Spaces of discrete shape and \(c\)-refinable maps that induce shape equivalences

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(Received Aug. 22, 1995)

Introduction.

In [15], following a Cantor completion process, the authors give a complete, non-Archimedean metric (or ultrametric) on the set of shape morphisms between two unpointed compacta (compact metric spaces) \(X, Y\), written \(Sh(X, Y)\). The ultrametric spaces so constructed allow to rediscover some of the more important invariants in shape theory and to introduce many others. It is clear that the construction given in [15] can be translated to the pointed case, consequently, as a particular case, we obtain a complete ultrametric that induces a norm on the shape groups of a compactum \(Y\).

Let \((X, x_0)\) and \((Y, y_0)\) be pointed compacta. We will assume \(Y\) to be embedded in the Hilbert cube \(Q\). Let \(i_{\varepsilon}:Y \rightarrow B(Y, \varepsilon)\) be the inclusion. For any pair \(f, g:(X, x_0)\rightarrow(Q, y_0)\) of maps, take \(F(f, g)=\inf\{\varepsilon>0:f \cong g\ \text{in}\ B(Y, \varepsilon)=Y_{\varepsilon}\}\) (\(\cong\) means the pointed homotopy relation).

It is clear that (pointed) approximative maps (see [3]) \(\{f_{k}\}:(X, x_0)\rightarrow(Y, y_0)\) correspond with \(F\)-Cauchy sequences and that (pointed) homotopic approximative maps are equivalent \(F\)-Cauchy sequences.

Given \(\alpha, \beta \in Sh((X, x_0), (Y, y_0))\) and \(F\)-Cauchy sequences \(\{f_{k}\}, \{g_{k}\}\) in the classes of \(\alpha, \beta\) respectively, the formula \(d(\alpha, \beta)=\lim_{k \rightarrow \infty}F(f_{k}, g_{k})\) produces a well defined complete, non-Archimedean metric in \(Sh((X, x_0), (Y, y_0))\) such that the composition of pointed shape morphisms induces uniformly continuous maps between the spaces involved. This fact provides many new pointed shape invariants (see [15] for details in the unpointed case).

**Proposition 1** ([15]). Given \(\alpha, \beta \in Sh((X, x_0), (Y, y_0))\), \(d(\alpha, \beta)<\varepsilon\) if and only if \(S(i_{\varepsilon}) \circ \alpha = S(i_{\varepsilon}) \circ \beta\), as pointed morphisms (\(S\) denotes the shape functor).

In order to simplify notation we suppress base points consistently until section 2.

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Key words and phrases: shape, calmness, AWNR, \(c\)-refinable map.

The authors have been supported by DGICYT, PB93-0454-C02-02.

Most of this work was done while the second author was visiting the Department of Mathematics of the University of Tennessee at Knoxville with a M.E.C. grant.
When we consider the special cases \( X=S^n, \ n\in\mathbb{N} \), we obtain an ultrametric on the shape groups \( \hat{\Pi}_n(Y) \) of a pointed compactum \( Y \). If, for any \( \alpha\in\hat{\Pi}_n(Y) \), we define \( \|\alpha\|=d(\alpha, 1) \) we have a norm such that

i) \( \|\alpha\beta\alpha^{-1}\|=\|\beta\| \) for any \( \alpha, \beta\in\hat{\Pi}_n(Y) \).

ii) \( \|\alpha\|=\|\alpha^{-1}\| \) for any \( \alpha\in\hat{\Pi}_n(Y) \).

iii) \( \|\cdot\| \) gives rise to a left and right invariant complete ultrametric in \( \hat{\Pi}_n(Y) \) given by \( d(\alpha, \beta)=\|\alpha\beta^{-1}\| \).

If \( X, Y \) are arbitrary topological spaces, let \( p:X\rightarrow X=(X_1, p_{12}, A) \) and \( q:Y\rightarrow Y=(Y_\mu, q_{\mu\mu'}, M) \) be HPol-expansions of \( X \) and \( Y \) respectively.

