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ON FANO THREEFOLDS OF TYPE $V_{22}$

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We classify rank-2 vector bundles with no intermediate cohomology on the
general prime Fano threefold of index 1 and genus 12. The structure of their
moduli spaces is given by means of a monad-theoretic resolution in terms of
exceptional bundles.

1. Introduction

The study of vector bundles with no intermediate cohomology, also called *arith-
metically Cohen–Macaulay* bundles (see Definition 2.1), has been taken up by
several authors. The well-known splitting criterion for projective spaces showed
by Horrocks [1964] has been generalized by Ottaviani [1987; 1989] to Grassman-
nians and quadrics. Kn"orrer [1987] proved that line bundles and spinor bundles
are the only ACM bundles on quadrics, while Buchweitz, Greuel and Schreyer
showed in [1987] that only projective spaces and quadrics admit a finite number
of equivalence classes of ACM bundles, up to twists.

On the other hand, the problem of classifying ACM bundles on special classes
of varieties has been studied in several papers. Arrondo and Costa [2000] took up
the case of prime Fano threefolds of index 2, while Faenzi [2005] considered the
case of the index-2 prime threefold $V_5$.

Madonna classified rank-2 ACM bundles on the quartic threefold [2000], and got
a numerical classification [2002] of the invariants of these bundles on any prime
Fano threefold $V_{2g-2}$ of index 1 and genus $g$, with $2 \leq g \leq 12$ and $g \neq 11$. In
particular, he conjectured that all the cases of this classification occur on every
such threefold $V_{2g-2}$.

For higher dimensional varieties, the case of $G(1, 4)$ has been studied in
[Arrondo and Graña 1999].

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In this paper we consider rank-2 ACM bundles on the general prime Fano threefold $X$ of index 1 and genus 12 (see Definition 2.3). Write the Chern classes of a sheaf $\mathcal{F}$ on $X$ as integers (see Section 2), and we denote by $\mathcal{F}_{c_1,c_2}$ a rank-2 sheaf $\mathcal{F}$ on $X$ with $c_1(\mathcal{F}) = c_1$ and $c_2(\mathcal{F}) = c_2$. The main result of this paper states:

**Theorem.** On the general $X$ as above, there exist the following vector bundles with no intermediate cohomology:

1. The bundle $\mathcal{F}_{-1,1}$ associated to a line contained in $X$;
2. The bundle $\mathcal{F}_{0,2}$ associated to a conic contained in $X$;
3. The bundle $\mathcal{F}_{-1,d}(1)$ associated to an elliptic curve of degree $d$, with $7 \leq d \leq 14$;
4. The bundle $\mathcal{F}_{0,4}(1)$ associated to a canonical curve of degree 26 and genus 14 contained in $X$;
5. The bundle $\mathcal{F}_{-1,15}(2)$ associated to a half-canonical curve $C_{59}$ of degree 59 and genus 60 contained in $X$.

These are the only possible indecomposable vector bundles with no intermediate cohomology on $X$, up to isomorphism and twists by line bundles.

The moduli space of semistable vector bundles with no intermediate cohomology is generically smooth, of dimension equal to 2 in Case (2), $2d - 14$ in Case (3), 5 in Case (4), and 16 in Case (5).

This gives a complete classification of ACM rank-2 bundles on the general Fano threefold $X$, together with a description of their moduli spaces. The main tools for proving the theorem are the study of elliptic curves in $X$ and the resolution of the diagonal on $X \times X$ obtained in [Faenzi 2006].

The paper is structured as follows: In Section 2 we state basic definitions and review some known facts concerning the threefold $X$. We also recall for the reader’s convenience the available descriptions of $X$, which we will use frequently.

In Section 3 we briefly consider lines and conics contained in $X$. We also give a monad-theoretic interpretation of the Hilbert scheme of lines and conics in $X$. In Sections 4 and 5 we take up the analysis of elliptic, canonical, and half-canonical curves in $X$ that give rise to vector bundles with no intermediate cohomology, proving their existence and describing their associated moduli spaces.

2. Preliminaries

Let $Y$ be a smooth projective threefold with $\text{Pic}(Y) \cong \mathbb{Z} = \langle \mathcal{O}_Y(1) \rangle$ and $\text{H}^1(\mathcal{O}(t)) = \text{H}^2(\mathcal{O}(t)) = 0$ for any $t \in \mathbb{Z}$. Following standard terminology, we have:
Definition 2.1. Given a sheaf $\mathcal{F}$ over $Y$, we say that $\mathcal{F}$ is ACM (arithmetically Cohen–Macaulay) if $H^p(Y, \mathcal{F}(t)) = 0$ for all $t \in \mathbb{Z}$ and $0 < p < 3$. Equivalently, we say that $\mathcal{F}$ has no intermediate cohomology.

We denote the dual of a vector bundle $\mathcal{F}$ by $\mathcal{F}^*$, and recall that if $\mathcal{F}$ has rank 2 then $\mathcal{F}^* \simeq \mathcal{F}(-c_1(\mathcal{F}))$.

We now review the Hartshorne–Serre correspondence between codimension-2 subvarieties and rank-2 vector bundles, originally introduced in [Serre 1963] and later considered by many authors; see for example [Hartshorne 1974; Vogelaar 1978; Okonek et al. 1980].

Definition 2.2. A complete subvariety $Z$ of $Y$ is called subcanonical if there exists a line bundle $\mathcal{O}(r)$ on $Y$ such that $\mathcal{O}(r)|_Z \cong \omega_Z$. Let $Z$ be a subcanonical locally complete intersection codimension-2 subvariety of $Y$. By [Okonek et al. 1980, Theorem 5.1.1], there exist a rank-2 vector bundle $\mathcal{F}_Z$ over $Y$ and a section $s_Z \in H^0(Y, \mathcal{F}_Z^2)$ such that $Z = \{s_Z = 0\}$, that is, $Z$ is the zero locus of $s_Z$. We say in this case that $\mathcal{F}_Z$ is associated to $Z$. We denote by $N_{Z,Y}$ the normal bundle of $Z$ in $Y$ and by $J_{Z,Y}$ the ideal sheaf of $Z$ in $Y$.

Under these hypotheses, we have the fundamental exact sequence

$$0 \rightarrow \det \mathcal{F}_Z \rightarrow \mathcal{F}_Z \rightarrow J_{Z,Y} \rightarrow 0$$

and the adjunction isomorphism

$$(\mathcal{F}_Z^*)|_Z \cong N_{Z,Y}.$$

Definition 2.3. A prime Fano threefold of index 1 and genus 12 is a 3-dimensional algebraic variety $X$ with $\text{Pic}(X) \cong \mathbb{Z} = (\mathcal{O}_X(1))$ and $\omega_X \cong \mathcal{O}_X(-1)$, and with $\deg \mathcal{O}_X(1) = 22$. Any such $X$ is rational. We have $h^0(\mathcal{O}_X(1)) = 14$, while $\text{CH}^i(X)$, the $i$-th Chow group of $X$, is isomorphic to $\mathbb{Z}$ for $i = 1, 2, 3$.

From now on, $X$ will denote a prime Fano threefold of index 1 and genus 12. We denote the Chern classes of a sheaf $\mathcal{F}$ on $X$ by integers $c_i \in \mathbb{Z}$, meaning that $c_1(\mathcal{F}) = c_1(\xi_1)$, where $\xi_i$ is the generator of $\text{CH}^i(X) \cong \mathbb{Z}$ for $i = 1, 2, 3$. Recall that $\xi_2$ is the class of a line in $X$.

Further, we define $\mu(\mathcal{F})$ as the rational number $c_1(\mathcal{F})/\text{rk} \mathcal{F}$. We say that a vector bundle $\mathcal{F}$ is normalized if $-\text{rk} \mathcal{F} < c_1(\mathcal{F}) \leq 0$. Equivalently, $\mathcal{F}$ is normalized if $-1 < \mu(\mathcal{F}) \leq 0$. Clearly, $\mu(\mathcal{F}_1 \otimes \mathcal{F}_2) = \mu(\mathcal{F}_1) + \mu(\mathcal{F}_2)$.

