Explicit Formulas for the 3-Jet Lift of a Matrix Group. Applications to Conformal Geometry

E. Aguirre-Dab and I. Sánchez-Rodríguez

Abstract:
The 3-jet lift $G^3$ of a matrix group $G$ is isomorphic, via a map that we give explicitly, to a semidirect product of $G$ itself and a nilpotent group built up from the first two prolongations of its Lie algebra. Using this isomorphism, we write down the formulas for the most natural representations of $G^3$, as well as for one additional representation of the 2-jet lift $G^2$ appearing when $G$ is of finite type 2. We apply these results to the case of the (linear) conformal group and we point out the geometric implications of these representations.

1 Introduction

The $r$-jet lift $G^r$ (with $r \in \mathbb{N}$, $r > 0$) of a Lie subgroup $G$ of the general linear group $GL(n, \mathbb{R})$ is the Lie group of $r$-jets at $0 \in \mathbb{R}^n$ of (local, 0-preserving) diffeomorphisms of $\mathbb{R}^n$ which preserve, in an obvious sense and up to order $(r-1)$, the canonical $G$-structure on $\mathbb{R}^n$.

The $r$-jet lift of $GL(n, \mathbb{R})$ itself is the structure group of the principal $r$-frame bundle $F^r(M)$ over any $n$-dimensional manifold $M$, and some of its properties have been studied in this context [5,8,10]. The second order frame bundle $F^2(M)$ has been extensively studied and explicit expressions concerning the 2-jet lift of $GL(n, \mathbb{R})$ are well known; see e.g. [3]. A proper use of such expressions leads [1] to a new proof of the first structure equation —first found in [10]— for connections in $F^2(M)$.

For a proper Lie subgroup $G$ of $GL(n, \mathbb{R})$, the group $G^r$ comes into play when one deals with “$(r-1)$-integrable” $G$-structures (“uniformly $(r-1)$-flat” in the sense of [4]), i.e. $G$-structures $P \subset LM$ having, over each point of $M$, contact of order $(r-1)$ with coordinate sections of the linear frame bundle $LM$ —let us remember that 1-integrable $G$-structures are those admitting symmetric (linear)
connections \([2,4]\). For each \((r-1)\)-integrable \(G\)-structure \(P\), one can define its \(r\)-jet lift \(P^r\) as the set of \(r\)-jets at \(0 \in \mathbb{R}^n\) of (local) diffeomorphisms of \(\mathbb{R}^n\) into \(M\) yielding such order of contact, and it turns out that \(P^r(M,G^r)\) becomes a principal subbundle of \(F^r(M)\).

Consider the Lie algebra \(g\) of \(G\). It may happen that the so-called “\(r\)-prolongation” \(g_r\) vanishes. Remember that when \(g_r = 0\) and \(g_{r-1} \neq 0\), one says that \(G\) is of finite type \(r\). In the case \(g_r\) vanishes, \(G^{r+1}\) becomes isomorphic to \(G^r\), and \(G^r\) inherits some additional properties from the \((r+1)\) order, mainly an additional natural representation that it is in the background of the appearance, in some cases \([4]\), of Cartan connections on the \(r\)-jet lifts \(P^r\) of \((r-1)\)-integrable \(G\)-structures \(P\).

The (pseudo) orthogonal conformal group \(CO(n,R)\) is (for \(n \geq 3\)) of finite type 2 and every \(CO(n,R)\)-structure is 1-integrable. Both \(CO(n,R)\) and its 2-jet lift \(CO(n,R)^2\) can be considered subgroups of the full conformal group \(O(n+2,R)/Z_2\). A lot of work has been done in the study of 2-jet lifts of conformal structures in dimension \(\geq 3\), see e.g. \([6,7]\), mainly by specific methods of the conformal case. We are interested in a more systematic treatment of connections on 2-jet lifts \(P^2(M,G^2)\) of 1-integrable \(G\)-structures \(P\) when \(G\) is of finite type 2, and their relationships with Cartan connections when they exist. Such treatment needs an appropriate handling of the 3-jet lift of a general matrix group. To our knowledge, no explicit formulas are up to now available.

