ON THE ZETA-FUNCTION OF A POLYNOMIAL AT INFINITY

S.M. Gusein–Zade, I. Luengo, A. Melle–Hernández

Abstract. We use the notion of Milnor fibres of the germ of a meromorphic function and the method of partial resolutions for a study of topology of a polynomial map at infinity (mainly for calculation of the zeta-function of a monodromy). It gives effective methods of computation of the zeta-function for a number of cases and a criterium for a value to be atypical at infinity.

§1.- Introduction

The main idea of the paper is to bring together methods of [7] and [8] for computing the zeta-function of the monodromy at infinity of a polynomial. Let P be a complex polynomial in $(n+1)$ variables. It defines a map from $\mathbb{C}^{n+1}$ to $\mathbb{C}$ which also will be denoted by P. It is known ([13]) that there exists a finite set $B(P) \subset \mathbb{C}$ such that the map P is a C^∞ locally trivial fibration over its complement. The monodromy transformation h of this fibration corresponding to the loop $z_0 \cdot \exp(2\pi i\tau)$ $(0 \le \tau \le 1)$ with $\|z_0\|$ big enough is called the *geometric monodromy at infinity* of the polynomial P. Let h_* be its action in the homology groups of the fibre (the level set) $\{P = z_0\}$.

Definition. The *zeta-function of the monodromy at infinity* of the polynomial P is the rational function

$$\zeta_P(t) = \prod_{q\ge 0} \{\det [\, id - t\, h_*|_{H_q(\{P=z_0\};\mathbb{C})}]\}^{(-1)^q}.$$

Remark 1. We use the definition from [2], which means that the zeta-function defined this way is the inverse of that used in [1].

The degree of the zeta-function (the degree of the numerator minus the degree of the denominator) is equal to the Euler characteristic χ_P of the (generic) fibre $\{P = z_0\}$. Formulae for the zeta-functions at infinity for certain polynomials were given in particular in [6], [9].

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§2.- Zeta-function of a polynomial via zeta-functions of meromorphic germs

A polynomial function $P : \mathbb{C}^{n+1} \to \mathbb{C}$ defines a meromorphic function P on the projective space $\mathbb{CP}^{n+1}$. At each point x of the infinite hyperplane $\mathbb{CP}^n_\infty$ the germ of the meromorphic function P has the form $\dfrac{F(u, x_1, \ldots, x_n)}{u^d}$ where $u, x_1, \ldots, x_n$ are local coordinates such that $\mathbb{CP}^n_\infty = \{u = 0\}$, F is the germ of a holomorphic function, and d is the degree of the polynomial P.

In [8], for a meromorphic germ $f = \frac{F}{G}$, there were defined two Milnor fibres (the zero and the infinite ones), two monodromy transformations, and thus two zeta-functions $\zeta^0_f(t)$ and $\zeta^\infty_f(t)$. Let $\zeta^\bullet_{P,x}(t)$ ($\bullet = 0$ or ∞) be the corresponding zeta-function of the germ of the meromorphic function P at the point $x \in \mathbb{CP}^n_\infty$.

For the aim of convinience, in [8] we considered only meromorphic germs $f = \frac{F}{G}$ with $F(0) = G(0) = 0$. At a generic point of the infinite hyperplane $\mathbb{CP}^n_\infty$ the meromorphic function P has the form $\frac{1}{u^d}$. For a germ of the form $f = \frac{1}{G}$ with $G(0) = 0$, it is reasonable to give the following definition: its infinite Milnor fibre coincides with the (usual) Milnor fibre of the holomorphic germ G and its zero Milnor fibre is empty. Thus $\zeta^0_f(t) = 1$ and $\zeta^\infty_f(t) = \zeta_G(t)$. According to this definition, for the germ $\frac{1}{u^d}$, its infinite zeta-function is equal to $(1 - t^d)$.