We refer to [Huybrechts and Lehn 1997] for the definition of (semi)stability (in the sense of Mumford and Takemoto). A stable bundle $\mathcal{F}$ with $\mu(\mathcal{F}) < 0$ satisfies $h^0(\mathcal{F}) = 0$. Recall that, by Hoppe’s criterion, since Pic($X$) is generated by $\mathcal{O}_X(1)$, a rank-2 bundle $\mathcal{F}$ on $X$ is stable if $h^0(\mathcal{F}(t)) = 0$, for the only integer $t$ such that $c_1(\mathcal{F}(t)) = 0$ or $c_1(\mathcal{F}(t)) = -1$. See for example [Okonek et al. 1980, Lemma 1.2.5].
From the Hirzebruch–Riemann–Roch formula for a vector bundle on $X$ of rank $r$ and Chern classes $c_i$, we obtain:

$$6\chi(\mathcal{F}(s)) = 22s^3r + 11s^2(3r+6c_1) + s(23r+66c_1-6c_2+66c_1^2)$$
$$+ 6r + 23c_1 - 3c_2 - 3c_1c_2 + 33c_1^2 + 3c_3 + 22c_1^3.$$

Given a smooth projective variety $Y$ equipped with a very ample line bundle $\mathcal{O}_Y(1)$, for any integer $r$ and string $c$ with $c_i \in \text{CH}^i(Y)$ (identified with integers whenever possible), we write $M_Y(r; c)$ for the moduli space of rank-$r$ semistable sheaves on $Y$ with Chern classes $c_i$.

By virtue of the exact sequence (1), the Hilbert polynomial and the Chern classes of $\mathcal{F}_Z$ are determined by the Hilbert polynomial of $Z$. Denote by $P[Z]$ the Hilbert polynomial of $Z$ with respect to the polarization $\mathcal{O}_Y(1)$, and by $\text{Hilb}_Y[Z]$ the Hilbert scheme of closed subschemes of $Y$ with Hilbert polynomial $P[Z]$ (see [Huybrechts and Lehn 1997, Page 41]). Further, if $Z$ is a curve of degree $d$ and genus $g$ contained in $X$, we denote $\text{Hilb}_Y[Z](X)$ by $\mathcal{H}_{d,g}(X)$.

If the bundle $\mathcal{F}_Z$ is stable, the Hartshorne–Serre correspondence provides a morphism

$$(3) \quad \tau : \text{Hilb}_Y[Z](X) \to M_X(2; c_1(\mathcal{F}_Z), c_2(\mathcal{F}_Z)), \quad [Z] \mapsto [\mathcal{F}_Z].$$

We next recall some of the available constructions of the threefold $X$. We also sketch the description of four fundamental vector bundles $E, U, Q, K$ of respective ranks $2, 3, 4, 5$ and defined over $X$.

We refer to [Mukai 1992, 2004; Schreyer 2001; Faenzi 2006] for proofs and more details.

**Nets of dual quadrics and twisted cubics.** Let $k$ be an algebraically closed field. Let $A \simeq k^4$ and $B \simeq k^3$ be $k$-vector spaces, and let $R(A) = k[A]$ and $R(B) = k[B]$ be polynomial algebras. Let $S^dA = R(A)_d$ be the $d$-th symmetric power of the vector space $A$.

Given a twisted cubic $\Gamma$, we have $P_{[\Gamma]}(t) = \chi(\mathcal{O}_\Gamma(t)) = 3t + 1$. We consider the Hilbert scheme $\text{Hilb}_{3t+1}(\mathbb{P}(A))$ of closed subschemes of $\mathbb{P}(A)$ with Hilbert polynomial $3t + 1$, and we define the variety $H$ to be the irreducible component of $\text{Hilb}_{3t+1}(\mathbb{P}(A))$ containing the rational normal cubics in $\mathbb{P}(A)$, as constructed in [Ellingsrud et al. 1987]. Given a twisted cubic $[\Gamma] \in H$, we denote by $J_\Gamma$ the ideal sheaf of $\Gamma$ in $\mathbb{P}(A)$. The open subset $H_c$ consisting of points that are arithmetically Cohen–Macaulay embeds in $G(k^3, S^2 A)$ by means of the vector bundle $U_H$ whose fiber over $[\Gamma] \in H_c$ is $\text{Tor}_{1}^{R[A]}(R[A]/J_\Gamma, k) \simeq k^3$. Equivalently, we associate to any $[\Gamma] \in H$ the net of quadrics in $\mathbb{P}(A)$ vanishing on $\Gamma$. 

Definition 2.4. A net of dual quadrics $\Psi$ (parametrized by $B$) in $\mathbb{P}(A)$ is defined as a surjective map $\Psi : S^2A \to B$. Let $V_\Psi = \ker \Psi$. Given a general net $\Psi$, we define:

\[ X_\Psi = \{ [\Gamma] \in H \subset \text{Hilb}_{3r+1}(\mathbb{P}^3) \mid \Psi(H^0(J_\Gamma(2))) = 0 \} \]

\[ = \{ [\Gamma] \in H \subset \text{Hilb}_{3r+1}(\mathbb{P}^3) \mid H^0(J_\Gamma(2)) \subset V_\Psi \}. \]

We define the bundle $U$ on $X = X_\Psi$ as the restriction of $U_H$ to $X$.

Definition 2.5. Let $\Psi$ be a general net of dual quadrics and $X = X_\Psi$. There is a rank-2 vector bundle $E$ on $X$ defined by $E|_{V_\Psi} = \text{Tor}_{rA}^2(R[A]/J_\Gamma, k)_3 \simeq k^2$. Equivalently, we associate to any $[\Gamma] \in H$ its space of first-order syzygies.

Lemma 2.6 [Faenzi 2006, Lemma 6.3]. The bundle $E^*$ is globally-generated and ACM, with $h^0(E^*) = 8$. Consider the rank-6 bundle $E' = \ker(H^0(E^*) \otimes \mathcal{O} \to E^*)$. The bundle $E'$ is also stable and ACM.

Plane quartics. Let $B$ be a 3-dimensional $k$-vector space and $F \in S^4B$ a plane quartic. Set $\tilde{\mathbb{P}}^2 = \mathbb{P}(B^*)$. Take the Hilbert scheme $\text{Hilb}_6(\tilde{\mathbb{P}}^2)$ of zero-dimensional length 6 closed subschemes of $\tilde{\mathbb{P}}^2$. Define the subvariety of $\text{Hilb}_6(\tilde{\mathbb{P}}^2)$ consisting of polar hexagons to $F$.

\[ X_F = \{ \Lambda=(f_1, \ldots, f_6) \in \text{Hilb}_6(\tilde{\mathbb{P}}^2) \mid f_1^4 + \cdots + f_6^4 = F \}. \]

Lemma 2.7 [Mukai 2004; Schreyer 2001]. For a general $F$, the variety $X_F$ is a prime Fano threefold of index 1 and genus 12. Given a net of dual quadrics $\Psi$, there exists a quartic form $F$ such that $X_F \simeq X_\Psi$.

Definition 2.8. Let $F$ be a general plane quartic and let $X = X_F$. There is a rank-5 vector bundle $K$ on $X_F$ defined over an element $\Lambda=(f_1, \ldots, f_6) \in X_F$ by $K_\Lambda = (f_1^4, \ldots, f_6^4)/F$. The bundle $K^*$ is stable and ACM, with $h^0(K^*) = 14$ and $c_1(K) = -2$ [Faenzi 2006, Lemma 6.1 and 6.2].

Remark 2.9. Under the hypothesis of Lemma 2.7, there is a natural isomorphism $V_\Psi \simeq S^3B/F(B^*)$, where we consider $F$ as a map $B^* \to S^3B$ taking an element $\delta \in B^*$ to the cubic form $\delta(F)$ (apolarity action). We set $V_F = S^3B/F(B^*)$.

The fiber $U$ over an element $\Lambda=(f_1, \ldots, f_6) \in X_F$ is naturally identified with $\langle f_1^4, \ldots, f_6^4 \rangle/F(B^*)$. The global sections of $U^*$ and $K^*$ are then identified with $V_F = S^3B/F(B^*)$ and $S^4B/F$, respectively. An element $\delta$ of $B^*$ gives a map $S^4B \to S^3B$ by the apolarity action and, therefore, a homomorphism $\delta : K \to U$.

