Dynamics of a reaction-diffusion equation with a discontinuous nonlinearity

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Abstract

We study the nonlinear dynamics of a reaction-diffusion equation where the nonlinearity presents a discontinuity. We prove the upper semicontinuity of solutions and of the global attractor with respect to smooth approximations of the nonlinear term. We also give a complete description of the set of fixed points and study their stability. Finally, we analyze the existence of heteroclinic connections between the fixed points, obtaining information on the fine structure of the global attractor.

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1 Introduction

In this paper we consider a reaction-diffusion equation having a discontinuous nonlinear term. The usual way to treat this kind of nonlinearities consists in joining the points in the discontinuity with a vertical line transforming the nonlinearity into a graph. Thus, the equation transforms into a differential inclusion instead of a differential equation. Equations of such a type appear in models of physical interest (see, for example, [4], [16], [17]).

In particular, we are interested in the following equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + f(u) \in H_0(u) + \omega u, \text{ on } \Omega \times (0,T), \\ u\,|_{\partial\Omega} = 0, \\ u(x,0) = u_0(x), \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open subset with smooth boundary, $\omega \geq 0$, $f : \mathbb{R} \to \mathbb{R}$ is a continuous non-decreasing function, and

$$H_0(u) = \begin{cases} -1, \text{ if } u < 0, \\ [-1,1], \text{ if } u = 0, \\ 1, \text{ if } u > 0 \end{cases}$$

is the Heaviside function.

A natural and useful way for treating this inclusion consists in writing it in an abstract form by using subdifferential maps. We note that the multivalued map $H_0(u)$ is in fact the subdifferential of the absolute value $|u|$. Thus, the equation can be written as

$$\frac{du}{dt} + \partial\psi^1(u) - \partial\psi^2(u) \ni 0,$$

where $\partial\psi^i$ are some subdifferential maps defined below, see (2.1).

One important property of such equations is the lack of uniqueness of the Cauchy problem. Nevertheless, the asymptotic behavior and the qualitative properties of the solutions can be studied by using multivalued semiflows instead of semigroups (see [12]). The existence of a global compact connected attractor for such equations, under suitable conditions, is proved in [19], [20].

Our aim in this paper is twofold. On one hand, if we assume

$$\omega < \lambda_1, \tag{1.2}$$

where λ_1 is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$, we approximate the nonlinear function H_0 by smooth ones H_ε and study the convergence of the solutions and also of the global attractors. In this way we prove the upper semicontinuity of the global attractors of the approximations $\mathcal{A}_\varepsilon$ with respect to the attractor of the original problem $\mathcal{A}_0$, see Section 3.

The approximation of attractors for multivalued semiflows and differential inclusions has been studied before in [9], [10]. In [10] some types of approximatios for differential inclusions with upper semicontinuous right-hand sides are studied, proving the upper (and in some cases lower) semicontinuity of the global attractor. However, the smooth approximations H_ε that we consider in this paper are not included in any of the cases treated in [10].

On the other hand, we give a detailed description of the structure of the global attractor $\mathcal{A}_0$ in the case where $f \equiv 0$ and $\omega = 0$ and the problem is one dimensional. In Section 4, we first prove that equation (1.1) has an infinite, but countable, number of equilibria $v_0 = 0$, $v_1^{\pm}$, $v_2^{\pm}$, ..., $v_k^{\pm}$, ..., which can be ordered using a natural energy (or Lyapunov function) $E(u)$, see Section 5:

$$E(v_1^+) = E(v_1^-) < E(v_2^+) = E(v_1^2) < ... < E(v_k^+) = E(v_k^-) < ... < E(v_0).$$

We prove then, in Section 6, that $v_1^{\pm}$ are asymptotically stable fixed points, and all the other ones are unstable. The fixed point $v \equiv 0$ possesses the following remarkable property: for any fixed point v_k different from 0 there exists a solution, $u(t)$, with initial value $u(0) = 0$, such that $u(t)$ converges to v_k as $t \to +\infty$. Note that the existence of a Lyapunov function implies that the global attractor can be described completely by the equilibria and the heteroclinic connections between them. The natural question is then to establish which connections actually exist. In the case of uniqueness of solutions, this question has already been studied for reaction–diffusion equations, see, for example, [5], [6], [13], [14].

In the present case, the attractors of the approximations, $\mathcal{A}_\varepsilon$, correspond to a Chafee-Infante problem for which all existing connections are known, [8]. The natural conjecture is that the connections are the same when we pass to the limit case, that is, that a connection exists from the fixed point v to the fixed point v^* if $E(v) > E(v^*)$. Of course, since the energy is decreasing, no connections can exist if $E(v) \leq E(v^*)$. In fact, we will show that, in a sense, it is natural to expect that (1.1) is equivalent to a Chafee-Infante problem that has has undergo all the typical bifurcation cascade of these type of problems, [7], and thus all connections should be present.

In Section 8, using the results on the approximation of the fixed points obtained in Section 7, we have given a partial answer to this question. In fact, we prove that a heteroclinic connection exists from v_0 to $v_k^{\pm}$, $\forall k \geq 1$, from $v_k^{\pm}$ to $v_{k-1}^{\pm}$, $\forall k \geq 2$, and from $v_k^{\pm}$ to $v_1^{\pm}$, $\forall k \geq 1$. Also, we have proved that, if there exists a connection from $v_k^{\pm}$ to $v_i^{\pm}$, $k > i \geq 1$, then it can be replicated to a connection from $v_{nk}^{\pm}$ to $v_{ni}^{\pm}$, for all natural n.

2 Preliminaries

Through the paper, we denote by $\|\cdot\|$ the norm of the space $L^2(\Omega)$. Note that (1.1) can be rewritten in the abstract form

$$\left\{ \begin{array}{c} \dfrac{\partial u}{\partial t} + \partial \psi^1 (u) - \partial \psi^2 (u) \ni 0, \\ u(0) = u_0, \end{array} \right. \tag{2.1}$$

where $\partial \psi^i$, $i = 1, 2$, are the subdifferentials of the proper, convex, lower semicontinuous functions $\psi^i : L^2(\Omega) \to]-\infty, +\infty]$ (see [3]):

$$\psi^1(u) = \left\{ \begin{array}{c} \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \int_\Omega \int_0^u f(s)\, ds dx, \text{ if } u \in H_0^1(\Omega), \int_0^u f(s)\, ds \in L^1(\Omega), \\ +\infty, \text{ otherwise,} \end{array} \right.$$

$$D\left(\partial \psi^1\right) = \left\{ u \in H^2(\Omega) \cap H_0^1(\Omega) : \ f(u) \in L^2(\Omega) \right\},$$

$$\partial \psi^1(u) = -\Delta u + f(u),$$

$$\psi^2(u) = \left\{ \begin{array}{c} \int_\Omega \left(\omega \frac{u^2}{2} + \int_0^u H_0(s)\, ds \right) dx, \text{ if } \int_0^u H_0(s)\, ds \in L^1(\Omega), \\ +\infty, \text{ otherwise,} \end{array} \right.$$

$$D\left(\partial \psi^2\right) = \left\{ u \in L^2(\Omega) : \text{there exists } y \in L^2(\Omega),\ y(x) \in H_0(u(x)) + \omega u(x), \text{ a.e. on } \Omega \right\}$$

and

$$\partial \psi^2(u) = \left\{ y \in L^2(\Omega),\ y(x) \in H_0(u(x)) + \omega u(x), \text{ a.e. on } \Omega \right\}.$$

We note that $\int_0^u H_0(s)\, ds = |u|$. It is easy to see also that $D(\partial \psi^2) = L^2(\Omega)$.

Definition 2.1 *The function $u(\cdot) \in C([0,T], L^2(\Omega))$ is called a strong solution of (2.1) if:*

1. *$u(0) = u_0$;*
2. *$u(\cdot)$ is absolutely continuous on $(0, T)$;*
3. *There exist a function $g(t) \in \partial \psi^2(u(t))$, a.e. on $(0, T)$, such that*

$$\frac{du(t)}{dt} - \Delta u + f(u) - g(t) = 0, \ \text{a.e. } t \in (0, T) \tag{2.2}$$

or, alternatively,

$$\frac{du(t)}{dt} - \Delta u + f(u) - h(t) = \omega u, \ \text{a.e. } t \in (0, T), \tag{2.3}$$

where, for a.e. $t > 0, x \in \Omega$, $h(t, x) \in H_0(u(t, x))$, $h(t, \cdot) \in L^2(\Omega)$, and the equalities are understood in the sense of the space $L^2(\Omega)$.

For each $u_0 \in L^2(\Omega)$ and $T > 0$ there exists at least one strong solution $u(\cdot)$ of (1.1) such that $g(\cdot) \in L^2(0,T;L^2(\Omega))$. Note that in fact $h \in L^\infty(0,T;L^\infty(\Omega))$ and that this is true for any solution such that $h(t)$ is measurable. Each solution can be extended to the whole semiline $t \geq 0$, so that they are global. Let $\mathcal{D}(u_0)$ be the set of all strong solutions defined on $[0,+\infty)$ and such that $g(\cdot) \in L^2_{loc}(0,T;L^2(\Omega))$. Then the multivalued map into the parts of $L^2(\Omega)$, $G_0 : \mathbb{R}_+ \times L^2(\Omega) \to P(L^2(\Omega))$,

$$G_0(t,u_0) = \underset{u\in\mathcal{D}(u_0)}{\cup} u(t)$$

is a strict multivalued semiflow, that is, for any $t_i \geq 0$, $i=1,2$, $G_0(t_1+t_2,u_0) = G_0(t_2, G_0(t_1,u_0))$ and $G_0(0,\cdot) = Id$. Moreover, the set $G_0(t,u_0)$ is compact and the semiflow is upper semicontinuous, that is

$$dist\left(G_0(t,y_k), G_0(t,y)\right) \to 0, \quad \text{if} \quad y_k \to y.$$

See [19, Theorem 4, Lemmas 1, 2 and 6] for the proof of these facts.

Let us consider the problem

$$\begin{cases} \dfrac{du}{dt} - \Delta u + f(u) = l(t), \\ \quad u(0) = u_0 \in L^2(\Omega). \end{cases} \tag{2.4}$$

It is clear that any $u \in \mathcal{D}(u_0)$ is a strong solution of (2.4) with $l = g$. We shall need the following regularity result, see e.g. [3, p.189].

Proposition 2.2 *For any $l(\cdot) \in L^2(0,T;L^2(\Omega))$, $u_0 \in L^2(\Omega)$, there exists a unique strong solution of inclusion (2.4) such that*

$$u(\cdot) \in C([0,T], L^2(\Omega)), \sqrt{t}\frac{du}{dt} \in L^2(0,T;L^2(\Omega)),\ \psi^1(u(\cdot)) \in L^1(0,T),$$

and $\psi^1(u(t))$ is absolutely continuous on $[\delta,T]$, for all $\delta > 0$.

If $u_0 \in D(\psi^1)$, then $\dfrac{du}{dt} \in L^2(0,T;L^2(\Omega))$ and $\psi^1(u) \in L^\infty(0,T)$. If $u_0, v_0 \in L^2(\Omega)$, then

$$\|u(t) - v(t)\| \leq \|u_0 - v_0\|,\ \textit{for all}\ t \in [0,T].$$

Remark 2.3 *Note that, from comparison arguments, if $f(0) \leq 1$ then if $u_0 \geq 0$ in Ω then there exists a solution of (1.1) which satisfies*

$$\frac{du(t)}{dt} - \Delta u + f(u) = 1 + \omega u,$$

since 0 is a subsolution. On the other hand, if $f(0) \leq -1$, since we can compare solutions of (2.3) from below with

$$\frac{du(t)}{dt} - \Delta u + f(u) = -1 + \omega u,$$

we have that the solution of (1.1) is unique an strictly positive for $t > 0$.

Analogous results can be derived for $u_0 \leq 0$, if $f(0) \geq -1$ and $f(0) \geq 1$ respectively.

The set $\mathcal{A}$ is called a global attractor for the multivalued semiflow G_0 if it attracts any bounded subset B of $L^2(\Omega)$, i.e.

$$dist\left(G_0\left(t,B\right),\mathcal{A}\right)\to 0, \text{ as } t\to+\infty,$$

where $dist\left(C,A\right)=\sup_{c\in C}\inf_{a\in A}\|c-a\|$ is the Hausdorff semidistance, and it is negatively semi-invariant, i.e.

$$\mathcal{A}\subset G_0\left(t,\mathcal{A}\right), \text{ for all } t\geq 0.$$

Note that since f is monotone, we have $(f(s)-f(0))s\geq 0$ for all $s\in\mathbb{R}$. Hence we have, for every $y_2\in H_0\left(s\right)$, $s\in\mathbb{R}$

$$-f(s)s+y_2s+\omega s^2\leq\omega s^2+D|s|, \tag{2.5}$$

with $D=|f(0)|+1$. Hence if we assume

$$\omega<\lambda_1, \tag{2.6}$$

where λ_1 is the first eigenvalue of $-\Delta$ in $H_0^1\left(\Omega\right)$, then the results in [19, Theorem 4] prove that G_0 has the global compact invariant (i.e. $\mathcal{A}=G_0\left(t,\mathcal{A}\right), \forall t\geq 0$) attractor $\mathcal{A}$. Moreover, $\mathcal{A}$ is a connected set [20]. See [1] for the case of singled–valued equations and Corollary 3.2 below.

3 Upper semicontinuity of the global attractor

Let us consider now a parameterized non-decreasing family of functions $H_\varepsilon\in C^1\left(\mathbb{R}\right)$, $\varepsilon>0$, such that

$$|H_\varepsilon\left(s\right)-H_0\left(s\right)|<\varepsilon, \text{ if } |s|>\varepsilon,$$
$$-1<H_\varepsilon\left(s\right)<1, \text{ for all } s,$$

and such that H_ε' is non-increasing for $u\geq 0$, and non-decreasing for $u\leq 0$. It follows from these conditions that

$$dist\left(Graph\left(H_\varepsilon\right),Graph\left(H_0\right)\right)\leq\varepsilon, \tag{3.1}$$

and also that $|H_\varepsilon'\left(s\right)|\leq C_\varepsilon$, for all s.