Let $\mathcal{S} = \{\Xi\}$ be a prestratification of the infinite hyperplane $\mathbb{CP}^n_\infty$ (that is a partitioning of $\mathbb{CP}^n_\infty$ into semi-analytic subspaces without any regularity conditions) such that, for each stratum Ξ of $\mathcal{S}$, the infinite zeta-function $\zeta^\infty_{P,x}(t)$ does not depend on x, for $x \in \Xi$. Let us denote this zeta-function by $\zeta^\infty_\Xi(t)$ and by χ^∞_Ξ its degree $\deg \zeta^\infty_\Xi(t)$. A straighforward repetition of the arguments from the proof of Theorem 1 in [7] gives

Theorem 1.

$$\zeta_P(t) = \prod_{\Xi \in \mathcal{S}} [\zeta^\infty_\Xi(t)]^{\chi(\Xi)},$$

$$\chi_P = \sum_{\Xi \in \mathcal{S}} \chi^\infty_\Xi \cdot \chi(\Xi).$$

Remark 2. One can write the formula for χ_P in the form of an integral with respect to the Euler characteristic

$$\chi_P = \int_{\mathbb{CP}^n_\infty} \chi^\infty_{P,x}\, d\chi$$

in the sense of Viro ([14]).

Remark 3. Let P_d be the (highest) homogeneous part of degree d of the polynomial P. Then at each point $x \in \mathbb{CP}^n_\infty \setminus \{P_d = 0\}$ the germ of the meromorphic function P is of the form $\frac{1}{u^d}$. The set $\Xi^n = \mathbb{CP}^n_\infty \setminus \{P_d = 0\}$ can be considered as the n-dimensional stratum of a partitioning. It brings the factor $(1 - t^d)^{\chi(\Xi^n)}$ into the zeta-function $\zeta_P(t)$.

§3.- Examples

3.1. Yomdin-at-infinity polynomials. This name was introduced in [4]. For a polynomial P be of the form $P = P_d + P_{d-k}+$ terms of lower degree, $k \geq 1$. Let us consider hypersurfaces in $\mathbb{CP}^n$ defined by $\{P_d = 0\}$ and $\{P_{d-k} = 0\}$. Let $\mathrm{Sing}(P_d)$ be the singular locus of the hypersurface $\{P_d = 0\}$ (including all points where $\{P_d = 0\}$ is not reduced). One says that P is a *Yomdine-at-infinity polynomial* if $\mathrm{Sing}(P_d) \cap \{P_{d-k} = 0\} = \emptyset$ (in particular it implies that $\mathrm{Sing}(P_d)$ is finite).

Y. Yomdin ([15]) has considered critical points of holomorphic functions which are local versions of such polynomials. He gave a formula for their Milnor numbers. The generic fibre (level set) of a Yomdin-at-infinity polynomial is homotopy equivalent to the bouquet of n-dimensional spheres ([5]). Its Euler characteristic χ_P (or rather the (global) Milnor number) has been determined in [4]. For $k = 1$, the zeta-function of such a polynomial has been obtained in [6].

Let $P(z_0, z_1, \ldots, z_n) = P_d + P_{d-k} + \ldots$ be a Yomdin-at-infinity polynomial. Let $\mathrm{Sing}(P_d)$ consist of s points $Q_1, \ldots, Q_s$. One has the following natural stratification of the infinite hyperplane $\mathbb{CP}^n_\infty$:

(1) the n-dimensional stratum $\Xi^n = \mathbb{CP}^n_\infty \setminus \{P_d = 0\}$;
(2) the $(n-1)$-dimensional stratum $\Xi^{n-1} = \{P_d = 0\} \setminus \{Q_1, \ldots, Q_s\}$;
(3) the 0-dimensional strata Ξ^0_i $(i = 1, \ldots, s)$, each consisting of one point Q_i.

The Euler characteristic of the stratum Ξ^n is equal to

$$\chi(\mathbb{CP}^n_\infty) - \chi(\{P_d = 0\}) = (n+1) - \chi(n, d) + (-1)^{n-1} \sum_{i=1}^{s} \mu_i,$$

where $\chi(n, d) = (n+1) + \frac{(1-d)^{n+1}-1}{d}$ is the Euler characteristic of a non-singular hypersurface of degree d in the complex projective space $\mathbb{CP}^n_\infty$, μ_i is the Milnor number of the germ of the hypersurface $\{P_d = 0\} \subset \mathbb{CP}^n_\infty$ at the point Q_i. At each point of the stratum Ξ^n, the germ of the meromorphic function P has (in some local coordinates $u, y_1, \ldots, y_n$) the form $\frac{1}{u^d}$ ($\mathbb{CP}^n_\infty = \{u = 0\}$) and its infinite zeta-function $\zeta^\infty_{\Xi^n}(t)$ is equal to $(1 - t^d)$.