Nets of alternating 2-forms. Let $V$ be a 7-dimensional $k$-vector space and $B$ a 3-dimensional one. Let $G$ be the Grassmannian $G(k^3, V)$. Define $U_G$ as the universal rank-3 subbundle, and $Q_G$ as the universal rank-4 quotient bundle on $G$. Let $\sigma$
be a section of $B^* \otimes \wedge^2 U^*_G$. Equivalently, $\sigma$ is a net of alternating 2-forms $\sigma \in B^* \otimes \wedge^2 V^*$.

**Definition 2.10.** Define $X_\sigma$ as the zero locus in $G$ of $\sigma \in B^* \otimes \wedge^2 V^*$. For a general $\sigma$, the variety $X_\sigma$ is a prime Fano threefold of index 1 and genus 12.

**Lemma 2.11** [Mukai 2004]. Given a general plane quartic $F$, there is a net of alternating 2-forms $\sigma_F$ such that $X_{\sigma_F} \cong X_F$.

From now on, we identify $X$ with $X_{\psi} \cong X_{F} \cong X_\sigma$, where $\psi$ is a general net of dual quadrics, $F$ is the quartic form provided by Lemma 2.7, and $\sigma$ is the net of alternating 2-forms given by Lemma 2.11. In particular, we fix the 3- and 4-dimensional $k$-vector spaces $B$ and $A$. Recall that, by Remark 2.9, we have $V \cong V_F \cong V_{\psi}$. We also notice that, under our hypotheses, $(U_G)|_X \cong (U_H)|_X$. Thus, we denote also by $U$ the restriction of the vector bundle $U_G$ to $X_\sigma$. We set $Q = (Q_G)|_X$.

**Lemma 2.12.** There are natural isomorphisms

\begin{align*}
(4) \quad & \text{Hom}(U, Q^*) \cong B, & \text{Hom}(E, U) \cong A^*, \\
(5) \quad & \text{Hom}(K, U) \cong B^*, & \text{Hom}(E, K) \cong A.
\end{align*}

Moreover, there are exact sequences

\begin{align*}
(6) \quad & 0 \rightarrow U \rightarrow V \otimes \mathcal{O} \rightarrow Q \rightarrow 0, \\
(7) \quad & 0 \rightarrow K \rightarrow B \otimes U \rightarrow Q^* \rightarrow 0, \\
(8) \quad & 0 \rightarrow \wedge^2 U \rightarrow A \otimes E \rightarrow K \rightarrow 0, \\
(9) \quad & 0 \rightarrow E \rightarrow \mathcal{O}^{\oplus 8} \rightarrow (E')^* \rightarrow 0.
\end{align*}

The Chern classes of these bundles are

\begin{align*}
c_1(E) &= -1, & c_2(E) &= 7, \\
c_1(U) &= -1, & c_2(U) &= 10, & c_3(U) &= -2, \\
c_1(Q^*) &= -1, & c_2(Q^*) &= 12, & c_3(Q^*) &= -4, \\
c_1(K) &= -2, & c_2(K) &= 40, & c_3(K) &= -20, \\
c_1(E') &= -1, & c_2(E') &= 15, & c_3(E') &= -8.
\end{align*}

**Proof:** The exact sequences (6) and (7), together with (4) and the first isomorphism in (5), are proved in [Faenzi 2006, Lemma 6.1]. The sequence (8) follows from [Faenzi 2006, Proposition 6.4], while (9) is Lemma 2.6. The second isomorphism in (5) follows from [Faenzi 2006, Corollary 6.8]. The Chern classes of $U$, $Q^*$ and $\wedge^2 U$ are easily computed by restriction from $\mathbb{G}(k^3, V)$. Finally, the Chern classes of $K$, $E$ and $E'$ follow from the exact sequences (7), (8) and (9). \qed
**Birational geometry.** We briefly sketch the birational geometry of $X$ following [Iskovskih 1978, 1989]. Fano’s double projection from a line is used, and we refer to [Iskovskikh and Prokhorov 1999] for a complete treatment.

Let $V$ be the del Pezzo threefold obtained by cutting $G(\mathbb{P}^1, \mathbb{P}^4) \subset \mathbb{P}^9$ with a general $\mathbb{P}^6 \subset \mathbb{P}^9$. Denote by $S$ a general hyperplane section of $V$.

It turns out that our $X$ is birational to $V$ under the double projection from a line contained in $X$. We will use this map to embed in $X$ some elliptic curves contained in $V$.

The divisor $S$ is a degree 5 del Pezzo surface, hence it is isomorphic to the blow up of $\mathbb{P}^2$ at 4 points $B_1, \ldots, B_4$. Further, we have $\omega^*_S \cong \mathcal{O}_S(1) \cong \mathcal{O}(3\ell - \sum b_i)$, where $\ell$ is the class of a line in $\mathbb{P}^2$ and $b_i$ is the exceptional divisor over the point $B_i$.

Recall that, by [Iskovskikh and Prokhorov 1999], the threefold $V$ contains a rational normal curve $C_0$ of degree 5 (restrict to $S$ and take the divisor $2\ell - b_1$). Furthermore, $C_0$ has exactly 3 chords $T_1, T_2, T_3$. Indeed, any chord of $C_0$ is contained in $S$, and the only lines in $S$ meeting $C_0$ in two points are of the form $\ell - b_i - b_j$ for $1 < i < j$.

Denote by $H$ the divisor associated to $\mathcal{O}_V(1)$. The linear system $3H - 2C_0$ defines a birational map $\varphi : V \dashrightarrow X$. Let $\tilde{X}$ be the variety obtained by blowing up $V$ along $C_0$ and then along the proper preimages of $T_1, T_2, T_3$. Denote by $\psi_1$ the contraction to $V$. There also exists a contraction $\psi_2 : \tilde{X} \to X$, and we have $\varphi \circ \psi_1 = \psi_2$.

**Definition 2.13.** Fix a general hyperplane section $S$ of $V$ and an isomorphism $S \to \text{Bl}_{B_1, \ldots, B_4}(\mathbb{P}^2)$ (there is a finite number of such isomorphisms). Let $b_i$ be the exceptional divisors of $S$ over $B_i$. For a given rational normal curve $C_0 \subset V$ with chords $\{T_1, T_2, T_3\}$, let $\{e_1, \ldots, e_5\} = S \cap C_0$ and $f_i = S \cap T_i$. On $S$, define $\mathcal{L} = 9\ell - 3\sum b_i - 2\sum e_j - \sum f_k$. We have $\varphi|_S = \varphi|_{\mathcal{L}}$, where $\varphi|_{\mathcal{L}}$ is the map associated to the linear system $|\mathcal{L}|$.

**Resolution of the diagonal.** We recall here the resolution of the diagonal on $X$ and the induced Beilinson theorem. We refer to [Gorodentsev 1990; Rudakov 1990; Drezet 1986] for the general setup on exceptional collections and mutations.

Define the collection $(G_3, \ldots, G_0) = (E, U, Q^*, \mathcal{O})$. This collection is strongly exceptional, that is, $\text{Ext}^p(G_j, G_i) = 0$ if $p > 0$ or $i > j$, as proved in [Kuznetsov 1996]. Further, we define the collection $(G_3, \ldots, G_0) = (E, K, U, \mathcal{O})$. The following lemma, proved in [Faenzi 2006, Theorem 7.2], states that these two collections fit together to give a resolution of $\mathcal{O}_\Delta$ over $X \times X$.

**Lemma 2.14.** For a general $X$, there exists a resolution of $\mathcal{O}_\Delta$ on $X \times X$ of the form:

$$0 \to G_3 \boxtimes G^3 \to \cdots \to G_0 \boxtimes G^0 \to \mathcal{O}_\Delta \to 0.$$
Any coherent sheaf $\mathcal{F}$ on $X$ is functorially isomorphic to the cohomology of a complex $\mathcal{C}^k$ whose terms are

$$\mathcal{C}^k = \bigoplus_{i-j=k} H^i(\mathcal{F} \otimes G^j) \otimes G_j.$$

Alternatively, $\mathcal{F}$ is functorially isomorphic to the cohomology of a complex $\mathcal{D}^k$ whose terms are

$$\mathcal{D}^k = \bigoplus_{i-j=k} H^i(\mathcal{F} \otimes G^j) \otimes G^j.$$

The following Castelnuovo–Mumford regularity for the collection $(G_3, \ldots, G_0)$ is a consequence of Lemma 2.14; see [Faenzi 2006, Corollary 7.4].