We consider the equation

$$\begin{cases}\dfrac{\partial u_\varepsilon}{\partial t}-\Delta u_\varepsilon+f\left(u_\varepsilon\right)-H_\varepsilon\left(u_\varepsilon\right)=\omega u_\varepsilon,\\ u_\varepsilon\mid_{\partial\Omega}=0,\\ u_\varepsilon\left(0\right)=u_0.\end{cases} \tag{3.2}$$

We note that the boundedness of H_ε' implies that $f\left(u\right)-H_\varepsilon\left(u\right)=f_\varepsilon\left(u\right)-\omega_\varepsilon u$, for some $\omega_\varepsilon\geq 0$ and a non-decreasing continuous map f_ε. Then (3.2) has a unique strong solution $u_\varepsilon\left(\cdot\right)$ for any $u_0\in L^2\left(\Omega\right)$ (see [3, p.189]). Hence, it defines a semigroup $G_\varepsilon:\mathbb{R}_+\times L^2\left(\Omega\right)\to L^2\left(\Omega\right)$.

Now we derive suitable uniform estimates on the solutions of (1.1) and (3.2), which will be used below. To accomplish this, note first that, analogously to (2.5), we have

$$-f(s)s+H_\varepsilon(s)s+\omega s^2\leq\omega s^2+D|s| \tag{3.3}$$

with $D=|f(0)|+1$, independent of ε.

Then we have the following result. Note that in the estimates below, when we refer to solutions of (1.1) we mean that the estimates are valid for all strong solutions in $\mathcal{D}\left(u_0\right)$, that is, for the multivalued semiflow G_0.

Theorem 3.1 *With the assumptions above, assume furthermore (2.6) holds true.*

Let $0 \leq \phi \in L^{\infty}(\Omega)$ *be the unique solution of*

$$\begin{cases} -\Delta\phi = \omega\phi + D & \text{in } \Omega \\ \phi = 0 & \text{on } \Gamma. \end{cases}$$

and let $u_0 \in B \subset L^2(\Omega)$ *a bounded set of initial data in (3.2).*

Then if u *denotes either a solution of (1.1) or (3.2) then we have:*
i) For $t \geq 0$

$$\|u(t)\| \leq C(B,t) + \|\phi\|$$

for some bounded, independent of ε*, decreasing function* $C(B,t) \to 0$ *as* $t \to \infty$.
ii) For $t \geq 1$

$$\|u(t)\|_{L^{\infty}(\Omega)} \leq C(B,t) + \|\phi\|_{L^{\infty}(\Omega)}$$

for some bounded, independent of ε*, decreasing function* $C(B,t) \to 0$ *as* $t \to \infty$*. Even more*

$$\limsup_{t\to\infty} |u(t,x,u_0)| \leq \phi(x), \quad \text{uniformly in} \quad x \in \Omega$$

and the limit above is uniform for $u_0 \in B$*. Moreover, if* $|u_0(x)| \leq \phi(x)$ *for all* $x \in \Omega$*, then* $|u(t,x,u_0)| \leq \phi(x)$ *for all* $x \in \Omega$ *and* $t \geq 0$.

In particular the ball in $L^{\infty}(\Omega)$ *of radius* $\|\phi\|_{L^{\infty}(\Omega)} + 1$ *is absorbing for (1.1) and (3.2). In the latter case, the entering time is independent of* ε.
iii) For every $1 \leq p < \infty$*,* $\delta > 0$ *and* $t \geq 1$

$$\|u(t)\|_{W^{2-\delta,p}(\Omega)} \leq C(B,\phi,t)$$

for some bounded, independent of ε*, decreasing function* $C(B,\phi,t) \to C(\phi)$ *as* $t \to \infty$.

Proof. We begin with the case of (3.2). Hence, from (3.3) and by standard comparison arguments we have that

$$|u_{\varepsilon}(t,x)| \leq U(t,x)$$

where U is a solution of the linear problem

$$\begin{cases} \dfrac{\partial U}{\partial t} - \Delta U = \omega U + D, \\ U\mid_{\partial\Omega} = 0, \\ U(0) = |u_0|. \end{cases} \tag{3.4}$$

Hence, U can be written as $U = z + \phi$ where

$$\begin{cases} \dfrac{\partial z}{\partial t} - \Delta z = \omega z, \\ z\mid_{\partial\Omega} = 0, \\ z(0) = |u_0| - \phi. \end{cases}$$

Thus, by standard smoothing estimates of the heat equation we have

$$\|z(t)\|_{L^{\infty}(\Omega)} \leq C\frac{e^{-\alpha t}}{t^{N/4}}\|z(0)\|_{L^2(\Omega)}$$

with $\alpha = \lambda_1 - \omega > 0$, since we assume (2.6). This and the fact that the heat equation preserves the sign of $z(0)$ concludes part i).

For part ii) notice that from (3.2) and the bounds above, we have that for $t \geq 1$,

$$\frac{\partial u_\varepsilon}{\partial t} - \Delta u_\varepsilon = h_\varepsilon(x,t)$$

with $h_\varepsilon(x,t) = -f(u_\varepsilon) + H_\varepsilon(u_\varepsilon) + \omega u_\varepsilon \in L^\infty(1,\infty; L^\infty(\Omega))$ and

$$\|h_\varepsilon\|_{L^\infty(\Omega)} \leq C_0(B,\phi,t)$$

for some function $C_0(B,\phi,t)$ as in the statement. Therefore, parabolic regularity gives the result

Finally, for the solutions of (1.1) note that according to Definition 2.1 and (2.5), we have

$$|u(t,x)| \leq U(t,x)$$

as well. Hence the same arguments as before conclude. ■

As a consequence we get the following result.

Corollary 3.2

Assume $\varepsilon_0 > 0$, $1 \leq p < \infty$ and $\delta > 0$. Then
i) For any bounded set $B \subset L^2(\Omega)$, the set

$$\bigcup_{0\leq\varepsilon\leq\varepsilon_0} \{G_\varepsilon(t,B),\ t \geq 0\} \subset L^2(\Omega)$$

is bounded in $L^2(\Omega)$.
ii) For any bounded set $B \subset L^2(\Omega)$, the set

$$\bigcup_{0\leq\varepsilon\leq\varepsilon_0} \{G_\varepsilon(t,B),\ t \geq 1\} \subset W^{2-\delta,p}(\Omega)$$

is precompact in $W^{2-\delta,p}(\Omega)$.
iii) There exists an absorbing ball for (1.1) and (3.2) in $W^{2-\delta,p}(\Omega)$. In particular (3.2) has a compact invariant attractor $\mathcal{A}_\varepsilon$ in $L^2(\Omega)$.

The attractors $\mathcal{A}_\varepsilon$ for (3.2) and $\mathcal{A}_0$ for (1.1) attract bounded sets of $L^2(\Omega)$ in the norm of $W^{2-\delta,p}(\Omega)$.
iv) The union of the attractors

$$\bigcup_{0\leq\varepsilon\leq\varepsilon_0} \mathcal{A}_\varepsilon \subset W^{2-\delta,p}(\Omega)$$

is precompact in $W^{2-\delta,p}(\Omega)$.

Remark 3.3 *Note that in both Theorem 3.1 and Corollary 3.2 the results remain true if we take $t \geq t_0$, for any $t_0 > 0$, instead of $t \geq 1$.*

Now we prove a result on the continuity of the semiflows as $\varepsilon \to 0$.

Theorem 3.4 *Assume $1 \le p < \infty$ and $\delta > 0$ and let (2.6) hold.*
Then for any $t > 0$ and any compact set $K \subset L^2(\Omega)$ we have

$$dist\left(G_\varepsilon(t, K), G_0(t, K)\right) \to 0, \text{ as } \varepsilon \to 0,$$

where the distance is taken in the norm of $W^{2-\delta,p}(\Omega)$.

If $u_{\varepsilon 0} \to u_0$ in $L^2(\Omega)$ as $\varepsilon \to 0$, then for any $T > 0$ there exists a subsequence ε_n such that u_{ε_n} converges to some $u \in \mathcal{D}(u_0)$ in $C([0,T], L^2(\Omega))$.

Proof. Note that it is enough to prove the result for the distance in $L^2(\Omega)$. Once this is done, the compactness results in Corollary 3.2 above conclude.

Suppose the opposite, that is, there exist $\delta > 0$, $\varepsilon_n \to 0$ and $u_0^n \in K$ such that

$$dist\left(G_{\varepsilon_n}(t, u_0^n), G_0(t, K)\right) > \delta, \text{ for all } n.$$

Let $u_n(t) = G_{\varepsilon_n}(t, u_0^n)$. Define the sequences $g_n(t) = H_{\varepsilon_n}(u_n(t)) + \omega u_n(t)$ and $h_n(t) = H_{\varepsilon_n}(u_n(t))$. Note that from point i) in Theorem 3.1 it follows that

$$\|u_n(t)\| \le C_0, \text{ for all } t \ge 0,$$

so that

$$\|g_n(t)\| \le C_1 + \omega \|u_n(t)\| \le C_2, \text{ for a.e. } t \ge 0.$$

Also, using similar arguments as in [19, p.722] one can prove that

$$\left\| \sqrt{t} \frac{du_n(t)}{dt} \right\|_{L^2(0,T;L^2(\Omega))} \le C_3.$$

Hence, there exists a subsequence, that we still denote the same, such that $u_n \to u$ and $\sqrt{t}\frac{du_n}{dt} \to \sqrt{t}\frac{du}{dt}$ weakly in $L^2(0,T;L^2(\Omega))$. Moreover, Ascoli-Arzelá theorem implies that for any fixed $M > 0$ we have $u_n \to u$ in $C\left(\left[\frac{1}{M}, T\right], L^2(\Omega)\right)$ and u is absolutely continuous on $\left[\frac{1}{M}, T\right]$. Furthermore, the compactness results in Corollary 3.2 imply that $u_n \to u$ in $C\left(\left[\frac{1}{M}, T\right], L^\infty(\Omega) \cap H_0^1(\Omega)\right)$.

Also, note that $g_n = h_n + \omega u_n$ converges to some $g = h + \omega u \in L^\infty(0,T;L^2(\Omega))$ weakly star in $L^\infty(0,T;L^2(\Omega))$ and weakly in $L^2(0,T;L^2(\Omega))$, where $h \in L^\infty(0,T;L^\infty(\Omega))$. On the other hand, since $-\Delta u_n + f(u_n) = -\frac{du_n}{dt} + g_n$, then $\sqrt{t}(-\Delta u_n + f(u_n))$ converges to $l(t) = \sqrt{t}\left(-\frac{du}{dt} + g\right)$ weakly in $L^2(0,T;L^2(\Omega))$. Finally, the convergence for u_n above implies that $f(u_n) \to f(u)$ in $C\left(\left[\frac{1}{M}, T\right], L^\infty(\Omega)\right)$. Hence, we find at once that u satisfies

$$\frac{du}{dt} - \Delta u(t) + f(u(t)) = g(t) = h(t) + \omega u, \text{ a.e. on } (0,T).$$

Now, we show that $h(t) \in H_0(u(t))$, a.e. in $(0,T)$. For this we shall prove first that for a.e. $x \in \Omega$ and $s \in (0,T)$

$$dist\left(h_n(s,x), H_0(u(s,x))\right) \to 0, \text{ as } n \to \infty.$$

Indeed, if $u(s,x)=0$, then $h_n(s,x)=H_{\varepsilon_n}(u_n(s,x))\in[-1,1]=H_0(u(s,x))$, $\forall n$, so that the result is evident. If $u(s,x)<0$, then

$$dist\,(h_n(s,x),H_0(u(s,x)))=|H_{\varepsilon_n}(u_n(s,x))+1|\to 0,\ \text{as } n\to\infty.$$

Finally, if $u(s,x)>0$, then

$$dist\,(h_n(s,x),H_0(u(s,x)))=|H_{\varepsilon_n}(u_n(s,x))-1|\to 0,\ \text{as } n\to\infty.$$

Now, Proposition 1.1 in [18] implies that for a.e. $t\in(0,T)$

$$h(t)\in\bigcap_{n\geq 0}\overline{co}\bigcup_{k\geq n}h_k(t).$$

Then $h(t)=\lim_{n\to\infty}y_n(t)$ strongly in $L^2(\Omega)$, where $y_n(t)=\sum_{i=1}^M\lambda_i h_{k_i}(t)$, $\sum_{i=1}^M\lambda_i=1$, $k_i\geq n$. We note that for any $t\in[0,T]$ and a.e. $x\in\Omega$ we can find $n(\varepsilon,x,t)$ such that if $k\geq n$, then $dist\,(h_k(t,x),H_0(u(t,x)))\leq\varepsilon$. Therefore,

$$dist\,(y_n(t,x),H_0(u(t,x)))\leq\sum_{i=1}^M\lambda_i dist\,(h_{k_i}(t,x),H_0(u(t,x)))\leq\varepsilon.$$

Hence, since we can assume that for a.e. $(t,x)\in(0,T)\times\Omega$, $y_n(t,x)\to h(t,x)$, it follows that $h(t,x)\in H_0(u(t,x))$.

It remains to check that u is continuous as $t\to 0^+$. Let $\widehat{u}$ be the unique solution of

$$\begin{cases}\dfrac{du}{dt}-\Delta u+f(u)=\omega u,\\ u\,|_{\partial\Omega}=0,\\ u(0)=u_0,\end{cases}$$

and let $v_n(t)=u_n(t)-\widehat{u}(t)$. In a standard way, using that f is monotone, we can prove that

$$\frac{d}{dt}\|v_n\|^2\leq K_1+K_2\|v_n\|^2\leq K,$$

so that

$$\|v_n(t)\|^2\leq\|v_n(0)\|^2+Kt=Kt.$$

Hence, $\|u(t)-\widehat{u}(t)\|^2=\lim_{n\to\infty}\|v_n(t)\|^2\leq Kt$, for $t>0$, and

$$\|u(t)-u_0\|\leq\|u(t)-\widehat{u}(t)\|+\|\widehat{u}(t)-u_0\|<\delta,$$

as soon as $t<\varepsilon(\delta)$.

We have proved that $u(\cdot)$ is a strong solution of (1.1). Since $u_n(t)\to u(t)\in G_0(t,K)$ we have obtained a contradiction.

Note that at the same time we have established the second statement. ■

Now we are ready to prove the following result on the uppersemicontinuity of the attractors

Theorem 3.5 *Assume* $1\leq p<\infty$ *and* $\delta>0$ *and let (2.6) hold.*

Then $dist\,(\mathcal{A}_\varepsilon,\mathcal{A}_0)\to 0$, *as* $\varepsilon\to 0$ *where the distance is taken in the norm of* $W^{2-\delta,p}(\Omega)$.