At each point of the stratum Ξ^{n-1}, the germ of the polynomial P has (in some local coordinates $u, y_1, \ldots, y_n$) the form $\frac{y_1}{u^d}$. Its infinite zeta-function $\zeta^\infty_{\Xi^{n-1}}(t)$ is equal to 1 and thus it does not contribute a factor to the zeta-function of the polynomial P.

At a point Q_i $(i = 1, \ldots, s)$, the germ of the meromorphic function P has the form $\varphi(u, y_1, \ldots, y_n) = \dfrac{g_i(y_1, \ldots, y_n) + u^k}{u^d}$, where g_i is a local equation of the hypersurface $\{P_d = 0\} \subset \mathbb{CP}^n_\infty$ at the point Q_i. Thus μ_i is its Milnor number.

To compute the infinite zeta-function $\zeta^\infty_\varphi(t)$ of the meromorphic germ φ, let us consider a resolution $\pi : (\mathcal{X}, \mathcal{D}) \to (\mathbb{C}^n, 0)$ of the singularity g_i, i.e., a proper modification of $(\mathbb{C}^n, 0)$ which is an isomorphism outside the origin in $\mathbb{C}^n$ and such that, at each point of the exceptional divisor $\mathcal{D}$, the lifting $g_i \circ \pi$ of the function g_i to the space $\mathcal{X}$ of the modification has (in some local coordinates) the form $y_1^{m_1} \cdot \ldots \cdot y_n^{m_n}$ $(m_i \geq 0)$.

Let us consider the modification $\widetilde{\pi} = id \times \pi : (\mathbb{C}_u \times \mathcal{X}, 0 \times \mathcal{D}) \to (\mathbb{C}^{n+1}, 0) = (\mathbb{C}_u \times \mathbb{C}^n, 0)$ of the space $(\mathbb{C}^{n+1}, 0)$ – the trivial extension: $(u, x) \mapsto (u, \pi(x))$. Let $\widetilde{\varphi} = \varphi \circ \widetilde{\pi}$ be the lifting of the meromorphic function φ to the space $\mathbb{C}_u \times \mathcal{X}$ of the modification $\widetilde{\pi}$. Let $\mathcal{M}^\infty_{\widetilde{\varphi}} = \widetilde{\pi}^{-1}(\mathcal{M}^\infty_\varphi)$ ($\mathcal{M}^\infty_\varphi$ is the infinite Milnor fibre of the germ φ) be the local level set of the meromorphic function $\widetilde{\varphi}$ (close to the infinite one). In the natural way one has the (infinite) monodromy $h^\infty_{\widetilde{\varphi}}$ acting on $\mathcal{M}^\infty_{\widetilde{\varphi}}$ and its zeta-function $\zeta^\infty_{\widetilde{\varphi}}(t)$.

Theorem 2.
$$\zeta^{\infty}_{\widetilde{\varphi}}(t) = (1-t^{d-k})^{\chi(\mathcal{D})-1}\,\zeta^{\infty}_{\varphi}(t).$$

Proof. The infinite monodromy transformation of the function $\widetilde{\varphi}$ can be described in the following way. Let $h^{\infty}_{\varphi} : \mathcal{M}^{\infty}_{\varphi} \to \mathcal{M}^{\infty}_{\varphi}$ be the infinite monodromy transformation of the germ φ. One can suppose that it preserves the intersection of the Milnor fibre $\mathcal{M}^{\infty}_{\varphi}$ with the line $\mathbb{C}_u \times \{0\}$. There it coincides with the infinite monodromy transformation of the restriction $\varphi|_{\mathbb{C}_u\times\{0\}} = \frac{u^k}{u^d}$ of the germ φ to this line, i.e., with a cyclic permutation of $(d-k)$ points. The zeta-function of a cyclic permutation of $(d-k)$ points is equal to $(1-t^{d-k})$. The projection $\widetilde{\pi} : \mathcal{M}^{\infty}_{\widetilde{\varphi}} \to \mathcal{M}^{\infty}_{\varphi}$ is an isomorphism outside $\mathcal{M}^{\infty}_{\varphi}\cap(\mathbb{C}_u\times\{0\})$, the preimage of each point from $\mathcal{M}^{\infty}_{\varphi}\cap(\mathbb{C}_u\times\{0\})$ is isomorphic to the exceptional divisor $\mathcal{D}$. This means that the transformation (the diffeomorphism) $h^{\infty}_{\widetilde{\varphi}} : \mathcal{M}^{\infty}_{\widetilde{\varphi}} \to \mathcal{M}^{\infty}_{\widetilde{\varphi}}$ can be constructed in such a way that it preserves $\widetilde{\pi}^{-1}(\mathcal{M}^{\infty}_{\varphi}\cap(\mathbb{C}_u\times\{0\}))$ and acts on it by a cyclic permutation of $(d-k)$ copies of $\mathcal{D}$. The zeta-function of this transformation of $\{(d-k)\text{ points}\}\times\mathcal{D}$ is equal to $(1-t^{d-k})^{\chi(\mathcal{D})}$. The result follows from the *multiplication property* of the zeta-function of a transformation (see [2] p. 94). □