**Corollary 2.15.** Let $\mathcal{F}$ be a coherent sheaf on $X$. If $H^p(G_p \otimes \mathcal{F}) = 0$ for $p > 0$, then $\mathcal{F}$ is globally-generated.

**Vector bundles with no intermediate cohomology.** Recall from the introduction that a rank-2 vector bundle $\mathcal{F}$ with $c_1(\mathcal{F}) = c_1$ and $c_2(\mathcal{F}) = c_2$ is denoted by $\mathcal{F}_{c_1,c_2}$. Similarly, a curve of genus $g$ and degree $d$ is denoted by $C^d_g$.

**Lemma 2.16** (Madonna). The only possible classes of indecomposable normalized rank-2 ACM vector bundles on $X$ are, up to isomorphism:

1. the unstable bundle $\mathcal{F}_{-1,1}$ associated to a line in $X$;
2. the semistable bundle $\mathcal{F}_{0,2}$ associated to a conic in $X$;
3. the stable bundle $\mathcal{F}_{-1,d}(1)$ associated to an elliptic curve $C^d_1$ contained in $X$, with $7 \leq d \leq 14$;
4. the stable bundle $\mathcal{F}_{0,4}(1)$ associated to a canonical curve $C^{26}_{14}$ in $X$;
5. the stable bundle $\mathcal{F}_{-1,15}(2)$ associated to a half-canonical curve $C^{59}_{60}$ contained in $X$.

In each case, the smallest $t \in \mathbb{Z}$ with $h^0(\mathcal{F}(t)) \neq 0$ is the one stated.

**Proof.** We refer to [Madonna 2002] for the full proof, with the exception of the condition $d \geq 7$ in (3), which we prove at the end of Section 4. We nonetheless sketch here the Madonna’s main argument.

Considering the first twist $\mathcal{F}_{c_1,c_2}$ of $\mathcal{F}$ by a nonzero global section $s$, one proves easily that $Z = \{s = 0\}$ is a connected curve of arithmetic genus $1 + 1/2(c_1c_2 - c_2)$ and degree $c_2$. Therefore $c_1 \geq 1 - 2/c_2 \geq -1$, so $\mathcal{F}$ is stable except for $c_1 = -1$ or $c_1 = 0$, which correspond, respectively, to Cases (1) and (2).

For $c_1 = 1$, we end up in Case (3) and, making use of (1), it is easy to check that $d \leq 14$. 

For \( c_1 > 1 \), we find \( h^p(\mathcal{F}_{c_1,c_2}(-1)) = 0 \) and \( h^p(\mathcal{F}_{c_1,c_2}(-2)) = 0 \) for any \( p \). Take the following polynomial equations in the variables \( c_1 \) and \( c_2 \):

\[
\begin{align*}
\chi(\mathcal{F}_{c_1,c_2}(-1)) &= 0, \\
\chi(\mathcal{F}_{c_1,c_2}(-2)) &= 0.
\end{align*}
\]

When \( c_1 > 1 \), we find Cases (4) and (5) as the only solutions. \( \square \)

3. Lines and conics

It is classically known that \( X \) contains a one-dimensional family of lines and a two-dimensional family of smooth conics; see [Iskovskikh and Prokhorov 1999, Propositions 4.2.2 and 4.2.5] and references therein. Denote a line in \( X \) by \( C_0^1 \) and a conic in \( X \) by \( C_0^2 \). We will just provide resolutions of the sheaf \( \mathcal{O}_{C_0^1}(-1) \) and of the bundle \( \mathcal{F}_{c_0} \) with respect to the collection \( (G_3, \ldots, G_0) \). This will give a straightforward description of the Hilbert schemes of lines and conics in \( X \).

**Lemma 3.1.** The sheaf \( \mathcal{O}_{C_0^1}(-1) \) admits the resolution

\[
0 \rightarrow E \rightarrow K \xrightarrow{\alpha_{C_0^1}} U \rightarrow \mathcal{O}_{C_0^1}(-1) \rightarrow 0.
\]

The map \( \alpha_{C_0^1} \in \text{Hom}(K,U) \cong B^* \) degenerates along a line \( C_0^1 \) if and only if it lies in the discriminant quartic curve \( \det \Psi^T \subset \mathbb{P}^2 = \mathbb{P}(B^*) \). In particular, the Hilbert scheme of lines in \( X \) is isomorphic to the curve \( \det \Psi^T \).

**Proof.** Clearly, we have \((G_j)_{C_0^1} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-j) \). Hence, \( h^1(G_j \otimes \mathcal{O}_{C_0^1}(-1)) = 1 \) for \( j = 1, 2, 3 \). By Lemma 2.14, the sheaf \( \mathcal{O}_{C_0^1}(-1) \) admits the resolution (10).

It is known from [Schreyer 2001, Theorem 6.1] that the Hilbert scheme of lines in \( X \) is isomorphic to the curve \( \det \Psi^T \). We nonetheless sketch here a simpler argument.

From (5) follows the isomorphism \( \text{Hom}(K,U) \cong B^* \). The application of the functor \( \text{Hom}(E,-) \) to a morphism \( \alpha : K \rightarrow U \) corresponds, under the morphism \( \Psi^T : B^* \rightarrow S^2 A^* \), to the linear map \( \alpha \mapsto \Psi^T(\alpha) \). That is, \( \alpha \) is taken by \( \text{Hom}(E,-) \) to a linear map \( \Psi^T(\alpha) : A \rightarrow A^* \). Since both \( \text{Hom}(E,K) \otimes E \rightarrow K \) and \( \text{Hom}(E,U) \otimes E \rightarrow U \) are epimorphisms, it follows that \( \text{Hom}(E,\alpha) \) is surjective if and only if \( \alpha \) is surjective. This fails to hold precisely when \( \alpha \) lies in the discriminant curve \( \det \Psi^T \), in which case there is a unique map \( E \rightarrow \ker \alpha \).

This map is an isomorphism. By a Hilbert polynomial computation, \( \text{coker} \alpha \) is isomorphic to \( \mathcal{O}_{C_0^1}(-1) \). \( \square \)

**Lemma 3.2 (Takeuchi).** Through any point in \( X \) there exists a finite number of conics contained in \( X \). The Hilbert scheme of conics in \( X \) is isomorphic to \( \mathbb{P}(B) \).
Proof. The first statement is proved in [Takeuchi 1989]. One may also consult [Iskovskikh and Prokhorov 1999, Lemma 4.2.6].

For any conic \( C^2_0 \) in \( X \), there exists an exact sequence

\[
(11) 0 \longrightarrow U \longrightarrow Q^* \longrightarrow J_{C^2_0, X} \longrightarrow 0.
\]

Any homomorphism \( U \rightarrow Q^* \) degenerates along a conic. Since \( \text{Hom}(U, Q^*) \simeq B \), the lemma is proved.

Corollary 3.3. The set of stable points in the moduli space \( M_X(2; 0, 2) \) is empty. The set of semistable points is isomorphic to \( \mathbb{P}^2 = \mathbb{P}(B) \). The bundle \( \mathcal{F}_{0,2} \) of Lemma 2.16, Case (2), admits the resolution

\[
0 \longrightarrow U \longrightarrow Q^* \oplus \mathcal{O} \longrightarrow \mathcal{F}_{0,2} \longrightarrow 0.
\]

Proof. Since the bundle \( \mathcal{F}_{0,2} \) admits a unique global section \( s \), and since \( s \) vanishes along a conic \( C^2_0 \), there exists an isomorphism between \( M_X(2; 0, 2) \) and \( \text{Hilb}_{t+1}(X) \cong \mathbb{P}^2 \), the Hilbert scheme of conics contained in \( X \). The bundle \( \mathcal{F}_{0,2} \) is strictly semistable for \( c_1(\mathcal{F}) = 0 \).

