DISTINGUISHED SUBSETS AND COMPLEMENTED COPIES OF C₀ IN VECTOR SEQUENCE SPACES

Fernando Bombal
Departamento de Análisis Matemático.
Universidad Complutense. 28040 Madrid (Spain)


Recall that a subset \( K \) of a Banach space \( E \) is said to be limited (resp., Dunford-Pettis) if for every weak* null (resp., weakly null) sequence \( (x_n^*) \) in the topological dual \( E^* \) of \( E \), the following holds:

\[
\lim_{n \to \infty} \sup \{|x_n^*(x)| : x \in K\} = 0.
\]

(See [BD] and [A].)

The classes \( \mathcal{Z}(E) \) and \( \mathcal{DP}(E) \) of limited and Dunford-Pettis subsets of \( E \), respectively, are formed by bounded sets, and they are preserved by continuous linear images, linear combinations, closed absolutely convex hulls and passing to subsets. Also the following relationships hold

\[
\mathcal{X}(E) \subset \mathcal{Z}(E) \subset \mathcal{DP}(E) \subset \mathcal{W}_{\mathcal{E}}(E)
\]

(\( \mathcal{X}(E), \mathcal{W}(E) \) and \( \mathcal{W}_{\mathcal{E}}(E) \) stand for the relatively norm compact, weakly relatively compact and weakly conditionally compact subsets of \( E \), respectively (recall that \( A \in \mathcal{W}_{\mathcal{E}}(E) \) if every sequence in \( A \) has a weakly Cauchy subsequence. Note that the proof of \( \mathcal{Z}(E) \subset \mathcal{W}_{\mathcal{E}}(E) \) given in [DR] also works to show that \( \mathcal{DP}(E) \subset \mathcal{W}_{\mathcal{E}}(E) \).

We are now interested in the study of the different classes of distinguished subsets (\( \mathcal{T} \)) in spaces of vector sequences.

Let \( (E_n) \) be a sequence of Banach spaces and \( 1 \leq p \leq \infty \). We shall denote, as usual, by \( (\sum_{n=1}^{\infty} x_n)_{\ell^p} \) the space of all vector valued sequences \( x = (x_n) \) such that \( x_n \in E_n \) (\( n \in \mathbb{N} \)) and \( \|x\|_{\ell^p} = \sum_{n=1}^{\infty} \|x_n\|_{E_n}^p \) is finite, endowed with the Banach norm \( x \mapsto \|x\|_{\ell^p} \).

Reasoning as in the scalar case, it is very easy to prove the following (well known) proposition (see [BrD], Th 2 or [B], Prop. 8.)

**Proposition 1.** Let \( (E_n) \) be a sequence of Banach spaces and \( 1 \leq p \leq \infty \). For a
bounded subset $A \subseteq E = (\sum_{n=1}^{\infty} e_n)^p$, the following properties are equivalent:

a) $A \in \mathcal{W}(E)$ (resp. $A \in \mathcal{W}(E)$).

b) We have

i) For every $k \in \mathbb{N}$, $\pi_k(A) \in \mathcal{W}(E_k)$ (resp., $\mathcal{W}(E_k)$)

and

ii) If $p = 1$, the following condition holds:

$$\lim_{n \to \infty} \sup \left\{ \sum_{k} \| \pi_k(x) \| : x \in A \right\} = 0.$$ 

For the other classes in (†), we have

Proposition 2. Let $(E_n)$ be a sequence of Banach spaces, $\mathcal{W}$ any of the classes $\mathcal{X}$, $\mathcal{Z}$ or $\mathcal{DP}$, and $1 \leq p < \infty$. For a bounded subset $A \subseteq E = (\sum_{n=1}^{\infty} e_n)^p$, the following properties are equivalent:

a) $A \in \mathcal{H}(E)$.

b) We have

i) For each $k \in \mathbb{N}$, $\pi_k(A) \in \mathcal{H}(E_k)$.

ii) $\lim_{n \to \infty} \sup \left\{ \sum_{k} \| \pi_k(x) \| : x \in A \right\} = 0$.

Many important properties of a Banach space $E$ are (or can be) defined by the coincidence of two classes of distinguished subsets of $E$. For example:

- $E$ has the Dunford–Pettis Property if $\mathcal{W}(E) = \mathcal{DP}(E)$ ([A]).
- $E$ has the Gelfand–Phillips Property if $\mathcal{Z}(E) = \mathcal{X}(E)$ ([BD]).
- $E$ has the Schur Property if $\mathcal{X}(E) = \mathcal{W}(E)$.
- $E$ is weakly sequentially complete if $\mathcal{W}(E) = \mathcal{W}(E)$.
- $E$ has the $\mathcal{DP}^*$ Property if $\mathcal{DP}(E) \subseteq \mathcal{W}(E)$ ([L]).

Hence propositions 1 and 2 can be interpreted as stability results for such properties when passing from the sequence $(E_n)$ to the corresponding $\ell_p$-sum:

Corollary 3. Let $(E_n)$ be a sequence of Banach spaces and $F_p = (\sum_{n=1}^{\infty} e_n)^p$ ($1 \leq p < \infty$).

a) On $F_1$ two of the classes $\mathcal{W}$, $\mathcal{W}$, $\mathcal{DP}$, $\mathcal{Z}$ or $\mathcal{X}$ coincide if and only if they coincide on every $E_n$. In particular, $F_1$ is weakly sequentially complete (resp., has the Schur property, the Dunford–Pettis property, the Gelfand–Phillips property or the $\mathcal{DP}^*$-property) if and only if so does every $E_n$.

b) On $F_p$ ($1 \leq p < \infty$) two of the classes $\mathcal{X}$, $\mathcal{Z}$ or $\mathcal{DP}$ coincide if and only if they coincide on every $E_n$. Also $W(F_p) = W(F_p)$ if and only if $W(E_n) = W(E_n)$, for every $n$. In particular, $F_p$ is weakly sequentially complete (resp., has the Gelfand–Phillips property or the $\mathcal{DP}^*$ property)
if and only if so does every $E_n$.

Limited sets are especially useful for detecting complemented copies of $c_0$, due to the following result:

Lemma 4. ([SL], [E1]) A Banach space contains a complemented copy of $c_0$ if and only if it contains a non limited sequence $(x_n)$, equivalent to the unit $c_0$-basis.

By using the above result, Emmanuele proved in [EZ] that if $\mu$ is a non purely atomic, finite measure and $E$ contains a (non necessarily complemented) copy of $c_0$, the space $L_p(\mu, E)$ ($1 \leq p < \infty$) of all $E$-valued Bochner $\mu$-integrable functions, contains always a complemented copy of $c_0$. For a purely atomic measure, the situation is completely different, as we shall see as a consequence of the following result:

Theorem 5. Let $(E_n)$ be a sequence of Banach spaces, $1 \leq p < \infty$ and $F_p = (\sum_1^\infty E_n)$, Then $F_p$ contains a complemented copy of $c_0$ if and only if there exists some $n \in N$ such that $E_n$ contains a complemented copy of $c_0$.

Corollary 6. Let $\mu$ be a $\sigma$-finite, purely atomic measure, $1 \leq p < \infty$ and $E$ a Banach space. Then $L_p(\mu, E)$ contains a complemented copy of $c_0$ if and only if $E$ contains a complemented copy of $c_0$.

REFERENCES