Take \( Sh(Z, X)=(Sh(Z, X_\lambda), p_{\lambda\lambda'}, A) \) and \( Sh(Z, Y)=(Sh(Z, Y_\mu), q_{\mu\mu'}, M) \), for any space \( Z \). In [16] we generalize the construction for arbitrary spaces, by giving to \( Sh(X, Y) \) the inverse limit topology as inverse limit in Top of \( \{Sh(X, Y_\lambda)\}_{\lambda\in A} \) where \( Sh(X, Y_\lambda) \) is assumed to have the discrete topology for any \( \lambda\in A \). Using these spaces, we will show in section 2 a generalization of a theorem of Kato [11], [12]. We prove that any \( c \)-refinable map \( f:X\rightarrow Y \) is a shape equivalence provided the induced morphism \( S(f)\in Sh(X, Y) \) is isolated. It is not difficult to see that \( S(f) \) is isolated if \( Y \) is calm or AWNR (because \( Sh(X, Y) \) is discrete) see [4], [2] and [27].

Returning to the compact framework, it is well known that out of pointed (compact) connected polyhedra there is a countable set \( \{P_n: n\in\mathbb{N}\} \) containing one of each pointed homotopy type. Consider the inverse system \( \{P_n, p_n, n\in\mathbb{N}\} \) where \( p_n: P_{n+1} \rightarrow P_n \) is the constant (pointed) map. Let \( (W, *) \) be the pointed internally movable connected space obtained by applying the star-construction, see [21] or [20] page 185, to the above inverse sequence.

The space \( W \) is useful because the uniform topological type of \( Sh(W, X) \) characterizes the shape of \( X \), provided \( X \) is pointed movable. More precisely, in [18] it is shown that a shape morphism \( F:X\rightarrow Y \) between connected pointed compacta is a shape equivalence if and only if the induced map \( F^*:Sh(W, X)\rightarrow Sh(W, Y) \) is a bi-uniform homeomorphism. Similar results can be obtained, in the unpointed case, by using the spaces introduced in [17].

Above considerations raise naturally what we are going to study here. The reader is referred to the text of [7] and [20] for information about shape theory.

1. Spaces of discrete shape.

**Definition 1.** A pointed compactum \( X \) has discrete shape if \( Sh(W, X) \) is uniformly discrete, i.e. there is \( \varepsilon>0 \) such that if \( \alpha, \beta\in Sh(W, X) \) and \( d(\alpha, \beta)<\varepsilon \) then \( \alpha=\beta \).
PROPOSITION 2. Let $X, Y$ be pointed compacta. If $\text{Sh}(X) \leq \text{Sh}(Y)$ and $Y$ has discrete shape then $X$ has discrete shape. Consequently, the property of having discrete shape is a shape invariant.

PROOF. It is a consequence of the fact that if $\text{Sh}(X) \leq \text{Sh}(Y)$ then $\text{Sh}(W, X)$ is a uniform retract of $\text{Sh}(W, Y)$. □

PROPOSITION 3. Let $X$ be a pointed compactum, then $X$ has discrete shape provided $X$ is calm or $X \in \text{AWNR}$.

PROPOSITION 4. Let $X$ be a pointed movable compactum. $X$ is a pointed FANR if and only if $X$ has discrete shape.

PROOF. It suffices to show that any pointed movable compactum $X$ having discrete shape is calm. Take any $\delta>0$. Let $0<\epsilon<\delta$ as in \[\text{Definition 1}\]. For any $0<\epsilon_{1}<\epsilon$ we consider $\epsilon_{2}<\epsilon_{1}$ such that $S(i_{B(X, \epsilon_{2}).B(X, \epsilon_{1})})=i_{X.B(X, \epsilon_{1})} \circ r$ for some shape morphism $r : B(X, \epsilon_{2}) \to X$.

Let $K$ be any polyhedron and let $f, g : K \to B(X, \epsilon_{2})$ be pointed $H$-maps such that $S(i_{B(X, \epsilon_{2}).B(X, \epsilon_{1})}) \circ f=S(i_{B(X, \epsilon_{2}).B(X, \epsilon_{1})}) \circ g$. Consider $H$-maps $\alpha : K \to W$ and $\beta : W \to K$ such that $\beta \circ \alpha=1_{K}$.