Proof. Let $\eta > 0$ be fixed and ε_0 sufficiently small. Then for sufficiently large T we have

$$dist\left(G_0(T, \overline{\bigcup_{0\leq\varepsilon\leq\varepsilon_0} \mathcal{A}_\varepsilon}), \mathcal{A}_0\right) \leq \eta/2.$$

Now from Theorem 3.4, we have, for sufficiently small ε_0

$$dist\left(G_\varepsilon(T, \overline{\bigcup_{0\leq\varepsilon\leq\varepsilon_0} \mathcal{A}_\varepsilon}), G_0(T, \overline{\bigcup_{0\leq\varepsilon\leq\varepsilon_0} \mathcal{A}_\varepsilon})\right) \leq \eta/2.$$

Hence, by the triangle inequality, we get

$$dist\,(\mathcal{A}_\varepsilon, \mathcal{A}_0) \leq dist\left(G_\varepsilon(T, \overline{\bigcup_{0\leq\varepsilon\leq\varepsilon_0} \mathcal{A}_\varepsilon}), \mathcal{A}_0\right) \leq \eta.$$

■

As a consequence of the last theorem and Corollary 3.2, we shall prove the convergence of solutions of the approximations in the space $C\left([0,T], W^{2-\delta,p}(\Omega)\right)$.

Corollary 3.6 *If $u_{\varepsilon 0} \to u_0$ in $L^2(\Omega)$, where $u_{\varepsilon 0} \in \mathcal{A}_\varepsilon$, $u_0 \in \mathcal{A}$, then for any $T > 0$ there exists a subsequence ε_n such that u_n converges to some $u \in \mathcal{D}(u_0)$ in $C\left([0,T], W^{2-\delta,p}(\Omega)\right)$, $1 \leq p < \infty, \delta > 0$.*

Proof. We know from Theorem 3.4 that there exists a subsequence such that $u_{\varepsilon_n} \to u \in \mathcal{D}(u_0)$ in $C([0,T], L^2(\Omega))$. Then the results follows from the fact that, in view of Corollary 3.2, $\bigcup_{0\leq\varepsilon\leq\varepsilon_0} \mathcal{A}_\varepsilon$ is precompact in $W^{2-\delta,p}(\Omega)$. ■

4 Fixed points

¿From now on we will consider equation (1.1) with $n = 1$, $\Omega = (0,1)$ and $f \equiv 0$, which reads

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} - H_0(u) \ni \omega u, \text{ on } (0,1) \times (0,T), \\ u(0,t) = u(1,t) = 0, \\ u(x,0) = u_0(x). \end{cases} \tag{4.1}$$

We shall assume in the sequel that $0 \leq \omega < \pi^2$. We note that the last condition implies that the global attractor exists (see Section 2).

Our aim is to give a complete description of the set of fixed points in this particular case. A fixed point (stationary point or equilibria) is a constant in time solution v. It is clear from the definition of a strong solution that v satisfies $-\dfrac{\partial^2 v}{\partial x^2} = \xi + \omega v$, where $\xi \in L^2(0,1)$ and $\xi(x) \in H_0(v(x))$ for a.e. $x \in (0,1)$. We note also that $v \in H^2(0,1) \cap H_0^1(0,1)$, and then v, v' are absolutely continuous on $[0,1]$, [3].

First let $\omega = 0$. The fixed points $v(x)$ satisfy the system

$$\begin{cases} y' \in -H_0(v), \\ v' = y, \\ v(0) = v(1) = 0. \end{cases} \tag{4.2}$$

The solution of system (4.2) is $y^2 + 2|v| = C$, which gives us two families of parabolic functions. Obviously, $v_0 \equiv 0$ is a fixed point. We note that $v(x)$ satisfies

$$\begin{aligned} v'' &= -1, \text{ if } v > 0, \\ v'' &= 1, \text{ if } v < 0. \end{aligned}$$

Then

$$v(x) = -\frac{x^2}{2} + Bx + D, \text{ if } v > 0, \tag{4.3}$$

$$v(x) = \frac{x^2}{2} + B'x + D', \text{ if } v < 0. \tag{4.4}$$

We note that if (4.3)-(4.4) are local solutions such that $v(0) = 0$, then $D = D' = 0$. Further, we need to satisfy $v(1) = 0$. If we consider the case $v > 0$, then the "time" for going from the initial point $(v(0), y(0)) = \left(0, \sqrt{C}\right)$ to the point $\left(\frac{C}{2}, 0\right)$, that is a quarter loop, is $T = \int_0^{\frac{C}{2}} \frac{du}{\sqrt{C-2u}} = \sqrt{C}$. By symmetry the "time" for going from the initial point $(v(0), y(0)) = \left(0, \sqrt{C}\right)$ to the point $\left(0, -\sqrt{C}\right)$, that is a half loop, is $T_0 = 2T = 2\sqrt{C}$. The case $v < 0$ is identical. In order to satisfy the boundary conditions in (4.2) we need that either $(v(1), y(1)) = \left(0, \sqrt{C}\right)$ or $(v(1), y(1)) = \left(0, -\sqrt{C}\right)$. Concatenating the functions (4.3)-(4.4) we obtain solutions of (4.2) if $T_0 = \frac{1}{n}$ for any $n \geq 1$, that is, if $C = C_n = \frac{1}{4n^2}$. We can easily check that these solution belong to $H^2(0,1) \cap H_0^1(0,1)$ and then are strong solutions of (4.2). Hence there exists an infinite number of fixed points. We note also that if v_n^+ (respectively v_n^-) is a fixed point such that $v'(0) > 0$ (respectively $v'(0) < 0$) and $C_n = \frac{1}{4n^2}$, then $v_n^+\left(\frac{1}{n}\right) = 0$ (respectively $v_n^-\left(\frac{1}{n}\right) = 0$), so that in the interval $\left[0, \frac{1}{n}\right]$ the functions v_n^+, v_n^- are given by (4.3)-(4.4) with $B = \frac{1}{2n}$, $D = 0$ (respectively $B' = -\frac{1}{2n}$, $D' = 0$). Finally, we obtain the following fixed points:

$$\begin{aligned}
v_0 &\equiv 0 \\
v_1^+(x) &= -\frac{x^2}{2} + \frac{x}{2}, \quad v_1^-(x) = -v_1^+(x) \\
v_2^+(x) &= \begin{cases} -\frac{x^2}{2} + \frac{x}{4}, \text{ if } 0 \leq x \leq \frac{1}{2}, \\ \frac{\left(x-\frac{1}{2}\right)^2}{2} - \frac{x-\frac{1}{2}}{4}, \text{ if } \frac{1}{2} \leq x \leq 1 \end{cases}, \quad v_2^-(x) = -v_2^+(x) \\
&\vdots \\
v_n^+(x) &= \begin{cases} -\frac{x^2}{2} + \frac{x}{2n}, \text{ if } 0 \leq x \leq \frac{1}{n}, \\ \vdots \\ -\frac{\left(x-\frac{k}{n}\right)^2}{2} + \frac{x-\frac{k}{n}}{2n}, \text{ if } \frac{k}{n} \leq x \leq \frac{k+1}{n}, k \text{ is even} \\ \frac{\left(x-\frac{k}{n}\right)^2}{2} - \frac{x-\frac{k}{n}}{2n}, \text{ if } \frac{k}{n} \leq x \leq \frac{k+1}{n}, k \text{ is odd}, \\ k = 0, ..., n-1 \end{cases}, \quad v_n^-(x) = -v_n^+(x)
\end{aligned}$$

Next we ensure that there are no other fixed points.

Lemma 4.1 *For any* $(v_0, y_0) \in \mathbb{R}^2$ *the Cauchy problem*

$$\begin{cases} y' \in -H_0(v) \\ v' = y \\ v(0) = v_0, y(0) = y_0 \end{cases} \tag{4.5}$$

has a unique strong solution in some interval $[0, \delta]$.

Proof. By a strong solution we mean an absolutely continuous function $(v(x), y(x))$ satisfying (4.5) for a.e. $x \in [0, \delta]$. The existence is clear from the previous results, so that we prove only the uniqueness.

Consider first the case where $(v_0, y_0) \neq (0, 0)$. If $v_0 \neq 0$ (say $v_0 > 0$) and v_1, v_2 are two solutions, then the continuity of any solution implies the existence of an interval $[0, \delta]$ such that $v_i(x) > 0$, for $x \in [0, \delta]$. In such a case $H_0(v_i(x)) = 1$, so that $(y_1 - y_2)' = 0$ and then $y_1 = y_2$ on $[0, \delta]$. Hence, $(v_1 - v_2)' = 0$ and $v_1 = v_2$ on $[0, \delta]$. If $v_0 = 0$, $y_0 \neq 0$ (say $y_0 > 0$) and v_1, v_2 are two solutions, then the continuity of y_i gives $y_i(x) > 0$ on some $[0, \delta]$. Since $v_i' = y_i$, this implies that $v_i(x) > 0$ on $(0, \delta]$, and now we argue as before to obtain $v_1 = v_2$ on $[0, \delta]$.

Finally, let $(v_0, y_0) = (0, 0)$. Let $v(x)$ be a solution not equal to $v_0 \equiv 0$. Then in some interval $(0, \delta]$ we have either $v(x) > 0$ or $v(x) < 0$. If, say, $v(x) > 0$, then $H_0(v(x)) = 1$ and $v'' = -1$. Hence, $v(x) = -\frac{x^2}{2} < 0$ on $(0, \delta]$, which is a contradiction. A similar argument is valid for $v(x) < 0$. ■

Remark 4.2 *We have proved that there exists an infinite but countable number of fixed points and have found them explicitly. We note also that the fixed points* v_n^+, v_n^- *have exactly* $n - 1$ *zeros on* $(0, 1)$.

Next we observe the following self-similarity property of the solutions. We note that if take v_1^+ *and define the function*

$$v(x) = \begin{cases} \frac{1}{4} v_1^+(2x), & x \in \left[0, \frac{1}{2}\right], \\ -\frac{1}{4} v_1^+\left(2\left(x - \frac{1}{2}\right)\right), & x \in \left[\frac{1}{2}, 1\right], \end{cases}$$

then $v(x) = v_2^+(x)$. *In the same way we have that for any* $v_k^\pm$, $k \geq 1$, *and* n *we have*

$$v_{nk}^\pm(x) = \begin{cases} \frac{1}{n^2} v_k^\pm(nx), & \text{if } \; x \in \left[0, \frac{1}{n}\right] \\ -\frac{1}{n^2} v_k^\pm\left(n\left(x - \frac{1}{n}\right)\right), & \text{if } \; x \in \left[\frac{1}{n}, \frac{2}{n}\right], \\ \vdots \end{cases}$$

The case $0 < \omega < \pi^2$ is rather similar and we omit the details. In this case the fixed points are the following:

$$
\begin{aligned}
v_0 &\equiv 0 \qquad (4.6)\\
v_1^+(x) &= \frac{1}{\omega}\cos\left(\sqrt{\omega}x\right)+\frac{1-\cos\left(\sqrt{\omega}\right)}{\omega\sin\left(\sqrt{\omega}\right)}\sin\left(\sqrt{\omega}x\right)-\frac{1}{\omega},\\
v_1^-(x) &= -v_1^+(x)\\
v_2^+(x) &= \begin{cases} \frac{1}{\omega}\cos\left(\sqrt{\omega}x\right)+\frac{1-\cos\left(\frac{\sqrt{\omega}}{2}\right)}{\omega\sin\left(\frac{\sqrt{\omega}}{2}\right)}\sin\left(\sqrt{\omega}x\right)-\frac{1}{\omega} & \text{, if } \quad 0\le x\le\frac{1}{2},\\ -\frac{1}{\omega}\cos\left(\sqrt{\omega}\left(x-\frac{1}{2}\right)\right)-\frac{1-\cos\left(\frac{\sqrt{\omega}}{2}\right)}{\omega\sin\left(\frac{\sqrt{\omega}}{2}\right)}\sin\left(\sqrt{\omega}\left(x-\frac{1}{2}\right)\right)+\frac{1}{\omega}, & \text{if } \frac{1}{2}\le x\le 1 \end{cases},\\
v_2^-(x) &= -v_2^+(x)\\
&\vdots\\
v_n^+(x) &= \begin{cases} \frac{1}{\omega}\cos\left(\sqrt{\omega}x\right)+\frac{1-\cos\left(\frac{\sqrt{\omega}}{n}\right)}{\omega\sin\left(\frac{\sqrt{\omega}}{n}\right)}\sin\left(\sqrt{\omega}x\right)-\frac{1}{\omega} \quad \text{, if } \quad 0\le x\le\frac{1}{n},\\ \vdots\\ \frac{1}{\omega}\cos\left(\sqrt{\omega}\left(x-\frac{k}{n}\right)\right)+\frac{1-\cos\left(\frac{\sqrt{\omega}}{n}\right)}{\omega\sin\left(\frac{\sqrt{\omega}}{n}\right)}\sin\left(\sqrt{\omega}\left(x-\frac{k}{n}\right)\right)-\frac{1}{\omega},\\ \text{if } \frac{k}{n}\le x\le\frac{k+1}{n}, k \text{ is even}\\ -\frac{1}{\omega}\cos\left(\sqrt{\omega}\left(x-\frac{k}{n}\right)\right)-\frac{1-\cos\left(\frac{\sqrt{\omega}}{n}\right)}{\omega\sin\left(\frac{\sqrt{\omega}}{n}\right)}\sin\left(\sqrt{\omega}\left(x-\frac{k}{n}\right)\right)+\frac{1}{\omega},\\ \text{if } \frac{k}{n}\le x\le\frac{k+1}{n}, k \text{ is odd},\\ k=0,...,n-1 \end{cases},\\
v_n^-(x) &= -v_n^+(x)
\end{aligned}
$$

Remark 4.3 *Observe that for each $n\in\mathbb{N}$, in the interval $\frac{k}{n}\le x\le\frac{k+1}{n}$, $k=0,\dots,n-1$, v_n^+ is the unique solution of*

$$
\begin{cases} -\dfrac{\partial^2 u}{\partial x^2}=(-1)^k+\omega u,\\ u\left(\frac{k}{n}\right)=u\left(\frac{k+1}{n}\right)=0 \end{cases}
$$

and $v_n^-(x)=-v_n^+(x)$. Hence its sign is given by $(-1)^k$.

Also, note that, for example v_2^+ satisfies $v(x)=-v\left(x-\frac{1}{2}\right)=-v(1-x)$, for $x\in\left[\frac{1}{2},1\right]$, with analogous symmetries for the other equilibria.

