting uniqueness result in the following sense: if we have a solution of the limit problem which is hyperbolic (in the sense that the linearized problem does not have zero as an eigenvalue) then there exists a unique solution of the peturbed problem nearby this solution, see Theorem 2.2 and Theorem 2.4.

This paper ir organized as follow. In section 2, we provide appropirate definitions on the domains considered and state the main results. In section 3, we go over several technical results. We provide a proof of the uppersemicontinuity of solutions in Section 4 and the lowersemicontinuity is shown in Section 5. Finally the convergence of the spectra of the linearizations is treated in Section 6. We include at the end a short Appendix with some known results on convergence of resolvent operators defined in different spaces.

2 Setting and main results

In this section we will clarify the setting of the problem and will state the main results of the paper. As a matter of fact, the setting described in this section, specially the part related to hypothesis **(H)** below is similar to the one from [3]. The part containing hypothesis **(I)** is different from [3].

We consider a family of smooth, bounded domains $\Omega_\epsilon \subset \mathbb{R}^N$, $N \geq 2$, for $0 \leq \epsilon \leq \epsilon_0$, for some $\epsilon_0 > 0$ fixed and we regard Ω_ϵ as a perturbation of the fixed domain $\Omega \equiv \Omega_0$. We consider the following condition on the domain

(H) i) for all $K \subset \Omega$, K compact, there exists $\epsilon(K) > 0$ such that $K \subset \Omega_\epsilon$ for $0 < \epsilon < \epsilon(K)$.

ii) There exists a finite open cover $\{U_i\}_{i=0}^m$ of Ω such that $\overline{U}_0 \subset \Omega$, $\partial\Omega \subset \cup_{i=1}^m U_i$ and for each $i = 1, \dots, m$, there exists a Lipschitz diffeomorphism $\Phi_i : Q_N \to U_i$, where

$Q_N = (-1,1)^N \subset \mathbb{R}^N$, such that

$$\Phi_i(Q_{N-1} \times (-1,0)) = U_i \cap \Omega,$$

$$\Phi_i(Q_{N-1} \times \{0\}) = U_i \cap \partial\Omega.$$

Moreover, we assume that $\overline{\Omega}_\epsilon \subset \cup_{i=0}^m U_i$ and for each $i = 1, \dots, m$ there exist Lipschitz functions $\rho_{i,\epsilon} : Q_{N-1} \to (-1,1)$ such that $\Phi_i^{-1}(U_i \cap \partial\Omega_\epsilon)$ is the graph of $\rho_{i,\epsilon}$. This means $U_i \cap \partial\Omega_\epsilon = \Phi_i(\{(x', \rho_{i,\epsilon}(x')), x' \in Q_{N-1}\})$, where we denote $(x_1, \dots, x_{N-1})$ by x'. We assume that $\rho_{i,\epsilon} \to 0$, when $\epsilon \to 0$, $i = 1, \dots, m$, uniformly in Q_{N-1}.

We observe that if $\Omega \subset \Omega_\epsilon$, then condition **(H)** i) is satisfied.

Condition **(H)** i) implies that there exists a nonincreasing function $\beta(\epsilon)$, with $\beta(\epsilon) \to 0$ when $\epsilon \to 0$, such that, if

$$K_\epsilon = \{x \in \Omega : d(x, \partial\Omega) > \beta(\epsilon)\}, \tag{2.1}$$

then $K_\epsilon \subset \Omega_\epsilon$. Moreover, by condition **(H)** ii) we have that

$$\lim_{\epsilon \to 0} |\Omega_\epsilon \setminus K_\epsilon| = 0.$$

We consider the following mappings:

$$T_{i,\epsilon} : Q_N \to Q_N,$$

defined by

$$T_{i,\epsilon}(x', s) = \begin{cases} (x', s + s\rho_{i,\epsilon}(x') + \rho_{i,\epsilon}(x')), & \text{for } s \in (-1,0) \\ (x', s - s\rho_{i,\epsilon}(x') + \rho_{i,\epsilon}(x')), & \text{for } s \in [0,1). \end{cases}$$

Also,

$$\Phi_{i,\epsilon} := \Phi_i \circ T_{i,\epsilon} : Q_N \to U_i$$

$$\Psi_{i,\epsilon} := \Phi_i \circ T_{i,\epsilon} \circ \Phi_i^{-1} : U_i \cap \Omega \to U_i \cap \Omega_\epsilon,$$

and, for $-1 < \eta < 1$, we also denote by

$$\begin{aligned} \phi_{i,\epsilon}^\eta : Q_{N-1} &\to U_i \cap \Omega_\epsilon \\ x' &\to \Phi_{i,\epsilon}(x', \eta) \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} \phi_{i,0}^\eta : Q_{N-1} &\to U_i \cap \Omega \\ x' &\to \Phi_i(x', \eta) \end{aligned} \tag{2.3}$$

Notice that $\phi_{i,\epsilon}^0$ and ϕ_i^0 are local parameterization of $\partial\Omega_\epsilon$ and $\partial\Omega$, respectively. Furthermore, observe that all the maps above are Lipschitz, although the Lipschitz constant may not be bounded as $\epsilon \to 0$.

As we mentioned in [3], boundary integrals over $\partial\Omega_\epsilon$ can be expressed, using standard partition of unity and localization arguments, as a sum of boundary integrals over $\partial\Omega_\epsilon \cap U_i$,

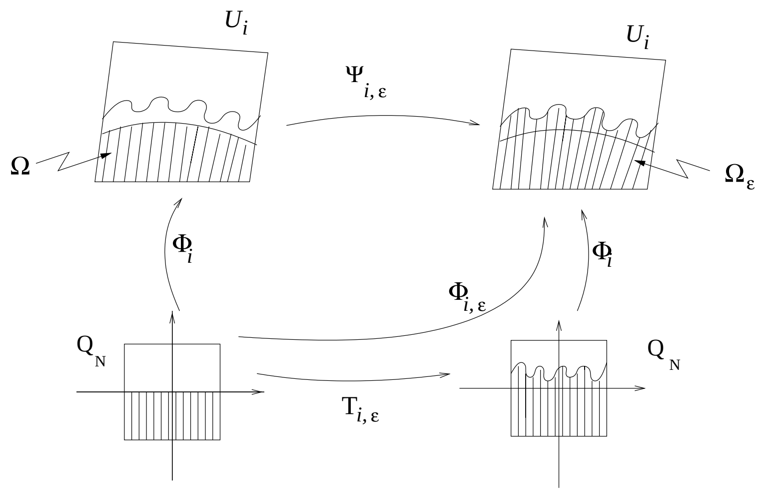

Figure 1: A representation of functions Φ_i, Ψ_i, T_i, etc..

$i = 1, \ldots, m$. Moreover, this boundary integrals over $\partial\Omega_\epsilon \cap U_i$ can be written, via a change of variables, as

$$\int_{\partial\Omega_\epsilon \cap U_i} u(x)dS = \int_{[-1,1]^{N-1}} u(\phi^0_{i,\epsilon}(x', 0))J\phi^0_{i,\epsilon}(x')dx'$$

where

$$J\phi = \sqrt{\sum_{j=1}^{N} (det(\mathrm{Jac}\,\phi)_j)^2}$$

and $(\mathrm{Jac}\,\phi)_j$ is the jacobian matrix without the j-th row. We observe that $J\phi$ measures locally the deformation of $(-1, 1)^{N-1} \times \{0\}$ in his image.

The behavior of $J\phi_{i\epsilon}$ as $\epsilon \to 0$ will be very important to decide the behavior of the solutions of (2.5) as $\epsilon \to 0$. As a matter of fact, our main hypothesis in this paper is the following:

(I) For each $t > 1$ the set $\{x \in (-1, 1)^{N-1} \times \{0\} : |J\phi^0_{i,\epsilon}(x)| \leq t\}$ satisfies that its $(N-1)$–dimensional measure goes to zero as $\epsilon \to 0$, for all $i = 1, \ldots, m$.

Remark 2.1. *Recall that in [3] the main hypothesis we use was the following:*

(F) *i)* $\|\nabla\rho_{i,\epsilon}\|_{L^\infty} \leq C$, *with* C *independent of* ϵ, $i = 1, \ldots, m$, *and*

ii) For each $i = 1, \ldots, m$, *there exists a function* $\gamma_i \in L^\infty(Q_{N-1})$ *such that*

$$J\phi^0_{i,\epsilon} \equiv |\frac{\partial\phi^0_{i,\epsilon}}{\partial x_1} \wedge \ldots \wedge \frac{\partial\phi^0_{i,\epsilon}}{\partial x_{N-1}}| \overset{\epsilon\to 0}{\longrightarrow} \gamma_i, \quad w - L^1(Q_{N-1}) \tag{2.4}$$

where $v_1 \wedge \ldots \wedge v_{N-1}$ is the exterior product of the $(N-1)$ vectors $v_1, \ldots, v_{N-1} \in \mathbb{R}^N$.

In [3] was proved that if condition **(F)** *is satisfied then the boundary condition of the limit problem is $\frac{\partial u}{\partial n} + \gamma g(x,u) = 0$.*

With respect to the equations, we will be interested in studying the behavior of the solutions of the elliptic equation with nonlinear boundary condition of the type,

$$\begin{cases} -\Delta u + u = f(x,u) & \text{in } \Omega_\epsilon, \\ \dfrac{\partial u}{\partial n} + g(x,u) = 0 & \text{on } \partial\Omega_\epsilon, \end{cases} \tag{2.5}$$

where the nonlinearities $f : U \times I\!R \to I\!R$, $g : U \times I\!R \to I\!R$ are continuous in both variables and C^2 in the second one. Moreover, U is a bounded domain containing $\bar{\Omega}_\epsilon$, for all $0 \leq \epsilon \leq \epsilon_0$.

For $0 < \epsilon \leq \epsilon_0$, We will denote by $\mathcal{E}_\epsilon = \{u_\epsilon \in H^1(\Omega_\epsilon) : u_\epsilon \text{ is a solution of (2.5)}\}$, $\mathcal{E}_{\epsilon,R} = \{u_\epsilon \in \mathcal{E}_\epsilon : \|u_\epsilon\|_{L^\infty(\Omega_\epsilon)} \leq R\}$ and $\mathcal{E}_{\epsilon,R^-} = \{u_\epsilon \in \mathcal{E}_\epsilon : \|u_\epsilon\|_{L^\infty(\Omega_\epsilon)} < R\}$

Our main results are stated in the following theorems.

Theorem 2.2. *Assume that hypotheses* **(H)** *and* **(I)** *are satisfied and moreover that the nonlinearity g satisfies a dissipative condition of the type:*

$$\exists\, b > 0,\ d \geq 1,\ \ s.t.\ \ g(x,s)s \geq b|s|^{d+1}, \forall\, |s| \leq R+1,\ x \in U. \tag{2.6}$$

Let u_ϵ^*, $0 < \epsilon \leq \epsilon_0$, be a family of solutions of problem (2.5) satisfying $\|u_\epsilon^*\|_{L^\infty(\Omega_\epsilon)} \leq R$, that is $u_\epsilon^* \in \mathcal{E}_{\epsilon,R}$, for some constant $R > 0$ independent of ϵ. Then,
i) There exists a subsequence, still denoted by u_ϵ^*, and a function $u_0^* \in \mathcal{E}_{0,R}$, that is $u_0^* \in H_0^1(\Omega)$ with $\|u_0^*\|_{L^\infty(\Omega)} \leq R$, solution of the problem

$$\begin{cases} -\Delta u + u = f(x,u) \ \ in\ \Omega, \\ u = 0 \ \ in\ \partial\Omega, \end{cases} \tag{2.7}$$

with the property that $\|u_\epsilon^* - u_0^*\|_{H^1(\Omega_\epsilon)} + \|u_\epsilon^* - u_0^*\|_{L^\infty(\Omega_\epsilon)} \to 0$ as $\epsilon \to 0$.
ii) If $d = 1$ in (2.6) and if the equilibrium point u_0^* is hyperbolic, in the sense that $\lambda = 0$ is not an eigenvalue of the linearized problem of (2.7) around u_0^*, then, there exists a $\delta > 0$ small such that problem (2.5) has one and only one solution $u_\epsilon^* \in \mathcal{E}_{\epsilon,R^-}$ satisfying $\|u_\epsilon^* - u_0^*\|_{H^1(\Omega_\epsilon)} \leq \delta$ for ϵ small enough.

Remark 2.3. *Observe that since g is a continuous function, condition (2.6) implies that $g(x,0) \equiv 0$ for all $x \in U$*

We will also be able to prove the spectral convergence of the linearizations around the equilibrium points. Observe that if u_ϵ^* is a solution of (2.5) then, the spectra of the linearization of (2.5) around u_ϵ^* is given by the eigenvalue problem

$$\begin{cases} -\Delta w + w - \partial_u f(x,u_\epsilon^*)w = \lambda w & \text{in } \Omega_\epsilon, \\ \dfrac{\partial w}{\partial n} + \partial_u g(x,u_\epsilon^*)w = 0 & \text{on } \partial\Omega_\epsilon. \end{cases} \tag{2.8}$$

Similarly, if u_0^* is a solution of (2.7), then the spectra of its linearization is given by the eigenvalue problem

$$\begin{cases} -\Delta w + w - \partial_u f(x, u_0^*)w = \lambda w & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.9}$$

Notice that both problems, (2.8) and (2.9), are selfadjoint and of compact resolvent. Hence, the eigenvalues of (2.8) are given by a sequence $\{\lambda_n^\epsilon\}_{n=1}^\infty$, ordered and counting their multiplicity, with $\lambda_n^\epsilon \to +\infty$ as $n \to +\infty$. Similarly the eigenvalues of (2.9) are also given by a sequence $\{\lambda_n^0\}_{n=1}^\infty$ with $\lambda_n^0 \to +\infty$.