We have that

\[S(i_{B(X, \epsilon_{2}).B(X, \epsilon_{1})}) \circ S(i_{B(X, \epsilon_{2}).B(X, \epsilon_{1})}) \circ f \circ \beta = S(i_{B(X, \epsilon_{2}).B(X, \epsilon_{1})}) \circ S(i_{B(X, \epsilon_{2}).B(X, \epsilon_{1})}) \circ g \circ \beta.\]

Then,

\[S(i_{B(X, \epsilon_{2}).B(X, \epsilon_{1})}) \circ S(i_{B(X, \epsilon_{2}).B(X, \epsilon_{1})}) \circ r \circ f \circ \beta = S(i_{B(X, \epsilon_{2}).B(X, \epsilon_{1})}) \circ S(i_{B(X, \epsilon_{2}).B(X, \epsilon_{1})}) \circ r \circ g \circ \beta.\]

Consequently, $d(r \circ f \circ \beta, r \circ g \circ \beta)<\epsilon$ and $r \circ f \circ \beta=r \circ g \circ \beta$.

It follows that $r \circ f=r \circ f \circ \beta \circ \alpha=r \circ g \circ \beta \circ \alpha=r \circ g$.

Therefore,

\[S(i_{B(X, \epsilon_{2}).B(X, \epsilon_{1})}) \circ f = S(i_{B(X, \epsilon_{2}).B(X, \epsilon_{1})}) \circ r \circ f = S(i_{B(X, \epsilon_{2}).B(X, \epsilon_{1})}) \circ r \circ g = S(i_{B(X, \epsilon_{2}).B(X, \epsilon_{1})}) \circ g.\]

□

REMARKS. $\text{Sh}(W, X)$ contains isometric copies of all shape groups $\tilde{\Pi}_{n}(X), n \in \mathbb{N}$. Then if $X$ has discrete shape it follows that $\tilde{\Pi}_{n}(X), n \in \mathbb{N}$ are uniformly discrete topological groups such that $\epsilon>0$ as in \[\text{Definition 1}\] does not depend on $n \in \mathbb{N}$. Using Baire's Theorem and the homogeneity of these groups we have that they are discrete if and only if they are countable. Therefore, if $sd(X)<\infty$, the assumption of $\text{Sh}(W, X)$ to be discrete is very strong and can be much weakened \[\text{[20]}\].
Since \(Sh(X, Y)\) is separable we have that if \(Sh(X, Y)\) is discrete then it is countable. As we said before, the converse is also true for the shape groups. A natural question is whether \(Sh(X, Y)\) countable implies \(Sh(X, Y)\) discrete. In the unpointed case there are very easy examples showing that this implication does not hold. In fact \(Sh(*, X) = \square X\), the space of components of \(X\), ([15]).

Then, if \(X = \{1/n : n \in \mathbb{N}\} \cup \{0\}\), \(Sh(W, X)\) is countable but it is not discrete. However, in the pointed case, it seems more difficult to find examples. Anyway, Corollary 3 will provide one of them. It will be given a pointed movable compactum \(X\) such that \(Sh(W, X)\) is countable but \(X\) has not discrete shape.

**Theorem 1.** Let \(T\) be the Taylor's compactum, [26]. It follows that \(Sh(W, T) = *\). In particular \(T\) has discrete shape.

The proof of above theorem depends on two previous results.

**Theorem 2.** Let \(F: X \rightarrow Y\) be a shape morphism that is a weak shape equivalence; then, for any compact connected pointed polyhedron \(P\), \(F\) induces an isomorphism \(Sh(P, X) \rightarrow Sh(P, Y)\) in pro-Top (pro-Set).

**Proof.** We assume \(F\) to be represented by a level preserving morphism \((f_{\lambda})\).

Let \(P\) be a compact connected polyhedron, \(dim P = m < n\). Using Lemma 1.4 in [10], see also [14], for any \(\lambda\) there exists \(\theta(\lambda, n) \geq n\) and a map \(h\) making the following diagram commutative, up to pointed homotopy (\(M(f)\) denotes the reduced mapping cylinder of \(f\))

\[
\begin{array}{ccc}
X_{\lambda} & \leftarrow & M(f_{\theta(\lambda, n)})^{n} \cup X_{\theta(\lambda, n)} \\
\downarrow f_{n} & & \downarrow i \\
Y_{\lambda} & \leftarrow & M(f_{\theta(\lambda, n)}).
\end{array}
\]

By the cellular approximation theorem,

\[j^{*}: Sh(P, M(f_{\theta(\lambda, n)})^{n} \cup X_{\theta(\lambda, n)}) \rightarrow Sh(P, M(f_{\theta(\lambda, n)})^{n})\]

is bijective.

