On M-Spaces and Banach Spaces

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Abstract

We define in this paper the concept of C-space, related with M-spaces and Banach spaces. We obtain various properties on these spaces and propose some open problems.

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1 Introduction

There exist three causes that motive this new paper. First, an early theorem of Corson, also the concept of M-space (defined by K.Morita), and finally a paper on Banach spaces by the author:

The Corson Theorem. [3] For any covering $\mathcal{U}$ of a infinite dimensional reflexive Banach space $B$, where $\mathcal{U}$ is formed by bounded, convex sets, there is not a point $x$ in $B$ such that each neighborhood of $x$ meets only finitely many members of $\mathcal{U}$, i.e., $\mathcal{U}$ is not locally finite.

In our paper [6], we study some problems related to the Corson Theorem. In particular, we proved that: "For every $r \geq 0$, there exits an open covering of $c_0$, which is locally finite and is formed by balls of radius $r$".

We will use in this paper the concept of M-space:

Definition 1. [7] A paracompact space $X$ is called a M-space if there is some perfect map from $X$ onto some metric space.
2 Main results.

Definition 2. Let $X$ be a topological space. We will say that $X$ is a $C$-space if there is some Banach space $E$ and some perfect map $f$ from $X$ onto $E$ such that exists a locally finite covering of $X$ formed by pre-images of open balls of radius 1 by the map $f$.

Remarks. 1. If $X$ is a $C$-space then $X$ is a paracompact $M$-space.
2. $c_0$, $E_{\infty}$, $IR^n$ are $C$-spaces.

Proposition 1. Let $X$ be a topological space, $E$ be a Banach space and $f$ be a perfect map from $X$ onto $E$. Then

$$V = \{ f^{-1}(B_1(x_j))| j \in J \} \text{ is a locally finite covering of } X, \text{ if and only if } \{ B_1(x_j)| j \in J \} \text{ is a locally finite covering of } E.$$

Proof. ($\Rightarrow$) If $V$ covers $X$ also $\{ B_1(x_j)| j \in J \}$ covers $E$, because $f$ is onto.

For each $z \in E$ and each $x \in f^{-1}(z)$ there exists an open neighborhood $U^z_x$ of $x$, such that meets only finitely members of $V$. Then $\{ U^z_x| x \in f^{-1}(z) \}$ is an open covering of $f^{-1}(z)$, and $f^{-1}(z) \subset \cup_{k=1}^{r} U^x_z$ (for some $x_1,\ldots,x_r \in f^{-1}(z)$) because $f$ is a perfect map.

Since $f$ is closed, there exists an open neighborhood $W^z$ of $z$ such that $f^{-1}(W^z \cap \cup_{k=1}^{r} U^x_z)$. Then, $f^{-1}(W^z)$ meets only finitely members of $V$, and also $W^z$ meets only finitely members of $\{ B_1(x_j)| j \in J \}$.

($\Leftarrow$) If $\{ B_1(x_j)| j \in J \}$ covers $E$, then $V$ covers $X$.

For each $x \in X$ there exists an open neighborhood $V^{f(x)}$ of $f(x)$ such that meets only finitely members of $\{ B_1(x_j)| j \in J \}$. Clearly, $f^{-1}(V^{f(x)})$ is an open neighborhood of $x$ and meets only finitely members of $V$.

Corollary 1. Let $X$ be a topological space. Then, $X$ is a $C$-space if and only if there exists a Banach space $E$ that has a locally finite open covering formed by balls of fixed radius, and a perfect map $f$ from $X$ onto $E$.

Corollary 2. For each compact space $K$, we have that $c_0 \times K$ is a $C$-space.

Proof. Since the projection map $p_1$ is a perfect map from $c_0 \times K$ onto $c_0$.

Corollary 3. For each compact space $K$, we have that $IR^{1N} \times K$ is a $C$-space.
Proof. It follows from the above Corollary, because $c_0$ is homeomorphic to $\mathbb{IR}^N$ (theorems of Kadec [5] and Anderson [1]).

Proposition 2. Let $X$ be a topological space. If $X$ is separable and $C$-space, then it is homeomorphic to some closed subset of $\mathbb{IR}^N \times I^N$.

Proof. Since $X$ is a separable $C$-space, there is some separable Banach space $E$ and some perfect map from $X$ onto $E$. From [8, Theorem 2] it follows that $X$ is homeomorphic to a closed subset in $E \times I^N$. Finally, therems of Kadec [5] and Anderson [1] yield the conclusion.

3 Open problems.

1. Let $X$ be a topological space. Have we that $X$ is a $C$-space if and only if $X$ is homeomorphic to a closed subset of $\mathbb{IR}^N \times I^N(X)$? (where $w(X)$ is the weight of $X$).

2. Have the normed spaces whith locally finite coverings by balls analogous properties to totally bounded spaces?

4 References


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