Theorem 2.4. *In the conditions of Theorem 2.2, if $u_\epsilon^* \in \mathcal{E}_{\epsilon,R}$ for $0 \le \epsilon \le \epsilon_0$, $\|u_\epsilon^* - u_0^*\|_{H^1(\Omega_\epsilon)} \to 0$ and there exists a constant $\tilde{b} > 0$ such that $\partial_u g(x, u_\epsilon^*(x)) \ge \tilde{b}$, then the eigenvalues and eigenfunctions of the linearization of (2.5) around u_ϵ^* converge to the eigenvalues and eigenfunctions of the linearizations of (2.7) around u_0^*. That is, for each fixed $n \in N$, $\lambda_n^\epsilon \to \lambda_n^0$, as $\epsilon \to 0$. Moreover, if we denote by $\{\varphi_n^\epsilon\}_{n=1}^\infty$ a set of orthonormal eigenfunctions associated to $\{\lambda_n^\epsilon\}_{n=1}^\infty$, then for each sequence $\epsilon_k \to 0$ there is another subsequence, that we still denote by ϵ_k, and a set of orthonormal eigenfunctions $\{\varphi_n^0\}_{n=1}^\infty$ associated to $\{\lambda_n^0\}_{n=1}^\infty$, such that, for all $n \in N$, we have $\|\varphi_n^{\epsilon_k} - \varphi_n^0\|_{H^1(\Omega_{\epsilon_k})} \to 0$ as $\epsilon_k \to 0$.*

Remark 2.5. *Since in Theorem 2.2 we are concerned with solutions satisfying a uniform bound of the type $\|u_\epsilon\|_{L^\infty(\Omega_\epsilon)} \le R$, we may modify the nonlinearities f and g outside the region $|u| \le R$ without modifying any of these solutions. Hence, we may perform a cut-off in the nonlinearities f and g in such a way that*

$$|f(x,u)| + |\partial_u f(x,u)| \le M, \quad x \in B, \quad u \in \mathbb{R} \tag{2.10}$$

$$|g(x,u)| + |\partial_u g(x,u)| \le M, \quad x \in U, \quad u \in \mathbb{R} \tag{2.11}$$

and we may also assume that the cut-off is performed so that the following also holds

$$g(x,s)s \ge b|s|, \forall\, |s| \ge R+1,\, x \in U. \tag{2.12}$$

3 Some technical results

Before stating and proving some technical results, let us make some general comments on the setting of the problem. This will help us to have a clear picture of the difficulties that we will encounter below.

i). Since problems (2.5) and (2.7) are not posed in the same domain, we will need to devise a way to compare functions $u_\epsilon \in H^1(\Omega_\epsilon)$ with functions $u \in H^1(\Omega)$. An appropriate way to accomplish this task is through the notion of E-convergence. This concept was considered initially by F. Stummel (see the works [21, 22, 23]) which he denoted as discrete convergence. We also refer to [24, 7] and references therein, for a detailed study of this notion and its applications to differential equations. For the sake of completeness we have included Appendix A, where we explore the basic fact about E-convergence. Also, we refer to [4] for another nontrivial example where this concept of E-convergence has been used to study a domain perturbation problem.

Hence, we consider the operator $E_\epsilon : H^1(\Omega) \to H^1(\Omega_\epsilon)$, which is defined as $E_\epsilon = R_\epsilon \circ E$, where $E : H^1(\Omega) \to H^1(\mathbb{R}^N)$ is an extension operator and R_ϵ is the restriction operator from functions defined in $\mathbb{R}^N$ to functions defined in Ω_ϵ. Observe that we also have $E_\epsilon : L^p(\Omega) \to L^p(\Omega_\epsilon)$ for all $1 \le p \le \infty$ and that in each case we have $\|E_\epsilon u\|_{X_\epsilon} \to \|u\|_{X_0}$ where $X_\epsilon = H^1(\Omega_\epsilon)$ or $L^p(\Omega_\epsilon)$, $\epsilon \ge 0$.

As it is stated in Appendix A we will say that $u_\epsilon \xrightarrow{E} u$ if $\|u_\epsilon - E_\epsilon u_0\|_{H^1(\Omega_\epsilon)} \to 0$, see Definition A.1. Also we have a notion of weak E-convergence, which is defined as follows: $u_\epsilon \xrightharpoonup{E} u$ if $(u_\epsilon, w_\epsilon)_{H^1(\Omega_\epsilon)} \to (u_0, w_0)_{H^1(\Omega_0)}$ for any sequence $w_\epsilon \xrightarrow{E} w_0$, see Definition A.2. Moreover, as it can be seen in Appendix A there is a number of results on E-convergence that is applicable to our situation.

ii). The operator E_ϵ allows us to pass a function from Ω to Ω_ϵ, but we still need a mechanism to pass a function from Ω_ϵ to Ω, which is usually accomplished with the use of extension operators from Ω_ϵ to $\mathbb{R}^N$ and restricting it to Ω. But, as a fundamental difference with respect to the analysis and techniques used in [3], since **(I)** is satisfied, we cannot use that the norm of the extension operators from Ω_ϵ to $\mathbb{R}^N$, Sobolev embeddings in Ω_ϵ and trace operators over $\partial\Omega_\epsilon$ are uniformly bounded in ϵ. In particular the extension operator P_{Ω_ϵ} defined in Proposition 4.1 in [3] and the operators $P_{\Omega_\epsilon, U}$ defined in Remark 4.2 in [3], which were extensively used in that article to pass a function from Ω_ϵ to Ω, are of little use in the present one. Therefore, we define another operator $\hat{E}_\epsilon : H^1(\Omega_\epsilon) \to H^1(\Omega)$, which although it is not properly speaking an extension operator it will play the role of it. Recall the definition of the set $K_\epsilon \subset \Omega \cap \Omega_\epsilon$ given by (2.1), $K_\epsilon = \{x \in \Omega : d(x, \partial\Omega) > \beta(\epsilon)\}$, where $\beta(\epsilon)$ is nonincreasing and $\beta(\epsilon) \to 0$. In particular, K_ϵ is a smooth Lipschitz deformation of Ω satisfying hypothesis **(H)** and **(F)** (see Remark 2.1 and [3]). Hence, for this family K_ϵ we have the operators $P_{K_\epsilon, \Omega} : H^1(K_\epsilon) \to H^1(\Omega)$ given by $P_{K_\epsilon, \Omega} = R_\Omega \circ P_{K_\epsilon}$, where R_Ω is the restriction operator to the open set Ω and P_{K_ϵ} is the extension operator from $H^1(K_\epsilon)$ to $H^1(\mathbb{R}^N)$, for more details see Section 4 in [3]. Let us define $\hat{E}_\epsilon : H^1(\Omega_\epsilon) \to H^1(\Omega)$,

$$\hat{E}_\epsilon(u_\epsilon) = P_{K_\epsilon, \Omega} \circ R_{K_\epsilon}(u_\epsilon) = P_{K_\epsilon, \Omega}(u_{\epsilon|K_\epsilon}) \tag{3.1}$$

where R_{K_ϵ} is the restriction operator to K_ϵ.

iii). If we have a close look at hypothesis **(H)** we see that the domains Ω_ϵ are a particular case of Example 5.1 of [5] and we may apply the results of [5] to our case. In particular, we have that Proposition 2.3 of [5] holds true and in particular we have the following two facts:

$$\text{If } \|u_\epsilon\|_{H^1(\Omega_\epsilon)} \le M, \text{ then } \|u_\epsilon\|_{L^2(\Omega_\epsilon \setminus K_\epsilon)} \to 0, \text{ as } \epsilon \to 0. \tag{3.2}$$

$$\text{If } \tau_\epsilon \equiv \min\{\frac{\|\nabla\psi_\epsilon\|^2_{L^2(\Omega_\epsilon)}}{\|\psi_\epsilon\|^2_{L^2(\Omega_\epsilon)}} : \psi_\epsilon \in H^1(\Omega_\epsilon), \psi_\epsilon \equiv 0, \text{ in } K_\epsilon\} \text{ then } \tau_\epsilon \to +\infty, \text{ as } \epsilon \to 0 \tag{3.3}$$

Moreover, we would like to recall an important inequality due to V. Maz'ja, which can be found in [17]: there exists a constan $C = C(N, |\Omega|) > 0$ such that for any funciton $u \in H^1(\Omega)$ we have

$$\|u\|_{L^p(\Omega)} \leq C(\|u\|_{H^1(\Omega)} + \|u\|_{L^2(\partial\Omega)}), \quad \forall\, 1 \leq p \leq \frac{2N}{N-1} \tag{3.4}$$

Observe that the power of Maz'ja's inequality lies in the fact that the constant C does not depend directly on the domain but only through its measure. Therefore, it is easy to see that (3.4) can be applied to our case if just hypothesis **(H)** holds. We will use this inequality below.

We are ready now to state and prove some important technical lemmas. With these two operators E_ϵ and $\hat{E}_\epsilon$ we can show,

Lemma 3.1. *Assume* **(H)** *is satisfied and let $u_\epsilon, v_\epsilon, w_\epsilon \in H^1(\Omega_\epsilon)$ a family of functions such that $\|u_\epsilon\|_{H^1(\Omega_\epsilon)}$, $\|v_\epsilon\|_{H^1(\Omega_\epsilon)}$, $\|w_\epsilon\|_{H^1(\Omega_\epsilon)} \leq M$. Then,*

(i) There exists a subsequence, denote by u_{ϵ_k}, and $u_0 \in H^1(\Omega)$ such that $u_{\epsilon_k} \xrightharpoonup{E} u_0$ and $\|u_\epsilon - E_\epsilon u_0\|_{L^2(\Omega_\epsilon)} \to 0$.
(ii) If there exists $u_0 \in H^1(\Omega)$ such that $\hat{E}_\epsilon u_\epsilon := \hat{u}_\epsilon$ converges weakly to u_0 in $H^1(\Omega)$ then $u_\epsilon \xrightharpoonup{E} u_0$.
(iii) If $v_\epsilon \xrightharpoonup{E} v$, $w_\epsilon \xrightharpoonup{E} w$ and $\|V_\epsilon\|_{L^\infty} \leq K$, $V_\epsilon \to V_0$ weakly in $L^2(\Omega)$, then

$$\int_{\Omega_\epsilon} V_\epsilon v_\epsilon w_\epsilon \to \int_\Omega V_0 v w \tag{3.5}$$

In particular, (3.5) holds for the case $v_\epsilon = E_\epsilon v$ $w_\epsilon = E_\epsilon w$.

Proof. (i) With a standard argument we can get a subsequence u_{ϵ_k} and a function $u_0 \in H^1(\Omega)$ such that $u_{\epsilon_k|_K} \rightharpoonup u_{0|_K}$ $w - H^1(K)$ and $\|u_{\epsilon_k} - u_0\|_{L^2(K)} \to 0$ for all $K \subset\subset \Omega$.

Following the argument from Lemma 4.3 (i) in [3] we obtain that $u_{\epsilon_k} \xrightharpoonup{E} u_0$.

Moreover, we can get a nested sequence of sets K_{ϵ_k}, where K_ϵ is defined in (2.1) such that $\cup K_{\epsilon_k} = \Omega$ and $\|u_{\epsilon_k} - u_0\|_{L^2(K_{\epsilon_k})} \to 0$. Hence

$$\|u_{\epsilon_k} - E_{\epsilon_k} u_0\|^2_{L^2(\Omega_{\epsilon_k})} \leq \|u_{\epsilon_k} - u_0\|^2_{L^2(K_{\epsilon_k})} + \|u_{\epsilon_k}\|^2_{L^2(\Omega_{\epsilon_k} \setminus K_{\epsilon_k})} + \|E_\epsilon u_0\|^2_{L^2(\Omega_{\epsilon_k} \setminus K_{\epsilon_k})}$$

But $\|u_{\epsilon_k} - u_0\|^2_{L^2(K_{\epsilon_k})} \to 0$ as it has been shown above, and $\|E_\epsilon u_0\|^2_{L^2(\Omega_{\epsilon_k} \setminus K_{\epsilon_k})} \to 0$ since u_0 is a fixed function and $|\Omega_\epsilon - K_\epsilon| \to 0$.

The fact that $\|u_\epsilon\|_{L^2(\Omega_\epsilon \setminus K_\epsilon)} \to 0$ follows from (3.2).

(ii) Since $\|u_\epsilon\|_{H^1(\Omega_\epsilon)} \leq M$ by Proposition A.3 it is sufficient to prove that for all $v \in H^1(\Omega)$ we have $(u_\epsilon, E_\epsilon v)_{H^1(\Omega_\epsilon)} \to (u, v)_{H^1}$. In fact,

$$\begin{aligned}(u_\epsilon, E_\epsilon v)_{H^1(\Omega_\epsilon)} - (u_0, v)_{H^1(\Omega)} &= (\hat{u}_\epsilon - u_0, v)_{H^1(\Omega)} + (u_\epsilon, E_\epsilon v)_{H^1(\Omega_\epsilon \setminus K_\epsilon)} \\ &\quad - (\hat{u}_\epsilon, v)_{H^1(\Omega \setminus K_\epsilon)}\end{aligned}$$

Since $|\Omega_\epsilon \setminus K_\epsilon| \to 0$, $|\Omega \setminus K_\epsilon| \to 0$, $\hat{u}_\epsilon \to u_0$ weakly in $H^1(\Omega)$ and $\|\hat{u}_\epsilon\|_{H^1(\Omega\setminus K_\epsilon)} \leq \tilde{M}\|u_\epsilon\|_{H^1(\Omega_\epsilon)}$, it follows that $(u_\epsilon, E_\epsilon v)_{H^1(\Omega_\epsilon)} \to (u_0, v)_{H^1(\Omega)}$.