Then, we have a map \(g^{*}: Sh(P, Y_{\theta(\lambda, n)}) \rightarrow Sh(P, X_{\lambda})\) such that the diagram

\[
\begin{array}{ccc}
Sh(P, X_{\lambda}) & \leftarrow & Sh(P, X_{\theta(\lambda, n)}) \\
\downarrow f^{*} & & \downarrow g^{*} \\
Sh(P, Y_{\lambda}) & \leftarrow & Sh(P, Y_{\theta(\lambda, n)}).
\end{array}
\]

commutes.
Now, from Morita's characterization of isomorphisms in pro-categories, \[14\], we have that $F$ induces an isomorphism $Sh(P, X) \rightarrow Sh(P, Y)$.

**PROPOSITION 5.** Let $F: X \rightarrow Y$ be a shape morphism between connected pointed compacta such that $F^*: Sh(P, X) \rightarrow Sh(P, Y)$ in injective, for every connected compact pointed polyhedron $P$; then, $F^*: Sh(W, X) \rightarrow Sh(W, Y)$ is injective.

**PROOF.** Given $\varepsilon > 0$, using the local contractibility of $B(X, \varepsilon)$, it is easy to check that if $a, b: W \rightarrow X$ are shape morphism such that $F \circ a = F \circ b$ then $d(a, b) < \varepsilon$.

**PROOF OF THEOREM 1.** Using \[26\], we have a CE-map $f: T \rightarrow Q$. Consequently, $S(f)$ is a weak shape equivalence. From Theorem 2 and Proposition 5, we have that $S(f)$ induces an injective map

$$S(f)^*: Sh(W, T) \rightarrow Sh(W, Q) = *.$$

Next corollaries point out that even though $Sh(W, X)$ is uniformly discrete $X$ does not need being an AWNR neither a calm space.

**COROLLARY 1.** Let $T$ be the Taylor's compactum. $T$ is not AWNR but $Sh(W, T) = *$.

**COROLLARY 2.** Consider $\{T_j, j \in \mathbb{N}\}$ to be a family of copies of the Taylor's compactum. Then, $\Pi_{j \in \mathbb{N}} T_j$ is a non calm compactum such that $Sh(W, \Pi_{j \in \mathbb{N}} T_j) = *$.

**Theorem 1** also allows to state the next corollary.

**COROLLARY 3.** There exists a pointed movable compactum $T'$ such that $Sh(W, T')$ is countable but $T'$ has not discrete shape.

**PROOF.** It suffices to take the space $T'$ obtained by applying the star-construction of Overton-Segal \[21\] to the inverse sequence associated with $T$.

Note that from Theorem 2 and Proposition 5 we have,

**COROLLARY 4.** For any pointed compactum $X$, $pro-\Pi_k(X) = *$ for every $k \in \mathbb{N}$, implies $Sh(W, X) = *$.

2. **c-refinable maps that induce shape equivalences.**

In this section we will work with (unpointed) arbitrary topological spaces. In \[11\] (\[12\]) H. Kato proved that any refinable map $r: X \rightarrow Y$ between compacta induces a shape equivalence $S(r): X \rightarrow Y$ provided $Y \in FANR (Y$ is calm) ($S$ denotes the shape functor). Recently J. M. R. Sanjurjo \[22\], gave an intrinsic description of the shape category of compacta by using upper-semicon-
tinuous multivalued maps. This approach allowed him to give an alternative
proof of the result of Kato. The authors in [19] extended the upper-semicon-
tinuous multivalued maps approach to shape to the class of paracompacta by
means of resolutions theory. Simultaneously Z. Čerin has given, see [4], by
using the cofinite Čech expansion and non-upper-semicontinuous multivalued
maps, an intrinsic description of the shape category for arbitrary topological
spaces. In this paper, we apply later useful description to prove in a short
way, by general topology methods, a rather general result in the realm of
arbitrary topological spaces dealing with c-refinable maps (see [13]).