In this paper we give (Lemma) a map which shows the isomorphism between \(G^3\) and a semidirect product of \(G\) itself and a nilpotent group built up from the first two prolongations of the Lie algebra \(g\) of \(G\). This map allows us to write down (Theorem) explicit formulas for the most natural representations of \(G^3\) and for the additional representation of \(G^2\) appearing when \(G\) is of finite type 2. When \(G\) is the (linear) conformal group \(CO(n,R)\), one can check that this additional representation is the restriction to \(CO(n,R)^2\) of the adjoint representation of the full conformal group \(O(n+2,R)/Z_2\). This last representation of \(CO(n,R)^2\) controls the behaviour of conformal Cartan connections on the 2-jet lift of a \(CO(n,R)\)-structure.

2 Basic definitions

For each \(v \in \mathbb{R}^n\), let \(D_{0,v}(\mathbb{R}^n)\) be the set of diffeomorphisms \(\varphi\) between open subsets of \(\mathbb{R}^n\), containing 0 and \(v\) respectively, such that \(\varphi(0) = v\). The \(r\)-jet \(j^r_0(\varphi)\) is the equivalence class of \(\varphi\) in \(D_{0,v}(\mathbb{R}^n)\) induced by the equivalence relation

\[
\varphi \sim \varphi' \iff D^s\varphi|_0 = D^s\varphi'|_0, \quad 1 \leq s \leq r.
\]

The set \(G^r(n) := D_{0,0}(\mathbb{R}^n) / \sim = \{j^r_0(g) : g \in D_{0,0}(\mathbb{R}^n)\}\) is a Lie group, the
group of $r$-jets, with product law
\[ j^0_0(g) \cdot j^0_0(g') := j^0_0(g \circ g') \]
and differentiable structure such that the bijection
\[ (u_0, u_1, \ldots, u_{r-1}) : \quad G^r(n) \rightarrow GL(n, \mathbb{R}) \times S^2(n) \times \cdots \times S^r(n) \]
\[ j^0_0(g) \mapsto (Dg|_0, D^2g|_0, \ldots, D^r g|_0) \]
becomes a global chart; we denote by $S^r(n)$ the subspace of symmetric $r$-linear maps in $L^r(n) \equiv L(\mathbb{R}^n \times \cdots \times \mathbb{R}^n, \mathbb{R}^n)$. This bijection induces a product law on $GL(n, \mathbb{R}) \times S^2(n) \times \cdots \times S^r(n)$ which can be explicitly computed by applying the chain rule to the derivatives of the composition of functions. The canonical Lie group isomorphism $G^1(n) \rightarrow GL(n, \mathbb{R})$, $j^0_0(g) \mapsto Dg|_0$, will be understood in what follows.

The set $F^r(\mathbb{R}^n) := \{j^0_0(\varphi) : \varphi \in D_{0, \nu}(\mathbb{R}^n), \ \nu \in \mathbb{R}^n\}$ is a (trivial) $G^r(n)$-principal bundle over $\mathbb{R}^n$ [5], the bundle of $r$-frames over $\mathbb{R}^n$, with projection $\pi^r : F^r(\mathbb{R}^n) \rightarrow \mathbb{R}^n$, $j^0_0(\varphi) \mapsto \varphi(0)$, $G^r(n)$-right action $R^r_{j^0_0(g)}j^0_0(\varphi) \equiv j^0_0(\varphi) \cdot j^0_0(g) := j^0_0(\varphi \circ g)$, and (global) trivialization
\[ \Psi^r : F^r(\mathbb{R}^n) \rightarrow \mathbb{R}^n \times G^r(n), \quad j^0_0(\varphi) \mapsto (\varphi(0), j^0_0(T_{\varphi(0)} \circ \varphi)), \quad T_\nu \text{ being the translation by } \nu \in \mathbb{R}^n. \] The canonical principal fibre bundle isomorphism $F^1(\mathbb{R}^n) \rightarrow LR^n$, $j^1_0(\varphi) \mapsto (\varphi \circ 0 : \mathbb{R}^n \rightarrow T_{\varphi(0)} \mathbb{R}^n)$, will be understood in what follows.