For $\bar{m} = (m_1, m_2, \ldots, m_n)$ with integer $m_1 \geq m_2 \geq \ldots \geq m_n \geq 0$, let $S_{\bar{m}}$ be the set of points of the exceptional divisor $\mathcal{D}$ of the resolution π at which the lifting of the germ g_i has the form $y_1^{m_1}\cdot\ldots\cdot y_n^{m_n}$; for $m \geq 1$, let S_m be $S_{(m,0,\ldots,0)}$. By the formula of A'Campo ([1])

$$\zeta_{g_i}(t) = \prod_{m\geq 1}(1-t^m)^{\chi(S_m)}. \tag{1}$$

At a point $x \in \{0\}\times S_{\bar{m}} \subset \{0\}\times\mathcal{D}$, the lifting $\widetilde{\varphi} = \varphi\circ\widetilde{\pi}$ of the function φ has the local form $\dfrac{y_1^{m_1}\cdot\ldots\cdot y_n^{m_n}+u^k}{u^d}$. Thus, for fixed $\bar{m}$, the infinite zeta-function $\zeta^{\infty}_{\widetilde{\varphi},x}(t)$ of the germ of the meromorphic function $\widetilde{\varphi}$ at a point x from $\{0\}\times S_{\bar{m}}$ is one and the same. It can be determined by the Varchenko type formula from [8]. If there are more than one integers m_i different from zero, $\zeta^{\infty}_{\widetilde{\varphi},x}(t) = (1-t^{d-k})$. For $x \in \{0\}\times S_m$,

$$\zeta^{\infty}_{\widetilde{\varphi},x}(t) = (1-t^{d-k})(1-t^{\frac{m(d-k)}{g.c.d.(m,k)}})^{-g.c.d.(m,k)}.$$

According to Theorem 1

$$\zeta^{\infty}_{\widetilde{\varphi}}(t) = (1-t^{d-k})^{\chi(\mathcal{D})}\prod_{m\geq 1}\left(1-t^{\frac{m(d-k)}{g.c.d.(m,k)}}\right)^{-g.c.d.(m,k)\cdot\chi(S_m)}$$

and by Theorem 2

$$\zeta^{\infty}_{\varphi}(t) = (1-t^{d-k})\prod_{m\geq 1}\left(1-t^{\frac{m(d-k)}{g.c.d.(m,k)}}\right)^{-g.c.d.(m,k)\cdot\chi(S_m)}. \tag{2}$$

The zeta-function $\zeta_h(t)$ of a transformation $h : X \to X$ of a space X into itself determines the zeta-function $\zeta^k_h(t)$ of the k-th power h^k of the transformation h. In particular, if $\zeta_h(t) = \prod\limits_{m\geq 1}(1-t^m)^{a_m}$, then

$$\zeta^k_h(t) = \prod\left(1-t^{\frac{m}{g.c.d.(k,m)}}\right)^{g.c.d.(k,m)\cdot a_m}.$$

The formulae (1) and (2) mean that

$$\zeta_{\varphi}^{\infty}(t) = (1 - t^{d-k}) \left(\zeta_{g_i}^{k}(t^{d-k})\right)^{-1} \qquad (3).$$

Combining the computations for the stratification $\{\Xi^n, \Xi^{n-1}, \Xi_i^0\}$ of the infinite hyperplane $\mathbb{CP}^n_\infty$, one has

Theorem 3. *For a Yomdin-at-infinity polynomial $P = P_d + P_{d-k} + \ldots$, its zeta-function at infinity is equal to*

$$\zeta_P(t) = (1 - t^d)^{\chi(\Xi^n)} (1 - t^{d-k})^s \left(\prod_{i=1}^{s} \zeta_{g_i}^{k}(t^{d-k})\right)^{-1},$$

where $\chi(\Xi^n) = \frac{1-(1-d)^{n+1}}{d} + (-1)^{n-1} \sum_{i=1}^{s} \mu(g_i)$ and g_i is a local equation of the hypersurface $\{P_d = 0\} \subset \mathbb{CP}^n_\infty$ at its singular point Q_i.