(iii) We have the following

$$
\begin{aligned}
&|\int_{\Omega_\epsilon} V_\epsilon v_\epsilon w_\epsilon - \int_\Omega V_0 v w| \\
&\leq |\int_{\Omega_\epsilon} V_\epsilon v_\epsilon w_\epsilon - \int_{\Omega_\epsilon} V_\epsilon E_\epsilon v E_\epsilon w| + |\int_{\Omega_\epsilon} V_\epsilon E_\epsilon v E_\epsilon w - \int_\Omega V_0 v w| \\
&\leq |\int_{\Omega_\epsilon} V_\epsilon v_\epsilon w_\epsilon - \int_{\Omega_\epsilon} V_\epsilon E_\epsilon v E_\epsilon w| + |\int_{K_\epsilon} V_\epsilon E_\epsilon v E_\epsilon w - \int_{K_\epsilon} V_0 v w| \\
&\quad + |\int_{\Omega_\epsilon\setminus K_\epsilon} V_\epsilon E_\epsilon v E_\epsilon w| + |\int_{\Omega\setminus K_\epsilon} V_0 v w| \\
&\leq |\int_{\Omega_\epsilon} V_\epsilon v_\epsilon (w_\epsilon - E_\epsilon w)| + |\int_{\Omega_\epsilon} V_\epsilon E_\epsilon w (v_\epsilon - E_\epsilon v)| + |\int_{K_\epsilon} (V_\epsilon - V_0) v w| \\
&\quad + |\int_{\Omega_\epsilon\setminus K_\epsilon} V_\epsilon E_\epsilon v E_\epsilon w| + |\int_{\Omega\setminus K_\epsilon} V_0 v w|
\end{aligned}
$$

Since $\|V_\epsilon\|_{L^\infty} \leq K$, the convergence of v_ϵ to v in $L^2(\mathbb{R}^N)$ and the convergence of w_ϵ to w in $L^2(\mathbb{R}^N)$, we get that the first and second integrals go to zero. Using that $V_\epsilon \to V_0$ weakly in $L^2(\Omega)$ and $\|V_\epsilon\|_{L^\infty} \leq K$ then $V_\epsilon \to V_0$ weakly in $L^q(\Omega)$ for any $1 \leq q < \infty$ and the third integral goes to zero. Since $|\Omega_\epsilon \setminus K_\epsilon| \to 0$ and $|\Omega \setminus K_\epsilon| \to 0$ and using the boundedness of $V_\epsilon E_\epsilon v E_\epsilon w$ and $V_0 v w$ in L^r for some $r > 1$, we get the convergence of the fourth and fifth integrals to 0 ∎

Let us introduce some notation. For any $0 \leq \eta < 1$, we define $\Gamma_{i,\eta} := \Phi_i(Q_{N-1} \times \{-\eta\})$ and $(U_i \cap \Omega)_\eta = \Phi_i(Q_{N-1} \times (-1, -\eta))$. It follows from the conditions on the perturbations Ω_ϵ that for each $\eta > 0$, there exists $\epsilon_0 = \epsilon_0(\eta) > 0$ such that $\Gamma_{i,\eta} \subset U_i \cap K_\epsilon \subset U_i \cap \Omega_\epsilon$ and $(U_i \cap \Omega)_\eta \subset \Omega \cap \Omega_\epsilon$ for $\epsilon < \epsilon_0$.

Using this notation, we have

Lemma 3.2. *Assume* **(H)** *is satisfied. Let $w_\epsilon \in W^{1,p}(\Omega_\epsilon)$, $\|w_\epsilon\|_{W^{1,p}(\Omega_\epsilon)} \leq M$ for some $1 \leq p \leq \infty$. If $1 < p < \infty$, then for $1 \leq q \leq p$*

$$\lim_{\epsilon\to 0} \int_{(U_i\cap\Omega)_\eta} |w_\epsilon \circ \Psi_{i,\epsilon} - w_\epsilon|^q = 0, \tag{3.6}$$

and

$$\lim_{\epsilon\to 0} \int_{\Gamma_{i,\eta}} |w_\epsilon \circ \Psi_{i,\epsilon} - w_\epsilon|^q = 0. \tag{3.7}$$

If $p = 1$ then there exists $K > 0$ independent of ϵ such that

$$\int_{(U_i\cap\Omega)_\eta} |w_\epsilon \circ \Psi_{i,\epsilon} - w_\epsilon| \leq K. \tag{3.8}$$

Remark 3.3. *If we are in the situation of an exterior perturbation of the domain Ω, that is, $\Omega \subset \Omega_\epsilon$, Lemma 3.2 holds with $\eta = 0$.*

Proof. Assume first that $1 < p < \infty$. To prove statements (3.6) and (3.7) for $1 < q \le p$ it will be enough to show them for $q = p$, since using Hölder inequality we will show it for all $1 \le q \le p$. Let us show first (3.6).

$$\int_{(U_i\cap\Omega)_\eta} |w_\epsilon \circ \Psi_{i,\epsilon} - w_\epsilon|^p = \int_{Q_{N-1}} \int_{-1}^{-\eta} |w_\epsilon \circ \Psi_{i,\epsilon} \circ \Phi_i(x', s) - w_\epsilon \circ \Phi_i(x', s)|^p J\Phi_i(x', s) ds dx'$$

$$\le \|J\Phi_i\|_{L^\infty} \int_{Q_{N-1}} \int_{-1}^{-\eta} |w_\epsilon \circ \Phi_i \circ T_{i,\epsilon}(x', s) - w_\epsilon \circ \Phi_i(x', s)|^p ds dx'$$

$$= \|J\Phi_i\|_{L^\infty} \int_{Q_{N-1}} \int_{-1}^{-\eta} |w_\epsilon \circ \Phi_i(x', s + (s+1)\rho_{i,\epsilon}(x')) - w_\epsilon \circ \Phi_i(x', s)|^p ds dx'$$

$$= \|J\Phi_i\|_{L^\infty} \int_{Q_{N-1}} \int_{-1}^{-\eta} |\int_s^{s+(s+1)\rho_{i,\epsilon}(x')} \frac{\partial w_\epsilon \circ \Phi_i}{\partial x_N}(x', x_N) dx_N|^p) ds dx'$$

$$\le \|J\Phi_i\|_{L^\infty} \int_{Q_{N-1}}\int_{-1}^{-\eta} ((s+1)|\rho_{i,\epsilon}(x')|)^{\frac{p}{p'}} \int_s^{s+(s+1)\rho_{i,\epsilon}(x')} |\frac{\partial (w_\epsilon \circ \Phi_i)}{\partial x_N}(x', x_N)|^p dx_N ds dx'$$

$$\le \|J\Phi_i\|_{L^\infty} \|\rho_{i,\epsilon}\|_{L^\infty}^{\frac{p}{p'}} \int_{Q_{N-1}} \int_{-1}^{-\eta} \int_s^{s+(s+1)\rho_{i,\epsilon}(x')} |\frac{\partial w_\epsilon \circ \Phi_i}{\partial x_N}(x', x_N)|^p dx_N ds dx'$$

$$\le \|J\Phi_i\|_{L^\infty} \|\rho_{i,\epsilon}\|_{L^\infty}^{\frac{p}{p'}} \int_{Q_{N-1}} \int_{-1}^{\rho_{i,\epsilon}(x')} |\nabla (w_\epsilon \circ \Phi_i)(x', x_N)|^p dx_N dx'$$

$$\le N^{p+1} \|\rho_{i,\epsilon}\|_{L^\infty}^{\frac{p}{p'}} \|J\Phi_i\|_\infty \|D\Phi_i\|_\infty^p \int_{Q_{N-1}} \int_{-1}^{\rho_{i,\epsilon}(x')} |\nabla w_\epsilon(\Phi_i(x', x_N))|^p dx_N dx'$$

$$\le \frac{N^{p+1} \|\rho_{i,\epsilon}\|_{L^\infty}^{\frac{p}{p'}} \|J\Phi_i\|_\infty \|D\Phi_i\|_\infty^p}{\inf_{z\in Q_N} J\Phi_i(z)} \int_{Q_{N-1}} \int_{-1}^{\rho_{i,\epsilon}(x')} |\nabla w_\epsilon(\Phi_i(x))|^p J\Phi_i(x) dx ds$$

$$= \frac{N^{p+1} \|\rho_{i,\epsilon}\|_{L^\infty}^{\frac{p}{p'}} \|J\Phi_i\|_\infty \|D\Phi_i\|_\infty^p}{\inf_{z\in Q_N} J\Phi_i(z)} \int_{U_i\cap\Omega_\epsilon} |\nabla w_\epsilon(\xi))|^p d\xi$$

$$\le \frac{N^{p+2} \|\rho_{i,\epsilon}\|_{L^\infty}^{\frac{p}{p'}} \|J\Phi_i\|_\infty \|D\Phi_i\|_\infty^p}{\inf_{z\in Q_N} J\Phi_i(z)} M^p \le C \|\rho_{i,\epsilon}\|_{L^\infty}^{\frac{p}{p'}}$$

The fact that $\|\rho_{i,\epsilon}\|_{L^\infty} \to 0$ as $\epsilon \to 0$ proves (3.6) in case $p > 1$. If $p = 1$, then $p/p' = 0$ and (3.8) holds. The proof of (3.7) follows similar arguments. ∎

4 Uppersemicontinuity of solutions

In this section we will provide a proof of Theorem 2.2 (i).

First, we consider for $\epsilon > 0$ the equations

$$\begin{cases} -\Delta w_\epsilon + w_\epsilon = f_\epsilon(x), \text{ in } \Omega_\epsilon \\ \frac{\partial w_\epsilon}{\partial n} + g(x, w_\epsilon) = 0 \text{ on } \partial\Omega_\epsilon \end{cases} \tag{4.1}$$

and for $\epsilon = 0$

$$\begin{cases} -\Delta w + w = f(x), \text{ in } \Omega \\ w = 0 \text{ on } \partial\Omega \end{cases} \tag{4.2}$$

where $f_\epsilon \in L^2(\Omega_\epsilon)$, for $0 < \epsilon \leq \epsilon_0$, $f \in L^2(\Omega)$ and g satisfies condition (2.6) and (2.12). We also observe that since g satisfy these hypotheses, then, for each $f_\epsilon \in L^2(\Omega_\epsilon)$ there exists at least one solution of (4.1), although uniqueness is not guaranteed a priori.

Now we prove

Proposition 4.1. *If $\|f_\epsilon\|_{L^2(\Omega_\epsilon)} \leq C$ independent of ϵ, then, the family w_ϵ of solutions of (4.1) satisfies*

$$\|\nabla w_\epsilon\|^2_{L^2(\Omega_\epsilon)} + \|w_\epsilon\|^2_{L^2(\Omega_\epsilon)} + \int_{\partial\Omega_\epsilon} |w_\epsilon|^{d(w_\epsilon(x))} \leq K. \tag{4.3}$$

for some K independent of ϵ, where

$$d(s) = \begin{cases} d+1, & \text{if } |s| \leq R+1 \\ 1, & \text{if } |s| \geq R+1 \end{cases} \tag{4.4}$$

where d and R are defined in (2.6).

Proof. In fact, since w_ϵ satisfies equation (4.1), we get

$$\int_{\Omega_\epsilon} |\nabla w_\epsilon|^2 + \int_{\partial\Omega_\epsilon} g(x, w_\epsilon) w_\epsilon + \int_{\Omega_\epsilon} w_\epsilon^2 = \int_{\Omega_\epsilon} f_\epsilon w_\epsilon.$$

By Hölder and Young inequalities, we get that

$$\int_{\Omega_\epsilon} f_\epsilon w_\epsilon \leq \delta \|w_\epsilon\|^2_{L^2(\Omega_\epsilon)} + C_\delta \|f_\epsilon\|^2_{L^2(\Omega_\epsilon)}.$$

Moreover, from conditions (2.6) and (2.12) ge get that

$$\int_{\partial\Omega_\epsilon} g(x, w_\epsilon) w_\epsilon \geq b \int_{\partial\Omega_\epsilon} |w_\epsilon|^{d(w_\epsilon(x))}$$

Therefore, for $\delta < 1$,

$$\min\{1-\delta, b\} \left(\|\nabla w_\epsilon\|^2_{L^2(\Omega_\epsilon)} + \|w_\epsilon\|^2_{L^2(\Omega_\epsilon)} + \int_{\partial\Omega_\epsilon} |w_\epsilon|^{d(w_\epsilon(x))} \right) \leq C_\delta \|f_\epsilon\|^2_{L^2(\Omega_\epsilon)}.$$

Since we are assuming that f_ϵ is a bounded family in $L^2(\Omega_\epsilon)$, then we get (4.3). ∎

Let us show now the following key result.

Proposition 4.2. *If $\|f_\epsilon\|_{L^2(\Omega_\epsilon)} \le C$ and w_ϵ is a solution of* (4.1)*, then there exist subsequences $\{f_{\epsilon_k}\}$, $\{w_{\epsilon_k}\}$, and functions $f_0, w_0 \in L^2(\Omega)$, such that $\hat{E}_\epsilon f_{\epsilon_k} \rightharpoonup f_0$ in $L^2(\Omega)$ and $w_{\epsilon_k} \xrightarrow{E} w_0$. Moreover, $w_0 \in H_0^1(\Omega)$ and it is the solution of (4.2) with $f = f_0$.*

Proof. Since $\|f_\epsilon\|_{L^2(\Omega_\epsilon)} \le C$, we obtain a subsequence f_{ϵ_k} and a function $f_0 \in L^2(\Omega)$ such that $f_{\epsilon_k} \rightharpoonup f_0$ weakly in $L^2(\mathbb{R}^N)$ (understanding that we have extended by zero to all of $\mathbb{R}^N$ the functions f_ϵ and f_0). For the subsequence f_{ϵ_k}, consider a sequence w_{ϵ_k} formed by solutions of (4.1). By (4.3), the sequence w_{ϵ_k} is bounded in $H^1(\Omega_{\epsilon_k})$. By Lemma 3.1(i), we get that there exists a subsequence, which we again denote by w_{ϵ_k} and a function $w_0 \in H^1(\Omega)$ such that $w_{\epsilon_k} \xrightarrow{E} w_0$ and $\|w_{\epsilon_k} - E_{\epsilon_k} w_0\|_{L^2(\Omega_{\epsilon_k})} \to 0$.