3 The $r$-jet lift of a matrix group

Let $G$ be a Lie subgroup of $GL(n, \mathbb{R})$. The canonical $G$-structure $GR^n$ is the subset of the linear frame bundle $LR^n$ obtained by moving with $G$ the global section $\sigma_I$ of $\pi^1 : LR^n \rightarrow \mathbb{R}^n$ induced by the identity chart $I \equiv (u^1, u^2, \ldots, u^n)$ of $\mathbb{R}^n$
\[ GR^n := (\text{Im } \sigma_I) \cdot G. \]

Every diffeomorphism $\varphi : U \subset \mathbb{R}^n \rightarrow V \subset \mathbb{R}^n$ raises a principal bundle isomorphism $\bar{\varphi} : LR^n|_U \rightarrow LR^n|_V$, $\bar{j}^1_0(\psi) \equiv 1 \mapsto \bar{j}^1_0(\varphi \circ \psi) \equiv \varphi|_{\psi(0)} \circ 1$.

The $r$-jet lift $G^r$ of $G$ is defined as the subset of $G^r(n)$ induced by those diffeomorphisms $g \in D_{0,0}(\mathbb{R}^n)$ whose associated $\bar{g}$ takes $\sigma_I$ into a (local) section of $LR^n$ having $(r-1)$-order contact over $0 \in \mathbb{R}^n$ with some section of $GR^n$
\[ G^r := \{\bar{j}^r_0(g) \in G^r(n) : \exists \sigma \in \text{Sec}(GR^n) \text{ with } \bar{j}^r_0^{-1}(\bar{g} \circ \sigma_I) = \bar{j}^r_0^{-1}(\sigma \circ g)\}. \]

It can be described equivalently by
\[ G^r = \{\bar{j}^r_0(g) \in G^r(n) : \exists \kappa \in \mathcal{F}(R^n, G) \text{ with } Dg|_0 = \kappa(0) \] and $D^{s+1}g|_0 = D^s \kappa|_0$ (1 $\leq s \leq r-1)$;
we have identified $D^s\kappa_0 \in L(R^n \times \cdots \times R^n, gl(n, R))$ with the element of $L^{s+1}(n)$ given by $D^s\kappa_0(v_1, \ldots, v_{s+1}) \equiv D^s\kappa_0(v_1, \ldots, v_s)(v_{s+1})$. We have that $(GL(n, R))^r = G^r(n)$ and $G^1$ is canonically isomorphic to $G$.

Let us show up some algebraic properties of $G^r$:

(i) For every $0 < r' < r$, the projection $\beta^{r,r'} : G^r \to G^{r'}$, $\beta_0^r(g) \mapsto \beta_0^{r'}(g)$, is a Lie group homomorphism and the invariant Lie subgroup $G^{r,r'} \equiv Ker \beta^{r,r'}$ is nilpotent. In particular, $G^{r,r-1}$ ($r > 1$) is abelian.

(ii) For $r > 1$, one sees very easily the existence of the following splitting exact sequence

$$1 \to G_{r,1} \hookrightarrow G^r \xrightarrow{\beta} G \to 1,$$

with $\hookrightarrow$ the inclusion $(G_{r,1} := \{ j_0^r(g) \in G^r : Dg|_0 = 1 \} = \{ j_0^r(Dg|_0^{-1} \circ g) : j_0^r(g) \in G^r \})$, $\beta(j_0^r(g)) := Dg|_0$ and $\gamma(a) := j_0^r(a)$.

Thus $G^r$ is canonically isomorphic with the semidirect product $G \rtimes G_{r,1}$, associated to the homomorphism of $G$ into $\text{Aut}(G_{r,1})$ given by

$$a \mapsto (j_0^r(g) \mapsto j_0^r(a \circ g \circ a^{-1}),$$

and this isomorphism is defined by the map

$$X : G^r \to G \rtimes G_{r,1},

j_0^r(g) \mapsto (Dg|_0, j_0^r(Dg|_0^{-1} \circ g)).$$

Remark The following sequence is also exact

$$1 \to G^{r,r-1} \hookrightarrow G^r \to G^{r-1} \to 1,$$

however it does not split, unless $r = 2$ (in which case both sequences are the same) or $G^{r,r-1} = \{0\}$ (trivial case); thus, in general, one cannot write $G^r \simeq G^{r-1} \rtimes G^{r,r-1}$ (the group $G^{r-1}$ has no natural structure as a subgroup of $G^r$).