3.2. Let $(n+1)$ be equal to 3, $P = P_d + P_{d-k} + \ldots$, $\{P_d = 0\}$ is a curve in $\mathbb{CP}^2_\infty$. Let $C_1^{q_1} + \ldots + C_r^{q_r}$ be its decomposition into irreducible components. Let $\{P_d = 0\}_{red}$ be the reduced curve $C_1 + \ldots + C_r$ and let $\mathrm{Sing}(\{P_d = 0\}_{red})$ consist of s points $\{Q_1, \ldots, Q_s\}$. Suppose that:

(1) the curve $\{P_{d-k} = 0\}$ is reduced;
(2) $Q_i \notin \{P_{d-k} = 0\}$, $(i = 1, \ldots, s)$;
(3) for each j with $q_j > 1$, the curves C_j and $\{P_{d-k} = 0\}$ intersect transversally, i.e., the set $C_j \cap \{P_{d-k} = 0\}$ consists of $d_j(d-k)$ different points ($d_j = \deg C_j$).

The generic fibre of the polynomial P is homotopy equivalent to the bouquet of 2-dimensional spheres. In this case the number of these spheres is equal to $\mu(P) = \dim_{\mathbb{C}} \mathbb{C}[x,y,z]/Jac(P)$ and is equal to

$$(d-1)^3 - k \cdot \Big(\chi(\{P_d = 0\}) + d(\,2d - \tilde{d} - 3)\Big) + k^2 \cdot (d - \tilde{d}),$$

where $\tilde{d} = d_1 + \ldots + d_r$ is the degree of the (reduced) curve $\{P_d = 0\}_{red}$, [4]. Let us consider the following partitioning of the infinite hyperplane $\mathbb{CP}^2_\infty$:

(1) the 0-dimensional stratum Ξ_i^0 consisting of one point Q_i each $(i = 1, \ldots, s)$;
(2) the 0-dimensional stratum $\Lambda_j^0 = C_j \cap \{P_{d-k} = 0\}$, for each $j = 1, \ldots, r$;
(3) the 1-dimensional stratum $\Xi_j^1 = C_j \setminus (\{Q_i\} \cup \Lambda_j^0)$, for each $j = 1, \ldots, r$;
(4) the 2-dimensional stratum $\Xi^2 = \mathbb{CP}^2_\infty \setminus \{P_d = 0\}$.

At each point of the stratum Ξ^2, the germ of the meromorphic function P has the form $\frac{1}{u^d}$ ($\mathbb{CP}^2_\infty = \{u = 0\}$). Its infinite zeta-function is equal to $(1 - t^d)$. The Euler characteristic $\chi(\Xi^2)$ of the stratum Ξ^2 is equal to

$$\chi(\mathbb{CP}^2_\infty) - \chi(\{P_d = 0\}) = 3 - 3\tilde{d} + \tilde{d}^2 - \sum_{i=1}^{s} \mu_i,$$

where μ_i is the Milnor number of the (reduced) curve $\{P_d = 0\}_{red}$ at the point Q_i

At each point of the stratum Ξ_j^1, the germ of the meromorphic function P has the form $\frac{y_1^{q_j}+u^k}{u^d}$. Its infinite zeta-function can be determined by the Varchenko type formula from [8] and is equal to

$$(1-t^{d-k})(1-t^{\frac{q_j(d-k)}{g.c.d.(q_j,k)}})^{-g.c.d.(q_j,k)}.$$

The Euler characteristic of the stratum Ξ_j^1 is equal to

$$\chi(C_j) - d_j(d-k) - \sharp\{C_j \cap \{Q_i : i = 1, \dots, s\}\}.$$

At each point of the stratum Λ_j^0, the germ of the meromorphic function P has the form $\frac{y_1^{q_j}+u^k y_2}{u^d}$. Its infinite zeta-function is equal to 1.

