A NOTE ON PROJECTIONS OF REAL ALGEBRAIC VARIETIES

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We prove that any regularly closed semialgebraic set of \( R^n \), where \( R \) is any real closed field and regularly closed means that it is the closure of its interior, is the projection under a finite map of an irreducible algebraic variety in some \( R^{n+k} \). We apply this result to show that any clopen subset of the space of orders of the field of rational functions \( K = R(X_1, \ldots, X_n) \) is the image of the space of orders of a finite extension of \( K \).

1. Introduction. Motzkin shows in [M] that every semialgebraic subset of \( R^n \), \( R \) an arbitrary real closed field, is the projection of an algebraic set of \( R^{n+1} \). However, this algebraic set is in general reducible, and we ask whether it can be found irreducible.

This turns out to be closely related to the following problem, proposed in [E-L-W]: let \( K = R( X_1, \ldots, X_n ) \), \( X_1, \ldots, X_n \) indeterminates, and let \( X_K \) be the space of orders of \( K \) with Harrison's topology. If \( E|K \) is an ordered extension of \( K \), let \( \varepsilon_{E|K} \) be the restriction map between the space of orders, \( \varepsilon_{E|K} : X_E \to X_K : P \mapsto P \cap K. \) Which clopen subsets of \( X_K \), that is, closed and open in Harrison's topology, are images of \( \varepsilon_{E|K} \) for suitable finite extension of \( K \)?

In this note we prove that every regularly closed semialgebraic subset \( S \subset R^n \) — \( S \) is the closure in the order topology of its inner points — is the projection of an irreducible algebraic set of \( R^{n+k} \) for some \( k \geq 1 \). Actually we prove more: the central locus of the algebraic set, i.e., the closure of its regular points, covers the whole semialgebraic \( S \). This allows us to prove that there exists an irreducible hypersurface in \( R^{n+1} \) whose central locus projects onto \( S \). As a consequence we prove that for every clopen subset \( Y \subset X_K \) there is a finite extension \( E \) of \( K \) such that \( \text{im}(\varepsilon_{E|K}) = Y \).

2. In what follows \( R \) will be a real closed field and \( \pi \) will always denote the canonical projection of some \( R^{n+k} \) onto the first \( n \) coordinates.

Let \( S \) be a semialgebraic closed subset of \( R^n \). Then \( S \) can be written in the form (cf. [C-C] [R]):

\[
S = \bigcup_{i=1}^{p} \{ x \in R^n : f_i(x) \geq 0, \ldots, f_{ir}(x) \geq 0 \}, \quad f_{ij} \in R[X_1, \ldots, X_n].
\]
Now, since if \( f = g \cdot h \) we have
\[
\{ f \geq 0 \} = \{ h \geq 0, g \geq 0 \} \cup \{ -h \geq 0, -g \geq 0 \},
\]
by decomposing each \( f_{i,j} \) in irreducible factors, we may assume that all of
the \( f_{i,j} \) are irreducible. Finally, by the distributive law, we write
\[
S = \bigcap_{(i_1,\ldots,i_p) \in \{1,\ldots,r\}^p} \left[ \{ f_{i_11} \geq 0 \} \cup \cdots \cup \{ f_{i_p1} \geq 0 \} \right].
\]

For the sake of simplicity, we order the set of \( p \)-tuples \( (i_1,\ldots,i_p) \) from 1 till \( m = r^p \). Thus we have
\[
(2.0.1) \quad S = S_1 \cap \cdots \cap S_m,
\]
where
\[
S_i = \{ f_{i1} \geq 0 \} \cup \cdots \cup \{ f_{ip} \geq 0 \}, \quad i = 1,\ldots,m,
\]
and \( f_{k,i} \) irreducible for all \( k = 1,\ldots,p; i = 1,\ldots,m \).