By looking at curves in $G^r$ through the identity $j_0^r(I)$, it is easy to see [4] that the Lie algebra $g^r$ of $G^r$ must be isomorphic — as a linear space — to the direct sum $\sum_{s=0}^{r-1} g_s$, $g_s$ being the $s$-prolongation of $g$, defined as $g_0 := g$ and, for $s > 0$, as the vector space

$$g_s := \{ t \in S^{s+1}(n) : t(v_1, \ldots, v_s, \cdot) \in g, \forall v_1, \ldots, v_s \in R^n \} = L(R^n, g_{s-1}) \cap S^{s+1}(n)$$

(ordinarily $(gl(n, R))_s = S^{s+1}(n)$; moreover: $g_s = 0 \Rightarrow g_{s+1} = 0$).

One can also prove [4, 6, 8] that the Lie algebra $g^r$ of $G^r$ is isomorphic to the
Given by 

\[ [t_p, t'_q]_0 := \frac{(p + q + 1)!}{p!(q + 1)!} S(t_p(t'_q(\cdot, q^{+1}, \cdot), \cdot, \cdot)) - \frac{(p + q + 1)!}{(p + 1)! q!} S(t'_q(t_p(\cdot, q^{+1}, \cdot), q, \cdot)), \]

if \( p + q \leq r - 1 \)

\[ [t_p, t'_q]_0 := 0, \quad \text{if } p + q > r - 1, \]

where \( t_p \in g_p, t'_q \in g_q, p, q \geq 0 \) (thus \( [t_p, t'_q]_0 \in g_{p+q} \)), and the symmetrizer \( S \) is given by (\( \forall t \in L^s(n) \) and \( \forall v_1, \ldots, v_s \in \mathbb{R}^n \)):

\[ S(t)(v_1, \ldots, v_s) := \frac{1}{s!} \sum_{\sigma \in P_s} t(v_{\sigma(1)}, \ldots, v_{\sigma(s)}). \]

It follows that the abelian group \( G^{r,r-1} \) \((r > 1)\) is isomorphic to the vector group \( g_{r-1} \). If \( g_{r-1} = 0 \), the homomorphism \( \beta^{r,r-1} : G^r \to G^{r-1} \) is a Lie group isomorphism.

**Remarks.**

(i) For \( r > 1 \), consider the Lie algebra \((\sum_{s=0}^{r-1} g_s, [\cdot, \cdot]_0^r)\), with \([\cdot, \cdot]_0^r\) being simply the restriction of \([\cdot, \cdot]_0^r\). One always has

\[ \left( \sum_{s=0}^{r-1} g_s, [\cdot, \cdot]_0^r \right) = g \oplus \left( \sum_{s=0}^{r-1} g_s, [\cdot, \cdot]_0^r \right) \] (semidirect sum).

Moreover — and corresponding to the above Remark on groups —

\((\sum_{s=0}^{r-2} g_s, [\cdot, \cdot]_0^{r-1})\) closes as a Lie subalgebra of \( (\sum_{s=0}^{r-1} g_s, [\cdot, \cdot]_0^r) \) if and only if \( r = 2 \) or \( g_{r-1} = 0 \) (trivial case).

(ii) The Lie bracket \([\cdot, \cdot]_0^r\) on \( g^r\) can be extended in a natural way to some kind of “bracket” \([\cdot, \cdot]_{r-1}^r\) on \( \mathbb{R}^n + g^r\), defined as follows

\[ [\cdot, \cdot]_{r-1}^r_{g^r \times g^r} := [\cdot, \cdot]_0^r \]

\[ [v, v']_{r-1} := 0 \]

\[ [t_p, v]_{r-1}^r := t_p(v) := -[v, t_p]_{r-1}, \]

where \( v, v' \in \mathbb{R}^n, \ t_p \in g_p, \ p \geq 0 \). Certainly \( \mathbb{R}^n + g^r \) always closes under \([\cdot, \cdot]_{r-1}^r\); however, the Jacobi-identity is fulfilled — only then it becomes possible to define the Lie algebra \( \mathbb{R}^n + g^r \equiv (\mathbb{R}^n + g^r, [\cdot, \cdot]_{r-1}^r) \) — if and only if either \( r = 1 \) (we get the affine algebra \( \mathbb{R}^n \otimes g \)) or the prolongation \( g_r \) vanishes.