In this case, the exact sequence (1) reads

\[
(12) 0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{F}_{0,2} \longrightarrow J_{C^2_0, X} \longrightarrow 0.
\]

Since \( \text{Ext}^1(Q^*, \mathcal{O}) = 0 \), any morphism \( Q^* \rightarrow J_{C^2_0, X} \) lifts to a morphism \( Q^* \rightarrow \mathcal{F}_{0,2} \). Considering the map \( \mathcal{O} \rightarrow \mathcal{F}_{0,2} \) in the exact sequence (12) and lifting the projection \( Q^* \rightarrow J_{C^2_0, X} \) in the exact sequence (11), we obtain a surjective bundle map \( Q^* \oplus \mathcal{O} \rightarrow \mathcal{F}_{0,2} \) whose kernel is isomorphic to \( U \). This provides the desired resolution. □

4. Elliptic curves

In this section we prove the existence in \( X \) of elliptic curves with the properties required by Case (3) of Lemma 2.16. In particular, the degree of these curves varies from 7 to 14. The case \( 7 \leq d \leq 13 \) is considered in Proposition 4.1, while the case \( d = 14 \) is considered in Proposition 4.4. In the latter we also deal with the case \( d = 15 \), which we will need in Section 5.

Proposition 4.1. On the general variety \( X \), there exist smooth elliptic curves \( C^d_1 \) of any degree \( d \), for \( 7 \leq d \leq 13 \). The curve \( C^d_1 \) is contained in exactly \( 14 - d \) independent hyperplanes.

We will construct smooth elliptic curves in \( X \) by means of the birational map \( \varphi : V = V_5 \dashrightarrow X \) of page 207.

Lemma 4.2. Let \( S = S_5 \) be a fixed hyperplane section of \( V \), and fix the notations from page 207. The irreducible component \( \mathbb{H}_{5t+1} \) of the Hilbert scheme
Hilb\(_{5r+1}(V)\) containing smooth rational normal quintics in \(V\) has dimension 10 at a general \([C_0^5]\). There is a dominant map \(\xi : \text{Hilb}_{5r+1}(V) \to \text{Hilb}_5(\mathbb{P}^2)\) defined by \(\xi : [C_0^5] \mapsto e_1 + \cdots + e_5\).

**Proof.** Set \(C = C_0^5\). First, notice that by the Riemann–Roch formula we have \(\expdim(\mathcal{F}_{\text{Hilb}_{5r+1}(V), [C]})) = 10\), because \(\deg N_{C, V} = 10\) and so \(\chi(N_{C, V}) = 10\). Since \(C \subset S\), we have the exact sequence of normal bundles

\[
0 \to N_{C, S} \to N_{C, V} \to (N_{S, V})_{|C} \to 0.
\]

Now, by computing \((2\ell - b_1)^2 = 3\), after the identification \(C \simeq \mathbb{P}^1\) we get \(N_{C, S} \simeq \mathcal{O}_{\mathbb{P}^1}(3)\) and obtain an exact sequence

\[
0 \to \mathcal{O}_{\mathbb{P}^1}(3) \to N_{C, V} \to \mathcal{O}_{\mathbb{P}^1}(5) \to 0.
\]

Therefore \(h^0(N_{C, V}) = \chi(N_{C, V}) = 10\), so \(\text{Hilb}_{5r+1}(V)\) is smooth and 10-dimensional.

Let \(\mathbb{P}(\mathcal{H}^0(V, \mathcal{O}_V(1))) = \mathbb{P}^6\). Notice that, once we fix the hyperplane section \(S\), for any curve \(C\) the intersection \(C \cap S\) gives 5 points spanning \(\mathbb{P}^4 \subset \mathbb{P}^6\). Conversely, given any \(\mathbb{P}^4 \subset \mathbb{P}^6\), there is a curve \(C\) such that the spaces \((C)\) and \((S)\) span \(\mathbb{P}^6\).

Fixing \(S\) thus provides a birational map \(\text{Hilb}_{5r+1}(V) \dashrightarrow \mathcal{G}(\mathbb{P}^4, \mathbb{P}^6)\).

Since \(\dim \text{Hilb}_{5r+1}(V) = \dim \text{Hilb}_5(\mathbb{P}^2) = 10\), we have to prove that the map \(\xi\) is generically finite. So we fix \(e = (e_1, \ldots, e_5)\) and consider the space \(\mathbb{P}_e^4 = \langle e_1, \ldots, e_5 \rangle\). Varying a hyperplane section \(S'\) of \(V\) in the pencil of hyperplanes containing \(\mathbb{P}_e^4\), we obtain a ruled surface \(S'_e\) consisting of exceptional lines in \(S'\) of type \(e'_1\). The ruled surface \(S'_e\) is not a cone, for there are finitely many lines through any point in \(V\) (see [Iskovskikh and Prokhorov 1999, page 64] and [Furushima and Nakayama 1989]). Thus, its dual variety is a hypersurface in \(\mathbb{P}^6\).

Given a curve \(C \subset S'\), write \(C = 2\ell - b'_1\). We have \(\xi(C) = e_1 + \cdots + e_5\) if and only if there is a hyperplane section \(S' = \mathbb{P}^5 \cap V\) with \(\mathbb{P}^5 \supset \mathbb{P}^4_e\) and such that \(\mathbb{P}^5\) contains the curve of class \(2\ell - b_1\). This happens if and only if the hyperplane \(\mathbb{P}^5\) is tangent to the ruled surface \(S'_1\). Being the dual variety of the hypersurface \(S'_1\), it intersects the general pencil of \(\mathbb{P}^5\)'s containing \(\mathbb{P}^4_b\) in a finite set of points. \(\square\)

**Lemma 4.3.** Let \(S\) be a fixed hyperplane section and fix notation as in Definition 2.13. Define the linear systems

\[
\begin{align*}
L_9 &= 4\ell - 2b_1 - 2b_2 - 2b_3 - b_4 - e_1 - e_2 - e_3 - \sum f_j, \\
L_{10} &= 5\ell - 2\sum b_j - 2e_1 - e_2 - e_3 - \sum f_j, \\
L_{11} &= 4\ell - 2b_1 - 2b_2 - b_3 - b_4 - e_1 - e_2 - \sum f_j, \\
L_{12} &= 5\ell - 2\sum b_j - 2e_1 - e_2 - \sum f_j, \\
L_{13} &= 4\ell - 2b_1 - 2b_2 - b_3 - b_4 - e_1 - \sum f_j.
\end{align*}
\]
Each \( L_d \) has positive dimension and contains a smooth element \( \tilde{C}_1^d \). The curve \( \varphi(\tilde{C}_1^d) \) is a smooth elliptic curve in \( X \) of degree \( d \), contained in precisely \( 14 - d \) independent hyperplanes.

**Proof.** The linear systems \( L_j \) have positive dimension, as can be seen by counting parameters. Indeed, it suffices to compute the expected dimension of the linear system of curves in \( \mathbb{P}^2 \) passing through assigned points and with prescribed nodes.

For odd (even) \( d \), the system \( L_d \) contains a smooth element \( \tilde{C}_1^d \) if and only if there exists an irreducible plane quartic with nodes only at \( B_1 \) and \( B_2 \) (respectively, an irreducible plane quintic with nodes only at \( B_1, \ldots, B_6 \) and at the point in \( \mathbb{P}^2 \) corresponding to \( e_1 \)). It suffices to project an elliptic normal quartic (quintic) in \( \mathbb{P}^3 (\mathbb{P}^4) \) from a general point (line) to obtain such a curve.

The degree of \( \varphi(\tilde{C}_1^d) \) is easily computed to be \( d = L_d \cdot L \), where \( L \) is the linear system of Definition 2.13.

Since any elliptic curve of degree \( d \leq 13 \) is contained in a hyperplane section \( S \) of \( X \), we have that

\[
h^0(L_{d^c}^{c_1} \cdot X(1)) = h^0(L_{d^c}^{c_1} \cdot S_{22}(1)) + 1.
\]

Using the map \( \varphi \) and the fixed isomorphism \( S \rightarrow \text{Bl}_{B_1, \ldots, B_4}(\mathbb{P}^2) \), we get

\[
h^0(L_{d^c}^{c_1} \cdot S_{22}(1)) = h^0(\mathbb{P}^2, L - L_d).
\]

It is then enough to compute the dimension of the following linear systems on \( \mathbb{P}^2 \):

\[
\begin{align*}
(13) \quad & L - L_0 = 5\ell - b_1 - b_2 - 2b_3 - 2b_4 - e_1 - e_2 - e_3 - 2e_4 - 2e_5, \\
(14) \quad & L - L_{10} = 4\ell - \sum b_i - e_2 - e_3 - 2e_4 - 2e_5, \\
(15) \quad & L - L_{11} = 5\ell - b_1 - b_2 - 2b_3 - 2b_4 - e_1 - e_2 - e_3 - 2e_4 - 2e_5, \\
(16) \quad & L - L_{12} = 4\ell - \sum b_i - e_2 - e_3 - 2e_4 - 2e_5, \\
(17) \quad & L - L_{13} = 5\ell - b_1 - b_2 - 2b_3 - 2b_4 - e_1 - e_2 - e_3 - 2e_4 - 2e_5.
\end{align*}
\]

Using Lemma 4.2, we can compute the dimension of these linear systems by choosing the points corresponding to the \( e_i \)'s in a Zariski open set of \( \text{Hilb}_5(\mathbb{P}^2) \). Notice that \( \expdim(L - L_d) = 13 - d \), so we need only check that \( \expdim(L - L_d) = \dim(L - L_d) \). This we can do using Cremona transformations on \( \mathbb{P}^2 \).