2.1. **Proposition.** Let \( f_1,\ldots,f_p \) be irreducible polynomials in \( R[X_1,\ldots,X_n] \). Then there exists an irreducible polynomial \( F(T, X_1,\ldots,X_n) \in R[X_1,\ldots,X_n, T] \) such that if \( V = \{ \bar{x} \in R^{n+1}; F(\bar{x}) = 0 \} \) then
\[
\pi(V) = \{ f_1 \geq 0 \} \cup \cdots \cup \{ f_p \geq 0 \}.
\]

2.2. **Remark.** In particular if \( \{ f_j > 0 \} \neq \emptyset \) for some \( j \), then \( \dim V = \dim S = n \) and therefore \( R[X_1,\ldots,X_n, T]/(F) \) is a real domain. Thus \( V \) is an irreducible hypersurface of \( R^{n+1} \) which projects onto \( S \).

**Proof of 2.1.** Set \( S = \{ f_1 \geq 0 \} \cup \cdots \cup \{ f_p \geq 0 \} \). The cases \( S = R^n \), \( S = \emptyset \) and \( p = 1 \) are trivial. So, we assume \( S \) proper and \( p \geq 2 \). Also, if for some \( f_i \) we have \( \{ f_i \geq 0 \} \subset \bigcup_{j \neq i} \{ f_j \geq 0 \} \), we just omit it, so that we may suppose the expression of \( S \) irredundant in this sense. To prove the proposition we shall exhibit an irreducible polynomial \( F(T, X_1,\ldots,X_n) \in R[X_1,\ldots,X_n, T] \) such that the set \( F = 0 \) projects onto \( S \). Let us say a single word about how this (rather messy) polynomial comes out. We first seek an irreducible hypersurface in \( R^{n+1} \) which projects over \( \{ X_1 \geq 0 \} \cup \cdots \cup \{ X_p \geq 0 \} \). The hypersurface defined by clearing denominators in
\[
X_p = \frac{T^2(T^2 - 2X_1)}{T^2 - X_1} + \cdots + \frac{T^2(T^2 - 2X_{p-1})}{T^2 - X_{p-1}}
\]
verifies this property. Thus, we substitute the $X_i$'s by the $f_i$'s and we check that we can modify a bit the equation above so that it keeps irreducible.

Precisely, consider the algebraic subset $V$ of $R^{n+1}$ defined by the polynomial $F(T, X_1, \ldots, X_n)$ obtained by clearing denominators in the equation

$$f_p = \frac{T^2(T^2 - \lambda_1 f_1)}{T^2 - \lambda_2 f_1} + \sum_{i=2}^{p-1} \frac{T^2(T^2 - 2f_i)}{T^2 - f_i}$$

where $\lambda_1, \lambda_2 \in R$, $0 < \lambda_2 < \lambda_1$. That is, if we set:

$$Q(T, X) = \prod_{i=2}^{p-1} (T^2 - f_i),$$

$$Q_i(T, X) = Q(T, X)/(T^2 - f_i) \quad (i = 2, \ldots, p - 1)$$

then

$$F(T, X) = Qf_p(T^2 - \lambda_2 f_1) - QT^2(T^2 - \lambda_1 f_1)$$

$$- (T^2 - \lambda_2 f_1) \sum_{i=2}^{p-1} T^2(T^2 - 2f_i)Q_i.$$  

We claim that $\pi(V) = S$. Indeed, let $a \in S$. If $f_i(a) = 0$ for some $i = 1, \ldots, p - 1$, then it is immediate that the point $(a, 0) \in V$. So we restrict ourselves to the case $f_i(a) \neq 0$ for all $i = 1, \ldots, p - 1$. Now notice that the graph of the functions (in the plane)

$$Y = \frac{T^2(T^2 - 2f_i(a))}{T^2 - f_i(a)} \quad (i = 2, \ldots, p - 1)$$

as well as

$$Y = \frac{T^2(T^2 - \lambda_1 f_1(a))}{T^2 - \lambda_2 f_1(a)} \quad (0 < \lambda_2 < \lambda_1)$$

look like Figure 1 if $f_i(a) < 0$ (resp. $f_i(a) < 0$) and like Figure 2 if $f_i(a) > 0$ (resp. $f_i(a) > 0$, where we have to change $\sqrt{2f_i(a)}$ and $\sqrt{f_i(a)}$ by $\sqrt{\lambda_1 f_1(a)}$ and $\sqrt{\lambda_2 f_1(a)}$).