Besides the adjoint representation, \( G^r \) has another natural representation; let us show how this one comes out.

The \( r \)-jet lift \( G^r(\mathbb{R}^n) \) of \( G\mathbb{R}^n \) is defined as the subset of \( F^r(\mathbb{R}^n) \) induced by those diffeomorphisms \( \varphi \in \mathcal{D}_{0,v}(\mathbb{R}^n) \), \( v \in \mathbb{R}^n \), whose associated \( \tilde{\varphi} \) takes \( \sigma_t \) into
a (local) section of $L \mathbb{R}^n$ having $(r - 1)$-order contact over $\varphi(0) \in \mathbb{R}^n$ with some section of $G \mathbb{R}^n$

$$G^r(\mathbb{R}^n) := \{ j_0^r(\varphi) \in F^r(\mathbb{R}^n) : \exists \sigma \in \text{Sec}(G \mathbb{R}^n) \text{ with } j_0^{r-1}(\tilde{\varphi} \circ \sigma_1) = j_0^{r-1}(\varphi \circ \varphi) \}.$$ 

One sees very easily that $G^r(\mathbb{R}^n)$ is a (trivial) $G^r$-principal bundle over $\mathbb{R}^n$, in fact a reduction of $F^r(\mathbb{R}^n)$ (see section 2).

The canonical isomorphism $G^1 \simeq G$ yields a natural representation of $G^1$ in $\mathbb{R}^n$. For $r > 1$, the above construction of $G^r(\mathbb{R}^n)$ leads to a natural representation $A^{G^r}$ of $G^r$ in the vector space $\mathbb{R}^n + g^{r-1}$, as follows:

(i) Every diffeomorphism $g : U \to V$ in $D_{0,0}(\mathbb{R}^n)$ raises a diffeomorphism $T^{-1}_g : F^{r-1}(\mathbb{R}^n)|_U \to F^{r-1}(\mathbb{R}^n)|_V$, $j_0^{r-1}(\psi) \mapsto j_0^{r-1}(g \circ \psi \circ g^{-1})$ (it is not a principal bundle isomorphism!). One can prove [4] that the tangent map $T^{-1}_g|_{j_0^{r-1}(I)}$, which depends in fact on $j_0^r(g)$, preserves the subspace $T_{j_0^{r-1}(I)}(G^{r-1}(\mathbb{R}^n))$ of $T_{j_0^{r-1}(I)}(F^{r-1}(\mathbb{R}^n))$; thus we can write

$$T^{-1}_g|_{j_0^{r-1}(I)} : T_{j_0^{r-1}(I)}(G^{r-1}(\mathbb{R}^n)) \to T_{j_0^{r-1}(I)}(G^{r-1}(\mathbb{R}^n)).$$

(ii) The global trivialization (see section 2) $\Psi^{r-1} : G^{r-1}(\mathbb{R}^n) \to \mathbb{R}^n \times G^{r-1}$ and the associated canonical identification $T_{j_0^{r-1}(I)}G^{r-1}(\mathbb{R}^n) \simeq \mathbb{R}^n + g^{r-1}$ lead to the (injective) representation

$$A^{G^r} : G^r \to GL(\mathbb{R}^n + g^{r-1})$$

$$j_0^r(g) \mapsto (T^{-1}_g|_{j_0^{r-1}(I)} : \mathbb{R}^n + g^{r-1} \to \mathbb{R}^n + g^{r-1}).$$

This representation of $G^r$ induces a representation $a^{g^r}$ of the Lie algebra $g^r$, given by

$$a^{g^r} \equiv A^{g^r}|_{j_0^r(I)} : g^r \to gl(\mathbb{R}^n + g^{r-1}).$$

Let us consider what happens if $g_{r-1} = 0$. In that case, the above cited Lie group isomorphism $\beta^{r,r-1} : G^r \to G^{r-1}$ gives an obvious representation

$$\overline{A}^{G^{r-1}} := A^{G^r} \circ (\beta^{r,r-1})^{-1} : G^{r-1} \to GL(\mathbb{R}^n + g^{r-1}).$$