Remark 4.4 *We note that when ω approaches and crosses the value π^2 the fixed points $v_1^\pm$ suffer a bifurcation to infinity and disappear. The same occurs for $v_2^\pm$ when ω cross $4\pi^2$, for $v_3^\pm$ when ω cross $9\pi^2$, etc.*

5 A Lyapunov function

Let us define the continuous function $E:H_0^1(0,1)\to\mathbb{R}$ by

$$
E(u)=\frac{1}{2}\int_0^1\left|\frac{\partial u}{\partial x}\right|^2dx-\int_0^1\left(|u|+\frac{\omega}{2}u^2\right)dx=\psi^1(u)-\psi^2(u). \qquad (5.1)
$$

Let $u(t)$ be an arbitrary solution of inclusion (4.1). We note that by the regularity of the solutions, Proposition 2.2, $E(u(t)):(0,+\infty)\to\mathbb{R}$ is a continuous function. We note also that if $u_0 \in H_0^1$, then $E(u(t))$ is continuous on $[0,+\infty)$. It follows also that $\frac{du}{dt}$ belongs to $L^2(\delta, T; L^2(0,1))$, for any $\delta > 0$. Then Lemma 2.1 in [3, p.189] implies that $E(u(t))$ is absolutely continuous on $[\delta, +\infty)$ and $\frac{1}{2}\frac{d}{dt}\left\|\frac{\partial u}{\partial x}\right\|^2 = \left(-\frac{\partial^2 u}{\partial x^2}, \frac{du}{dt}\right), \left(g(t), \frac{du}{dt}\right) = \frac{d}{dt}\psi^2(u(t))$, for a.e. $t \in [\delta, +\infty)$, where $g(t) \in \partial\psi^2(u(t))$.

Then we can take the derivative of $E(u(t))$ with respect to t to obtain

$$\frac{dE(u(t))}{dt} = \int_0^1 \left(-\frac{\partial^2 u}{\partial x^2} - g(t)\right)\frac{du}{dt}dx = -\int_0^1 \left|\frac{\partial u}{\partial t}\right|^2 dx.$$

The following properties follow:

1. $E(u(t))$ is continuous on $(0,+\infty)$;
2. $E(u(t)) \leq E(u(s))$, if $t \geq s > 0$;
3. If $E(u(t)) = E(u(0))$, for some $t > 0$, then $u(\tau) = u(0)$ for any $\tau \in [0,t]$, i.e. $u(0) = v$ is a fixed point.

Such a function is called energy or Lyapunov function in the literature. Finally note that, using Young's and Poincaré inequalities we have that E is bounded below if (2.6) is satisfied, that is, if $\omega < \pi^2$.

We note first that easy computations give

$$\begin{aligned} E(0) &= 0, \quad E\left(v_n^+\right) = E\left(v_n^-\right) = -\frac{1}{24n^2}, \qquad \text{for } n \geq 1, \text{ if } \omega = 0, \\ E(0) &= 0, \quad E\left(v_n^+\right) = E\left(v_n^-\right) = \frac{n}{\omega\sqrt{\omega}}\left(\frac{\cos\left(\frac{\sqrt{\omega}}{n}\right) - 1}{\sin\left(\frac{\sqrt{\omega}}{n}\right)}\right) + \frac{1}{2\omega}, \qquad \text{if } 0 < \omega < \pi^2. \end{aligned}$$

We note that the function $n \mapsto \frac{n}{\omega\sqrt{\omega}}\left(\frac{\cos\left(\frac{\sqrt{\omega}}{n}\right)-1}{\sin\left(\frac{\sqrt{\omega}}{n}\right)}\right)$ is strictly increasing and

$$\lim_{n\to+\infty} \frac{n}{\omega\sqrt{\omega}}\left(\frac{\cos\left(\frac{\sqrt{\omega}}{n}\right) - 1}{\sin\left(\frac{\sqrt{\omega}}{n}\right)}\right) = -\frac{1}{2\omega}.$$

Hence,

$$\begin{aligned} E\left(v_1^+\right) &= E\left(v_1^-\right) < E\left(v_2^+\right) = E\left(v_2^-\right) < ... < E\left(v_n^+\right) = E\left(v_n^-\right) < ... < E(0) = 0, \\ \lim_{n\to+\infty} E\left(v_n^+\right) &= 0. \end{aligned} \tag{5.2}$$

For any solution $u(t)$ we define the omega limit set

$$\omega(\{u(t)\}) = \left\{y : u(t_k) \to y \text{ in } L^2(0,1), \text{ where } t_k \to +\infty\right\},$$

which is non-empty.

Lemma 5.1 *$\omega(\{u(t)\}) \subset Z$, where Z is the set of fixed points. Moreover, $\omega(\{u(t)\}) = z \in Z$, and $u(t) \to z$, as $t \to +\infty$, in $H_0^1(0,1)$.*

Proof. Since E is non-increasing along the solutions, and bounded below, we deduce that $E(u(t))$ converges to some $l \in \mathbb{R}$ as $t \to +\infty$.

Also, from the compactness in Corollary 3.2, $\omega(\{u(t)\})$ is non empty. Therefore, if $y \in \omega(\{u(t)\})$, then there exists a subsequence (again denoted by $u(t_k)$) such that $u(t_k) \to y$ in H_0^1. The continuity of E in the space H_0^1 implies that $E(y) = l$. Hence, $E(y) = l$, for all $y \in \omega(\{u(t)\})$. Let us prove that $y \in Z$. Fix $t > 0$. Since $u(t_k) \to y$ and the multivalued map $G_0(t, \cdot)$ is upper semicontinuous (see Section 2), we have

$$dist\left(G_0(t, u(t_k)), G_0(t, y)\right) \to 0, \text{ as } t_k \to +\infty.$$

It is known that $u(t + t_k) \in G_0(t, u(t_k))$ (see [19, p.718]), and then $dist\left(u(t + t_k), G_0(t, y)\right) \to 0$. The compactness of $G_0(t, y)$ implies the existence of $z \in G_0(t, y)$ and a subsequence $u\left(t + t_{k_j}\right)$ such that $u\left(t + t_{k_j}\right) \to z$. Hence, $E(z) = l = E(y)$ and $y \in Z$.

Now we repeat exactly the same proof of [2, Proposition 4.1] to obtain that $\omega(\{u(t)\})$ is connected. $\omega(\{u(t)\})$ is a closed set and it is contained in the global attractor, so that it is compact. If it is not connected, then $\omega(\{u(t)\}) = \omega_1 \cup \omega_2$, where ω_i are non-empty compact disjoint sets. We take open disjoint sets U_i such that $\omega_i \subset U_i$. Since $u(t)$ is a continuous function, there must exist $t_j \to +\infty$ such that $u(t_j) \notin U_1 \cup U_2$. But the set $\{u(t_j)\}$ is precompact in $L^2(0,1)$, and then there exists a subsequence $u(t_{j_k})$ converging to some $z \notin U_1 \cup U_2$. But $z \in \omega(\{u(t)\})$, which is a contradiction.

Finally, we note that the set Z is countable. Therefore, the only possibility is that $\omega(\{u(t)\})$ consists of one point of Z. The point z attracts $u(t)$ in H_0^1, because in other case there would exist a sequence $u(t_j)$ such that $\|u(t_j) - z\|_{H_0^1} > \varepsilon$, and then we could obtain a subsequence converging to z in H_0^1, a contradiction. ■

6 Stability of fixed points

We shall study now the stability of the fixed points in the case $\omega = 0$. We say that a fixed point v of (4.1) is stable in the Banach space Y if for any $\rho > 0$ there exists $\delta > 0$ such that if $\|u_0 - v\|_Y < \delta$, then $\|u(t) - v\|_Y < \rho$, for any $t \geq 0$ and any solution $u(\cdot) \in \mathcal{D}(u_0)$ (i.e. $\|y - v\|_Y < \rho$, for any $t \geq 0$, $y \in G_0(t, u_0)$). In other words, if the initial data is in a sufficiently small neighborhood of the fixed point, then all the solutions remain uniformly in a given neighborhood of the fixed point. In other case we say that it is unstable. We say that the fixed point v is asymptotically stable if it is stable and there exists $\delta > 0$ such that $\lim_{t \to +\infty} dist\left(G(t, u_0), v\right) = 0$, if $\|u_0 - v\|_Y < \delta$.

We shall prove that the fixed points v_1^+, v_1^- are the unique stable equilibria.

We start proving the following result which is independent of the dimension of the problem.

Proposition 6.1 *Assume (2.6) and $f \equiv 0$. Then (1.1) has a unique positive fixed point that we denote by v_1^+ and for any nonnegative initial data u_0 there exists a solution of (1.1) that converges to v_1^+ as $t \to +\infty$.*

In particular, this implies that the trivial solution $v_0 = 0$ is unstable.

The same results holds true for negative solutions.

Proof. With the notations of Theorem 3.1 note that positive equilibria must satisfy

$$\begin{cases} -\Delta u = \omega u + 1 & in\ \Omega \\ u = 0 & on\ \Gamma. \end{cases}$$

which has a unique solution, since (2.6) is assumed. Let $U(t)$ be the unique solution to

$$\begin{cases} \dfrac{\partial U}{\partial t} - \Delta U = \omega U + 1, \\ U \mid_{\partial\Omega} = 0, \\ U(0) = u_0 \geq 0, \end{cases}$$

which is strictly positive for $t > 0$ and it is also a solution of (1.1). Hence, U can be written as $U = z + \phi$, where

$$\begin{cases} \dfrac{\partial z}{\partial t} - \Delta z = \omega z, \\ z \mid_{\partial\Omega} = 0, \\ z(0) = u_0 - \phi. \end{cases}$$

Since z converges to 0 (again we use that $\omega < \lambda_1$), the result follows. ■

We return now to problem (3.2).

Theorem 6.2 *(Stability in $H_0^1(0,1)$) If $\omega = 0$ the fixed points v_1^+, v_1^- are stable in $H_0^1(0,1)$.*

Proof. We take $v_1^+(x) = -\frac{x^2}{2} + \frac{x}{2}$ and put $v_1 = v_1^+$. The case v_1^- is similar. We note that $\frac{d^2}{dx^2} v_1 = -1$ and then for any solution of (4.1) with $u_0 \in H_0^1(0,1)$ we have that $z(t) = u(t) - v_1$ satisfies

$$\begin{cases} \dfrac{\partial z}{\partial t} - \dfrac{\partial^2 z}{\partial x^2} = g(t) - 1, \text{ on } (0,1) \times (0,\infty), \\ z(0,t) = z(1,t) = 0, \\ z(x,0) = u_0(x) - v_1(x) \in H_0^1(0,1), \end{cases} \tag{6.1}$$

where $g(t) \in H_0(u(t))$, for a.e. t, and $g \in L^\infty(0,\infty; L^\infty(0,1))$. It is known that $z \in C([0,T], H_0^1(0,1))$, for any $T > 0$, [15].

Let $\|z_0\|_{H_0^1} = \sqrt{\int_0^1 \left| \dfrac{dz_0}{dx} \right|^2 dx} < \frac{\rho}{2}$, $\rho > 0$. Let t_1 be the largest number such that $\|z(t)\|_{H_0^1} \leq \rho$, for all $t \leq t_1$. We shall prove that if ρ is small enough, then $t_1 = +\infty$.

Since $z(x,t) = z(0,t) + \int_0^x \dfrac{\partial z}{\partial s} ds$ ($z(\cdot,t)$ is absolutely continuous on x), we obtain that $|z(x,t)| \leq \|z(t)\|_{H_0^1} \leq \rho$, for any $t \in [0,t_1]$, $x \in [0,1]$. The function $u(x,t)$ is then positive if $v_1(x) > \rho$, so that $g(t,x) - 1 = 0$. If $\rho < \frac{1}{8}$, solving $\frac{-x^2+x}{2} = \rho$ we can easily compute that $v_1(x) > \rho$ whenever $x \in [4\rho, 1-4\rho]$.

For $x \in [0, 4\rho]$ we have

$$|z(x,t)| \leq \int_0^{4\rho} \left| \frac{\partial z}{\partial s} \right| ds \leq \sqrt{4\rho} \, \|z(t)\|_{H_0^1} \leq 2\rho^{\frac{3}{2}},$$

and the same is valid for $x \in [1-4\rho, 1]$. Again the function $u(x,t)$ is positive for any $x \in [0,4\rho]$ such that $v_1(x) > 2\rho^{\frac{3}{2}}$. Now solving $\frac{-x^2+x}{2} = 2\rho^{\frac{3}{2}}$ we obtain that if $\rho < \frac{1}{16^{\frac{2}{3}}}$, then this is true for $x \in \left[8\rho^{\frac{3}{2}}, 1 - 8\rho^{\frac{3}{2}}\right]$. Hence, $u(x,t) > 0$, for any $x \in \left[8\rho^{\frac{3}{2}}, 1 - 8\rho^{\frac{3}{2}}\right]$, $t \in [0,t_1]$, and $g(x,t) - 1 = 0$, for a.e. $(x,t) \in \left(8\rho^{\frac{3}{2}}, 1 - 8\rho^{\frac{3}{2}}\right) \times (0,t_1)$. We note that $\rho < \frac{1}{8}$ implies $8\rho^{\frac{3}{2}} < 4\rho$.