Thus, the range of the function

$$(2.1.2) \quad Y = \frac{T^2(T^2 - \lambda_1 f_1(a))}{T^2 - \lambda_2 f_1(a)} + \sum_{i=2}^{p-1} \frac{T^2(T^2 - 2f_i(a))}{T^2 - f_i(a)}$$

is either the whole line $R$ if $f_i(a) > 0$ for some $i = 1, \ldots, p - 1$, or $Y \geq 0$ if $f_i(a) < 0$ for all $i = 1, \ldots, p - 1$. Since in this case we have $f_p(a) \geq 0$
(by the very definition of $S$), it is clear that for any $a \in S$ there exists $t \in \mathbb{R}$ such that $(t, f_p(a))$ verifies (2.1.2). Obviously this means that the point $(a, t) \in V$ and so $a \in \pi(V)$. This shows $S \subset \pi(V)$.

The converse is immediate, for, if $a \notin S$ then $f_i(a) < 0$ for all $i = 1, \ldots, p$. But, by the definition of $V$, $(a, t) \in V$ and $f_i(a) < 0, \ldots, f_{p-1}(a) < 0$, imply $f_p(a) \geq 0$, and so $a \notin \pi(V)$ if $a \notin S$.

Finally, the following Lemma 2.3 shows that there exist $\lambda_1, \lambda_2$, $0 < \lambda_2 < \lambda_1$, such that $F(T, X_1, \ldots, X_n)$ is irreducible, what concludes the proof of 2.1.
2.3. **Lemma.** Let $f_1, \ldots, f_p$, $p \geq 2$ be irreducible polynomials in $\mathbb{R}[X_1, \ldots, X_n]$, such that $S = \{ f_1 \geq 0 \} \cup \cdots \cup \{ f_p \geq 0 \}$ is irredundant (i.e. $\{ f_i \geq 0 \} \subset \bigcup_{j \neq i} \{ f_j \geq 0 \}$ for all $i$) and $S$ is neither $\mathbb{R}^n$ nor empty. Then there exist $\lambda_1, \lambda_2 \in \mathbb{R}, 0 < \lambda_2 < \lambda_1$, such that the polynomial $F(T, X)$ defined in (2.1.1) is irreducible.

**Proof.** The result is a consequence of Bertini's theorem\(^1\). To see this, we write $F(T, X)$ in the form

$$F(T, X) = P_0 + \lambda_1 P_1 + \lambda_2 P_2,$$

where

\[
P_0 = Qf_p T^2 - QT^4 - T^4 \sum_{i=2}^{p-1} (T^2 - 2f_i) Q_i,
\]

\[
P_1 = Qf_1 T^2,
\]

\[
P_2 = f_1 T^2 \sum_{i=2}^{p-1} (T^2 - 2f_i) Q_i - Qf_1 f_p.
\]

Now, if $C = R(\sqrt{-1})$, set

$$Z = \{ (x, t) \in C^{n+1} : P_0(x, t) = P_1(x, t) = P_2(x, t) = 0 \}$$

and consider $\phi: C^{n+1} \setminus Z \to \mathbf{P}_2(C)$ defined by

$$\phi(x_1, \ldots, x_n, t) = (P_0(x, t), P_1(x, t), P_2(x, t)).$$

Let $\Lambda$ be the set of points $(\lambda_1, \lambda_2) \in C^2$ such that $\{ P_0 + \lambda_1 P_1 + \lambda_2 P_2 = 0 \}$ is irreducible and non-singular (as a subvariety of $C^{n+1} \setminus Z$). Then Bertini's theorem (cf. [H], pag. 275) assures that $\Lambda$ contains a Zariski open subset of $C^2$ provided that

(a) $\dim(\text{im} \phi) = 2$.

