On a conjecture of W. Veys

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Abstract. In this work we show a singular $K3$ surface S of $\mathbb{P}^3$ such that $\chi(\mathbb{P}^3 \setminus S) = 0$ producing a counter-example to a conjecture of Veys.

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In this journal W. Veys [V, p. 547] stated the following

Conjecture. *Let C_i, $1 \leq i \leq r$, be irreducible hypersurfaces in $\mathbb{P}^n(\mathbb{C})$. If*

$$(-1)^n \chi(\mathbb{P}^n \setminus \bigcup_{i=1}^{r} C_i) \leq 0$$

then all C_i have Kodaira dimension $-\infty$.

This conjecture was proven for $n = 2$ by A.J. de Jong and J. Steenbrink in [JS]. Also, R. Gurjar and A. Parameswaran [GP] gave an independent proof. In this note we give a counterexample to this conjecture for $n = 3$.

As W. Veys has pointed out to us, this problem is related with the monodromy conjecture of the topological and Igusa zeta–functions of singularities of hypersurface, see [V].

Counterexample to the conjecture of Veys. Let $D \subset \mathbb{P}^2$ be a reduced projective plane curve of degree d, with singularities $\operatorname{Sing}(D) = \{P_1, \ldots, P_s\}$. The Euler characteristic of D is given by the well-known formula

$$\chi(D) = 3d - d^2 + \sum_{i=1}^{s} \mu(D, P_i),$$

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where $\mu(D, P_i)$ is the Milnor number of the curve D at P_i. Recall that a *simple singularity of type* $\mathbb{A}_n$ means a singularity locally defined by $v^2 + u^{n+1} = 0$, and so it has Milnor number equal to n.

Let C_1 and C_2 be the projective plane curves of degree 3 in $\mathbb{P}^2$ given as the zero locus of the homogeneous polynomials $f_1(x, y, z) := x^3 + y^2z - 3x^2z$ and $f_2(x, y, z) := x^3 + y^2z - 3x^2z + 4z^3$. Both cubics have only one singular point, which is of type $\mathbb{A}_1$; choosing as above homogeneous coordinates $(x : y : z)$ in $\mathbb{P}^2$ the curve C_1 is singular at $P_1 = (0 : 0 : 1)$ and C_2 is singular at $P_2 = (2 : 0 : 1)$. Then $\chi(C_1) = \chi(C_2) = 1$.

Let $C = C_1 \cup C_2$ be the corresponding sextic which has three singular points, namely P_1, P_2 and $P_3 = C_1 \cap C_2 = (0 : 1 : 0)$; also C has at P_3 a simple singularity of type $\mathbb{A}_{17}$. It means again that the Euler characteristic $\chi(C) = 1$.

Let X be the normal compact complex surface obtained as a double covering of $\mathbb{P}^2$ branched along C. The surface X has only a finite (three) number of rational singularities. Let $\sigma : \widetilde{X} \to X$ be the canonical resolution of these singularities. Then $\widetilde{X}$ is the minimal resolution of the double covering of the projective plane ramified on a sextic curve having only simple singularities: $\widetilde{X}$ is a $K3$ surface, see [BPV, pp. 182–183]. Then the Kodaira dimension $\kappa(\widetilde{X})$ of $\widetilde{X}$ is 0. Since the Kodaira dimension is a birational invariant, $\kappa(X) = 0$ (e.g. see [H, p. 421]).

The idea is to find a surface $S \subset \mathbb{P}^3$ birationally equivalent to X such that $\chi(\mathbb{P}^3 \setminus S) = 0$, or, equivalently, $\chi(S) = 4$. This last equivalence is a consequence of:

Additivity Principle. *Let Y be an analytic variety. Let $\{S_i\}_{i\in I}$ be a finite pre-stratification of Y, i.e., I is a finite set, $Y = \coprod_{i\in I} S_i$ and S_i are locally closed analytic subvarieties of Y for $i \in I$. Then*

$$\chi(Y) = \sum_{i\in I} \chi(S_i).$$

Let $S \subset \mathbb{P}^3$ be the irreducible surface given by the zero locus of the homogeneous polynomial of degree 6

$$\begin{aligned} f_6(x, y, z) &= f_1(x, y, z) f_2(x, y, z) + w^2 z^4 \\ &= (x^3 + y^2z - 3x^2z)(x^3 + y^2z - 3x^2z + 4z^3) + z^4w^2. \end{aligned}$$

It can be easily seen that S and X are birationally equivalent: Take the affine chart (x, y) of $\mathbb{P}^2$; the affine part A of C is the disjoint union of two nodal cubics with only one (common) place at infinity each one; the equation of A is $f_1(x, y, 1) f_2(x, y, 1) = 0$; therefore, $\chi(A) = 0$. Take the double covering $\check{S}$ of $\mathbb{C}^2$ ramified on A. If we denote by $f(x, y) = 0$ the equation of A, then $\check{S}$ is defined in $\mathbb{C}^3$ with coordinates (x, y, w) as $f_1(x, y, 1) f_2(x, y, 1) + w^2 = 0$. We obtain that X is birationally equivalent to $\check{S}$, because of the birational equivalence of $\mathbb{C}^2$ and $\mathbb{P}^2$; S is the hypersurface of $\mathbb{P}^3$ defined by the homogeneisation of

$f_1(x, y, 1) f_2(x, y, 1) + w^2$ in z, and as $\check{S}$ is birationally equivalent to S then $\kappa(S) = 0$.

Let us compute $\chi(S)$ with the help of the following partition of S into two disjoint parts: $S_1 := S \cap \{z = 0\}$ and $S_2 := S \cap \{z \neq 0\}$.

Since $S_1 = \{x = z = 0\} \cong \mathbb{P}^1$, we have $\chi(S_1) = 2$.

We can identify S_2 with $\check{S}$. Let us consider the double covering $\pi : \check{S} \to \mathbb{C}^2$, defined by $(x, y, w) \mapsto (x, y)$. Recall that $\pi_{|\pi^{-1}(A)} \; \pi^{-1}(A) \to A$ is an isomorphism and $\pi_{|\check{S}\setminus\pi^{-1}(A)} \; \check{S} \setminus \pi^{-1}(A) \to \mathbb{C}^2 \setminus A$ is an unramified double covering. Then by the standard properties of coverings, the Additivity Principle and the fact that $\chi(A) = 0$:

$$\begin{aligned}\chi(S_2) = \chi(\check{S}) &= \chi(\check{S} \setminus \pi^{-1}(A)) + \chi(\pi^{-1}(A)) = 2\chi(\mathbb{C}^2 \setminus A) + \chi(A)\\ &= 2 \cdot 1 + 0 = 2.\end{aligned}$$

In particular, $\chi(S) = 4$.

We have found in this way several other examples of surfaces disproving the conjecture of Veys, this one being the simplest. In all cases, the Kodaira dimension $\kappa(C_i)$ of each irreducible component is zero. It will be interesting to know which is the upper bound for this Kodaira dimension $\kappa(C_i)$ under the topological hypothesis of the Euler characteristic of the complement verifying

$$(-1)^n \chi(\mathbb{P}^n \setminus \bigcup_{i=1}^{r} C_i) \leq 0.$$

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