We use now similar arguments as in [13] to prove that, as a matter of fact, $w_0 \in H_0^1(\Omega)$. Since $w_0 \in H^1(\Omega)$ is a fixed function, we observe that given $\beta > 0$ small, there exists $\eta_0 > 0$ such that, for $\eta < \eta_0$, we have

$$| \int_{U_i \cap \partial\Omega} w_0 - \int_{\Gamma_{i,\eta}} w_0 | \le \beta. \tag{4.5}$$

Moreover, by Lemma 3.2, we get that for each $\beta > 0$ fixed and for $\eta < \eta_0$ also fixed, there exists $\epsilon_0 > 0$ such that

$$\int_{\Gamma_{i,\eta}} |w_\epsilon \circ \Psi_{i,\epsilon} - w_\epsilon| \le \beta, \qquad \text{for } 0 < \epsilon < \epsilon_0. \tag{4.6}$$

Moreover, the trace operator from $H^1(K_\eta)$ to $L^2(\Gamma_{i,\eta})$ is continuous and compact, then $w_{\epsilon|\Gamma_{i,\eta}}$ converges to $w_{0|\Gamma_{i,\eta}}$ in $L^2(\Gamma_{i,\eta})$. Hence, we can choose an even smaller ϵ_0 such that

$$\int_{\Gamma_{i,\eta}} |w_\epsilon - w_0| \le \beta \quad \text{ for } 0 < \epsilon < \epsilon_0. \tag{4.7}$$

Putting together (4.5), (4.6) and (4.7), we obtain that for $0 < \epsilon \le \epsilon_0$,

$$\int_{U_i \cap \partial\Omega} |w_0| \le 3\beta + \int_{\Gamma_{i,\eta}} |w_\epsilon \circ \Psi_{i,\epsilon}| \tag{4.8}$$

Consider now the sets $A_t^\epsilon = \{x' \in Q_{N-1} : J\phi_{i,\epsilon}^0(x'_{N-1}) \le t\}$ and $B_t^\epsilon = \{x' \in Q_{N-1} : J\phi_{i,\epsilon}^0(x'_{N-1}) > t\}$, so that $Q_{N-1} = A_t^\epsilon \cup B_t^\epsilon$, $A_t^\epsilon \cap B_t^\epsilon = \emptyset$ and, by hypothesis **(I)**, $|A_t^\epsilon| \to 0$ as $\epsilon \to 0$. Moreover

$$\int_{\Gamma_{i,\eta}} |w_\epsilon \circ \Psi_{i,\epsilon}| = \int_{\phi_{i,0}^{-\eta}(A_t^\epsilon)} |w_\epsilon \circ \Psi_{i,\epsilon}| + \int_{\phi_{i,0}^{-\eta}(B_t^\epsilon)} |w_\epsilon \circ \Psi_{i,\epsilon}|$$

But, for all $1 < p < \infty$, we have

$$\int_{\phi_{i,0}^{-\eta}(A_t^\epsilon)} |w_\epsilon \circ \Psi_{i,\epsilon}| \le \Big(\int_{\Gamma_{i,\eta}} |w_\epsilon \circ \Psi_{i,\epsilon}|^p \Big)^{\frac{1}{p}} [\mathcal{H}_{N-1}(\phi_{i,0}^{-\eta}(A_t^\epsilon))]^{\frac{1}{p'}}$$

where $\mathcal{H}_{N-1}$ is the $(N-1)$-dimensional Haussdorf measure.

Taking into account that $\|w_\epsilon\|_{H^1(K_\eta)} \le C$ and using Lemma 3.2 and trace theorems, we have, for $1 < p$ small,

$$\Big(\int_{\Gamma_{i,\eta}} |w_\epsilon \circ \Psi_i|^p\Big)^{\frac{1}{p}} \le \Big(\int_{\Gamma_{i,\eta}} |w_\epsilon \circ \Psi_i - w_\epsilon|^p\Big)^{\frac{1}{p}} + \Big(\int_{\Gamma_{i,\eta}} |w_\epsilon|^p\Big)^{\frac{1}{p}} \le C.$$

Since $\mathcal{H}_{N-1}(\Phi_i(A_t^\epsilon, -\eta)) \le C|A_t^\epsilon|_{N-1} \to 0$ as $\epsilon \to 0$ by **(I)** then,

$$\int_{\phi_{i,0}^{-\eta}(A_t^\epsilon)} |w_\epsilon \circ \Psi_{i,\epsilon}| \to 0, \text{ as } \epsilon \to 0.$$

Let us show now that

$$\int_{\phi_{i,0}^{-\eta}(B_t^\epsilon)} |w_\epsilon \circ \Psi_{i,\epsilon}| \le c(\eta) + C\int_{B_t^\epsilon} |w_\epsilon \circ \phi_{i,\epsilon}^0(x')|dx' \tag{4.9}$$

where $c(\eta) \to 0$ as $\eta \to 0$. For this,

$$\int_{\phi_{i,0}^{-\eta}(B_t^\epsilon)} |w_\epsilon \circ \Psi_{i,\epsilon}| = \int_{B_t^\epsilon} |w_\epsilon \circ \Psi_{i,\epsilon} \circ \phi_{i,0}^{-\eta}(x')| J\phi_{i,0}^{-\eta} dx'$$

But from the definition of $\phi_{i,0}^{-\eta}$ given in (2.2) we get that $J\phi_{i,0}^{-\eta} \le C$ independent of η for η small. Hence

$$\int_{\phi_{i,0}^{-\eta}(B_t^\epsilon)} |w_\epsilon \circ \Psi_{i,\epsilon}| \le C\int_{B_t^\epsilon} |w_\epsilon \circ \Psi_{i,\epsilon} \circ \phi_{i,0}^{-\eta}(x')|dx'$$

$$\le C\int_{B_t^\epsilon} |w_\epsilon \circ \Psi_{i,\epsilon} \circ \phi_{i,0}^{-\eta}(x') - w_\epsilon \circ \Psi_{i,\epsilon} \circ \phi_{i,0}^{0}(x')|dx' + C\int_{B_t^\epsilon} |w_\epsilon \circ \Psi_{i,\epsilon} \circ \phi_{i,0}^{0}(x')|dx'$$

$$= C\int_{B_t^\epsilon} |w_\epsilon \circ \Phi_i(x', -\eta + (1-\eta)\rho_{i,\epsilon}(x')) - w_\epsilon \circ \Phi_i(x', \rho_{i,\epsilon}(x'))|dx' + C\int_{B_t^\epsilon} |w_\epsilon \circ \phi_{i,\epsilon}^0(x')|dx'$$

$$\le C\int_{B_t^\epsilon}\int_{-\eta+(1-\eta)\rho_{i,\epsilon}(x')}^{\rho_{i,\epsilon}(x')} |\frac{d}{ds}(w_\epsilon \circ \Phi_i)(x', s)|dsdx' + C\int_{B_t^\epsilon} |w_\epsilon \circ \phi_{i,\epsilon}^0(x')|dx'$$

Applying Hölder inequality to the first integral we get

$$\le C(\eta)\|w_\epsilon\|_{H^1(\Omega_\epsilon)} + C\int_{B_t^\epsilon} |w_\epsilon \circ \phi_{i,\epsilon}^0(x')|dx'$$

where $C(\eta) \le C|B_t^\epsilon|\,|\eta|(1 + \max_i\{\|\rho_{i,\epsilon}\|_{L^\infty}\}) \max_i\{\|D\Phi_i\|_{L^\infty}\} \le \tilde{C}|\eta| \to 0$ as $\eta \to 0$. This shows (4.9). Let us show now that

$$\int_{B_t^\epsilon} |w_\epsilon \circ \phi_{i,\epsilon}^0(x')|dx' \le Ct^{-1/(d+1)}, \text{ for all } \epsilon \in (0, \epsilon_0) \tag{4.10}$$

For this, decompose the set $B_t^\epsilon = C_t^\epsilon \cup D_t^\epsilon$ where $C_t^\epsilon = \{x' \in B_t^\epsilon : |w_\epsilon(\phi_{i,\epsilon}^0(x'))| \le R+1\}$ and $D_t^\epsilon = \{x' \in B_t^\epsilon : |w_\epsilon(\phi_{i,\epsilon}^0(x'))| > R+1\}$, so that

$$\int_{B_t^\epsilon} |w_\epsilon \circ \phi_{i,\epsilon}^0(x')|dx' = \int_{C_t^\epsilon} |w_\epsilon \circ \phi_{i,\epsilon}^0(x')|dx' + \int_{D_t^\epsilon} |w_\epsilon \circ \phi_{i,\epsilon}^0(x')|dx'$$

Hence, by (4.3),

$$\begin{aligned}
\int_{C_t^\epsilon} |w_\epsilon \circ \phi_{i,\epsilon}^0(x')|dx' &\le |C_t^\epsilon|^{\frac{d+1}{d}} (\int_{C_t^\epsilon} |w_\epsilon \circ \phi_{i,\epsilon}^0|^{d+1})^{\frac{1}{d+1}} \\
&\le \frac{|C_t^\epsilon|^{\frac{d+1}{d}}}{t^{\frac{1}{d+1}}} (\int_{C_t^\epsilon} |w_\epsilon \circ \phi_{i,\epsilon}^0|^{d+1} J\phi_{i,\epsilon}^0 dx')^{\frac{1}{d+1}} \\
&\le \frac{|C_t^\epsilon|^{\frac{d+1}{d}}}{t^{\frac{1}{d+1}}} (\int_{U_i \cap \partial\Omega_\epsilon} |w_\epsilon|^{d(w_\epsilon)})^{\frac{1}{d+1}} \le Ct^{-\frac{1}{d+1}}.
\end{aligned}$$

and

$$\int_{D_t^\epsilon} |w_\epsilon \circ \phi_{i,\epsilon}^0(x')|dx' \le \frac{1}{t} \int_{D_t^\epsilon} |w_\epsilon \circ \phi_{i,\epsilon}^0| J\phi_{i,\epsilon}^0 dx' \le \frac{1}{t} \int_{U_i \cap \partial\Omega_\epsilon} |w_\epsilon|^{d(w_\epsilon)} \le Ct^{-1} \le Ct^{-\frac{1}{d+1}}.$$

where we have used that $|w_\epsilon| \ge R+1$ in D_t^ϵ. This shows (4.10).

Since t can be chosen arbitrarily large and putting together all inequalities above, we get that

$$\int_{\partial\Omega \cap U_i} |w_0| = 0, \quad i = 1, 2, \ldots, n$$

which implies that $w_0 \in H_0^1(\Omega)$.

In order to show that w_0 satisfy the equation (4.2), we consider $\theta \in \mathcal{D}(\Omega)$. Multiplying (4.1) by θ and integrating, we get for ϵ small enough such that $\text{supp}(\theta) \subset \Omega_\epsilon$,

$$\int_\Omega \nabla w_\epsilon \nabla \theta + \int_\Omega w_\epsilon \theta = \int_\Omega f_\epsilon \theta,$$

using that $w_\epsilon \to w_0$ weakly in $H^1(\text{supp}(\theta))$ and strongly in $L^2(\text{supp}(\theta))$ and $f_\epsilon \to f_0$ weakly in $L^2(\Omega)$, we get that w_0 is a weak solution of (4.2).

Now, we prove that $w_{\epsilon_k} \xrightarrow{E} w_0$. In order to do this, we prove the convergence of the norms $\|w_\epsilon\|_{H^1(\Omega_\epsilon)} \to \|w_0\|_{H^1(\Omega)}$ and apply Proposition A.4. In fact, since w_ϵ is the solution of (4.1) for f_ϵ, then

$$\|w_\epsilon\|^2_{H^1(\Omega_\epsilon)} = \int_{\Omega_\epsilon} f_\epsilon w_\epsilon - \int_{\partial\Omega_\epsilon} g(x, w_\epsilon) w_\epsilon \le \int_{\Omega_\epsilon} f_\epsilon w_\epsilon$$

where we have used that $g(x,u)u \ge 0$. Now, using the convergence of f_{ϵ_k} to f_0 weak in $L^2(\Omega)$, the convergence of w_{ϵ_k} to w_0 in $L^2(K)$, for all $K \subset\subset \Omega$ and the boundedness of f, we get that

$$\int_{\Omega_\epsilon} f_\epsilon w_\epsilon \to \int_\Omega f_0 w_0 = \|w_0\|_{H^1(\Omega)}.$$

Hence, we obtain that $\lim_{\epsilon\to 0}\|w_\epsilon\|^2_{H^1(\Omega_\epsilon)} \leq \|w_0\|^2_{H^1(\Omega)}$. By Proposition A.4, we get $w_\epsilon \xrightarrow{E} w_0$.

This complete the proof. ∎

Remark 4.3. *If we consider (4.1) with $g(x,u) = \mu_\epsilon(x)u$ with $\mu_\epsilon(x)$ a potential with $\mu_\epsilon(x) \geq \mu_0 > 0$ for all ϵ, then the conclusions of Proposition 4.2 also holds true in this case.*

In order to show the L^∞ convergence of the solutions we will need the following useful result.