Since (6.1) is a particular case of problem (2.4) with $l(t) = g(t) - 1 \in \partial\psi^2(u(t)) - 1$, $\psi^2(u) = \int_0^1 |u| \, dx$, we can use Proposition 2.2 and Lemma 2.1 in [3, p.189] to have that

$\frac{1}{2}\frac{d}{dt}\left\|\frac{\partial z}{\partial x}\right\|^2 = \left(-\frac{\partial^2 z}{\partial x^2}, \frac{dz}{dt}\right)$ and $\left(g(t) - 1, \frac{dz}{dt}\right) = \left(g(t), \frac{du}{dt}\right) - \left(1, \frac{dz}{dt}\right) = \frac{d}{dt}\left(\psi^2(u(t)) - (1, z(t))\right)$. Hence, multiplying (6.1) by $\dfrac{dz}{dt}$ we have

$$\left\|\frac{dz}{dt}\right\|^2 + \frac{1}{2}\frac{d}{dt}\left\|\frac{\partial z}{\partial x}\right\|^2 = \int_0^1 (g(t) - 1)\frac{dz}{dt}dx = \frac{d}{dt}\int_0^1 (|u(t)| - z(t))\, dx.$$

Integrating over $(0, t)$ we get

$$\frac{1}{2}\left\|\frac{\partial z}{\partial x}(t)\right\|^2 \leq \frac{1}{2}\left\|\frac{\partial z}{\partial x}(0)\right\|^2 + \int_0^1 (|u(t)| - z(t) - |u(0)| + z(0))\, dx.$$

Since $|u(x,t)| = u(x,t) = v_1(x) + z(x,t)$, $|u(x,t)| = u(x,0) = v_1(x) + z(x,0)$, for all $x \in \left[8\rho^{\frac{3}{2}}, 1 - 8\rho^{\frac{3}{2}}\right]$, $t \in [0, t_1]$, we have to consider the last integral only in the intervals $I_1 = \left[0, 8\rho^{\frac{3}{2}}\right]$ and $I_2 = \left[1 - 8\rho^{\frac{3}{2}}, 1\right]$. But on the other hand, $|z(x,t)| \leq 2\rho^{\frac{3}{2}}$, and $|v_1(x)| = v_1(x) \leq 4\rho^{\frac{3}{2}}$ in $I_1 \cup I_2$. Hence,

$$|u(x,t)| = |z(x,t) + v_1(x)| \leq C_1\rho^{\frac{3}{2}}, \text{ for all } x \in I_1 \cup I_2, t \in [0, t_1].$$

Finally,

$$\left\|\frac{\partial z}{\partial x}(t)\right\|^2 \leq \left\|\frac{\partial z}{\partial x}(0)\right\|^2 + C_2\rho^{\frac{3}{2}}\left(\int_0^{8\rho^{\frac{3}{2}}} 1dx + \int_{1-8\rho^{\frac{3}{2}}}^1 1dx\right) = \|z_0\|^2_{H_0^1} + C_3\rho^3.$$

Choosing $\rho \leq \frac{1}{4C_3}$, we have

$$\|z(t)\|^2_{H_0^1} \leq \frac{\rho^2}{4} + \frac{\rho^2}{4} < \rho^2, \text{ for all } t \in [0, t_1].$$

Since $t_1 < +\infty$ gives a contradiction, it follows that $t_1 = +\infty$. ■

Now we have:

Theorem 6.3 *(Stability in $L^2(0,1)$) If $\omega = 0$ the fixed points v_1^+, v_1^- are stable in $L^2(0,1)$.*

Proof. Put $v_1 = v_1^+$. Multiplying (6.1) by z and using $|g(x,t) - 1| \leq 2$ we have

$$\frac{1}{2}\frac{d}{dt}\|z(t)\|^2 + \|z(t)\|^2_{H_0^1} \leq 2\int_0^1 |z|\, dx \leq 2\|z(t)\|. \tag{6.2}$$

Differentiating we get $\frac{d}{dt}\|z(t)\| \leq 2$ and then

$$\|z(t)\| \leq \|z_0\| + 2t \tag{6.3}$$

$$\int_0^t \|z(s)\|^2_{H_0^1} ds \leq \frac{1}{2}\|z_0\|^2 + 2\int_0^t \|z(s)\|\, ds \leq \frac{1}{2}\|z_0\|^2 + 2t\|z_0\| + 2t^2. \tag{6.4}$$

Further, we take a sequence $z_0^n \in H_0^1$ converging to z_0 in L^2. If $z_n(t)$ are the solutions of (6.1) with $z_n(0) = z_0^n$, then Proposition 2.2 gives $z_n \to z$ in $C([0,T], L^2(0,1))$ for any $T > 0$. Multiplying (6.1) by $t\dfrac{dz_n}{dt}$ we have

$$t\left\|\frac{dz_n}{dt}\right\|^2 + \frac{t}{2}\frac{d}{dt}\|z\|_{H_0^1}^2 \le 2t\left\|\frac{dz_n}{dt}\right\| \le \frac{t}{2}\left\|\frac{dz_n}{dt}\right\|^2 + 2t.$$

Integrating by parts on $(0,T)$ and using (6.4) we obtain

$$\int_0^T \frac{t}{2}\left\|\frac{dz_n}{dt}\right\|^2 dt + \frac{T}{2}\|z_n(T)\|_{H_0^1}^2 \le T^2 + \frac{1}{2}\int_0^T \|z\|_{H_0^1}^2 dt$$
$$\le 2T^2 + T\|z_0\| + \frac{1}{4}\|z_0\|^2.$$

Passing to the limit we have

$$\|z(T)\|_{H_0^1}^2 \le 4T + 2\|z_0\| + \frac{1}{2T}\|z_0\|^2. \tag{6.5}$$

For a given $\rho > 0$ we can choose σ and T such that $4T + 2\sigma + \frac{\sigma^2}{2T} \le \frac{\rho^2}{4}$, $\sigma + 2T \le \rho$. Then if $\|z_0\| \le \sigma$, (6.3) and (6.5) imply $\|z(t)\| \le \rho$, for all $t \in [0,T]$, $\|z(T)\|_{H_0^1} \le \frac{\rho}{2}$. Finally, choosing ρ small enough Theorem 6.2 gives $\|z(t)\|_{H_0^1} \le \rho$, for all $t \ge T$. Since the first eigenvalue of $-\frac{\partial^2}{\partial x^2}$ in $H_0^1(0,1)$ is $\lambda_1 = \pi^2$, we have

$$\|z(t)\| \le \frac{\|z(t)\|_{H_0^1}}{\pi} \le \rho, \text{ for any } t \ge T.$$

■

Hence we get:

Corollary 6.4
i) The solution $u(t) = v_1^+$ (respectively v_1^-) corresponding to $u_0 = v_1^+$ (respectively v_1^-) is unique.
ii) The equilibrium points $v_1^\pm$ are asymptotically stable.

Proof. The first part is immediate from stability. For the second note that $v_1^\pm$ are stable, isolated, and $\omega(u_0) = z \in Z$, for all u_0 (see Lemma 5.1), so that if $\left\|u_0 - v_1^\pm\right\| < \delta$, for δ small enough, then $\omega(u_0) = v_1^\pm$. ■

Remark 6.5 *This stability result is not difficult to prove also in the case $0 < \omega < \pi^2$ with some little changes in the proof.*

We shall prove further that the other fixed points are unstable.

Theorem 6.6 *If $\omega = 0$ for any $n \ge 2$ the fixed points v_n^+, v_n^- are unstable.*

Proof. As before we shall consider the case $v_n = v_n^+$. The other case is similar. We define the following approximations of v_n :

$$y_\varepsilon^n(x) = \begin{cases} -\frac{x^2}{2} + \frac{1+\varepsilon}{2n}x, & \text{if } 0 \le x \le \frac{1+\varepsilon}{n}, \\ v_n^+(x) + \frac{\varepsilon-\varepsilon^2}{2n^2}, & \text{if } \frac{1+\varepsilon}{n} \le x \le \frac{n-1-\varepsilon}{n}, \\ -\frac{x^2}{2} + \frac{2n-1-\varepsilon}{2n}x - \frac{n-1}{2n} + \frac{\varepsilon}{2n}, & \text{if } \frac{n-1-\varepsilon}{n} \le x \le 1, \end{cases} \quad \text{if } n \text{ is odd,}$$

$$y_{\varepsilon}^{n}(x)=\left\{\begin{array}{c}-\frac{x^{2}}{2}+\frac{1+\varepsilon}{2n}x, \quad \text{if } 0\leq x\leq\frac{1+\varepsilon}{n}, \\ v_{n}^{+}(x)+\frac{\varepsilon-\varepsilon^{2}}{2n^{2}}, \quad \text{if } \frac{1+\varepsilon}{n}\leq x\leq\frac{n-1+\varepsilon}{n}, \\ \frac{x^{2}}{2}-\frac{2n-1+\varepsilon}{2n}x+\frac{n-1}{2n}+\frac{\varepsilon}{2n}, \quad \text{if } \frac{n-1+\varepsilon}{n}\leq x\leq 1,\end{array}\right. , \quad \text{if } n \text{ is even,}$$

where $\varepsilon>0$. It is easy to see that $y_{\varepsilon}^{n}\in H_{0}^{1}$ and $y_{\varepsilon}^{n}\underset{\varepsilon\to0}{\to}v_{n}^{+}$ in L^{2}. We can compute that

$$\begin{aligned} E\left(y_{\varepsilon}^{n}\right) &= -\frac{1}{24n^{2}}-\frac{3\varepsilon^{2}}{4n^{3}}+\frac{\varepsilon^{3}}{n^{2}}-\frac{11\varepsilon^{3}}{4n^{3}}, \quad \text{if } n \text{ is odd,} \\ E\left(y_{\varepsilon}^{n}\right) &= -\frac{1}{24n^{2}}-\frac{\varepsilon^{2}}{4n^{3}}+\frac{\varepsilon^{3}}{n^{2}}-\frac{2\varepsilon^{3}}{n^{3}}, \quad \text{if } n \text{ is even,}\end{aligned}$$

so that $E\left(y_{\varepsilon}^{n}\right)<E\left(v_{n}^{+}\right)$ for sufficiently small ε.

Let $u_{\varepsilon}(t)$ be a solution with $u_{\varepsilon}(0)=y_{\varepsilon}^{n}$. Lemma 5.1 implies that $\omega\left(\left\{u_{\varepsilon}(t)\right\}\right)=z\in Z$. Moreover, since $E\left(u_{\varepsilon}(t)\right)$ is non-increasing, it is clear that $E(z)=\lim_{t\to+\infty}E\left(u_{\varepsilon}(t)\right)<E\left(v_{n}^{+}\right)$. Hence, $\omega\left(\left\{u_{\varepsilon}(t)\right\}\right)\subset\left\{v_{1}^{+},v_{1}^{-},...,v_{n-1}^{+},v_{n-1}^{-}\right\}$. Finally, if we denote

$$\rho=\frac{\min_{i\in\{1,\ldots,n-1\}}\left\{\left\|v_{i}^{+}-v_{n}^{+}\right\|,\left\|v_{i}^{-}-v_{n}^{+}\right\|\right\}}{2},$$

then for any $\sigma>0$ there exist y_{ε}^{n}, a solution $u_{\varepsilon}(t)$ and $T>0$ such that $u_{\varepsilon}(0)=y_{\varepsilon}^{n}$, $\left\|y_{\varepsilon}^{n}-v_{n}^{+}\right\|<\sigma$, and

$$\left\|u_{\varepsilon}(t)-v_{n}^{+}\right\|>\rho, \text{ for all } t\geq T,$$

so that v_{n}^{+} is unstable. ■

Finally, we shall study in further detail the instability of $v_{0}\equiv0$.

Theorem 6.7 *If $\omega=0$ for any v_{k}^{+} (respectively v_{k}^{-}) there exists a solution u_{k}^{+} (respectively u_{k}^{-}) with $u(0)=v_{0}=0$ such that $u_{k}^{+}(t)\to v_{k}^{+}$ (respectively $u_{k}^{-}(t)\to v_{k}^{-}$), as $t\to+\infty$.*

Remark 6.8 *It follows that there exist infinite solutions starting at $v_{0}=0$ and also that there is a heteroclinic connection from v_{0} to any v_{k}^{+} and v_{k}^{-}.*

Proof. Note that the case $k=1$ is given by Proposition 6.1. Let now $k=2$ and consider the equation

$$\left\{\begin{array}{c}\frac{\partial u}{\partial t}-\frac{\partial^{2}u}{\partial x^{2}}=1, \\ u(0,x)=0, \\ u(t,0)=u\left(t,\frac{1}{2}\right)=0.\end{array}\right. \tag{6.6}$$

The same arguments than in Proposition 6.1 applied now in the interval $(0,\frac{1}{2})$ proves that the unique solution of (6.6), $u_{1}(t)$, converges to v_{2}^{+} in $L^{2}\left(0,\frac{1}{2}\right)$. Analogously, replacing 1 by -1 in (6.6) and working in the interval to $\left(\frac{1}{2},1\right)$, we prove that the unique solution, $u_{2}(t)$, converges to v_{2}^{+} in $L^{2}\left(\frac{1}{2},1\right)$. We put $u(t,x)=u_{1}(t,x)$, if $x\in\left[0,\frac{1}{2}\right]$, $u(t,x)=u_{2}(t,x)$, if $x\in\left[\frac{1}{2},1\right]$. It is clear that $u(t)\to v_{2}^{+}$ in $L^{2}(0,1)$. We have to prove that $u(t)$ is a strong solution for any $T>0$. First we check that $u(\cdot)\in C\left([0,T],L^{2}(0,1)\right)$ is absolutely continuous in any compact subset $[a,b]\subset(0,T)$. Since $\|u(t)-u(s)\|^{2}=\int_{0}^{\frac{1}{2}}\left(u_{1}(t,x)-u_{1}(s,x)\right)^{2}dx+\int_{\frac{1}{2}}^{1}\left(u_{2}(t,x)-u_{2}(s,x)\right)^{2}dx$ and $u_{1}(t),u_{2}(t)$ are absolutely continuous in $[a,b]$ with respect to the spaces $L^{2}\left(0,\frac{1}{2}\right)$ and $L^{2}\left(\frac{1}{2},1\right)$, respectively, the result follows. Hence, $u(t)$ is a.e. differentiable in $(0,T)$ and it remains to check that $u(t)\in H^{2}(0,1)$ and satisfies (2.2) for a.e. $t\in(0,T)$ with $g(t,x)=1$, if

$x \in (0,1)$, $g(t,x) = -1$, if $x \in \left(\frac{1}{2}, 1\right)$. We note that $u_2(t,x) = -u_1\left(t, x - \frac{1}{2}\right) = -u_1(t, 1-x)$, for $x \in \left[\frac{1}{2}, 1\right]$, so that $\frac{\partial}{\partial x} u_2 \mid_{x=\frac{1}{2}^+} = \frac{\partial}{\partial x} u_1 \mid_{x=\frac{1}{2}^-}$, and then $u(t) \in C^1([0,1])$. In fact, since $\frac{\partial}{\partial x} u_1(t,x), \frac{\partial}{\partial x} u_1(t,x)$ are absolutely continuous in $\left[0, \frac{1}{2}\right]$ and $\left[\frac{1}{2}, 1\right]$, respectively, it follows that $\frac{\partial}{\partial x} u(t,x)$ is absolutely continuous in $[0,1]$ for a.e. $t \in (0,T)$, so that $u(t) \in H^2(0,1)$ for a.e. t. The last assertion is now evident.