In order to do this section as self-contained as possible we point out some of
the notions we will handle.

A normal covering of a topological space $Y$ is an open covering $\omega$ which
admits a partition of the unity subordinated to $\omega$. Normal coverings can also be
characterized as those admitting a sequence of open coverings $\omega \leq \omega_1 \leq \omega_2 \leq \omega_3 \cdots$
where the symbol $\leq$ stands for the star-refinement relation [1].

Two open coverings of $Y$ are said to be equivalent if they refine each other.
$\tilde{Y}$ will denote the collection of all normal coverings classes of a topological
space $Y$. By $\tilde{Y}$ we shall mean the family of all finite subsets $c \subset \tilde{Y}$ having,
respect the refinement relation, a maximal element $\varepsilon \in \tilde{Y}$.

Let $X, Y$ be topological spaces and $\alpha \in \tilde{X}, \beta \in \tilde{Y}$. A multivalued map
$F: X \to Y$ is said to be $(\alpha, \beta)$-small if for any $U \in \alpha$ there is a $V \in \beta$ such that
$F(U) \subset V$. We will say that $F$ is $\beta$-small if there exists $\alpha \in \tilde{X}$ such that $F$ is
$(\alpha, \beta)$-small.

Two multivalued maps $F, G: X \to Y$ are said to be $\beta$-homotopic, written
$F \approx G$, if there is a $\beta$-small map $H: X \times I \to Y$ such that $F \subset H(\cdot, 0)$ and $G \subset H(\cdot, 1)$. Note that $F \approx G$ and $G \approx T$ imply $F \approx T$ provided $\beta_1 \geq \beta$.

A multinet $F: X \to Y$ is a collection $F = \{F_d\}_{d \in \tilde{Y}}$ of multivalued functions
$F_d: X \to Y$ such that for every $\gamma \in \tilde{Y}$ there is a $\omega \leq \gamma$ with $F_d \approx G_d$ for any $d > \omega$.
Two multinet $F = \{F_d\}, G = \{G_d\}: X \to Y$ are homotopic if for every $\gamma \in \tilde{Y}$ there is
a $c \in \tilde{Y}$ with $F_d \approx G_d$ for any $d > c$.

In [5], Čerin defined the composition of homotopy classes of multinets
producing a category isomorphic to the shape category.

Given $[F]_\gamma \in Sh(X, Y)$ and $\gamma \in \tilde{Y}$ let $B([F], \gamma) = \{[G] \in Sh(X, Y) :$ there exists $c \in \tilde{Y}$ with $F_d \approx G_d$ for any $d > c\}$. It is readily seen that the family
$\{B([F], \gamma)\}_{\gamma \in \tilde{Y}}$ is a neighborhood system for the shape morphism $[F]: X \to Y$.
We will consider $Sh(X, Y)$ endowed with the induced topology. This topology
coincide with the topology obtained by giving to $Sh(X, Y)$ the inverse limit
topology as inverse limit in Top of $(Sh(X, Y_d))_{d \in \tilde{Y}}$ where $\{Y_d, q_{d+1}, A\}$ is
any HPol-expansion of $Y$ and $Sh(X, Y_d)$ is assumed to have the discrete topology
for any $d \in \tilde{A}$, see [16].
Before stating our result we recall that a surjective map $r: X \to Y$ is said to be $c$-refinable if for any normal coverings $\alpha$, $\beta$ of $X$ and $Y$ respectively, there is a closed and onto $(\alpha, \beta)$-refinement $s: X \to Y$, of $r$; i.e. $s$ and $r$ are $\beta$-near and for any $y \in Y$ there is $U_y \in \alpha$ such that $s^{-1}(y) \subseteq U_y$.

**Theorem 3.** Let $X, Y$ be topological spaces and let $r: X \to Y$ be a $c$-refinable map. Then $S(r)$ is a shape equivalence provided $S(r)$ is an isolated point in $Sh(X, Y)$.