Lemma 4.4. *Consider $M > 0$, let $\phi_\epsilon \in H^1(\Omega_\epsilon)$ be a solution of*

$$\begin{cases} -\Delta\phi_\epsilon + \phi_\epsilon = M, & \text{in } \Omega_\epsilon \\ \frac{\partial\phi_\epsilon}{\partial n} + g(x,\phi_\epsilon) = 0 & \text{on } \partial\Omega_\epsilon, \end{cases} \tag{4.11}$$

and let $\phi_0 \in H^1_0(\Omega)$ be the unique solution of

$$\begin{cases} -\Delta\phi_0 + \phi_0 = M, & \text{in } \Omega \\ \phi_0 = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.12}$$

Then, $\phi_\epsilon \xrightarrow{E} \phi_0$ and $\|\phi_\epsilon - \phi_0\|_{L^\infty(\Omega_\epsilon)} \to 0$ as $\epsilon \to 0$.

Proof. With maximum principles arguments and standard elliptic regularity theory, we get that $0 \leq \phi_\epsilon, \phi_0 \leq M$ and $\phi_\epsilon \in C^0(\bar{\Omega}_\epsilon)$, $\phi_0 \in C^0(\bar{\Omega})$. Also, applying Proposition 4.2 to (4.12), (4.12) by setting $f(x,u) \equiv M$, we get that $\phi_\epsilon \xrightarrow{E} \phi_0$.

Moreover, with a localization argument in Ω and applying elliptic regularity results again, it is easy to prove that for any compact set $K \subset \Omega$, we have $\|\phi_\epsilon - \phi_0\|_{C^0(K)} \to 0$ as $\epsilon \to 0$.

In particular, for any $\delta > 0$, we have $\phi_\epsilon \to \phi_0$ in $C(K_\delta)$, where K_δ is defined by (2.1). In particular, since $u_0^* = 0$ at $\partial\Omega$, we have that $\|\phi_0\|_{L^\infty(\Omega\setminus K_\delta)} \to 0$ as $\delta \to 0$. Hence, to show the L^∞ convergence of ϕ_ϵ to ϕ_0 it will be enough to show that

$$\limsup_{\epsilon\to 0}\|\phi_\epsilon\|_{L^\infty(\Omega_\epsilon\setminus K_\delta)} \leq \eta(\delta), \text{ and } \eta(\delta) \to 0, \text{ as } \delta \to 0. \tag{4.13}$$

Hence, let us fix $\eta > 0$ small and let us choose $\delta_0 > 0$ such that $\limsup_{\epsilon\to 0}\|\phi_\epsilon\|_{L^\infty(\partial K_\delta)} \leq \eta/2$ for all $0 < \delta \leq \delta_0$ and $\|\phi_0\|_{L^\infty(\Omega\setminus K_{\delta_0})} \leq \eta/2$.

Consider $A_\epsilon = \Omega_\epsilon \setminus K_\delta$, $\Gamma_\delta = \partial K_\delta$. Then ϕ_ϵ is solution of

$$\begin{cases} -\Delta\phi_\epsilon + \phi_\epsilon = M, & \text{in } A_\epsilon \\ \phi_\epsilon = \psi(x) \leq \eta & \text{on } \Gamma_\delta \\ \frac{\partial\phi_\epsilon}{\partial n} + g(x,\phi_\epsilon) = 0 & \text{on } \partial\Omega_\epsilon \end{cases} \tag{4.14}$$

Let $\theta_{\epsilon,k} = (\phi_\epsilon - k)^+$, for $\eta \leq k \leq M$. Observe that if $k > M$, $(\phi_\epsilon - k)^+ \equiv 0$ since $|\phi_\epsilon(x)| \leq M$ for all $x \in \bar{\Omega}_\epsilon$ and $0 < \epsilon \leq \epsilon_0$. Then, multiplying the equation (4.14) by $\theta_{\epsilon k}$ and integrating in A_ϵ, we get

$$\int_{A_\epsilon} |\nabla \theta_{\epsilon,k}|^2 + \int_{A_\epsilon} \theta_{\epsilon,k}^2 + \int_{\partial\Omega_\epsilon} g(x,\phi_\epsilon)\theta_{\epsilon,k} \leq M \int_{A_\epsilon} \theta_{\epsilon,k}$$

But, using (2.6), we have

$$\int_{\partial\Omega_\epsilon} g(x,\phi_\epsilon)\theta_{\epsilon,k} = \int_{\partial\Omega_\epsilon \cap \{\phi_\epsilon \geq k\}} g(x,\phi_\epsilon)\theta_{\epsilon,k} \geq \frac{1}{k} \int_{\partial\Omega_\epsilon \cap \{\phi_\epsilon \geq k\}} g(x,\phi_\epsilon)\phi_\epsilon \theta_{\epsilon,k}$$

$$\geq \frac{b}{k} \int_{\partial\Omega_\epsilon \cap \{\phi_\epsilon \geq k\}} |\phi_\epsilon|^{d+1}\theta_{\epsilon,k} \geq \frac{b}{k} k^d \int_{\partial\Omega_\epsilon \cap \{\phi_\epsilon \geq k\}} \phi_\epsilon \theta_{\epsilon,k} \geq bk^{d-1} \int_{\partial\Omega_\epsilon} \theta_{\epsilon,k}^2$$

from where we get

$$\min\{1, bk^{d-1}\} \left(\int_{A_\epsilon} |\nabla \theta_{\epsilon,k}|^2 + \int_{A_\epsilon} \theta_{\epsilon,k}^2 + \int_{\partial\Omega_\epsilon} \theta_{\epsilon,k}^2 \right) \leq M \int_{A_\epsilon} \theta_{\epsilon,k}$$

Observe that since $\eta \leq k \leq M$ and $d \geq 1$, we have a constant $\tilde{b} > 0$, independent of k, although may depend on η, such that $\min\{1, bk^{d-1}\} \geq \tilde{b}$.

Since $\theta_{\epsilon,k} = 0$ in Γ_δ, we can extend $\theta_{\epsilon,k}$ by zero in K_δ. Denoting by $\tilde{\theta}_{\epsilon,k}$ the extension and using Maz'ja inequality (3.4) in Ω_ϵ we get there exists C independent of ϵ such that

$$\|\tilde{\theta}_{\epsilon,k}\|^2_{L^{\frac{2N}{N-1}}(\Omega_\epsilon)} \leq C \int_{\Omega_\epsilon} \tilde{\theta}_{\epsilon k}.$$

Denoting $A_{\epsilon,k} = \{x \in A_\epsilon : \phi_\epsilon \geq k\}$, we have that

$$\int_{\Omega_\epsilon} \tilde{\theta}_{\epsilon,k} = \int_{A_{\epsilon,k}} \theta_{\epsilon,k} \leq |A_{\epsilon k}|^{\frac{N+1}{2N}} \|\theta_{\epsilon,k}\|_{L^{\frac{2N}{N-1}}(A_{\epsilon k})},$$

then

$$\|(\phi_\epsilon - k)^+\|_{L^1(A_{\epsilon k})} \leq C|A_{\epsilon k}|^{\frac{N+1}{N}} = \gamma_\epsilon |A_{\epsilon k}|^{1+\frac{1}{2N}}, \quad \eta < k \leq C_1$$

where $\gamma_\epsilon = C|A_{\epsilon k}|^{\frac{1}{2N}}$, and $\gamma_\epsilon \to 0$ since $\phi_\epsilon \to \phi_0$ in $L^2(\mathbb{R}^N)$, which implies $(\phi_\epsilon - k)^+ \to (\phi_0 - k)^+ \equiv 0$ in A_ϵ.

Using Lemma 5.1 in [16], we get that

$$\|\phi_\epsilon\|_{L^\infty(A_\epsilon)} \leq \eta + \tilde{\gamma}_\epsilon,$$

where $\tilde{\gamma}_\epsilon \to 0$, when $\epsilon \to 0$. This shows that $\limsup_{\epsilon\to 0} \|\phi_\epsilon\|_{L^\infty(\Omega_\epsilon \setminus K_\delta)} \leq \eta$ which proves the result. ∎

Now we can start providing a proof of our main results.

Proof of Theorem 2.2 i). Hence, let u_ϵ^*, $0 < \epsilon \leq \epsilon_0$, be a family of solutions of problem (2.5) satisfying $\|u_\epsilon^*\|_{L^\infty(\Omega_\epsilon)} \leq R$. We have that u_ϵ^* satisfies (4.1) with $f_\epsilon(x) = f(x, u_\epsilon^*(x))$ and since $\|u_\epsilon^*\|_{L^\infty(\Omega_\epsilon)} \leq R$, we get that $\|f_\epsilon\|_{L^2(\Omega_\epsilon)} \leq C$. Applying Proposition 4.1, u_ϵ^* satisfy (4.3). Again, by Lemma 3.1, we obtain a subsequence $u_{\epsilon_k}^*$ and a function $u_0^* \in H^1(\Omega)$ such that

$u^*_{\epsilon_k} \to u^*_0$ in $L^2(K)$, for all $K \subset\subset \Omega$, $u^*_{\epsilon_k} \xrightarrow{E} u^*_0$, and $f(x, u^*_{\epsilon_k}) \to f(x, u^*_0)$ weakly in $L^2(\mathbb{R}^N)$. Applying Proposition 4.2, we get that u^*_0 satisfy (2.7) with $f_0 = f(x, u^*_0)$ and $u^*_{\epsilon_k} \xrightarrow{E} u^*_0$. This shows that, extending u^*_0 by zero outside Ω, we have $\|u^*_{\epsilon_k} - u^*_0\|_{H^1(\Omega_\epsilon)} \to 0$.

To show the L^∞-convergence we proceed as follows. Applying standard elliptic regularity theory and taking into account the boundedness of f and g, we have, for fixed $0 \le \epsilon \le \epsilon_0$, that $u^*_\epsilon \in C^0(\bar{\Omega}_\epsilon)$

With an standard localization argument in Ω and applying elliptic regularity results, it is easy to prove that for any compact set $K \subset \Omega$, we have $u^*_\epsilon \to u^*_0$ in $C(K)$. In particular, for any $\delta > 0$, we have $u^*_\epsilon \to u^*_0$ in $C(K_\delta)$, where K_δ is defined by (2.1). Again by regularity arguments, we have that $u^*_0 \in C(\bar{\Omega})$ and in particular, since $u^*_0 = 0$ at $\partial\Omega$, we have that $\|u^*_0\|_{L^\infty(\Omega \setminus K_\delta)} \to 0$ as $\delta \to 0$. Hence, to show the L^∞ convergence of u^*_ϵ to u^*_0 it will be enough to show that

$$\limsup_{\epsilon \to 0} \|u^*_\epsilon\|_{L^\infty(\Omega_\epsilon \setminus K_\delta)} \le \eta(\delta), \text{ and } \eta(\delta) \to 0, \text{ as } \delta \to 0. \tag{4.15}$$

Since the nonlinearity f is bounded by (2.10) and g satisfies $g(x, s)s \ge 0$, applying comparison results, we get that $|u^*_\epsilon(x)| \le \phi_\epsilon(x)$, $x \in \Omega_\epsilon$ where ϕ_ϵ satisfies (4.11). Applying Lemma 4.4, we get that $\|\phi_\epsilon - \phi_0\|_{L^\infty(\Omega_\epsilon)} \to 0$ and since $\phi_0 \in C^0(\bar{\Omega})$ and $\phi_0 \equiv 0$ in $\partial\Omega$, we get (4.15). ∎

5 Lower Semicontinuity of solutions

In this section we give a proof of Theorem 2.2 (ii). Throughout this section we will assume that conditions **(H)** and **(I)** hold. Moreover we also assume that f satisfies (2.10) and g satisfies (2.11), (2.6) and (2.12). Moreover we will assume that in (2.11), $d = 1$.

We are dealing with solutions of (2.5) which lie in $\mathcal{E}_{\epsilon,R}$ and in particular they are bounded in the sup norm by the constant R. It is not difficult to show

Lemma 5.1. *For each $R > 0$ and $\rho > 0$ there exists $\epsilon = \epsilon(\rho, R) > 0$ such that for any $u^*_\epsilon \in \mathcal{E}_{\epsilon,R}$ we have $\|u^*_\epsilon\|_{L^\infty(\partial\Omega_\epsilon)} \le \rho$ for $0 < \epsilon \le \epsilon(\rho, R)$.*

Proof. From (2.10) and comparison principles, we know that $|u_\epsilon(x)| \le \phi_\epsilon(x)$, where ϕ_ϵ is the solution of (4.11). The result follows now by Lemma 4.4. ∎

Observe that since we are assuming $d = 1$ in (2.6), and since $g(x, 0) \equiv 0$ (see Remark 2.3), we get that $\partial_u g(x, 0) \ge b > 0$ and since $g : U \times \mathbb{R} \to \mathbb{R}$ is a continuous function and C^1 on its second variable, we have that there exists a $\rho > 0$ such that $\partial_u g(x, u) \ge b/2 > 0$ for $|u| \le 3\rho$ and all $x \in U$.