In the general case we have to repeat the same procedure but now in k intervals. See Remark 4.3. ■

7 Approximations of the fixed points

Let us consider now a parameterized family of functions $H_\varepsilon \in C^2(\mathbb{R})$, $\varepsilon > 0$, such that

$$\begin{aligned} H_\varepsilon(0) &= 0,\ H_\varepsilon'(0) = \frac{1}{\varepsilon}, \\ H_\varepsilon'(s) &> 0, -1 < H_\varepsilon(s) < 1,\ \forall s, \\ H_\varepsilon''(s) &< 0, \text{ if } s > 0,\ H_\varepsilon''(s) > 0, \text{ if } s < 0, \\ \lim_{s \to +\infty} H_\varepsilon(s) &= 1,\ \lim_{s \to -\infty} H_\varepsilon(s) = -1. \end{aligned} \tag{7.1}$$

Moreover, we shall assume that H_ε is odd and

$$H_\varepsilon(s) > g_\varepsilon(s), \text{ if } s > 0,\ H_\varepsilon(s) < g_\varepsilon(s), \text{ if } s < 0, \tag{7.2}$$

where

$$g_\varepsilon(s) = \begin{cases} 1 - \varepsilon, \text{ if } s \geq \varepsilon, \\ \frac{1-\varepsilon}{\varepsilon} s, \text{ if } -\varepsilon \leq s \leq \varepsilon, \\ -1 + \varepsilon, \text{ if } s \leq -\varepsilon. \end{cases}$$

It follows from these conditions that

$$dist\left(Graph\left(H_\varepsilon\right), Graph\left(H\right)\right) \leq \varepsilon. \tag{7.3}$$

The maps H_ε are approximations of the map $H_0(s)$ as $\varepsilon \to 0$. It is clear from (7.3) and (7.2) that if $s_n \to s > 0$ (respectively < 0), $\varepsilon_n \to 0$, then $H_{\varepsilon_n}(s_n) \to H_0(s) = 1$ (respectively -1). We study the Chafee-Infante like problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} = H_\varepsilon(u) = \lambda_\varepsilon f_\varepsilon(u), \text{ in } (0,1) \times (0,T), \\ u(0,t) = u(1,t) = 0, \\ u(x,0) = u_0(x), \end{cases} \tag{7.4}$$

where $\lambda_\varepsilon = \frac{1}{\varepsilon}$, $f_\varepsilon(u) = \frac{1}{\lambda_\varepsilon} H_\varepsilon(u)$. This equation is an approximation of (4.1) with $\omega = 0$ which is a particular case of (3.2).

It is well known that if $n^2\pi^2 < \lambda_\varepsilon \leq (n+1)^2 \pi^2$, $n \in \mathbb{N}$, then equation (7.4) has exactly $2n+1$ equilibria denoted by $v_{\varepsilon 0} \equiv 0$, $v_{\varepsilon k}^+, v_{\varepsilon k}^-$, $k = 1, ..., n$, with the following properties [7, p.121]:

1. $\frac{d}{dx} v_{\varepsilon k}^+(0) > 0$, $\frac{d}{dx} v_{\varepsilon k}^-(0) < 0$;

2. $v^+_{\varepsilon k}, v^-_{\varepsilon k}$ vanish $k-1$ times in $0<x<1$.

Furthermore, if $n^2\pi^2 < \lambda_\varepsilon < (n+1)^2\pi^2$ then all equilibria are hyperbolic.

The construction of the fixed points [7, p.121] and the fact that H_ε is odd gives us that the points in $(0,1)$ where $v^+_{\varepsilon k}, v^-_{\varepsilon k}$ vanish are $a_j = \frac{j}{k}$, $j = 1,...,k-1$, which coincide with the same points for v^+_k, see (4.6) and Remark 4.3. The functions $v^+_{\varepsilon k}$ (respectively $v^-_{\varepsilon k}$) are positive (respectively negative) in $\frac{2j}{k} < x < \frac{2j+1}{k}$, and negative (respectively positive) in $\frac{2j+1}{k} < x < \frac{2j+2}{k}$, $j = 0,..., \left[\frac{k-1}{2}\right]$, where $[\cdot]$ denotes the integer part of a number. Moreover, $v^+_{\varepsilon k} = -v^-_{\varepsilon k}$.

Now consider a sequence $\varepsilon_n \to 0$, where $n^2_{\varepsilon_n}\pi^2 < \lambda_{\varepsilon_n} < (n_{\varepsilon_n}+1)^2\pi^2$, with $n_{\varepsilon_n} \to +\infty$.

Lemma 7.1 *Let k be fixed. Then $v^+_{\varepsilon_n k}$ (respectively $v^-_{\varepsilon_n k}$) cannot converge to 0 in $H^1_0(0,1)$ as $\varepsilon_n \to 0$.*

Proof. Consider the interval $\left[0, \frac{1}{k}\right]$. Suppose that $v^+_{\varepsilon_n k} \to 0$ in $H^1_0(0,1)$. Then $v^+_{\varepsilon_n k} \to 0$ in $C([0,1])$.

The function $v^+_{\varepsilon_n k}$ has a unique maximum in that interval at some $a \in \left(0, \frac{1}{k}\right)$ and $\frac{d}{dx}v^+_{\varepsilon_n k}(a) = 0$ (in fact $a = \frac{1}{2k}$, but this is not important for our proof). Let $x_0(\varepsilon_n)$ be the first point where $v^+_{\varepsilon_n k}(x_0) = \varepsilon_n$ or $x_0(\varepsilon_n) = a$ if such a point does not exist.

We state that $x_0(\varepsilon_n) \to 0$, as $\varepsilon_n \to 0$. It is clear from (7.1) that the second derivative of $v^+_{\varepsilon_n k}$ is negative in $\left(0, \frac{1}{k}\right)$, and then $\dfrac{v^+_{\varepsilon_n k}(x_0)}{x_0(\varepsilon_n)} x \le v^+_{\varepsilon_n k}(x) \le \varepsilon_n$, for all $x \in [0, x_0]$. Hence, integrating first on (s,a) and then on $(0,x)$ with $x \le x_0$, and using (7.2), we have

$$\frac{d}{dx}v^+_{\varepsilon_n k}(s) = \int_s^a H_{\varepsilon_n}\left(v^+_{\varepsilon_n k}(\tau)\right) d\tau \tag{7.5}$$

$$\begin{aligned} v^+_{\varepsilon_n k}(x) &= \int_0^x \int_{x_0}^a H_{\varepsilon_n}\left(v^+_{\varepsilon_n k}(\tau)\right) d\tau ds + \int_0^x \int_s^{x_0} H_{\varepsilon_n}\left(v^+_{\varepsilon_n k}(\tau)\right) d\tau ds \qquad (7.6) \\ &\ge \int_0^x \int_s^{x_0} \frac{1-\varepsilon_n}{\varepsilon_n} v^+_{\varepsilon_n k}(\tau)\, d\tau ds \ge \frac{v^+_{\varepsilon_n}(x_0)}{x_0(\varepsilon_n)} \frac{1-\varepsilon_n}{\varepsilon_n} \int_0^x \int_s^{x_0} \tau d\tau ds. \end{aligned}$$

Hence

$$1 \ge \frac{1-\varepsilon_n}{\varepsilon_n}\left(\frac{x_0 x}{2} - \frac{x^3}{x_0 6}\right).$$

It follows that $x_0(\varepsilon_n) \to 0$, as $\varepsilon_n \to 0$.

Let $\delta > 0$ be such that $x_0(\varepsilon_n) \le \delta$. We note also that $v^+_{\varepsilon_n k}(x) \ge \varepsilon_n$, for all $x \in [x_0, a]$. Therefore, integrating (7.5) over (δ, x) with $x > \delta$ we have

$$v^+_{\varepsilon_n k}(x) \ge v^+_{\varepsilon_n k}(\delta) + \int_\delta^x \int_s^a (1-\varepsilon_n)\, d\tau ds$$

and passing to the limit we obtain a contradiction if $v^+_{\varepsilon_n k} \to 0$ in $C([0,1])$. For the sequence $v^-_{\varepsilon_n k}$ the proof is similar. The lemma is proved. ■

We are now ready to prove the following:

Lemma 7.2 *$v^+_{\varepsilon_n k}$ (respectively $v^-_{\varepsilon_n k}$) converges to v^+_k (respectively v^-_k) in $H^1_0(0,1)$ as as $\varepsilon_n \to 0$.*

Proof. Note that Corollary 3.2 applied to (7.4) implies that $v^+_{\varepsilon_n k}$ is bounded in, say, H^1_0. Moreover, it is easy to see that they are bounded in H^2. Hence, we may assume that $v^+_{\varepsilon_n k}$ converges to some v weakly in H^2, strongly in H^1_0 and in $C^1([0,1])$.

We have to check that v is a fixed point. It is clear that the functions $g_{\varepsilon_n} = H_{\varepsilon_n}\left(v^+_{\varepsilon_n k}\right)$ are bounded in $L^\infty(0,1)$. Passing to a subsequence we can then assume that g_{ε_n} converges to some g weakly in $L^2(0,1)$. It is clear that $-\dfrac{\partial^2 v}{\partial x^2} = g$ and the result will follow if we prove the inclusion $g(x) \in H_0(v(x))$ for a.e. $x \in (0,1)$. By Masur's theorem [21] there exist $z_m \in V_m = \text{conv}\left(\bigcup_{k \geq m}^{\infty} g_{\varepsilon_k}\right)$ such that $z_m \to g$, as $m \to \infty$, strongly in $L^2(0,1)$. Taking a subsequence we have $z_m(x) \to g(x)$, a.e. in $(0,1)$. Since $z_m \in V_m$, we get $z_m = \sum_{i=1}^{N_m} \lambda_i g_{\varepsilon_{k_i}}$,where $\lambda_i \in [0,1]$, $\sum_{i=1}^{N_m} \lambda_i = 1$ and $k_i \geq m$, $\forall i$. Now (7.3) implies that

$$dist\left(g_{\varepsilon_k}(x), H_0(v(x))\right) \to 0, \text{ as } k \to \infty, \text{ for a.e. } x.$$

Indeed, if $v(x) = 0$, then $g_{\varepsilon_k}(x) \in [-1,1] = H_0(v(x))$. If $v(x) > 0$, then

$$dist\left(g_{\varepsilon_k}(x), H_0(v(x))\right) = \left|H_{\varepsilon_k}\left(v_{\varepsilon_k}(x)\right) - 1\right| \to 0, \text{ as } k \to \infty.$$

The argument is similar if $v(x) < 0$. Hence, for any $\delta > 0$ and a.e. x there exists $m(x,\delta)$ such that

$$g_{\varepsilon_k}(x) \subset [a(x) - \delta, b(x) + \delta], \ \forall k \geq m,$$

where $[a(x), b(x)] = H_0(v(x))$. Hence, $z_m(x) \subset [a(x) - \delta, b(x) + \delta]$, as well. Passing to the limit we obtain

$$g(x) \in [a(x), b(x)], \text{ a.e. on } (0,1).$$

We shall prove further that $v = v^+_k$. We know that $v^+_{\varepsilon_n k}(x) > 0$, for all $x \in \left(0, \frac{1}{k}\right)$, and the convergence in the space $C([0,1])$ implies that $v(x) \geq 0$ in $\left[0, \frac{1}{k}\right]$ and $v(0) = v\left(\frac{1}{k}\right) = 0$. By (4.6) the only possibilities are $v = v^+_k$ or $v \equiv 0$. Lemma 7.1 gives us that $v = v^+_k$.

For the sequence $v^-_{\varepsilon_n k}$ the proof is similar. ■

Corollary 7.3 *$v^+_{\varepsilon k}$ (respectively $v^-_{\varepsilon k}$) converges to v^+_k (respectively v^-_k) in $H^1_0(0,1)$ as $\varepsilon \to 0^+$.*

8 Heteroclinic connections between fixed points

Let us consider now the important question about which heteroclinic connections can exist between the fixed points in the case $\omega = 0$. These connections give us some information about the structure of the global attractor.

Recall first that a complete trajectory is a function $u(t)$ defined on $(-\infty, \infty)$ such that $u(\cdot)$ is a strong solution of (4.1) on any interval $(-T, T)$. This implies that $u(t) \in G_0(t - \tau, u(\tau))$, for all $t \geq \tau$. Since the global attractor is invariant, it is easy to see that it is equal to the union of all complete bounded trajectories. Indeed, if $u_0 \in \mathcal{A}$, then we take an arbitrary solution $u_1(t)$, $t \geq 0$, with $u_1(0) = u_0$ and some points $u_i \in \mathcal{A}$, $i = -1, -2, \ldots$, such that $u_i \in G_0(1, u_{i+1})$. Then there exist solutions $u_i(t)$ defined on $[i, i+1]$ such that $u_i(i) = u_i$ and $u_i(i+1) = u_{i+1}$. Concatenating all these solutions we obtain a complete bounded trajectory lying on the global attractor. Hence, any $y \in \mathcal{A}$ belongs to a complete bounded trajectory.

Conversely, if $u(t)$ is a complete bounded trajectory, then it is clear that $B = \cup_{t\in(-\infty,+\infty)} u(t)$ satisfies $B \subset G_0(t,B)$ and

$$dist\left(G_0(t,B), \mathcal{A}\right) \to 0, \text{ as } t \to +\infty.$$

Therefore, $B \subset \mathcal{A}$. The same results are valid for the attractors $\mathcal{A}_\varepsilon$ defined in Section 3.

We have defined before the ω-limit set. For a complete trajectory we define as usual the α-limit set by

$$\alpha\left(\{u(t)\}\right) = \left\{y : u(t_k) \to y \text{ in } L^2(0,1), \text{ where } t_k \to -\infty\right\},$$

which is non-empty.

Lemma 8.1 *Let $u(t)$ be a bounded complete trajectory. Then $\alpha(\{u(t)\}) \subset Z$, where Z is the set of fixed points. Moreover, $\alpha(\{u(t)\}) = z \in Z$, and $u(t) \to z$, as $t \to -\infty$, in $H_0^1(0,1)$.*

Proof. We note that the global attractor $\mathcal{A}$ is precompact in H_0^1 by Corollary 3.2 and then $B = \cup_{t\in(-\infty,+\infty)} u(t)$ is precompact in H_0^1. The rest of the proof is the same as in Lemma 5.1 but taking the subsequences t_k converging to $-\infty$. ■

We have obtained in Lemmas 5.1 and 8.1 that every complete bounded trajectory satisfies

$$\begin{aligned} u(t) \underset{t\to+\infty}{\to} z_2 &\in Z \\ u(t) \underset{t\to-\infty}{\to} z_1 &\in Z. \end{aligned}$$

In such a case we say that there exists a heteroclinic connection from z_1 to z_2. In the sequel we shall denote this property by $z_1 \rightsquigarrow z_2$. Note that the compactness in Corollary 3.2 implies that although we are now considering the convergence in the phase space $L^2(0,1)$ to the fixed point, we also have convergence in stronger norms, e.g. in H_0^1.