Furthermore, if

(b) $P_0$, $P_1$ and $P_2$ are relatively prime, then $Z$ has codimension $\geq 2$, hence $\{ P_0 + \lambda_1 P_1 + \lambda_2 P_2 = 0 \}$ is irreducible in $C^{n+1}$.

Thus since open intervals of $R$ are Zariski-dense in $C$, the result follows at once if we prove (a) and (b). Let us begin with the second:

(b) Assume that $h(X, T)$ is an irreducible common factor of $P_0$, $P_1$ and $P_2$.

Then $h|P_1$ and so, we have $h = T, h = f_1$ or $h|Q$. Since $P_2(0, X) = (-1)^{p-1} \prod_{i=1}^{p} f_i \neq 0$, it follows that $T \nmid P_2$.

\(^1\)We want to thank Professor J. P. Serre who called our attention to Bertini's theorem in order to prove 2.3.
Now, suppose \( h = f_1 \). Since \( h | P_0 \), we have
\[
\begin{align*}
f_1 \left( Q_i' - T^2 Q - T^2 \sum_{i=2}^{p-1} (T^2 - 2f_i)Q_i \right).
\end{align*}
\]

In particular, setting \( T = 0, f_1 \left( (-1)^{p-2} \prod_{i=2}^p f_i \right) \), which implies, since \( f_1 \) is irreducible, that there exist \( a \in R \) and \( j \in \{2, \ldots, p\} \) such that \( f_1 = af_j \). But \( a > 0 \) means \( \{ f_1 \geq 0 \} = \{ f_j \geq 0 \} \), and \( S \) would not be irredundant, while \( a < 0 \) implies \( S = R^n \). Therefore \( h \neq f_1 \).

Finally, suppose \( h | Q \). Then, we have \( h = T^2 - f_j \) for some \( j = 2, \ldots, p - 1 \). Since \( h | P_0 \), we deduce
\[
\begin{align*}
h \left( \sum_{i=2}^{p-1} Q_i \cdot (T^2 - 2f_i) \right).
\end{align*}
\]

But \( h \) divides \( Q_i \) for all \( i \neq j \). Thus \( h | Q_j (T^2 - 2f_j) \) which is absurd. This ends the proof of (b).

(a) It is enough to check that there is no homogeneous polynomial \( H(Y_0, Y_1, Y_2) \in C[Y_0, Y_1, Y_2] - \{0\} \) such that \( H(P_0, P_1, P_2) = 0 \). Suppose the opposite and assume that \( H \) is of degree \( d \). Then
\[
\begin{align*}
H(Y_0, Y_1, Y_2) = \sum_{a+b+c=d} \alpha_{abc} Y_0^a Y_1^b Y_2^c.
\end{align*}
\]

We shall work on the lowest degree in \( T \) of the monomials \( P_0^a P_1^b P_2^c \). From (2.3.1) we get
\[
\begin{align*}
(2.3.3) \quad P_0^a P_1^b P_2^c = \left( \prod_{i=2}^{p-1} (-f_i) \right)^d \left( -1 \right)^c f_1^b + c f_a + c T^{2(a+b)} + T^{2(a+b)+1} G(X, T)
\end{align*}
\]

(where in the case \( p = 2 \) the first product is taken to be 1).

We will prove that \( \alpha_{abc} = 0 \) for all \( a, b, c \). Set \( h = a + b \). We work by induction on \( h \).