For Case (13), consider the Cremona transformation \( \gamma_9 \) associated to the linear system \( 2\ell - b_3 - b_4 - e_4 \). Any curve in \( L - L_0 \) touches a conic through \( b_3 - b_4 - e_4 \) in 4 points. Further, any curve in \( L - L_9 \) touches the line \( \langle B_3, B_4 \rangle \) (or, respectively, \( \langle B_4, e_4 \rangle \) or \( \langle B_3, e_4 \rangle \)) in a single further point \( e'_4 \) (or, respectively, \( b'_3 \) or \( b'_4 \)). Therefore, the linear system \( L - L_9 \) is mapped under \( \gamma_9 \) to \( 4\ell - b_1 - b_2 - b'_3 - b'_4 - e_1 - e_2 - e_3 - e'_4 - 2e_5 \). By Lemma 4.2, the points \( e_1, \ldots, e_5 \) lie in general position. The points \( b_i \) can be chosen generic, for we can define \( S \) to...
be the blow-up of $\mathbb{P}^2$ at a general 4-tuple of points. Since we now have a linear system of plane quartics with only one node and passing through 8 general points, we conclude that $h^0(\mathbb{P}^2, L - L_9) = 4$.

In Case (15), define $\gamma_{11}$ as the Cremona transformation associated to $2\ell - b_3 - b_4 - e_3$, sending $L - L_9$ to $4\ell - b_1 - b_2 - b_3' - b_4' - e_1 - e_2' - 2e_4 - 2e_5$. Take $\gamma_{11}' = \gamma_{2\ell - e_1 - e_2 - b_1}$. Then $\gamma_{11}' \circ \gamma_{11}$ sends $L - L_9$ to $3\ell - b_2 - b_3' - b_4' - e_1 - e_2 - e_3' - e_4'' - e_5''$. And 8 general points impose 8 linearly independent conditions on the 10-dimensional space of plane cubics.

In Case (17), put $\gamma_{13} = \gamma_{2\ell - b_1 - b_3 - e_2}$ and $\gamma_{13}' = \gamma_{2\ell - e_1 - e_2 - e_3}$. The linear system $L - L_{13}$ is mapped by $\gamma_{13}' \circ \gamma_{13}$ to $2\ell - b_2 - b_3 - b_4' - e_1 - e_2'$. Since there is no conic through 6 general points, we are done.

In Case (14), set $\gamma_{10} = \gamma_{2\ell - e_1 - e_2 - e_3}$. The lines $\langle e_3, e_4 \rangle$ and $\langle e_3, e_5 \rangle$ give rise to two extra points $e_1'$ and $e_5'$, so we compute $h^0(3\ell - \sum b_i - e_1 - e_4' - e_5') = 3$.

In Case (16), put $\gamma_{12} = \gamma_{2\ell - e_1 - e_3 - e_4}$. Here we have no extra points, and the statement follows since $h^0(2\ell - \sum b_i - e_2) = 1$. \hfill $\square$

**Proof of Proposition 4.1.** The curve $C_7^1$ exists according to [Kuznetsov 1996; Faenzi 2006]. In fact, it is just the zero locus of a general global section $s$ from $H^0(F^*) \simeq k^8$.

For $C_8^1$, consider a homomorphism $\alpha : K \to U$, with $\alpha \in \text{Hom}(K, U) \simeq B^*$. This morphism is surjective whenever $\alpha$ lies outside the discriminant curve $\det \Psi^T \subset \mathbb{P}(B^*)$ (see Lemma 3.1), so for a general $\alpha$ we get a rank-2 locally free sheaf $F_8 = \ker \alpha$. It follows easily from Lemma 2.12 that $c_1(F_8) = -1$ and $c_2(F_8) = 8$. Taking global sections of $F_8^*$ and using the identifications of Lemma 2.7, we get

$$H^0(F_8^*) \simeq \ker (\alpha : S^4 B/F \to S^3 B/F(B^*)).$$

For a general $\alpha$, this map is surjective, so $h^0(F_8^*) = 7$. Further, $F_8^*$ is globally-generated since $K^*$ is. Therefore, a general section of $F_8^*$ vanishes along the required curve $C_8^1$.

Finally, for $9 \leq d \leq 13$ the statement follows from Lemma 4.3. \hfill $\square$

**Proposition 4.4.** On the general variety $X$, there exists a smooth elliptic curve $C_1^d$ of degree $d$ for $d = 14$ or $d = 15$. In both cases, $C_1^d$ is nondegenerate.

**Proof:** It is well-known that there exist smooth elliptic normal curves of degree 7 in $V$. Nonetheless, we sketch a quick proof.

Denote by $U_V$ and $Q_V$ the universal rank-2 subbundle and the universal rank-3 quotient bundle on $\mathbb{G}(k^2, k^5)$, restricted to $V$. One proves that, for a general map $\alpha : U_V^{\oplus 2} \to (Q_V^*)^{\oplus 2}$, the sheaf $\text{coker} \alpha \otimes \mathcal{O}_V(1)$ is a globally-generated rank-2 bundle on $V$, whose general section vanishes on the required curve $D_7$.

Take now a hyperplane section $S$ and denote by $d_1, \ldots, d_t$ the intersection points of $D_7$ with $S$. Recall the notation from Definition 2.13. Choose a smooth curve
Consider \( d \) with 4.1 (1) 4.4 (1) \( d \) with Proposition 4.5 and \( c_2 \) over \( X \) associated to the elliptic curve \( C \). Recall that the vector bundles \( U_1981 \) of degree 15, for it intersects \( C^5_0 \) at 3 points with normal crossings. This curve is nondegenerate, since \( D_7 \) is nondegenerate as well.

Moving the hyperplane section \( S \) in \( \mathbb{P}^6 \), we can suppose that the point \( d_4 \) coincides with the point \( f_1 \). Taking again \( C^5_0 \in |2\ell - b_1 - d_1 - d_2 - d_3| \), we have that \( D_7 \) is now mapped by \( \varphi_{|\mathcal{X}|} \) to a nondegenerate smooth elliptic curve of degree 14; indeed, it intersects \( C^5_0 \) at 3 points and \( T_1 \) at 1 point, with normal crossings. \( \square \)

**Proposition 4.5.** Consider \( d \) with \( 7 \leq d \leq 15 \) and let \( F_d \) be the rank-2 vector bundle over \( X \) associated to the elliptic curve \( C^d_1 \) constructed above. We have \( c_1(F_d) = -1 \) and \( c_2(F_d) = d \). Furthermore, \( F_d \) is stable for any \( d \), is ACM when \( 7 \leq d \leq 14 \), and has \( h^0(F^*_d) = h^1(F^*_d) = 1 \) when \( d = 15 \).

**Proof.** Set \( C = C^d_1 \). The numerical invariants of the bundle \( F_d \) are obvious, while its stability follows at once from Hoppe's criterion.

By Serre duality and (1), one has \( h^2(F^*_d) = h^1(F_d(-1)) = h^1(F^*_d(-2)) = 0 \).

Taking twisted sections in the sequence (1), we get that \( F_d \) is ACM if and only if \( h^1(F_d(1)) = 0 \), that is, if and only if \( h^1(J_{C,X}(1)) = 0 \). Indeed, in this case the map \( H^0(\mathcal{O}_X(1)) \to H^0(\mathcal{O}_C(1)) \) is surjective. This implies that \( H^0(\mathcal{O}_X(t)) \to H^0(\mathcal{O}_C(t)) \) is surjective for all \( t \geq 1 \), so \( h^1(J_{C,X}(t)) = 0 \) for \( t \geq 1 \). After using (1), we get \( h^1(F_d(t)) = 0 \) for \( t \geq 1 \). For \( t \leq 0 \) this trivially holds as well, so, by Serre duality, \( F_d \) is ACM.