We note further that the global attractors consists of the fixed points and all the heteroclinic connections between them. Hence, if we know all the possible connections we have a complete description of the structure of the global attractor.

We shall consider further the family of approximations given in (7.4). We have seen that if $n^2\pi^2 < \lambda_\varepsilon < (n+1)^2\pi^2$, then this equation has exactly $2n+1$ hyperbolic fixed points denoted by $v_{\varepsilon 0} \equiv 0$, $v_{\varepsilon k}^+, v_{\varepsilon k}^-$, $k = 1, ..., n$, and it is well known [8] that

$$\begin{aligned} v_{\varepsilon 0} &\rightsquigarrow v_{\varepsilon k}^+, \quad v_{\varepsilon 0} \rightsquigarrow v_{\varepsilon k}^-, \\ v_{\varepsilon k}^+ &\rightsquigarrow v_{\varepsilon i}^+, \quad v_{\varepsilon k}^+ \rightsquigarrow v_{\varepsilon i}^-, \\ v_{\varepsilon k}^- &\rightsquigarrow v_{\varepsilon i}^+, \quad v_{\varepsilon k}^- \rightsquigarrow v_{\varepsilon i}^-, \end{aligned}$$

whenever $k > i \geq 1$, that is, there exists at least one heteroclinic connection from 0 to any other fixed point and from any $v_{\varepsilon k}^{\pm}$ to any $v_{\varepsilon i}^{\pm}$ if $k > i \geq 1$. Moreover, these are all the possible connections; no more bounded complete trajectories can exist. We note again that the complete trajectory $u_\varepsilon(t)$ converges to the fixed point in the space $H_0^1(0,1)$.

Using these known connections between fixed points in the approximations we shall prove some connections in the limit problem. Note that we have already shown in Theorem 6.7 that for any $k \geq 1$,

$$v_0 \rightsquigarrow v_k^{\pm}.$$

For the rest of the results, the Lyapunov function defined by (5.1) will play a crucial place in the proofs. We recall first that the inequalities (5.2) hold. Since the Lyapunov function E is not increasing with respect to t in every trajectory $u(t)$ it follows that any connection from $v_k^{\pm}$ to $v_i^{\pm}$ with $i \geq k \geq 1$ is forbidden. Then the question becomes if it is true that for $1 \leq i < k$ there exists at least one heteroclinic connection from $v_k^{\pm}$ to $v_i^{\pm}$ as it occurs in the approximations. In this direction note that in view of (7.4), and since we assume

$$n_{\varepsilon}^2 \pi^2 < \lambda_{\varepsilon} < (n_{\varepsilon}+1)^2 \pi^2, \quad \text{with} \quad n_{\varepsilon} \to +\infty, \tag{8.1}$$

one can expect that (4.1) has undergo all bifurcation cascade of the Chafee-Infante problem and thus all connections should be present. We cannot still give a complete answer to this question, but we shall prove some partial results.

First we shall obtain that there exist a connection from $v_k^{\pm}$ to $v_{k-1}^{\pm}$ for all $k \geq 2$. This result will follow from some preliminary lemmas.

Thus, assuming (8.1), let us consider the sequence of fixed points $v_{\varepsilon}^1, v_{\varepsilon}^2$, where $v_{\varepsilon}^1 = v_{\varepsilon 0} = 0$, or $v_{\varepsilon}^1 = v_{\varepsilon k}^+$ or $v_{\varepsilon}^1 = v_{\varepsilon k}^-$, and $v_{\varepsilon}^2 = v_{\varepsilon i}^+$ or $v_{\varepsilon}^2 = v_{\varepsilon i}^-$, with $k > i \geq 1$. Recall that from Lemma 7.2 and Corollary 7.3 we have $v_{\varepsilon k}^+ \to v_k^+$, $v_{\varepsilon k}^- \to v_k^-$ in H_0^1 for any $k \geq 1$ as $\varepsilon \to 0$ (and $v_{\varepsilon 0} = v_0 = 0$). We choose then an arbitrary sequence of bounded complete trajectories $u_{\varepsilon}(t)$ such that

$$\begin{aligned} u_{\varepsilon}(t) &\to v_{\varepsilon}^2 \quad \text{as } t \to +\infty, \\ u_{\varepsilon}(t) &\to v_{\varepsilon}^1 \quad \text{as } t \to -\infty, \end{aligned}$$

where the convergence is understood now in the space H_0^1.

In the sequel by v_k we denote a fixed point equal to v_k^+ or v_k^- for any $k \geq 1$ (of course $v_0 = 0$ for $k = 0$).

Lemma 8.2 *If there exist sequences $u_{\varepsilon_j}(t_j)$, $u_{\varepsilon_j}(\tau_j)$ such that $u_{\varepsilon_j}(t_j) \to v$, $u_{\varepsilon_j}(\tau_j) \to v^*$ in H_0^1, and v, v^* are fixed points, then either $E(v) \neq E(v^*)$ or $v = v^*$.*

Proof. We can assume without loss of generality that $t_j < \tau_j$ for all j. If $|t_j - \tau_j| \to 0$ then, eventually passing to a subsequence, the convergence of the functions

$$y_{\varepsilon_j}(t) = u_{\varepsilon_j}(t_j + t)$$

to some solution $y(\cdot) \in \mathcal{D}(v)$ in $C([0,T], H_0^1)$, $T > 0$ (see Corollary 3.6), implies that

$$u_{\varepsilon_j}(\tau_j) = y_{\varepsilon_j}(\tau_j - t_j) \to y(0) = v,$$

so that $v^* = v$.

If $|t_j - \tau_j| \not\to 0$, then passing to a subsequence we can assume that for some $\delta > 0$ we have

$$\delta + t_j < \tau_j, \text{ for all } j.$$

Let us define the set

$$\Lambda_{(t_j,\tau_j)} = \cap_{\varepsilon > 0} \overline{\cup_{\varepsilon_j < \varepsilon} \cup_{t_j \leq t \leq \tau_j} u_{\varepsilon_j}(t)}^{H_0^1},$$

where the closure is understood in the space H_0^1.

In the approximation problem (7.4) we have also the Lyapunov function (see [7, p.119]):

$$E_{\varepsilon}(u) = \int_0^1 \left(\frac{1}{2} \left| \frac{\partial}{\partial x} u(x) \right|^2 dx - \int_0^u H_{\varepsilon}(s)\, ds \right) dx.$$

It is easy to check that $E_\varepsilon(y_\varepsilon) \to E(y)$, if $y_\varepsilon \to y$ in H_0^1 as $\varepsilon \to 0$. For this, note that $\int_0^{y_\varepsilon(x)} H_\varepsilon(s)\,ds \to |y(x)|$, for all $x \in [0,1]$. Indeed, $y_\varepsilon \to y$ in $C([0,1])$ and then we have three cases. If $y(x)=0$, then $\left|\int_0^{y_\varepsilon(x)} H_\varepsilon(s)\,ds\right| \le |y_\varepsilon(x)| \to 0$. If $y(x)>0$, then

$$\left|\int_0^{y_\varepsilon(x)} H_\varepsilon(s)\,ds - y(x)\right| \le \left|\int_0^{\varepsilon} H_\varepsilon(s)\,ds\right| + \left|\int_\varepsilon^{y_\varepsilon(x)} H_\varepsilon(s)\,ds - y(x)\right|.$$

It follows from (7.2) that $H_\varepsilon(s) = 1 - f_\varepsilon(s)$, for $s \ge \varepsilon$, where $0 < f_\varepsilon(s) < \varepsilon$. Hence,

$$\left|\int_0^{y_\varepsilon(x)} H_\varepsilon(s)\,ds - y(x)\right| \le \varepsilon + |y_\varepsilon(x) - \varepsilon - y(x)| + \varepsilon(y_\varepsilon(x) - \varepsilon) \to 0.$$

For $y(x)<0$ the proof is similar. On the other hand, $\left|\int_0^{y_\varepsilon(x)} H_\varepsilon(s)\,ds\right| \le |y_\varepsilon(x)| \le C$, for all x. Thus Lebesgue's theorem implies $\int_0^{y_\varepsilon} H_\varepsilon(s)\,ds \to |y|$ in $L^1(0,1)$, and then $E_\varepsilon(y_\varepsilon) \to E(y)$.

It follows then that $E_{\varepsilon_j}\left(u_{\varepsilon_j}(t_j)\right) \to E(v)$, $E_{\varepsilon_j}\left(u_{\varepsilon_j}(\tau_j)\right) \to E(v^*)$. Since $E_{\varepsilon_j}\left(u_{\varepsilon_j}(\tau_j)\right) \le E_{\varepsilon_j}\left(u_{\varepsilon_j}(t_j+t)\right) \le E_{\varepsilon_j}\left(u_{\varepsilon_j}(t_j)\right)$, for all $t \in [0, \tau_j - t_j]$, we have that $E(v^*) \le E(y) \le E(v)$, for any $y \in \Lambda_{(t_j,\tau_j)}$.

Suppose that $v \ne v^*$ and $E(v) = E(v^*)$. Then

$$E(y) = E(v) = E(v^*), \text{ for all } y \in \Lambda_{(t_j,\tau_j)},$$

and there exists $y \in \Lambda_{(t_j,\tau_j)}$ different from v and v^*. Indeed, we take two open sets U_1, U_2 (in H_0^1) such that

$$v \in U_1, \quad v^* \in U_2, \quad U_1 \cap U_2 = \varnothing.$$

Since $u_{\varepsilon_j}(t_j) \in U_1$, $u_{\varepsilon_j}(\tau_j) \in U_2$, for $j \ge J$, and $u_{\varepsilon_j}(t)$ is continuous, there must exist $s_j \in (t_j, \tau_j)$ such that $u_{\varepsilon_j}(s_j) \notin U_1 \cup U_2$. Passing to a subsequence by Corollary 3.2 we obtain

$$u_{\varepsilon_j}(s_j) \to y \in \Lambda_{(t_j,\tau_j)},$$

where $y \notin U_1 \cup U_2$, $y \ne v$, $y \ne v^*$ (the limit is understood in the space H_0^1). Moreover, y cannot be a fixed point, because there are only two with the same value of the Lyapunov function, see (5.2).

Further, it is clear arguing as before that $s_j + \xi < \tau_j$, for some $\xi > 0$ and any j. The functions $w_{\varepsilon_j}(t) = u_{\varepsilon_j}(s_j + t)$ converge by Corollary 3.6 to some solution $\omega(\cdot) \in \mathcal{D}(y)$ in $C([0,\xi], H_0^1)$. It is evident that $w(t) \in \Lambda_{(t_j,\tau_j)}$, for any $t \in [0,\xi]$, so that $E(w(t)) = E(v) = E(v^*)$. This contradicts that y is not a fixed point and this contradiction proves the Lemma. ■

Corollary 8.3 *Let now*

$$\Lambda = \cap_{\varepsilon_0 > 0} \overline{\cup_{\varepsilon < \varepsilon_0} \cup_{-\infty \le t \le +\infty} u_\varepsilon(t)}^{H_0^1}.$$

If $y \in \Lambda$ is a fixed point such that $y \ne v_k$, $y \ne v_i$, then $E(v_i) < E(y) < E(v_k)$. The number of fixed points in Λ is finite if $v_k \ne v_0$.

Proof. In a similar way as in the proof of Lemma 8.2 we can check that $E(v_i) \le E(y) \le E(v_k)$, for all $y \in \Lambda$. If $u_{\varepsilon_j}(t_j) \to y \ne v_k$ is a fixed point, then choosing a sequence $u_{\varepsilon_j}(\tau_j) \to v_k$, $\tau_j > t_j$, and using Lemma 8.2 we obtain that $E(y) < E(v_k)$. The same argument is valid for v_i. The last statement is evident. ■

Now we are ready to prove the existence of a finite chain of complete bounded trajectories from v_k to v_i if $k > i \ge 1$.

Lemma 8.4 *Let $\omega = 0$ and let v_1 and v_2 be two fixed points different from $v_0 = 0$ such that $E(v_1) > E(v_2)$ (i.e. $v_1 = v_k$, $v_2 = v_i$, $k > i \geq 1$). Then there exist a finite number of fixed points $y_{-r}, ..., y_{-1}, y_0, y_1, ..., y_m$ such that*

$$E(v_1) > E(y_{-r}) > \cdots > E(y_0) > \cdots > E(y_m) > E(v_2),$$

$$v_1 \rightsquigarrow y_{-r} \rightsquigarrow \cdots \rightsquigarrow y_0 \rightsquigarrow y_1 \rightsquigarrow \cdots \rightsquigarrow y_m \rightsquigarrow v_2.$$

Proof. We choose the sequences $v_{1\varepsilon_j} = v_{\varepsilon_j k}, v_{2\varepsilon_j} = v_{\varepsilon_j i}$. From Lemma 7.2 it follows that

$$v_{1\varepsilon_j} \to v_1, \quad v_{2\varepsilon_j} \to v_2, \text{ as } \varepsilon_j \to 0, \text{ in } H_0^1.$$

Let $u_{\varepsilon_j}(t)$ be a complete trajectory of (7.4) such that $u_{\varepsilon_j}(t) \underset{t\to-\infty}{\to} v_{1\varepsilon_j}$, $u_{\varepsilon_j}(t) \underset{t\to+\infty}{\to} v_{2\varepsilon_j}$. Fix $T > 0$. Using Corollary 3.2 we obtain that up to a subsequence $u_{\varepsilon_j}(-T) \to y$ in H_0^1. Corollary 3.6 implies then that u_{ε_j} converges in $C([-T,T], H_0^1)$ to some solution u of (4.1) with $u(-T) = y$. We choose successive subsequences for $-2T, -3T...$ and by the standard diagonal procedure we obtain that a subsequence u_{ε_j} converges to a complete trajectory u of (4.1) in $C([-T,T], H_0^1)$ for any $T > 0$. We note also that the solution $u(t)$ is a bounded complete trajectory contained in the global attractor $\mathcal{A}_0$. Indeed, $u_{\varepsilon_j}(t) \in \mathcal{A}_{\varepsilon_j}$ and by Theorem 3.5, $dist\left(\mathcal{A}_{\varepsilon_j}, \mathcal{A}_0\right) \to 0$, so that $u(t) \in \mathcal{A}_0$ for all t. Lemmas 5.1, 8.1 imply the existence of two fixed points y_0, y_1 such that $u(t) \underset{t\to-\infty}{\to} y_0$, $u(t) \underset{t\to+\infty}{\to} y_1$.