If \( h = 0 \), then \( a = b = 0 \) and we have to prove that \( \alpha_{0,0,d} = 0 \). But the independent term of \( H(P_0, P_1, P_2) \) is \( \alpha_{0,0,d} \cdot (\prod_{i=1}^p f_i)^d \). Then \( \alpha_{0,0,d} = 0 \). Suppose \( \alpha_{a' b' c'} = 0 \) whenever \( a' + b' < h \). Then
\[
\begin{align*}
H(P_0, P_1, P_2) = \sum_{a+b+c=d \atop a+b \geq h} \alpha_{abc} P_0^a P_1^b P_2^c = T^{2h} M(T, X).
\end{align*}
\]

Since we have seen that \( P_0^a P_1^b P_2^c = T^{2(a+b)} \cdot R(T, X) \), the term of degree \( 2h \) in \( H(P_0, P_1, P_2) \) comes from those \( a, b, c \) such that \( a + b = h \) and its
coefficient is, after (2.3.3),
\[ \sum_{a+b+c=d \atop a+b=h} \alpha_{abc} (-1)^d \left( \prod_{i=2}^{p-1} f_i \right)^d \left( -1 \right)^c f_1^{b+c} f_p^{a+c}. \]
Thus, we obtain
\[ \sum_{i=0}^{h} \alpha_{i,h-i,d-h} f_1^{d-i} f_p^{d-h+i} = 0, \]
which implies
\[ \sum_{i=0}^{h} \alpha_{i,h-i,d-h} (f_p/f_1)^i = 0. \]
But, if \( \alpha_{i,h-i,d-h} \neq 0 \) for some \( i \), this means that \( f_p/f_1 \) is algebraic over \( C \), hence \( f_p = \lambda f_1, \lambda \in C \). Moreover, since \( f_1, f_p \in R[X_1, \ldots, X_n] \), we know that \( \lambda \in R \). Repeating a foregoing argument, \( \lambda > 0 \) means \( \{ f_1 \geq 0 \} = \{ f_p \geq 0 \} \) and \( \lambda < 0 \) means \( S = R^n \). Since both cases have been eliminated it follows \( \alpha_{abc} = 0 \) whenever \( a + b = 0 \) and the proof of the lemma is complete.

3. The main result. From now on, given an algebraic set \( V \), \( V_c \) will denote the set of central points of \( V \), that is the closure of the regular points of \( V \). We start with:

3.1. Definition. A semialgebraic subset \( S \) of \( R^n \) is regularly closed if \( \overline{S} \) is the closure of its inner points.

We are now ready to prove the following:

3.2. Theorem. Let \( S \subset R^n \) be a closed semialgebraic set of dimension \( n \). There exists a positive integer \( m \) and an irreducible \( n \)-dimensional algebraic set \( V \subset R^{n+m} \) such that

1. \( \pi: V \to R^n \) is finite,
2. \( \dot{S} \subset \pi(V) \subset S. \)

Moreover, if \( S \) is regularly closed then \( \pi(V_c) = \pi(V) = S. \)

Proof. We may assume \( S \) written in the form (2.0.1), i.e.
\[ S = S_1 \cap \cdots \cap S_m, \quad \text{with} \quad S_i = \{ f_{1i} \geq 0 \} \cup \cdots \cup \{ f_{pi} \geq 0 \} \]
and $f_{ki} \in \mathbb{R}[X_{1}, \ldots, X_{n}]$ irreducible for every $(i, k) \in \{1, \ldots, m\} \times \{1, \ldots, p\}$. We will find $V \subset \mathbb{R}^{n+m}$. To do that we work by induction on $m$.

For $m = 1$, let $V \subset \mathbb{R}^{n+1}$ be the hypersurface $F(T, X) = 0$ of Proposition 2.1 if $p > 1$ and $T^2 - f_1 = 0$ if $p = 1$. Notice that the leading coefficient of $F(T, X)$ as polynomial in $T$ is $1 - p$ (see 2.1.1) and consequently $\pi: V \rightarrow \mathbb{R}^n$ is finite. Since $\pi(V) = S$ condition (2) is trivially satisfied.