4 The 2-jet lift of $G$

We now give a short review of the 2-jet lift $G^2$ of a matrix group $G$. The material of this section is fairly well known, although our description shows up the semidirect product structure of $G^2(n)$, so the outcoming formulas are not the usual ones [3].
As the Abelian group $G^{2,1}(n) = \{ j_0^2(Dg)|_0^{-1} \circ g : j_0^2(g) \in G^2(n) \}$ is isomorphic via the map $u_1$ to the vector group $S^2(n)$, we get the isomorphism

$$(u_0, \tilde{u}_1) := (I, u_1) \circ \chi : G^2(n) \xrightarrow{\sim} GL(n, \mathbb{R}) \boxtimes S^2(n)$$

where the product law in $GL(n, \mathbb{R}) \boxtimes S^2(n)$ is given by

$$(a, t) \cdot (a', t') := (aa', a'^{-1}t(a', a') + t').$$

Restricting $(u_0, \tilde{u}_1)$ to

$$G^2 = \{ j_0^2(g) \in G^2(n) : \exists \kappa \in \mathcal{F}(\mathbb{R}^n, G) \text{ with } Dg|_0 = \kappa(0) \text{ and } D^2g|_0 = D\kappa|_0 \}$$

we get, just because $Dg|_0^{-1}D^2g|_0 = (\kappa^{-1}D\kappa)|_0 \in S^2(n) \cap L(\mathbb{R}^n, g)$, the isomorphism (which will be understood in what follows)

$$(u_0, \tilde{u}_1) : G^2 \rightarrow G \boxtimes g_1$$

$$(j_0^2(g) \mapsto (Dg|_0, Dg|_0^{-1}D^2g|_0).$$

Before looking into the representation $Ad^{G^2}$ and $\mathcal{A}^{G^2}$ of $G^2$, we want to introduce the following useful notations: $\forall t \in L^r(n) \simeq L(\mathbb{R}^n, L^{r-1}(n))$, $\forall a \in GL(n, \mathbb{R})$ and $\forall \alpha \in gl(n, \mathbb{R})$, we write

$$t_a \equiv at(a^{-1}, \ldots, a^{-1}) \in L^r(n)t^a$$

$$L^r(n)t^a \equiv \alpha - t(\alpha \cdot \cdot \cdot , \cdot \cdot \cdot ) - \ldots - t(\cdot \cdot \cdot , \alpha \cdot \cdot \cdot ) \in L^r(n).$$

Note that if we call $Q \in Hom(GL(n, \mathbb{R}), GL(L^r(n)))$, $Q(a) : t \rightarrow t_a$, and $q \in Hom(gl(n, \mathbb{R}), gl(L^r(n)))$, $q(\alpha) : t \rightarrow t^a$; it is very easy to see that $q = Q|_t$.

In the case $t \in S^2(n) \subset L^2(n) \simeq L(\mathbb{R}^n, gl(n, \mathbb{R}))$, it is easy to check the following properties:

(i) Both maps $t \mapsto t_a$ and $t \mapsto t^a$ belong to $gl(S^2(n))$,

(ii) $t_{ab} = (t_b)_a$ and $t^{[\alpha, \beta]} = (t^\beta)^\alpha - (t^\alpha)^\beta$,

(iii) $t_a(v) = at^{-1}a^{-1}a$ and $t^a(v) = [\alpha, t(v)] - t(\alpha v), \forall v \in \mathbb{R}^n$,

(iv) $t(v)(w) - t(w)(v) = 2[t(v), t(w)], \forall v, w \in \mathbb{R}^n$,

(v) $(ta)_a = (t_a)^{aa^{-1}}$.

Remark. We can rewrite the product law in $GL(n, \mathbb{R}) \boxtimes S^2(n)$ as follows:

$$(a, t) \cdot (a', t') = (aa', t_{a^{-1}} + t'), \text{ thus } (a, t)^{-1} = (a^{-1}, -t_a).$$
With this notations, the representations $\text{Ad}^{G^2}$ (adjoint) and $\mathcal{A}^{G^2}$ —and the corresponding representations $\text{ad}^{g^2}$ and $a^{g^2}$— have the form

$$\text{Ad}^{G^2} : G^2 \simeq G \otimes g_1 \rightarrow \text{Aut}(g \otimes g_1)$$

$$(a, t) \mapsto \begin{pmatrix} \text{Ad}_a^G & 0 \\ -(t \cdot) a & (\cdot)_a \end{pmatrix},$$