Hence, we can perform a cut-off in such a way that the new function, that we denote it by $\tilde{g}(x, u)$ satisfies

- $\tilde{g}(x, u) = g(x, u)$, for all $x \in \mathbb{R}^N$, $|u| \le 2\rho$
- $\partial_u \tilde{g}(x, u) > 0$ for all $x \in \mathbb{R}^N$ and all $u \in \mathbb{R}$.
- The function $\tilde{g}$ satisfies (2.6), (2.11) and (2.12)

Therefore, Lemma 5.1 tells us that if we consider problem (2.5) with $\tilde{g}$ instead of g we have that the set of solutions $\mathcal{E}_{\epsilon,R}$ with $0 < \epsilon \le \epsilon(\rho, R)$, coincide with the set of solutions for $\tilde{g}$. Equivalently, if we are focusing on the set $\mathcal{E}_{\epsilon,R}$ we may consider that the function g satisfies, besides the hypotheses above, that $\partial_u g(x,u) > 0$, $u \in \mathbb{R}$, $x \in U$. We will assume this for the rest of this Section.

An important consequence of the fact that $\partial_u g(x,u) > 0$ is that for each $f \in L^2(\Omega_\epsilon)$, we have a unique solution of (4.1), since if we have two solutions w_ϵ and $\tilde{w}_\epsilon$, denoting $v_\epsilon = w_\epsilon - \tilde{w}_\epsilon$, we have that v_ϵ, satisfies

$$\begin{cases} -\Delta v_\epsilon + v_\epsilon = 0, \text{ in } \Omega_\epsilon \\ \frac{\partial v_\epsilon}{\partial n} + \partial_u g(x, \theta_\epsilon(x)) v_\epsilon = 0 \text{ on } \partial\Omega_\epsilon \end{cases}$$

where $\theta_\epsilon(x)$ is an intermediate point between $w_\epsilon(x)$ and $\tilde{w}_\epsilon(x)$.
Using that $\partial_u g(x, \theta_\epsilon(x)) > 0$, we easily obtain that $v_\epsilon \equiv 0$, which implies the uniqueness result.

Hence, we can define the nonlinear continuous operator $B_\epsilon^{-1} : L^2(\Omega_\epsilon) \to H^1(\Omega_\epsilon)$, that maps f_ϵ to w_ϵ solution of (4.1), that is, $B_\epsilon^{-1} f_\epsilon = w_\epsilon$. We also define the operator B_0^{-1} : $L^2(\Omega_\epsilon) \to H_0^1(\Omega_0)$ by $B_0^{-1} f = w_0$, where w_0 is the unique solution of (4.2). We observe that B_ϵ^{-1}, $0 < \epsilon \le \epsilon_0$ is not a linear operator.

For the operator B_ϵ^{-1}, we have

Lemma 5.2. *For each $0 \le \epsilon \le \epsilon_0$, B_ϵ^{-1} is a continuous compact operator.*

Proof. If $\epsilon = 0$, B_0^{-1} is the resolvent of the Laplace operator with Dirichlet boundary conditions, which is known to be compact.

Moreover, for each $0 < \epsilon \le \epsilon_0$ fixed, elliptic regularity theory applied to (4.1) implies the compactness of B_ϵ^{-1}. ∎

Considering the family of mappings $f_\epsilon^e : H^1(\Omega_\epsilon) \to L^2(\Omega_\epsilon)$ given by $f_\epsilon^e(u_\epsilon) = f(\cdot, u_\epsilon(\cdot))$, by (2.10) the map f_ϵ^e is globally Lipschitz.

For $0 \le \epsilon < \epsilon_0$, we consider $B_\epsilon^{-1} \circ f_\epsilon^e : H^1(\Omega_\epsilon) \to H^1(\Omega_\epsilon)$. We observe that for $0 < \epsilon < \epsilon_0$, $(B_\epsilon^{-1} \circ f_\epsilon^e)(u_\epsilon) = w_\epsilon$, where w_ϵ is the unique solution of (4.1) with $f_\epsilon(x) = f(x, u_\epsilon(x))$. For $\epsilon = 0$, $(B_0^{-1} \circ f_0^e)(u_0) = w_0$ is the unique solution of (4.2) with $f_0 = f \circ u_0$

We have the following

Proposition 5.3. *The family of nonlinear operators $B_\epsilon^{-1} \circ f_\epsilon^e$ compactly converges to the operator $B_0^{-1} \circ f_0^e$.*

Proof. We first observe that for each ϵ, the map $B_\epsilon^{-1} \circ f_\epsilon^e$ is compact operator. This fact follows from the continuity of f_ϵ^e and by Lemma 5.2.

Now, we prove that if we consider a family $\{u_\epsilon\}_\epsilon$, such that $\|u_\epsilon\|_{H^1(\Omega_\epsilon)} \le C$ then there exists a subsequence of $w_\epsilon = B_\epsilon^{-1} \circ f_\epsilon^e(u_\epsilon)$ that E-converges.

Since $\|u_\epsilon\|_{H^1(\Omega_\epsilon)} \le C$, by the boundedness of $\hat{E}_\epsilon u_\epsilon$, we get a subsequence $\{\hat{E}_{\epsilon_k} u_{\epsilon_k}\}$ and a function $u_0 \in H^1(\Omega)$ such that $\hat{E}_{\epsilon_k} u_{\epsilon_k} \to u_0$ strongly in $L^2(\Omega)$ and weakly in $H^1(\Omega)$. In particular $\hat{E}_{\epsilon_k} f^e_{\epsilon_k}(u_{\epsilon_k}) \to f^e_0(u_0)$, strongly in $L^2(\Omega)$. Moreover, by the boundedness of the map f^e_ϵ we have that $\|f^e_\epsilon(u_\epsilon)\|_{L^2(\Omega_\epsilon)} \le C$. Applying Proposition 4.2, we can get another subsequence of ϵ_k, that we denote again by ϵ_k such that $w_{\epsilon_k} \xrightarrow{E} w_0$, where $w_0 = B_0^{-1} f^e_0(u_0) \in H^1_0(\Omega)$

To prove that the third condition of compact convergence is satisfied, that is, to prove that if $u_\epsilon \xrightarrow{E} u$ then $w_\epsilon \xrightarrow{E} w$, we follow similar arguments. ∎

Now we assume that a solution u_0^* of (2.7) is hyperbolic. Let us start with the following,

Corollary 5.4. *If u_0^* is a hyperbolic equilibrium point of (2.7) then there exists $\{u_\epsilon^*\}$ equilibrium points of (2.5) such that $\{u_\epsilon^*\}$ E-converges to u_0^*.*

Proof. Let u_0^* be a hyperbolic equilibrium point of (2.7), then it is isolated, that means there exists $\delta > 0$ such that u_0^* is the unique equilibrium point in $B(u_0^*, \delta)$ and his index $|ind(u_0^*, B_0^{-1} \circ f^e_0)| = 1$. Since $B_\epsilon^{-1} \circ f^e_\epsilon$ compactly converges to $B_0^{-1} \circ f^e_0$ then, by Theorem 5.3 in [4], we get that $|ind(B(E_\epsilon u_0^*, \delta), B_\epsilon^{-1} \circ f^e_\epsilon)| = 1$. Therefore, for each ϵ there exists at least one equilibrium points $\{u_\epsilon^*\}$ of (2.5) in $B(E_\epsilon u_0^*, \delta)$. ∎

Also, we will be able to prove that, under the conditions above, if u_0^* is a hyperbolic solution, there exist $\delta, \epsilon_0 > 0$, small enough, such that for $\epsilon < \epsilon_0$ there exits only one solution u_ϵ^* of (2.5), with $\|u_\epsilon^* - E_\epsilon u_0^*\| \le \delta$.

We start by considering the derivative of $B_\epsilon^{-1} \circ f^e_\epsilon$ in u_ϵ^*, that we denote by $d(B_\epsilon^{-1} f^e_\epsilon)(u_\epsilon^*)$. In fact, for each $0 < \epsilon < \epsilon_0$, the operator $d(B_\epsilon^{-1} f^e_\epsilon)(u_\epsilon^*)$ is given by $d(B_\epsilon^{-1} f^e_\epsilon)(u_\epsilon^*)(v_\epsilon) = z_\epsilon$, where z_ϵ is solution of

$$\begin{cases} -\Delta z + z = \partial_u f(x, u_\epsilon^*) v_\epsilon, \text{ in } \Omega_\epsilon \\ \frac{\partial z}{\partial n} + \partial_u g(x, u_\epsilon^*) z = 0, \text{ on } \partial\Omega_\epsilon \end{cases} \tag{5.1}$$

For $\epsilon = 0$, since B_0^{-1} is a linear operator we have that $d(B_0^{-1} \circ f^e_0)(u_0^*)(v) = z$, where z is the solution of

$$\begin{cases} -\Delta z + z = \partial_u f(x, u_0^*) v, \text{ in } \Omega \\ z = 0, \text{ on } \partial\Omega \end{cases} \tag{5.2}$$

Now, we prove that

Lemma 5.5. *Under the conditions above, for all $\eta > 0$, there exists $\delta = \delta(\eta) > 0$ and $\epsilon = \epsilon(\eta) > 0$, such that for all v_ϵ, $\|v_\epsilon\|_{H^1(\Omega_\epsilon)} \le \delta(\eta)$ and for all $0 < \epsilon \le \epsilon(\eta)$ we have*

$$\|B_\epsilon^{-1} \circ f^e_\epsilon(u_\epsilon^* + v_\epsilon) - B_\epsilon^{-1} \circ f^e_\epsilon(u_\epsilon^*) - dB_\epsilon^{-1} \circ f^e_\epsilon(u_\epsilon^*)(v_\epsilon)\|_{H^1(\Omega_\epsilon)} \le \eta \|v_\epsilon\|_{H^1(\Omega_\epsilon)}.$$

Proof. In fact, let $w_\epsilon = B_\epsilon^{-1} \circ f^e_\epsilon(u_\epsilon^* + v_\epsilon)$, $u_\epsilon^* = B_\epsilon^{-1} \circ f^e_\epsilon(u_\epsilon^*)$ and $z_\epsilon = d(B_\epsilon^{-1} \circ f^e_\epsilon(u_\epsilon^*))(v_\epsilon)$. We need to estimate $\|w_\epsilon - u_\epsilon^* - z_\epsilon\|_{H^1(\Omega_\epsilon)}$. If we denote $\chi_\epsilon = w_\epsilon - u_\epsilon^* - z_\epsilon$, then, using the equations χ_ϵ satisfies

$$\begin{cases} -\Delta \chi_\epsilon + \chi_\epsilon = f(x, u_\epsilon^* + v_\epsilon) - f(x, u_\epsilon^*) - \partial_u f(x, u_\epsilon^*) v_\epsilon, \text{ in } \Omega_\epsilon \\ \frac{\partial \chi_\epsilon}{\partial n} + g(x, w_\epsilon) - g(x, u_\epsilon^*) - \partial_u g(x, u_\epsilon) z_\epsilon = 0 \text{ on } \partial\Omega_\epsilon \end{cases} \tag{5.3}$$

Multiplying (5.3) by χ_ϵ and integrating, we get

$$\|\chi_\epsilon\|^2_{H^1(\Omega_\epsilon)} = \int_{\Omega_\epsilon} (f(x, u^*_\epsilon + v_\epsilon) - f(x, u^*_\epsilon) - \partial_u f(x, u^*_\epsilon) v_\epsilon)\chi_\epsilon$$

$$- \int_{\partial\Omega_\epsilon} (g(x, w_\epsilon) - g(x, u^*_\epsilon) - \partial_u g(x, u^*_\epsilon) z_\epsilon)\chi_\epsilon.$$

We first analyze the integral in Ω_ϵ. Using Hölder and Young inequalities we have

$$\begin{aligned}
&|\int_{\Omega_\epsilon} (f(x, u^*_\epsilon + v_\epsilon) - f(x, u^*_\epsilon) - \partial_u f(x, u^*_\epsilon) v_\epsilon))\chi_\epsilon| \\
&\quad \le \|f(x, u^*_\epsilon + v_\epsilon) - f(x, u^*_\epsilon) - \partial_u f(x, u^*_\epsilon) v_\epsilon)\|_{L^2(\Omega_\epsilon)} \|\chi_\epsilon\|_{L^2(\Omega_\epsilon)} \\
&\quad \le \tfrac{1}{2}\|f(x, u^*_\epsilon + v_\epsilon) - f(x, u^*_\epsilon) - \partial_u f(x, u^*_\epsilon) v_\epsilon)\|^2_{L^2(\Omega_\epsilon)} + \tfrac{1}{2}\|\chi_\epsilon\|^2_{L^2(\Omega_\epsilon)}
\end{aligned}$$

Moreover, using Lemma 4.2 from [5], we get

$$\begin{aligned}
|\int_{\Omega_\epsilon} (f(x, u^*_\epsilon + v_\epsilon) - f(x, u^*_\epsilon) - \partial_u f(x, u^*_\epsilon) v_\epsilon)\chi_\epsilon| \;&\le\; C(\frac{1}{\tau_\epsilon} + \delta^{\frac{2}{N}})\|v_\epsilon\|^2_{H^1(\Omega_\epsilon)} \\
&\quad + \frac{1}{2}\|\chi_\epsilon\|^2_{L^2(\Omega_\epsilon)},
\end{aligned}$$

where τ_ϵ is defined in (3.3) and it satisfies $\tau_\epsilon \to +\infty$.

For the boundary part, we have

$$\begin{aligned}
\int_{\partial\Omega_\epsilon} (g(x, w_\epsilon) - g(x, u^*_\epsilon) - \partial_u g(x, u^*_\epsilon) z_\epsilon)\chi_\epsilon \;&=\; \int_{\partial\Omega_\epsilon} \partial_u g(x, \tilde{w}_\epsilon)\chi^2_\epsilon \\
&\quad - \int_{\partial\Omega_\epsilon} (\partial_u g(x, u^*_\epsilon) - \partial_u g(x, \tilde{w}_\epsilon)) z_\epsilon \chi_\epsilon
\end{aligned}$$

where we have applied the mean value theorem and $\tilde{w}_\epsilon(x) \in (w_\epsilon(x), u^*_\epsilon(x))$ for all $x \in \partial\Omega_\epsilon$.