Suppose that $y_0 \neq v_1$.

It is clear that we can choose a subsequence $u_{\varepsilon_j}(\tau_j)$ converging to y_0 as $j \to \infty$. We take $\delta > 0$ small enough, so that in the closed ball $B_\delta(y_0)$ of radius δ (in the space H_0^1) there is no any other fixed point, and, moreover, the value of the Lyapunov function $E(y)$ of any point $y \in B_\delta(y_0)$ is different from the value of E of any other fixed point (excluding, of course, the pair of y_0) and there exists $\xi > 0$ such that

$$E(\underline{v}) < E(y_0) - \xi \leq E(y) \leq E(y_0) + \xi < E(\overline{v}), \text{ for all } y \in B_\delta(y_0),$$

where $\underline{v}$ (respectively $\overline{v}$) is the first fixed point in the scale (5.2) such that $E(\underline{v}) < E(y_0)$ (respectively $E(\overline{v}) > E(y_0)$).

On the other hand, since $u_{\varepsilon_j}(t) \underset{t\to-\infty}{\to} v_{1\varepsilon_j}$ and $v_{1\varepsilon_j} \notin B_\delta(y_0)$ (note that $v_{1\varepsilon_j} \to v_1 \neq y_0$ in H_0^1), there exist $t_j < \tau_j$ such that

$$\begin{aligned} \left\|u_{\varepsilon_j}(t_j) - y_0\right\|_{H_0^1} &= \delta, \\ \left\|u_{\varepsilon_j}(t) - y_0\right\|_{H_0^1} &> \delta, \text{ for all } t < t_j. \end{aligned}$$

Define $w_{\varepsilon_j}(t) = u_{\varepsilon_j}(t_j + t)$. Passing to a subsequence we obtain as before that w_{ε_j} converges to some bounded complete trajectory w in $C([-T,T], H_0^1)$, for all $T > 0$, and $u_{\varepsilon_j}(t_j) \to w(0) = w_0$, $\|w_0 - y_0\|_{H_0^1} = \delta$.

We shall prove that $w(t) \underset{t\to+\infty}{\to} y_0$. Let first $\tau_j - t_j \to +\infty$. Arguing as in Lemma 8.2 we obtain that for any $z \in \Lambda_{(t_j,\tau_j)} = \cap_{\varepsilon>0} \overline{\cup_{\varepsilon_j<\varepsilon} \cup_{t_j \leq t \leq \tau_j} u_{\varepsilon_j}(t)}^{H_0^1}$ the inequality

$$E(y_0) - \xi \leq E(z) \leq E(y_0) + \xi$$

holds. Since $w(t) \in \Lambda_{(t_j,\tau_j)}$, for any $t \in \mathbb{R}$, we have that the fixed point $\overline{z} = \lim_{t\to+\infty} w(t)$ satisfies the same inequality. Hence, either $\overline{z} = y_0$ or $\overline{z}$ is the pair of y_0 with the same value of the Lyapunov function. Again passing to a subsequence we can choose $t_j \le s_j \le \tau_j$ such that $u_{\varepsilon_j}(s_j) \to \overline{z}$ in H_0^1. Hence, Lemma 8.2 implies that $\overline{z} = y_0$.

Let now $|\tau_j - t_j|$ be bounded by a constant K. Passing to a subsequence $\tau_j - t_j \to s \ge 0$ and then $u_{\varepsilon_j}(\tau_j) = w(\tau_j - t_j) \to w(s)$. Therefore, $w(s) = y_0$.

Lemma 8.1 gives $w(t) \underset{t\to-\infty}{\to} y_{-1} \in Z$, with $E(y_{-1}) > E(y_0)$. We note also that $w(t) \underset{t\to+\infty}{\to} y_0$. In the second case (with $|\tau_j - t_j|$ bounded) we can put $w(t) = y_0$, for all $t \ge s$. We have obtained an heteroclinic connection from y_{-1} to y_0, i.e. $y_{-1} \rightsquigarrow y_0$.

If $y_{-1} \neq v_1$, repeating the same argument we obtain $y_{-2} \in Z$ with $E(y_{-2}) > E(y_{-1})$ such that $y_{-2} \rightsquigarrow y_{-1}$. Continuing in this way we obtain y_{-3}, y_{-4}, etc. By Corollary 8.3, $E(v_1) \le E(y_{-l}) \le E(v_2)$ and $E(v_1) = E(y_{-l})$ if and only if $v_1 = y_{-l}$. Since the sequence $a_l = E(y_{-l})$ is strictly increasing and the number of possible fixed points is finite, there must exist r such that $y_{-r-1} = v_1$. Hence, we have obtained the chain of connections

$$v_1 \rightsquigarrow y_{-r} \rightsquigarrow y_{-(r-1)} \rightsquigarrow \cdots \rightsquigarrow y_{-1} \rightsquigarrow y_0 \rightsquigarrow y_1.$$

In a similar way we obtain the chain of connections from y_1 to v_2. ■

As a consequence of Lemma 8.4 we obtain the following:

Theorem 8.5 *If $\omega = 0$ for any $n \ge 2$ the following heteroclinic connections exist:*

$$\begin{array}{rcl} v_n^+ & \rightsquigarrow & v_{n-1}^+, \quad v_n^+ \rightsquigarrow v_{n-1}^-, \\ v_n^- & \rightsquigarrow & v_{n-1}^+, \quad v_n^- \rightsquigarrow v_{n-1}^-. \end{array}$$

We shall prove further the following connection:

Theorem 8.6 *If $\omega = 0$, then $v_k^\pm \rightsquigarrow v_1^\pm$, for all $k > 1$.*

Proof. It is clear from the formulas of the fixed points that

$$\begin{array}{rcl} v_1^-(x) & \le & v_k^+(x) \le v_1^+(x), \\ v_1^-(x) & \le & v_k^-(x) \le v_1^+(x), \end{array}$$

for all $x \in [0,1]$ and $k > 1$, and also that there is not any other v_i such that $v_k(x) \le v_i(x)$ or $v_i(x) \le v_k(x)$ for all x. The same holds for (7.4).

Take, for example, $v_k = v_k^+$, $v_1 = v_1^+$. The other cases are similar. Take the fixed points of the approximations $v_{\varepsilon_j k}^+$ and $v_{\varepsilon_j 1}^+$. It is well known (see [7]) that the eigenfunction corresponding to the first eigenvalue of the linearization of (7.4) around $v_{\varepsilon_j k}^+$ is positive. Also note that from (8.1), this equilibrium is hyperbolic.

We claim now that in the unstable manifold of $v_{\varepsilon_j k}^+$ there exists $u_{\varepsilon_j}^0$ such that $u_{\varepsilon_j}^0(x) \ge v_{\varepsilon_j k}^+(x)$, for all $x \in (0,1)$.

Once this is shown we have then that the corresponding solution $u_{\varepsilon_j}(t) = u_{\varepsilon_j}(t, u_{\varepsilon_j}^0)$ satisfies $u_{\varepsilon_j}(x,t) \ge v_{\varepsilon_j k}^+(x)$, for all x and t, by monotonicity, and $u_{\varepsilon_j}(t) \underset{t\to-\infty}{\to} v_{\varepsilon_j k}^+$. Since $u_{\varepsilon_j}(x,t) \ge v_{\varepsilon_j k}^+(x)$ the only possibility is that $u_{\varepsilon_j}(t) \underset{t\to+\infty}{\to} v_{\varepsilon_j 1}^+$. Hence $u_{\varepsilon_j}(t)$ is a suitable complete trajectory that realizes the connection $v_{\varepsilon_j k}^+ \rightsquigarrow v_{\varepsilon_j 1}^+$.

Now, we know from Lemma 8.4 that

$$v_k^+ \rightsquigarrow y_{-r} \rightsquigarrow \ldots \rightsquigarrow y_m \rightsquigarrow v_1^+,$$

for some fixed points y_i. In the proof of that lemma it is shown that there exists a subsequence $u_{\varepsilon_j}(\tau_j)$ converging to y_i in $H_0^1(0,1)$ (and then in $C([0,1])$). Hence, $v_k^+(x) \leq y_i(x)$, for all x, so that either $y_i = v_k^+$ or $y_i = v_1^+$. It follows then that $v_k^+ \rightsquigarrow v_1^+$.

We now prove the claim above. Note that from Corollary 3.2, the local analysis around the hyperbolic equilibria $v_{\varepsilon_j k}^+$ can be set within a $W_0^{1,p}$ setting and the local invariant theorem in [7] applies. Denote by $\phi_{\varepsilon_j k}^1$ the first eigenfunction of the linearization around $v_{\varepsilon_j k}^+$. Then for $\mu \geq 0$ the point $v_{\varepsilon_j k}^+ + \mu\phi_{\varepsilon_j k}^1$ lies in the linear unstable manifold of $v_{\varepsilon_j k}^+$. Since the unstable manifold is tangent to the linear unstable manifold at $v_{\varepsilon_j k}^+$, for sufficiently small $\mu > 0$, there exists $u_{\varepsilon_j}^0 \in W_{loc}^u(v_{\varepsilon_j k}^+)$ such that $\|u_{\varepsilon_j}^0 - \left(v_{\varepsilon_j k}^+ + \mu\phi_{\varepsilon_j k}^1\right)\|_{W^{2-\delta,p}} = o(\mu)$, for any $\delta > 0$. In particular the above is true in the C^1 topology. Since $v_{\varepsilon_j k}^+ + \mu\phi_{\varepsilon_j k}^1$ is strictly above $v_{\varepsilon_j k}^+$ in $(0,1)$ and is a C^1 function, the claim is proved. ■

Finally, we shall obtain some more connections in the case $\omega = 0$ using a self-similarity property of the solutions. In fact note that we obtained in Remark 4.2 that for $\omega = 0$ the equilibria are self-similar. The same property is valid for any solution of the parabolic problem $u(t)$. In fact, if we define $u_n(t) = \frac{1}{n^2}u(n^2t, nx)$, for $x \in \left[0, \frac{1}{n}\right]$, $u_n(t) = -\frac{1}{n^2}u\left(n^2t, n\left(x - \frac{1}{n}\right)\right)$, for $x \in \left[\frac{1}{n}, \frac{2}{n}\right]$, and so on, then we obtain a new solution. This can be verified by a direct substitution on u_n in (4.1) with $\omega = 0$. Hence, if $u(t)$ is a complete trajectory going from v_k to v_i, $k > i \geq 1$, then $u_n(t)$ is a complete trajectory going from v_{nk} to v_{ni}.

We have proved the following results, which somehow extends Theorem 6.7:

Theorem 8.7 *Let $\omega = 0$. If $v_k \rightsquigarrow v_i$, $k > i \geq 1$, then $v_{nk} \rightsquigarrow v_{ni}$, for all $n \in \mathbb{N}$.*

This result allow us to obtain new heteroclinic connections from Theorems 8.5 and 8.6. For example, from $v_2 \rightsquigarrow v_1$ it follows $v_4 \rightsquigarrow v_2$, $v_6 \rightsquigarrow v_3$, etc.

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References

[1] J. M. Arrieta, A. N. Carvalho, A. Rodriguez-Bernal, Attractors for parabolic problems with nonlinear boundary conditions: uniform bounds, Comm. Partial Diff. Equations 25, 1-37 (2000).

[2] J.M. Ball, Continuity properties and global attractors of generalized semiflows and the Navier-Stokes equations, in "Mechanics: from theory to computation", Springer, New York, 2000, pp.447–474.

[3] Barbu V. "Nonlinear Semigroups and Differential Equations in Banach Spaces", Editura Academiei, Bucuresti, 1976.

[4] Feireisl E., Norbury J., Some existence and nonuniqueness theorems for solutions of parabolic equations with discontinuous nonlinearities, *Proc. Roy. Soc. Edinburgh*, **119A** (1991), 1-17.

[5] P. Brunovsky, B. Fiedler, Connecting orbits in scalar reaction diffusion equations, *Dynamics Reported,* **1** (1988), 57-89.

[6] C. Rocha, B. Fiedler, Heteroclinic orbits of semilinear parabolic equations, *J. Differential. Equations,* **125** (1996), 239-281.

[7] D. Henry, "Geometric theory of semilinear parabolic equations", Springer, Berlin, 1981.

[8] D.Henry, Some infinite-dimensional Morse-Smale systems defined by parabolic partial differential equations, *J. Differential Equations*, **59** (1985), 165-205.

[9] A.V. Kapustyan, V.S. Melnik, On global attractors of multivalued semidynamical systems and their approximations, *Kibernetika i Sistemny Analiz*, **5** (1998), 102-111.

[10] A.V. Kapustyan, J. Valero, Attractors of multivalued semiflows generated by differential inclusions and their approximations, *Abstr. Appl. Anal.*, **5** (2000), 33-46.

[11] J.L. Lions, "Quelques Méthodes de Résolution des Problèmes aux Limites non Linéares", Dunod, Gauthier Villars, 1969.

[12] Melnik V.S., Valero J., On attractors of multivalued semi-flows and differential inclusions, *Set-Valued Anal.*, **6** (1998), 83-111.

[13] C. Rocha, Examples of Attractors in Scalar Reaction-Diffusion Equations, *J. Differential Equations,* **73** (1988), 178-195.

[14] C. Rocha, Properties of the Attractor of a Scalar Parabolic PDE, *J. Dynamics Differential Equations,*. **3** (1991), 575-591.

[15] R. Temam, "Infinite-Dimensional Dynamical Systems in Mechanics and Physics", Springer-Verlag, New York, 1988.

[16] D. Terman, A free boundary problem arising from a bistable reaction-diffusion equation, *Siam J. Math. Anal.*, **14** (6)(1983), 1107-1129.

[17] D. Terman, A free boundary arising from a model for nerve conduction, *J. Differential Equations*, **58** (1985), 345-363.

[18] A.A. Tolstonogov, On solutions of evolution inclusions I, *Siberian Math. J.*, **33** (1992), 500-511.

[19] J. Valero, Attractors of parabolic equations without uniqueness, *J. Dynamics Differential Equations*, **13** (2001), 712-744.

[20] J. Valero, On the Kneser property for some parabolic problems, *Topology Appl.* (to appear).

[21] K. Yosida, "Functional Analysis", Springer-Verlag, Berlin, 1965.