Assume now that there exists an irreducible algebraic set $W' \subset \mathbb{R}^{n+m-1}$ of dimension $n$ verifying:

\begin{enumerate}
  \item $\pi: W' \rightarrow \mathbb{R}^n$ is finite
  \item $\hat{S}' \subset \pi(W') \subset S'$,
\end{enumerate}

where $S' = S_1 \cap \cdots \cap S_{m-1}$ (which has, of course, dimension $n$).

Let $\mathcal{I}(W') \subset \mathbb{R}[X_1, \ldots, X_n, T_1, \ldots, T_{m-1}]$ be the ideal of polynomials vanishing on $W'$ and consider the variety $W \subset \mathbb{R}^{n+m}$ defined by $\mathcal{I}(W') \cdot \mathbb{R}[X_1, \ldots, X_n, T_1, \ldots, T_{m-1}, T]$, where $T$ is a new variable. Obviously $W$ is irreducible and verifies the condition (ii) of (3.2.1).

Now let $F(T, X) = P_0 + \lambda_1 P_1 + \lambda_2 P_2 \in \mathbb{R}[X_1, \ldots, X_n, T]$ be the polynomial defined in (2.1.1) such that for any $\lambda_1, \lambda_2 \in \mathbb{R}$, $0 < \lambda_2 < \lambda_1$, the set $V'_{m}$ of zeros of $F$ (in $\mathbb{R}^{n+1}$) projects onto $S_m$. Let $V_{m}$ be the algebraic set of $\mathbb{R}^{n+m}$ defined by $F(T, X)$ considered as a polynomial in $\mathbb{R}[X_1, \ldots, X_n, T_1, \ldots, T_{m-1}, T]$. We have

$$\hat{S} \subset S_m \cap \hat{S}' \subset \pi(V_m \cap W) \subset S.$$ 

Set $Z = \{(x, t_1, \ldots, t_{m-1}, t) \in \mathbb{R}^{n+m}: P_0(x, t) = P_1(x, t) = P_2(x, t) = 0\}$. Since $P_0, P_1, P_2$ have no common factors (see proof of 2.3), it is $\text{codim}(\pi(Z)) \geq 1$. Let $H = \text{Sing}(W) \cup (Z \cap W)$. Then $\text{codim}(\pi(H)) \geq 1$, since by induction hypothesis $\text{dim} W' = n$. Let $C = \mathbb{R}(\sqrt{-1})$ be the algebraic closure of $\mathbb{R}$ and consider $\phi: W \setminus H \rightarrow \mathbb{P}_2(C)$ defined by $\phi(x, t_1, \ldots, t_{m-1}, t) = (P_0(x, t), P_1(x, t), P_2(x, t))$.

Since $W \setminus H$ is non-singular, Bertini's theorem applies assuring that the set of points $(\lambda_1, \lambda_2) \in C^2$ such that $$(W \setminus H) \cap \{(x, t_1, \ldots, t_{m-1}, t): P_0(x, t) + \lambda_1 P_1(x, t) + \lambda_2 P_2(x, t) = 0\}$$
is irreducible and non-singular (as a subvariety of $W \setminus H$) contains a Zariski open subset of $C^2$, provided that $\text{dim}(\text{im } \phi) = 2$.

Since $\pi(W)$ has non-empty interior, to prove that $\text{dim}(\text{im } \phi) = 2$ it is enough to show that $P_0$, $P_1$ and $P_2$ do not verify any homogeneous
polynomial. But this was shown in the proof of Lemma 2.3. Therefore there exist $\lambda_1, \lambda_2 \in R$, $0 < \lambda_2 < \lambda_1$, such that $V_m \cap (W \setminus H)$ is irreducible and nonsingular (in $W \setminus H$). Let $V$ be the irreducible component of $V_m \cap W$ which coincides with $V_m \cap (W \setminus H)$ on $W \setminus H$. Thus $\dim V \leq n$ and from $\text{codim}(\pi(H)) \geq 1$ it follows $\dim V = \dim(W \cap V_m) = n$.