Notice that, from Lemma 4.4, $\|u^*_\epsilon\|_{L^\infty(\partial\Omega_\epsilon)} \to 0$ as $\epsilon \to 0$. Moreover, since w_ϵ is the solution of

$$\begin{cases} -\Delta w_\epsilon + w_\epsilon = f(x, x, u^*_\epsilon + v_\epsilon), \text{ in } \Omega_\epsilon \\ \frac{\partial w_\epsilon}{\partial n} + g(x, w_\epsilon) = 0 \text{ on } \partial\Omega_\epsilon \end{cases}$$

and $|f(x, s)| \le C_1$ by (2.10), using maximum principles, we get $|w_\epsilon(x)| \le \phi_\epsilon$ where ϕ_ϵ is given by Lemma 4.4 with $M = C_1$. Hence, we also get that $\|w_\epsilon\|_{L^\infty(\partial\Omega_\epsilon)} \to 0$ as $\epsilon \to 0$. Therefore, we obtain that $\|\tilde{w}_\epsilon\|_{L^\infty(\partial\Omega_\epsilon)} \to 0$ as $\epsilon \to 0$. Since we are assuming that g is a C^2 function, then $g_u(x, \tilde{w}_\epsilon(x)) \ge b/2$ and $\|g_u(., \tilde{w}_\epsilon) - g_u(., u^*_\epsilon)\|_{L^\infty(\partial\Omega_\epsilon)} \le \rho(\epsilon)$ where $\rho(\epsilon) = \rho_\epsilon$ goes to 0, when $\epsilon \to 0$.

This implies that

$$\int_{\partial\Omega_\epsilon}(g(x,w_\epsilon)-g(x,u_\epsilon^*)-\partial_u g(x,u_\epsilon^*)z_\epsilon)\chi_\epsilon \;\geq\; \int_{\partial\Omega_\epsilon}\frac{b}{2}\chi_\epsilon^2-\rho(\epsilon)\int_{\partial\Omega_\epsilon}z_\epsilon\chi_\epsilon,$$

Now, by Hölder and Young Inequalities, we have $\int_{\partial\Omega_\epsilon} z_\epsilon\chi_\epsilon \leq \frac{1}{2}\int_{\partial\Omega_\epsilon} z_\epsilon^2 + \frac{1}{2}\int_{\partial\Omega_\epsilon}\chi_\epsilon^2$. Since $z_\epsilon = dB_\epsilon^{-1}\circ f_\epsilon^e(u_\epsilon^*)(v_\epsilon)$, by using Hölder and Young inequalities, we have that

$$\int_{\Omega_\epsilon}|\nabla z_\epsilon|^2+\int_{\Omega_\epsilon}z_\epsilon^2+\int_{\partial\Omega_\epsilon}\partial_u g(x,u_\epsilon^*)z_\epsilon^2=\int_{\Omega_\epsilon}\partial_u f(x,u_\epsilon^*)v_\epsilon z_\epsilon$$

$$\leq \|\partial_u f(x,u_\epsilon^*)\|_{L^\infty}(C_\beta\|v_\epsilon\|^2_{L^2(\Omega_\epsilon)}+\beta\|z_\epsilon\|^2_{L^2(\Omega_\epsilon)}).$$

Consider β small enough such that $\beta\|\partial_u f(x,u_\epsilon^*)\|_{L^\infty}<\frac{1}{2}$, we get

$$\frac{b}{2}\int_{\partial\Omega_\epsilon}z_\epsilon^2\leq\int_{\Omega_\epsilon}|\nabla z_\epsilon|^2+\frac{1}{2}\int_{\Omega_\epsilon}z_\epsilon^2+\int_{\partial\Omega_\epsilon}\partial_u g(x,u_\epsilon^*)z_\epsilon^2\leq\|\partial_u f(x,u_\epsilon^*)\|_{L^\infty}C_\beta\|v_\epsilon\|^2_{L^2(\Omega_\epsilon)}.$$

Finally, we get

$$\begin{aligned}
-\int_{\partial\Omega_\epsilon}(g(x,w_\epsilon)-g(x,u_\epsilon^*)-\partial_u g(x,u_\epsilon^*)z_\epsilon))\chi_\epsilon &\leq -\int_{\partial\Omega_\epsilon}\frac{b}{2}\chi_\epsilon^2+\rho_\epsilon\int_{\partial\Omega_\epsilon}z_\epsilon\chi_\epsilon\\
&\leq -\int_{\partial\Omega_\epsilon}\frac{b}{2}\chi_\epsilon^2+\frac{\rho_\epsilon}{2}\int_{\partial\Omega_\epsilon}z_\epsilon^2+\frac{\rho_\epsilon}{2}\int_{\partial\Omega_\epsilon}\chi_\epsilon^2\\
&\leq -\int_{\partial\Omega_\epsilon}\frac{b}{2}\chi_\epsilon^2+\frac{\rho_\epsilon}{b}\|\partial_u f(x,u_\epsilon^*)\|_{L^\infty}C_\beta\|v_\epsilon\|^2_{L^2(\Omega_\epsilon)}+\frac{\rho_\epsilon}{2}\int_{\partial\Omega_\epsilon}\chi_\epsilon^2\\
&\leq \frac{\rho_\epsilon}{b}\|\partial_u f(x,u_\epsilon^*)\|_{L^\infty}C_\beta\|v_\epsilon\|^2_{L^2(\Omega_\epsilon)},
\end{aligned}$$

where in the last inequality we consider ϵ small enough such that $\frac{\rho_\epsilon-b}{2}<0$. Hence

$$\|\chi_\epsilon\|^2_{H^1(\Omega_\epsilon)}\leq 2\left(C(\frac{1}{\tau_\epsilon}+\delta^{\frac{2}{N}})+\frac{\rho_\epsilon}{b}\|\partial_u f(x,u_\epsilon^*)\|_{L^\infty}C_\beta\right)\|v_\epsilon\|^2_{L^2(\Omega_\epsilon)}.$$

which proves the result. ∎

Proposition 5.6. *Under the conditions above, if u_ϵ^* is a family of solutions of (2.5) that E-converges to u_0^*, a solution of (2.7), then $d(B_\epsilon^{-1}f_\epsilon^e)(u_\epsilon^*)$ compactly converges to $d(B_0^{-1}f_0^e)(u_0^*)\equiv B_0^{-1}(f_0^e)'(u_0^*)$.*

Proof. Let v_ϵ such that $\|v_\epsilon\|_{H^1(\Omega_\epsilon)}\leq K$. In this case, $z_\epsilon = dB_\epsilon^{-1}\circ f_\epsilon^e(u_\epsilon^*)v_\epsilon$ is uniform bounded. In fact,

$$\|z_\epsilon\|^2_{H^1(\Omega_\epsilon)}=\int_{\Omega_\epsilon}\partial_u f(x,u_\epsilon^*)v_\epsilon z_\epsilon-\int_{\partial\Omega_\epsilon}\partial_u g(x,u_\epsilon^*)z_\epsilon^2\leq\int_{\Omega_\epsilon}\partial_u f(x,u_\epsilon^*)v_\epsilon z_\epsilon$$

where we use that $\partial_u g(x,u) > 0$. Hence

$$\|z_\epsilon\|^2_{H^1(\Omega_\epsilon)} \leq \int_{\Omega_\epsilon} \partial_u f(x,u^*_\epsilon) v_\epsilon z_\epsilon \leq \|\partial_u f(x,u^*_\epsilon)\|_{L^\infty(\Omega_\epsilon)} \|v_\epsilon\|_{L^2(\Omega_\epsilon)} \|z_\epsilon\|_{L^2(\Omega_\epsilon)}. \tag{5.4}$$

Using that $z_\epsilon = dB_\epsilon^{-1} \circ f^e_\epsilon(u^*_\epsilon) v_\epsilon$ and v_ϵ are uniform bounded, by the same arguments that in Proposition 4.2, we prove that there exists subsequences z_{ϵ_k} and v_{ϵ_k}, and $z, v \in H^1(\Omega)$ such that v_{ϵ_k} E-converges weakly to v and z_{ϵ_k} E-converges weakly to z and strongly $L^2(\Omega)$ and z satisfy

$$-\Delta z + z = \partial_u f(x, u^*_0) v.$$

Again, by the same arguments in the Proposition 4.2, we prove that z_ϵ converges to 0 in $L^2(\partial\Omega)$ and z satisfy the Dirichlet boundary condition, that is, z satisfy (5.2).

Now, we prove the E-convergence. In fact, we observe that by (5.4),

$$\|z_\epsilon\|^2_{H^1(\Omega_\epsilon)} \leq \int_{\Omega_\epsilon} \partial_u f(x,u^*_\epsilon) v_\epsilon z_\epsilon.$$

By Lemma 3.1 (iii), we get

$$\int_{\Omega_\epsilon} \partial_u f(x,u^*_\epsilon) v_\epsilon z_\epsilon \to \int_{\Omega} \partial_u f(x,u^*_0) v z = \|z\|^2_{H^1(\Omega)},$$

then

$$\limsup_{\epsilon\to 0} \|z_\epsilon\|^2_{H^1(\Omega_\epsilon)} \leq \|z\|^2_{H^1(\Omega)}.$$

By Proposition A.4, we get z_ϵ E-converges to z.

Using the same arguments we prove the third condition of compact convergence. ∎

We are ready now to complete the proof of Theorem 2.2.

Proof of Theorem 2.2 (ii): Since u^*_0 is a hyperbolic solution and $B_\epsilon^{-1} \circ f^e_\epsilon$ converges compactly to $B_0^{-1} \circ f^e_0$, by Corollary 5.4, we have that there exists u^*_ϵ solution of (2.5) such that $\|u^*_\epsilon - E_\epsilon u^*_0\| \leq \delta$. Now we prove that u^*_ϵ is unique. In fact, u_ϵ is a solution of (2.5) if and only if u_ϵ is a fixed point of $B_\epsilon^{-1} \circ f^e_\epsilon$. We prove that if $u_\epsilon \neq u^*_\epsilon$

$$\|u_\epsilon - B_\epsilon^{-1} \circ f^e_\epsilon(u_\epsilon)\| > 0,$$

By assumption, $dB_\epsilon^{-1} f_\epsilon(u^*_\epsilon)$ converges compactly to $dB_0^{-1} \circ f^e_0(u^*_0)$ and u^*_0 is a hyperbolic solution that implies $\|(I - dB_\epsilon^{-1} \circ f^e_\epsilon(u^*_\epsilon))^{-1}\| \leq M$, M independent of ϵ, then there exists $\eta > 0$ such that $\|(I - dB_\epsilon^{-1} \circ f^e_\epsilon(u^*_\epsilon))(z_\epsilon)\| \geq \eta \|z_\epsilon\|$. Thus,

$$\begin{aligned}
&\|u_\epsilon - B_\epsilon^{-1} \circ f^e_\epsilon(u_\epsilon)\|_{H^1(\Omega_\epsilon)} \\
&\quad \geq \|u_\epsilon - u^*_\epsilon - dB_\epsilon^{-1} \circ f^e_\epsilon(u^*_\epsilon)(u_\epsilon - u^*_\epsilon)\|_{H^1(\Omega_\epsilon)} \\
&\quad -\|B_\epsilon^{-1} \circ f^e_\epsilon(u_\epsilon) - B_\epsilon^{-1} \circ f^e_\epsilon(u^*_\epsilon) - dB_\epsilon^{-1} \circ f^e_\epsilon(u^*_\epsilon)(u_\epsilon - u^*_\epsilon)\|_{H^1(\Omega_\epsilon)}.
\end{aligned}$$

By Lemma 5.5, we have that there exists $0 < \delta_0 < 1$ such that

$$\|B_\epsilon^{-1} \circ f_\epsilon^e(u_\epsilon) - B_\epsilon^{-1} \circ f_\epsilon^e(u_\epsilon^*) - dB_\epsilon^{-1} \circ f_\epsilon^e(u_\epsilon^*)(u_\epsilon - u_\epsilon^*)\|_{H^1(\Omega_\epsilon)} \leq \eta/2\|(u_\epsilon - u_\epsilon^*)\|_{H^1(\Omega_\epsilon)},$$

for $\|u_\epsilon - u_\epsilon^*\|_{H^1(\Omega_\epsilon)} \leq \delta_0$. Using this, we get that

$$\|u_\epsilon - B_\epsilon^{-1} \circ f_\epsilon^e(u_\epsilon)\|_{H^1(\Omega_\epsilon)} \geq \eta/2\|u_\epsilon - u_\epsilon^*\|_{H^1(\Omega_\epsilon)}.$$

and this completes the proof. ∎

6 Spectral continuity

In this section, we prove the convergence of the eigenvalues and eigenfunctions of the linearizations around stationary solutions as $\epsilon \to 0$. We define the following linear operators:

$$A_\epsilon : D(A_\epsilon) \subset L^2(\Omega_\epsilon) \to L^2(\Omega_\epsilon)$$

given by $A_\epsilon u_\epsilon = -\Delta u_\epsilon + u_\epsilon$ where $D(A_\epsilon) = \{u \in H^2(\Omega_\epsilon) : \frac{\partial u_\epsilon}{\partial n} + g_u(x, u_\epsilon^*)u_\epsilon = 0 \text{ in } \partial\Omega_\epsilon\}$ and $\tilde{h}_\epsilon : H^1(\Omega_\epsilon) \to H^{-\alpha}(\Omega_\epsilon)$, $\frac{1}{2} < \alpha < 1$ defined by

$$\langle \tilde{h}_\epsilon(w_\epsilon), \phi_\epsilon \rangle = \int_{\Omega_\epsilon} V_\epsilon(x) w_\epsilon \phi_\epsilon.$$