At a point Q_i, the germ of the meromorphic function P has the form $\frac{g_i(y_1,y_2)+u^k}{u^d}$, where $\{g_i = 0\}$ is the local equation of the (non-reduced) curve $\{P_d = 0\}$ at the point Q_i. Its infinite zeta-function is equal to

$$(1-t^{d-k})\left(\zeta_{g_i}^k(t^{d-k})\right)^{-1}.$$

Remark 4. We can not apply the formula (3) directly since the singularity of the germ g_i is, in general, not isolated. However, it is not difficult to see that, actually, the proof of this formula uses only the fact that the singularity of the germ g_i can be resolved by a modification which is an isomorphism outside the origin. This is so for a curve singularity.

Thus one obtains

$$\zeta_P(t) = (1-t^d)^{\chi(\Xi^2)}(1-t^{d-k})^{(3\tilde{d}-\tilde{d}^2-\tilde{d}(d-k)+\sum\mu_i)} \times$$
$$\times \prod_{j=1}^{r}\left(1-t^{\frac{q_j(d-k)}{g.c.d.(q_j,k)}}\right)^{-g.c.d.(q_j,k)\cdot\chi(\Xi_j^1)} \cdot \prod_{i=1}^{s}\left(\zeta_{g_i}^k(t^{d-k})\right)^{-1}.$$

§4.- On the bifurcation set of a polynomial map

As we have mentioned, a polynomial map $P : \mathbb{C}^{n+1} \to \mathbb{C}$ defines a locally trivial fibration over the complement to a finite set in $\mathbb{C}$. The minimal set $B(P)$ with this property is called the bifurcation set of P. The bifurcation set consists of critical values of the polynomial P (in the affine part) and of atypical ("critical") values at infinity.

In order to consider a level set $\{P = c\}$, one can substitute the polynomial P by the polynomial $(P-c)$ and consider the zero level set. Thus let us consider the zero level set $V_0 = \{P = 0\} \subset \mathbb{C}^{n+1}$ of the polynomial P. Let us suppose that the level set V_0 of the polynomial P has only isolated singular points (in the affine part $\mathbb{C}^{n+1}$). For $\rho > 0$, let B_ρ be the open ball of radius ρ centred at the origin in $\mathbb{C}^{n+1}$ and $S_\rho = \partial B_\rho$ be the $(2n+1)$-dimensional sphere of radius ρ with the centre at the origin. There exists $R > 0$ such that, for all $\rho \geq R$, the sphere S_ρ is transversal to the level set $V_0 = \{P = 0\}$ of the polynomial map P. The restriction $P|_{\mathbb{C}^{n+1}\setminus B_R} : \mathbb{C}^{n+1} \setminus B_R \to \mathbb{C}$ of the function P to the complement of the ball B_R defines a C^∞ locally trivial fibration over a punctured neighbourhood of the origin in $\mathbb{C}$. The loop $\varepsilon_0 \cdot \exp(2\pi i\tau)$ $(0 \leq \tau \leq 1$, $\|\varepsilon_0\|$ small enough) defines the monodromy transformation $h : V_{\varepsilon_0} \setminus B_R \to V_{\varepsilon_0} \setminus B_R$. Let us denote its zeta-function $\zeta_h(t)$ by $\zeta^0_{\infty}(t)$. We use the following definition.

Definition. The value 0 is *atypical at infinity* for the polynomial P if the restriction $P|_{\mathbb{C}^{n+1}\setminus B_R}$ of the function P to the complement of the ball B_R is not a C^∞ locally trivial fibration over a neighbourhood of the origin in $\mathbb{C}$.

Remark 5. This definition does not depend on a choice of coordinates, i.e., it is invariant with respect to polynomial diffeomorphisms of the space $\mathbb{C}^{n+1}$. One can see that an atypical at infinity value is atypical, i.e. it belongs to the bifurcation set $B(P)$ of the polynomial P. Moreover the bifurcation set $B(P)$ is the union of the set of critical values of the polynomial P (in $\mathbb{C}^{n+1}$) and of the set of values atypical at infinity in the described sense. If the singular locus of the level set $V_0 = \{P = 0\}$ is not finite, the value 0 hardly can be considered as typical at infinity. Thus, one should consider this definition as a (possible) general definition of a value atypical at infinity. In fact the same definition was used in [10].