Since the morphisms $\pi: W' \to R^n$ and $\pi: V_m \to R^n$ are finite so is $\pi: V_m \cap W \to R^n$, which implies the finiteness of $\pi: V \to R^n$. Whence $\pi(V)$ is closed in $R^n$. Obviously $\pi(V) \subset S$. Let us see that $\hat{S} \subset \pi(V)$. Let $x \in \hat{S}$ and let $U \subset \hat{S}$ be a strong open neighborhood of $x$. Since $\text{codim}(\pi(H)) \geq 1$, we deduce that $U \cap (\hat{S} \setminus \pi(H)) \neq \emptyset$. Take $y \in U \cap (\hat{S} \setminus \pi(H))$. Then $y \in \pi(W') \cap \pi(V_m')$. Pick $(t_1, \ldots, t_{n-1}) = t' \in R^{n-1}$ and $t \in R$ such that $(y, t') \in W'$ and $(y, t) \in V_m'$. We have $(y, t', t) \in (W \cap V_m') \setminus H \subset V$. Hence $U \cap \pi(V) \neq \emptyset$ and since $\pi(V)$ is closed we conclude that $\hat{S} \subset \pi(V)$, what proves the first part of the theorem.

Finally, assume that $S$ is regularly closed. First of all notice that, since $\pi$ is finite, $\pi(V_c)$ is a closed semialgebraic subset of $R^n$ (see [B], page 170). From $\hat{S} \subset \pi(V)$ it follows that $\hat{S} \subset \pi(V_c)$. For let $x \in \hat{S} \setminus \pi(V_c)$ and let $U \subset \hat{S}$ be a strong open neighborhood of $x$ such that $U \cap \pi(V_c) = \emptyset$. Thus $U \subset \pi(V \setminus V_c)$; but $\dim \pi(V \setminus V_c) < n = \dim U$, contradiction. Therefore we have $\hat{S} \subset \pi(V_c) \subset \pi(V) \subset S$. Taking into account once more that both $\pi(V_c)$ and $\pi(V)$ are closed and that $S$ is regularly closed, it follows at once by taking closures that $\pi(V_c) = \pi(V) = S$ and Theorem 3.1 is complete.

3.3. COROLLARY. Let $S \subset R^n$ be a regularly closed semialgebraic set. Then there exists an irreducible algebraic hypersurface $\tilde{V} \subset R^{n+1}$ such that $\pi(\tilde{V}_c) = S$.

Proof. Let $V \subset R^{n+m}$ be the irreducible algebraic variety constructed in 3.2, and let $C = R[X_1, \ldots, X_n, x_{n+1}, \ldots, x_{n+m}]$ be its coordinate ring. Then $\pi(V_c) = \pi(V) = S$ and $C$ is integral over $A = R[X_1, \ldots, X_n]$. Let $t = \lambda_1 X_{n+1} + \cdots + \lambda_m X_{n+m}$, $\lambda_i \in R$, be a primitive element of $R(V)$ over $R(X_1, \ldots, X_n)$ and let $\tilde{V}$ be the hypersurface of $R^{n+1}$ with coordinate ring $B = R[X_1, \ldots, X_n, t]$. Then we have the following diagram,

\[
\begin{array}{ccc}
V & \xrightarrow{\pi} & \tilde{V} \\
\downarrow & & \downarrow \\
S & \xrightarrow{\pi} & \end{array}
\]
where all the morphisms are finite, \( \pi \) represents the projection on the first \( n \) coordinates, and \( \rho \) induces a birational isomorphism. Therefore \( \rho(V_c) = \bar{V}_c \) (see [D-R], 2.9) and we get \( \pi(\bar{V}_c) = S \).