We observe that w_ϵ is a solution of

$$\begin{cases} \Delta w_\epsilon - w_\epsilon = V_\epsilon(x) w_\epsilon, & \text{on } \Omega_\epsilon \\ \frac{\partial w_\epsilon}{\partial n} + g_u(x, u_\epsilon^*(x)) w_\epsilon = 0 \text{ in } \partial\Omega_\epsilon \end{cases} \tag{6.1}$$

if and only if $A_\epsilon^{-1} \circ \tilde{h}_\epsilon(w_\epsilon) = w_\epsilon$

For $\epsilon = 0$, we consider

$$A_0 : D(A_0) \subset L^2(\Omega_0) \to L^2(\Omega_0)$$

given by $A_0 u_0 = -\Delta u_0 + u_0$ where $D(A_0) = \{u \in H^2(\Omega_0) : u_0 = 0 \text{ in } \partial\Omega_0\}$ and $\tilde{h}_0 : H^1(\Omega_0) \to H^{-\alpha}(\Omega_0)$, $\frac{1}{2} < \alpha < 1$ defined by

$$\langle \tilde{h}_0(w_0), \phi_0 \rangle = \int_{\Omega_0} V_0(x) w_0 \phi_0.$$

We observe that w_0 is a solution of

$$\begin{cases} \Delta w_0 - w_0 = V_0(x) w_0, & \text{on } \Omega_0 \\ u_0 = 0 \text{ in } \partial\Omega_0 \end{cases} \tag{6.2}$$

if and only if $A_0^{-1} \circ \tilde{h}_0(w_0) = w_0$

Since

$$A_\epsilon^{-1} \circ \tilde{h}_\epsilon(v_\epsilon) = d(B_\epsilon^{-1} \circ f_\epsilon^e)(u_\epsilon^*)(v_\epsilon),$$

by Proposition 5.6, $A_\epsilon^{-1} \circ \tilde{h}_\epsilon$ compact converges to $A_0^{-1} \circ \tilde{h}_0$

Finally, we prove Theorem 2.4

Proof of Theorem 2.4: We observe that, by Proposition 4.2 and Remark 4.3, A_ϵ^{-1} converges compactly to A_0^{-1}. Since $(A_\epsilon + \tilde{h}_\epsilon)^{-1} = (I + A_\epsilon^{-1}\tilde{h}_\epsilon)^{-1} A_\epsilon^{-1}$ and $A_\epsilon^{-1}\tilde{h}_\epsilon$ converges compactly to $A_0^{-1}\tilde{h}_0$, we have by Proposition 4.13 in [4], that $(A_\epsilon + \tilde{h}_\epsilon)^{-1}$ converges compactly to $(A_0 + \tilde{h}_0)^{-1}$. Using Proposition A.10, we obtain the result. ∎

A On E-convergence

In this appendix we are going to develop the basic tools that we are going to use to compare the solutions of two problems defined in different spaces. We will refer to [**?**, 24, 7] for a general theory and to [3, 4] for a concrete application.

In our setting we will have a family of Hilbert spaces, H_ϵ, $0 < \epsilon \leq \epsilon_0$ and we also have a "limitting" Hilbert space H. We denote by $(.,.)_\epsilon$ the inner product in H_ϵ, and by $(.,.)$ the inner product in H.

We consider $E_\epsilon \in L(H, H_\epsilon)$ a family of operators $E_\epsilon : H \to H_\epsilon$, such that

$$\|E_\epsilon u\|_{H_\epsilon} \to \|u\|_H$$

when $\epsilon \to 0$.

Definition A.1. *A sequence of elements $\{u_\epsilon\}, u_\epsilon \in H_\epsilon, \epsilon > 0$, is said to be E-convergent to $u \in H$ iff $\|u_\epsilon - E_\epsilon u\|_{H_\epsilon} \to 0$ as $\epsilon \to 0$; we write this as $u_\epsilon \xrightarrow{E} u$.*

Definition A.2. *A sequence of elements $\{u_\epsilon\}, u_\epsilon \in H_\epsilon, \epsilon > 0$, is said to be E-weakly convergent to $u \in H$ iff for all w_ϵ E-convergent to w implies $(w_\epsilon, u_\epsilon)_\epsilon \to (w, u)$, with $u \in H$, when $\epsilon \to 0$. We denote by $u_\epsilon \xrightharpoonup{E} u$.*

We observe that if $u_\epsilon \xrightharpoonup{E} u$ then for all $w \in H$, $(E_\epsilon w, u_\epsilon)_\epsilon \to (w, u)$ when $\epsilon \to 0$.

The following propositions can simplify the analysis of these convergences.

Proposition A.3. *If $\|u_\epsilon\|_\epsilon \leq K$ and for all $w \in H$, $(E_\epsilon w, u_\epsilon)_\epsilon \to (w, u)$ with $u \in H$ when $\epsilon \to 0$, then $u_\epsilon \xrightharpoonup{E} u$.*

Proof. Let $w_\epsilon \xrightarrow{E} w$, $(w_\epsilon, u_\epsilon)_\epsilon = (w_\epsilon - E_\epsilon w, u_\epsilon)_\epsilon + (E_\epsilon w, u_\epsilon)_\epsilon \to (w, u)$. ∎

Proposition A.4. *If $u_\epsilon \xrightharpoonup{E} u$ and $\limsup_{\epsilon \to 0} \|u_\epsilon\|_\epsilon \leq \|u\|$ then $u_\epsilon \xrightarrow{E} u$.*

Proof. Since $0 \leq \|u_\epsilon - E_\epsilon u\|_{H_\epsilon}^2 = \|u_\epsilon\|_{H_\epsilon}^2 - 2(u_\epsilon, E_\epsilon u)_\epsilon + \|E_\epsilon u\|_{H_\epsilon}^2$ we get the result. ∎

We now can introduce the notion of compactness and convergence of operators in variable spaces.

Definition A.5. *A sequence of elements $\{u_n\}, u_n \in H_{\epsilon_n}, n \in \mathbb{N}$, is said to be E-precompact if for any subsequence $\{u_{n'}\}$ there exist a subsequence $\{u_{n''}\}$ and $u \in H$ such that $u_{n''} \xrightarrow{E} u$, as $n'' \to \infty$. A family $\{u_\epsilon\}$, $\epsilon \in (0,1]$ is said precompact if for each sequence $\{u_{\epsilon_n}\}$, with $\epsilon_n \to 0$, is E-precompact.*

Definition A.6. *We say that a family of operators $T_\epsilon \in L(H_\epsilon)$, $\epsilon \in (0,1]$, converges to $T \in L(H)$ as $\epsilon \to 0$ if $T_\epsilon u_\epsilon \xrightarrow{E} Tu$, whenever $u_\epsilon \xrightarrow{E} u \in H$. We denote this by $T_\epsilon \xrightarrow{EE} T$.*

Definition A.7. *We say that a family of compact operators $T_\epsilon : H_\epsilon \to H_\epsilon$, $\epsilon \in (0,1]$ converges compactly to a compact $T : H \to H$ if for any family u_ϵ with $\|u_\epsilon\|_\epsilon$ bounded, the family $\{T_\epsilon u_\epsilon\}$ is E-precompact and $T_\epsilon \xrightarrow{EE} B$. We write $T_\epsilon \xrightarrow{CC} T$.*

An important result on convergence of fixed points is the following:

Theorem A.8. *Let $T_\epsilon : H_\epsilon \to H_\epsilon$ be a family of compact operators such that $T_\epsilon \xrightarrow{CC} T$. Let u_ϵ be a fixed point of T_ϵ such that $\|u_\epsilon\|_{H_\epsilon}$ is uniformly bounded. Then, there exists a subsequence u_{ϵ_k} and $u \in H$ with $u = Tu$ such that $u_{\epsilon_k} \xrightarrow{E} u$.*

Proof. Since $\|u_\epsilon\|_{H_\epsilon}$ is uniformly bounded, by Definition A.7, $T_\epsilon u_\epsilon$ is E-precompact. Thus, there is a sequence u_{ϵ_k}, and an element $u \in H$ such that $T_{\epsilon_k} u_{\epsilon_k} \xrightarrow{E} u$. Hence, $u_{\epsilon_k} = T_{\epsilon_k} \xrightarrow{E} u$ and by compact convergence, $T_{\epsilon_k} u_{\epsilon_k} \xrightarrow{E} Tu$. That is, $u = Tu$. ∎

In the case where the operators involved are linear, we have some important results.

Lemma A.9. *Assume that $T_\epsilon \in \mathcal{L}(H_\epsilon)$ converges compactly to $T \in \mathcal{L}(H)$ as $\epsilon \to 0$. Then,*
i) $\|T_\epsilon\|_{\mathcal{L}(H_\epsilon)} \leq C$ for some constant C, independent of ϵ.
ii) Assume that $\mathcal{N}(I + T) = \{0\}$ then, there exists an $\epsilon_0 > 0$ and $M > 0$ such that

$$\|(I + T_\epsilon)^{-1}\|_{\mathcal{L}(H_\epsilon)} \leq M, \quad \forall \epsilon \in [0, \epsilon_0]. \tag{A.1}$$

Proof. This result is exactly Lemma 4.7 in [4]. For the sake of completeness and since the proof is short, we include it here.
i) If the norms are not bounded, then we can choose a sequence of $\epsilon_n \to 0$ and $u_{\epsilon_n} \in H_{\epsilon_n}$ with $\|u_{\epsilon_n}\|_{H_{\epsilon_n}} = 1$ such that $\|T_{\epsilon_n} u_{\epsilon_n}\| \to +\infty$. But this is in contradiction with the compact convergence of T_ϵ given in Definition A.7.
ii) Since T_ϵ is compact for every $\epsilon \in [0,1]$, the estimate (A.1) is equivalent to say that

$$\|(I + T_\epsilon)u_\epsilon\|_{H_\epsilon} \geqslant \frac{1}{M}, \quad \forall \epsilon \in [0, \epsilon_0] \text{ and } \forall u_\epsilon \in H_\epsilon \text{ with } \|u_\epsilon\| = 1.$$

Suppose that this is not true; that is, suppose that there is a sequence $\{u_n\}$, with $u_n \in U_{\epsilon_n}$, $\|u_n\| = 1$ and $\epsilon_n \to 0$ such that $\|(I + T_{\epsilon_n})u_n\| \to 0$. Since $\{T_{\epsilon_n} u_n\}$ has a convergent subsequence, which we again denote by $\{T_{\epsilon_n} u_n\}$, to u, $\|u\| = 1$, then $u_n + T_{\epsilon_n} u_n \to 0$ and $u_n \to -u$. This implies that $(I + T)u = 0$ contradicting our hypothesis. ∎

In many instances, the operators T_ϵ will be inverses of certain differential operators A_ϵ. Therefore, let us assume that we have operators $A_\epsilon : D(A_\epsilon) \subset H_\epsilon \to H_\epsilon$, with well defined inverses and denote by $T_\epsilon = A_\epsilon^{-1} : H_\epsilon \to H_\epsilon$.

One important implication of the compact convergence of linear operators is the convergence of the spectra and of the spectral projections. Since the operators involved are compact, then the spectrum is discrete and the convergence of the spectra will mean the pointwise convergence of the eigenvalues. For the convergence of the spectral projections we need a concept of convergence of linear spaces. Hence, we will say that a family of subspaces $W_\epsilon \subset H_\epsilon$ E-converges to $W_0 \subset H$ and we will write it as $W_\epsilon \xrightarrow{E} W_0$, if $dist_{H_\epsilon}(B_{W_\epsilon}, E_\epsilon B_{W_0}) \to 0$ as $\epsilon \to 0$, where B_W is the unit ball of the space W and $dist_{H_\epsilon}$ is the symmetric Haussdorf distance of two sets in H_ϵ.

We can show,

Proposition A.10. *If $A_\epsilon : D(A_\epsilon) \subset H_\epsilon \to H_\epsilon$ is a closed operator, with compact resolvent and $0 \in \rho(A_\epsilon)$ and $A_0 : D(A_0) \subset H \to H$ is also closed, with compact resolvent and $0 \in \rho(A)$, then if $A_\epsilon^{-1} \xrightarrow{CC} A_0^{-1}$, then the eigenvalues and eigenfunctions of A_ϵ converge to the eigenvalues and eigenfunctions of A_0. That is, if $\bar{B}(\lambda_0, \rho_0) \subset \mathbb{C}$ lies in the resolvent set of A_0, then, there exists $\epsilon_0 = \epsilon_0(\lambda_0, \rho_0)$, such that the ball is also contained in the resolvent set of A_ϵ for all $0 < \epsilon \leq \epsilon_0$. Moreover, if $\lambda_0 \in \sigma(A_0)$, $\bar{B}(\lambda_0, \rho_0) \cap \sigma(A_0) = \{\lambda_0\}$ and W_0 is the generalized eigenspace associated to λ_0, then there exists $\epsilon_0 = \epsilon_0(\lambda_0, \rho_0) >$ such that for $0 < \epsilon \leq \epsilon_0$, $B(\lambda_0, \rho_0) \cap \sigma(A_\epsilon) = \{\lambda_1^\epsilon, \ldots, \lambda_{k(\epsilon)}^\epsilon\}$ and if W_ϵ =span$\{W_1^\epsilon, \ldots, W_{k(\epsilon)}^\epsilon\}$, where W_j^ϵ is the generalized eigenspace associated λ_j^ϵ, then dim(W_ϵ)=dim(W_0) and $W_\epsilon \xrightarrow{E} W_0$.*

Proof. For a proof of this result we refer to Lemma 4.8, Lemma 4.9 and Theorem 4.10 in [4]. ∎

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