Let $\mathcal{S}$ be a prestratification of the infinite hyperplane $\mathbb{CP}^n_\infty$ such that, for each stratum Ξ of $\mathcal{S}$, the zero zeta-function $\zeta^0_{P,x}(t)$ of the germ of the meromorphic function P at a point $x \in \mathbb{CP}^n_\infty$ does not depend on the point x, for $x \in \Xi$ (let it be $\zeta^0_\Xi(t)$ and let its degree be χ^0_Ξ).

Theorem 4.
$$\zeta^0_P(t) = \prod_{\Xi\in\mathcal{S}} [\zeta^0_\Xi(t)]^{\chi(\Xi)},$$
$$\chi(V_{\varepsilon_0} \setminus B_R) = \sum_{\Xi\in\mathcal{S}} \chi^0_\Xi \cdot \chi(\Xi).$$

The proof is essentially the same as that of Theorem 1. Since the Euler characteristic of the set $V_0 \setminus B_R$ is equal to 0, one has

Corollary 1. *If* $\zeta^0_P(t) \not\equiv 1$, *then the value* 0 *is atypical at infinity for the polynomial* P.

In several papers (see, e.g., [3], [11], [12]) there was considered an integer $\lambda_P(c)$ ($c \in \mathbb{C}$) such that

$$\chi(\{P = c\}) = \chi(\{P = c + \varepsilon\}) + (-1)^{n+1}(\sum \mu_i + \lambda_P(c)),$$

where μ_i are the Milnor numbers of the (isolated) singular points of the level set $\{P = c\} \subset \mathbb{C}^{n+1}$. Theorem 4 gives the following formula for this invariant:

Corollary 2.

$$\lambda_P(0) = (-1)^n \deg \zeta^0_P(t) = (-1)^n \sum_{\Xi\in\mathcal{S}} \chi^0_\Xi \cdot \chi(\Xi) \ \left(= (-1)^n \int_{\mathbb{CP}^n_\infty} \chi^0_{P,x}\, d\chi \right).$$

Example. Let $P(x, y, z) = x^a y^b (x^c y^d - z^{c+d}) + z$, $(ad - bc) \neq 0$, and let $D = \deg(P) = a+b+c+d$. The curve $\{P_D = 0\} \subset \mathbb{CP}^2_\infty$ consists on three components: the line $C_1 = \{x = 0\}$ with multiplicity a, the line $C_2 = \{y = 0\}$ with multiplicity b, and the reduced curve $C_3 = \{x^c y^d - z^{c+d} = 0\}$. Let $Q_1 = C_2 \cap C_3 = (1 : 0 : 0)$, $Q_2 = C_1 \cap C_3 = (0 : 1 : 0)$, $Q_3 = C_1 \cap C_2 = (0 : 0 : 1)$. At each point x of the infinite hyperplane $\mathbb{CP}^2_\infty$ except Q_1 and Q_2, one has $\zeta^0_{P,x}(t) = 1$. At the point Q_1, the germ of the meromorphic function P has the form $\dfrac{y^b(y^d - z^{c+d}) + zu^{D-1}}{\ldots}$

Its zero zeta-function can be obtained by the Varchenko type formula from [8]. If $(ad-bc)<0$, then $\zeta^0_{P,Q_1}(t)=1$. If $(ad-bc)>0$, then

$$\zeta^0_{P,Q_1}(t)=(1-t^{\frac{ad-bc}{G.C.D.}})^{G.C.D.},$$

where $G.C.D.=g.c.d(c,d)\cdot g.c.d.(\frac{ad-bc}{g.c.d(c,d)},D-1)$. At the point Q_2, we have just the symmetric situation. Finally

$$\zeta^0_P(t)=(1-t^{\frac{|ad-bc|}{G.C.D.}})^{G.C.D.}.$$

It means that the value 0 is atypical at infinity. In the same way $\zeta^0_{P-c}(t)=1$, for $c\neq 0$.

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Moscow State University, Faculty of Mathematics and Mechanics, Moscow, 119899, Russia.

E-mail address: `sabir@ium.ips.ras.ru`

Departamento de Álgebra, Universidad Complutense, Ciudad Universitaria s/n, E-28040 Madrid, Spain.

E-mail address: `iluengo@eucmos.sim.ucm.es`

Departamento de Geometría y Topología, Universidad Complutense, Ciudad Universitaria s/n, E-28040 Madrid, Spain.