**Proof.** Let $\gamma_0 \in \hat{Y}$ such that $B(S(r), \gamma_0) = |S(r)|$. Take $\gamma_1 \in \hat{Y}$ such that $\overline{\gamma_1} \approx \gamma_0$.

Let $c \in \overset{\circ}{X}$ and $\bar{c} \in \hat{X}$. Consider $\bar{d} \in \overset{\circ}{X}$ to be a 3-star-refinement of $\bar{c}$. More precisely, choose normal coverings $\bar{d} \supset \bar{d}_2 \supset \bar{d}_1 \supset \bar{c}$. Take $s: X \to Y$ be any $(\bar{d}, \gamma_1)$-refinement of $r$. We define $F_c: Y \to X$ by $F_c(y) = s^{-1}(y)$. Since $s$ is closed $(\{Y \setminus s(X \setminus U)\}_{U \in \bar{d}}$ is a normal covering of $Y$; hence $F_c$ is a $\bar{d}$-small multivalued map.

The base of the proof is the following fact:

**Claim.** If we start from different $(\bar{d}, \gamma_1)$-refinements of $r$ we obtain $\bar{c}$-homotopic multivalued maps.

Indeed, let $s_1, s_2: X \to Y$ be two $(\bar{d}, \gamma_1)$-refinements of $r$ and denote by $F^1_c$ and $F^2_c$ the corresponding $\bar{d}$-small multivalued maps obtained from $s_1$ and $s_2$ respectively.

Since $B(S(r), \gamma_0) = |S(r)|$ we have that for any (single-valued) map $f: X \to Y$ $\gamma_1$-near to $r$, $r \approx \bar{f}$ for every $\mu \in Y$. Consequently, $F^1_c \circ r \approx F^2_c \circ s_1 \supset Id_X$. A similar argument shows that $F^1_c \circ r \approx F^2_c \circ s_2 \supset Id_X$. Then, $F^1_c \circ r \approx F^2_c \circ r$.

Let $H: X \times I \to X$ be a $(\alpha, \bar{d}_2)$-small homotopy connecting $F^1_c \circ r$ and $F^2_c \circ r$. Choose a normal covering $\bar{d} \in \overset{\circ}{X}$ such that there is a stacking function, in the sense of [6] (page 358), $\bar{d} \to \{1, 2, 3, \ldots\}$ producing a refinement of $\alpha \in \overset{\circ}{X} \times I$.

Let $\beta \in \hat{Y}$ be a refinement of both $\{Y \setminus s_2(X \setminus U)\}_{U \in \bar{d}}$ and $\{Y \setminus s_1(X \setminus U)\}_{U \in \bar{d}}$ and take $\beta_i \in \hat{Y}$ such that $\overline{\beta} \approx \beta$.

Let $s': X \to Y$ any $(\bar{a}, \beta_i)$-refinement of $r$. Define a $\bar{a}$-small map $G: Y \to X$ by $G(y) = s'^{-1}(y)$. It follows that $r \circ G \approx \bar{g}$. Therefore, $F^1_c \approx F^2_c \circ r \circ G \approx F^2_c \circ r \circ G \approx F^2_c$. Consequently, $F^1_c \approx F^2_c$. This proves the claim.

Now it is a routine to check that $F = \{F_c\}: Y \to X$ is a multinet such that $S(r) \ast [F] = Id_Y$ and $[F] \ast S(r) = Id_X$.

**Remarks.** The assumption of $r$ to be isolated in $Sh(X, Y)$ holds, in particular, when $Sh(X, Y)$ is discrete. For example, if $Y$ is stable, for every topological space $X$ one has that $Sh(X, Y)$ is discrete. In the non necessarily movable context the same follows if $Y$ is calm.
Note that it is easy to produce examples showing that \(c\)-refinable maps can not be substituted by refinable maps in above theorem. In fact, if \(Y\) is any infinite trivial shape space and we denote by \(X\) the set \(Y\) endowed with the discrete topology, it is clear that \(Id : X \to Y\) is a refinable map that fails to be a shape equivalence.

ACKNOWLEDGEMENTS. The second author wants to express his gratitude to Professor J. Dydak for his hospitality.

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