This happens precisely when \( h^0(J_{C,X}(1)) = 14 - d \), so the conclusion follows from Propositions 4.1 and 4.4. \( \square \)

**Theorem 4.6.** For \( d \) with \( 8 \leq d \leq 15 \), the bundle \( F_d \) of Proposition 4.5 is isomorphic to the cohomology of a monad

\[
E^{\oplus d-8} \xrightarrow{\beta_d} K^{\oplus d-7} \xrightarrow{\alpha_d} U^{\oplus d-7}.
\]

For \( d = 7 \), the bundle \( F_7 \) is isomorphic to \( E \).

**Proof.** From Hirzebruch–Riemann–Roch we get the equalities

\[
\begin{align*}
\chi(Q^* \otimes F_d) &= d - 7, \\
\chi(U \otimes F_d) &= d - 7, \\
\chi(E \otimes F_d) &= d - 8.
\end{align*}
\]

Recall that the vector bundles \( U, Q^*, E \) and \( F_d \) are stable. Hence, by [Maruyama 1981, Theorem 1.14], any tensor product between them is also stable. This implies at once the vanishings

\[
\begin{align*}
h^0(Q^* \otimes F_d) &= 0, \\
h^0(U \otimes F_d) &= 0, \\
h^0(E \otimes F_d) &= 0.
\end{align*}
\]
Serre duality also yields

\begin{align}
(19) & \quad h^3(Q^* \otimes F_d) = h^0(Q \otimes F_d) = 0 \quad \text{because } \mu(Q \otimes F_d) = -1/4, \\
(20) & \quad h^3(U \otimes F_d) = h^0(U^* \otimes F_d) = 0 \quad \text{because } \mu(U^* \otimes F_d) = -1/6, \\
(21) & \quad h^3(E \otimes F_d) = h^0(E^* \otimes F_d) = 0 \quad \text{because } c_2(E) \neq c_2(F_d).
\end{align}

Here, (21) follows since \( \mu(E) = \mu(F_d) = -1/2 \), but \( c_2(E) = 7 \neq d = c_2(F_d) \), so \( \text{Hom}(E, F_d) = 0 \).

Consider the tensor product of the bundle \( F_d \) by the sequences (6), (9), and the dual of sequence (6). Since \( h^0(F_d) = 0 \) and \( h^1(F_d) = 0 \), we have

\[
\begin{align*}
h^1(Q^* \otimes F_d) &= h^0(U^* \otimes F_d) = 0 \quad \text{by (20)}, \\
h^1(U \otimes F_d) &= h^0(Q \otimes F_d) = 0 \quad \text{by (19)}, \\
h^1(E \otimes F_d) &= h^0((E')^* \otimes F_d).
\end{align*}
\]

The group \( H^0((E')^* \otimes F_d) \) vanishes as well, because \( E' \) is a stable bundle as well, and we have \( \mu((E')^* \otimes F_d) = -1/3 \). Summing up:

\[
h^2(Q^* \otimes F_d) = d - 7, \quad h^2(U \otimes F_d) = d - 7, \quad h^2(E \otimes F_d) = d - 8.
\]

This implies that \( F_d \) is isomorphic to the cohomology of a monad of form (18). Clearly, for \( d = 7 \) the above argument implies \( E \simeq F_7 \).

**Theorem 4.7.** Consider \( d \) with \( 7 \leq d \leq 15 \), and let \( X \) be general. Take the Hilbert scheme \( \mathcal{H}_{d,1}(X) \) of curves in \( X \) of degree \( d \) and arithmetic genus 1. At generic points, \( \mathcal{H}_{d,1}(X) \) is smooth of dimension \( d \) and the moduli space \( M_X(2; -1, d) \) is smooth of dimension \( 2d - 14 \).

**Proof:** Let \( Z = C^d_1 \) be a curve of degree \( d \) and arithmetic genus 1, contained in \( X \). Consider the vector bundle \( F_d \) associated to \( Z \).

Tensoring by \( F_d \) both the exact sequence (1) and the exact sequence defining \( Z \subset X \), we get, after using the isomorphism (2), the exact sequences

\[
\begin{align}
(22) & \quad 0 \rightarrow F_d \rightarrow \mathcal{E}nd(F_d) \rightarrow F_d^* \otimes J_{Z,X} \rightarrow 0, \\
(23) & \quad 0 \rightarrow F_d^* \otimes J_{Z,X} \rightarrow F_d^* \rightarrow N_{Z,X} \rightarrow 0.
\end{align}
\]

Taking global sections, we get \( h^2(X, \mathcal{E}nd(F_d)) = h^1(Z, N_{Z,X}) \). This means that \( M_X(2; -1, d) \) is unobstructed at \([F_d]\) if and only if \( \mathcal{H}_{d,1}(X) \) is unobstructed at \([Z]\).

Consider now the monad (18) given by **Theorem 4.6.** Denote by \( W^1_d \) the vector space \( H^2(Q^* \otimes F_d) \simeq k^{d-7} \), and by \( W^2_d \) the space \( H^2(U \otimes F_d) \simeq k^{d-7} \). An element \((m, n)\) of the group \( \text{SL}(W^1_d) \times \text{SL}(W^2_d) \) acts on \( \mathbb{P}(\text{Hom}(K, U) \otimes \text{Hom}(W^1_d, W^2_d)) \) by taking \( \alpha_d \) to \( n \circ \alpha_d \circ m^{-1} \). For a general \( \alpha_d \), this action is free. Taking now the
functor $\text{Hom}(E, -)$, we get a morphism:

$$
\text{Hom}(K, U) \otimes \text{Hom}(W_d^1, W_d^2) \rightarrow A^* \otimes A^* \otimes \text{Hom}(W_d^1, W_d^2).
$$

Recall from (5) that $\text{Hom}(K, U) \simeq B^*$. Hence, any element $\alpha_d$ in the vector space $\text{Hom}(K, U) \otimes \text{Hom}(W_d^1, W_d^2)$ can be seen as a map $W_d^1 \rightarrow W_d^2$ with entries in $B^*$. The morphism (24) takes the map $\alpha_d$ to a $4(d-7) \times 4(d-7)$ square matrix $W_d^1 \otimes A \rightarrow W_d^2 \otimes A^*$, whose entries are given by $\Psi^\top \otimes \text{id}_{W_d^1} \otimes \text{id}_{W_d^2}$. Denote this matrix by $\Psi^\top(\alpha_d)$ (see Lemma 3.1).

Consider the sheaf $\ker(\alpha_d : W_d^1 \otimes K \rightarrow W_d^2 \otimes U)$. The above discussion implies that there exists an injective map $\beta_d : E^{d-8} \hookrightarrow \ker \alpha_d$ if and only if $\text{rk} \Psi^\top(\alpha_d) \leq 4(d-7) - (d-8) = 3d - 20$. Since $F_d$ is stable and $h^2(E \otimes F_d) = d - 8$, there is a unique $\beta_d$ up to isomorphisms.

Summing up, around $[F_d]$ there exists an open neighborhood of an irreducible component of the moduli space $M(X; -1, d)$, isomorphic to the set

$$
M(d) = \{ [\alpha_d] \in \mathbb{P}(B^* \otimes \text{Hom}(W_d^1, W_d^2)) \mid \text{rk} \Psi^\top(\alpha_d) = 3d - 20 \}
$$

$$
\big/ \text{SL}(d-7) \times \text{SL}(d-7).
$$

For a sufficiently general $\Psi^\top : B^* \rightarrow A^* \otimes A^*$, the variety $M(d)$ admits smooth points, indeed it is obtained by cutting the smooth subset of the variety of $(3d-20)$-secant $(3d-19)$-spaces to the Segre image of $\mathbb{P}^{4d-27} \times \mathbb{P}^{4d-27}$ by a sufficiently general linear space.

It is easy to check that the dimension of $M(d)$ at a smooth point $[\alpha_d']$ is $2d - 14$. At the bundle $[F_d']$ corresponding to $[\alpha_d']$, the dimension of $M_X(2; -1, d)$ is also $2d - 14$. Thus, taking a section of the general bundle $F_d'$, we obtain a curve $(Z')$ with $h^1(N_{(Z', X)}) = 0$, so $h^0(N_{(Z', X)}) = d$. Therefore, the Hilbert scheme $\mathcal{H}_{d, 1}(X)$ is $d$-dimensional and smooth at $[(Z')]$.