3.4. REMARK. We still do not know whether a regularly closed semialgebraic subset of \( R^n \) is the projection of an irreducible hypersurface of \( R^{n+1} \). In case the answer is negative, is there a bound of the integer \( m \) which does not depend on \( S \) (i.e. an universal bound for all regularly closed semialgebraic subsets of \( R^n \))?.

4. Application to Harrison’s topology. Throughout this section \( K = \mathbb{R}(X_1, \ldots, X_n) \) will be a pure transcendental extension of \( \mathbb{R} \) of degree \( w \), and \( X(K) \) will denote its space of orders. If \( E \) is a formally real extension of \( K \), we will denote by \( \varepsilon_{E|K} \) the induced morphism between \( X(E) \) and \( X(K) \), namely

\[ \varepsilon_{E|K}: X(E) \to X(K): P \mapsto P \cap K. \]

A clopen subset \( Y \) of \( X(K) \) is a subset which is open and closed in the Harrison’s topology of \( X(K) \), i.e. the topology whose basis consists of the sets:

\[ H(f_1, \ldots, f_r) = \{ P \in X(K): f_1 \in P, \ldots, f_r \in P \}, \]

\( f_i \in \mathbb{R}[X_1, \ldots, X_n] \) for all \( i \).

Since \( X(K) \) with Harrison’s topology is compact ([P]), every clopen set \( Y \) can be written as a finite union of open basic sets:

\[ Y = H_1 \cup \cdots \cup H_p, \quad \text{where} \; H_i = H(f_{i1}, \ldots, f_{ir}). \]

Theorem 3.2 will be used to prove the following:

4.1. THEOREM. Let \( Y \) be any clopen set of \( X(K) \). Then there exists a finite extension \( E \) of \( K \) such that \( Y = \text{im} \; \varepsilon_{E|K} \).

Proof. Let \( Y = H_1 \cup \cdots \cup H_p, \; H_i = H(f_{i1}, \ldots, f_{ir}), \; f_{ki} \in \mathbb{R}[X_1, \ldots, X_n] \) for all \( (k, i) \in \{1, \ldots, r\} \times \{1, \ldots, p\} \). Define the semialgebraic associated to \( Y \) by

\[ Y^* = H_1^* \cup \cdots \cup H_p^* \]

where \( H_i^* = \{ x \in \mathbb{R}^n: f_{i1}(x) > 0, \ldots, f_{ir}(x) > 0 \} \). In [D-R] it is shown that the correspondence \( Y \to Y^* \) verifies that \( Y_1 = Y_2 \) if and only if \( \bar{Y}_1 = \bar{Y}_2 \), where \( \bar{Y} \) denotes the closure of \( Y \) in the strong topology of \( \mathbb{R}^n \).
Since \( Y \) is open, \( \overline{Y} \) is a regularly closed semialgebraic subset of \( \mathbb{R}^n \). Then 2.5 applies producing an \( n \)-dimensional irreducible algebraic set \( V \subset \mathbb{R}^{n+m} \) such that \( \pi(V) = \pi(V_c) = \overline{Y} \). In particular, \( \overline{\pi(V_c)} = \overline{Y} \). Since \( \dim V = n \), the function field \( E \) of \( V \) is a finite extension of \( K \) and \( R[X_1, \ldots, X_n] \rightarrow R[V] \) is integral since \( \pi: V \rightarrow \mathbb{R}^n \) is finite.

It follows immediately from [D-R] (Prop. 2.7) that \( \text{im} \, \varepsilon_{E|K} = Y \).

4.2. REMARK. In [E-L-W] is suggested that the characterization of those clopen subsets of the space of orders \( X_K \) of a field \( K \) which are the image of \( \varepsilon_{E|K} \) for some finite extension \( E|K \) could depend on topological properties of \( \varepsilon \) for finite extensions. However, since there are examples ([E-L-W]) of clopen sets which are not \( \text{im}(\varepsilon_{E|K}) \) for any \( E \), and after Theorem 4.1, it follows that such a characterization is not intrinsic to \( \varepsilon \) but depends on the base field \( K \).

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