End of the proof of Lemma 2.16. Consider a general hyperplane section $S_{22}$ of $X$. It is a K3 surface of Picard number $\rho(S_{22}) = 1$. Take $F_d$ as defined in Proposition 4.5. Restricting $F_d$ to $S_{22}$, we get a stable rank-2 vector bundle on $S_{22}$. The moduli space $M_{S_{22}}(2; -1, d)$ is then smooth and projective, of dimension $-\chi(\text{End}(S_{22}, F_d)) = 2$. It is immediate to check that $\dim M_{S_{22}}(2; -1, d) = 4d - 28$. Hence $d \geq 7$.

5. Canonical and half-canonical curves

We now prove the existence of the bundles from Cases (4) and (5) of Lemma 2.16. We deal with the latter case first.

**Half-canonical curves.** We prove the existence of a smooth half-canonical curve $C_{60}^{59}$ by a deformation argument.
Lemma 5.1. There exists a smooth curve $Z = C_{60}^{59}$ in $X$ of degree 59 and genus 60, given as the zero locus of a section of an ACM vector bundle $\mathcal{F}_{-1,15}(2)$. We have $\omega_Z \cong \mathcal{O}_X(2)|_Z$. The ACM bundle $\mathcal{F}_{-1,15}$ specializes to the non-ACM bundle $F_{15}$.

Proof. Recall from Proposition 4.4 that there exists an elliptic curve $C = C_1^{15}$ such that $h^1(J_{C,X}) = 1$ and $C$ is not contained in any hyperplane. According to Proposition 4.5, the vector bundle $F_{15}^*$ has a unique section vanishing along $C$.

By Theorem 4.6, the moduli space $M_X(2; -1, 15)$ is smooth and 16-dimensional at a general $[F_{15}]$. Consider the irreducible component of $M_X(2; -1, 15)$ that contains $[F_{15}]$ and take an open neighborhood of $[F_{15}]$ contained in this component. Pick a point $[F'_{15}]$ belonging to this neighborhood and represented by a stable bundle $F'_{15}$ not isomorphic to $F_{15}$.

Suppose $F'_{15}(1)$ has a nontrivial global section $s$. Recall that $h^0(F_{15}) = 0$ by stability. The zero locus of $s$ would be a curve $C'$ of degree 15 and arithmetic genus 1. Therefore, $s$ would give a point $[C']$ in $\mathcal{H}_{15,1}(X)$. The point $[C']$ could not coincide with $[C]$, for otherwise $J_{C',X} \cong J_{C,X}$ would yield $F'_{15} \cong F_{15}$.

Since $\mathcal{H}_{15,1}(X)$ is smooth of dimension 15 at $[C]$, the above discussion proves that the map $\tau : \mathcal{H}_{15,1}(X) \to M_X(2; -1, 15)$ is an open embedding at $[C]$, and that its image is the codimension-1 locus $\{F'_{15} \in M_X(2; -1, 15) \mid h^0(F'_{15}(1)) \neq 0\}$. Thus, for a general $[F'_{15}]$ we must have $h^0(F'_{15}(1)) = 0$.

Now, since $\chi(F_{15}(1)) = 0$, we also get $h^1(F'_{15}(1)) = 0$. We set $\mathcal{F}_{-1,15} = F'_{15}$, and then $\mathcal{F}_{-1,15}$ is ACM. Finally, by Castelnuovo–Mumford regularity, $\mathcal{F}_{-1,15}(2)$ is globally generated, so a general section vanishes along a smooth curve $Z$ with the required invariants.

Remark 5.2. Any ACM stable bundle of type $\mathcal{F}_{-1,15}$ is the cohomology of a monad of type (18) with $d = 15$. Indeed, it suffices to apply the proof of Theorem 4.6 to $\mathcal{F}_{-1,15}$.

Canonical curves. Here we will prove the existence of a smooth canonical curve in $X$ by exhibiting the bundle $\mathcal{F}_{0,4}$ of Lemma 2.16.

Lemma 5.3. Given a general homomorphism $\alpha : U^{\oplus 2} \to (Q^*)^{\oplus 2}$, the sheaf $\text{coker} \alpha$ is a vector bundle of type $\mathcal{F}_{0,4}$.

Proof. Let $W_1$ and $W_2$ be 2-dimensional vector spaces such that the domain and codomain of $\alpha$ are $W_1 \otimes U$ and $W_2 \otimes Q^*$. Let $p_1 : k \to W_1$ be an element of $\mathbb{P}(W_1)$ and $p_2 : W_2 \to k$ an element of $\mathbb{P}(W_2)$. To the pair $(p_1, p_2)$ we associate a map $U \to Q^*$ via the morphism

$$\eta_\alpha : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2 \cong \mathbb{P}(B), \quad (p_1, p_2) \mapsto (p_2 \otimes \text{id}_{Q^*}) \circ \alpha \circ (p_1 \otimes \text{id}_U).$$

For a general $\alpha$, the map $\eta_\alpha$ is a 2 : 1 cover. Suppose now that $\alpha$ is not injective, as a bundle map, at some given point $x$ of $X$. Then there exists $p_1 : k \to W_1$ such that,
for any $p_2 : W_2 \to k$, the map $\eta_\alpha(p_1, p_2)$ is zero over $x$. Equivalently, $x$ lies in the conic whose ideal is $\text{coker } \eta_\alpha(p_1, p_2)$. Since $\eta_\alpha$ is a finite map, this means that $x$ lies in the pencil of conics parameterized by $p_2 \in \mathbb{P}(W_2)$, thus contradicting Lemma 3.2. Therefore $\text{coker } \alpha$ is locally free and, by a straightforward computation, it has the required Chern classes.

From the exact sequence $0 \to U^{\oplus 2} \to (Q^*)^{\oplus 2} \to \mathcal{F}_{0,4} \to 0$, we see immediately that $h^0(\mathcal{F}_{0,4}) = 0$ and $h^1(\mathcal{F}_{0,4}(t)) = 0$ for any $t \in \mathbb{Z}$; indeed, $U$ and $Q^*$ are ACM bundles.

Therefore $\mathcal{F}_{0,4}$ is stable and ACM. Indeed, Serre duality gives $h^2(\mathcal{F}_{0,4}(t)) = h^1(\mathcal{F}_{0,4}(-1-t)) = 0$ for all $t \in \mathbb{Z}$. Finally, one can compute

\[ h^1(Q^* \otimes \mathcal{F}_{0,4}(1)) = 0, \quad h^2(U \otimes \mathcal{F}_{0,4}(1)) = 0, \quad h^3(E \otimes \mathcal{F}_{0,4}(1)) = 0. \]

By Corollary 2.15, we get that $\mathcal{F}_{0,4}(1)$ is globally generated, hence the zero locus of its general global section is the required canonical curve. □

Lemma 5.4. Any ACM stable vector bundle of type $\mathcal{F}_{0,4}$ is the cokernel of a map $\alpha : U^{\oplus 2} \to (Q^*)^{\oplus 2}$.

Proof. The argument is analogous to that of Theorem 4.6. We find $h^p(U \otimes \mathcal{F}_{0,4}) = 0$ for $p \neq 1$, $h^0(K \otimes \mathcal{F}_{0,4}) = 0$ for $p \neq 1$, and $h^p(E \otimes \mathcal{F}_{0,4}) = 0$ for all $p$. We conclude that $h^1(U \otimes \mathcal{F}_{0,4}) = -\chi(U \otimes \mathcal{F}_{0,4}) = 2$ and that $h^1(K \otimes \mathcal{F}_{0,4}) = -\chi(K \otimes \mathcal{F}_{0,4}) = 2$, so the statement follows from Lemma 2.14.

Remark 5.5. Summing up, we found that an open subset of a component of $M^X(2; 0, 4)$ is isomorphic to an open subset of the variety of Kronecker modules $\mathbb{P}(W_1^* \otimes W_2 \otimes B) / \text{SL}(W_1) \times \text{SL}(W_2)$, where $W_1$ and $W_2$ are 2-dimensional vector spaces. In particular, it is unirational and generically smooth of dimension 5.

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