or $\text{Ad}^{G^2}_{(a, t)}(\beta, s) = (a\beta a^{-1} , s_a - (t\beta)_a)$;

$$\text{ad}^{g^2} : g^2 \simeq g \otimes g_1 \rightarrow \text{Der}(g \otimes g_1)$$

$$(\alpha, t) \mapsto \begin{pmatrix} \text{ad}_\alpha^g & 0 \\ -t(\cdot) & (\cdot)_a \end{pmatrix},$$

or $\text{ad}^{g^2}_{(\alpha, t)}(\beta, s) = ([\alpha, \beta], s^a - t^\beta) = [(\alpha, t), (\beta, s)]^2_0$;

$$\mathcal{A}^{G^2} : G^2 \simeq G \otimes g_1 \rightarrow \text{GL}(\mathbb{R}^n + g)$$

$$(a, t) \mapsto \begin{pmatrix} a & 0 \\ 0 & \text{Ad}_a^G(t \cdot) \end{pmatrix},$$

or $\mathcal{A}^{G^2}_{(a, t)}(\nu, \beta) = (av, a(t(v) + \beta)a^{-1})$;

$$a^{g^2} : g^2 \simeq g \otimes g_1 \rightarrow \text{gl}(\mathbb{R}^n + g)$$

$$(\alpha, t) \mapsto \begin{pmatrix} \alpha & 0 \\ t(\cdot) & a^{ad}_\alpha^g \end{pmatrix},$$

or $a^{g^2}_{(\alpha, t)}(\nu, \beta) = (av, t(v) + [\alpha, \beta])$.

The case $g_1 = 0$. In that case, one has the isomorphism

$$G \rightarrow G^2 \simeq G \otimes \{0\}$$

$$a \mapsto (a, 0)$$

$$\overline{\mathcal{A}}^G : G \rightarrow \text{Aut}(\mathbb{R}^n \oplus g)$$

and

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & \text{Ad}_a^G \end{pmatrix},$$

or $\overline{\mathcal{A}}^G_a(\nu, \beta) = (av, a\beta a^{-1})$ (we write $\overline{\mathcal{A}}^G_a \in \text{Aut}(\mathbb{R}^n \oplus g)$ because $\overline{\mathcal{A}}^G$ preserves the bracket in the Lie algebra $\mathbb{R}^n \oplus g$).

5 The 3-jet lift of $G$

In analogous way as we did for $G^2$, we first describe $G^3$ as a semidirect product of $G$ itself and a (now non-abelian) nilpotent group builded up on the first two
prolongations $g_1$ and $g_2$ of its Lie algebra $g$.

The nilpotent group $G^{3,1}(n) = \{ j^3_0(g) \in G^3(n) : Dg|_0 = I \}$ is isomorphic —via $(u_1, u_2)$— to the group $(S^2(n) \times S^3(n), *)$, $*$ being the product law induced by applying the chain rule to the derivatives of the composition of functions

$$(t, T) \star (t', T') := (t + t', T + 3S(t) t' + T').$$

Now, instead of using the chart $(u_1, u_2)$, it will be more convenient (see the next lemma) to use the chart $(u_1, u_2 - \frac{3}{2} S(u_1(\cdot)u_1(\cdot)))$, which establishes an isomorphism between $G^{3,1}(n)$ and the group $(S^2(n) \times S^3(n), \circ)$, where $\circ$ stands for the product law

$$(t, T) \circ (t', T') := (t + t', T + \frac{1}{2}[t, t']^3_0 + T'),$$

and the bracket $[t, t']^3_0 := 3S(t(\cdot) t'(\cdot)) - 3S(t'(\cdot) t(\cdot))$ was given in section 3. In that way we get the isomorphism

$$(u_0, \bar{u}_1, \bar{u}_2) := (I, u_1, u_2 - \frac{3}{2} S(u_1(\cdot)u_1(\cdot))) \circ \chi : G^3(n) \longrightarrow GL(n,\mathbb{R}) \ltimes (S^2(n) \times S^3(n), \circ)$$

$$j^3_0(g) \longmapsto (Dg|_0, Dg|_0^{-1} D^2g|_0, Dg|_0^{-1} D^3g|_0 - \frac{3}{2} S(Dg|_0^{-1} D^2g|_0(\cdot) Dg|_0^{-1} D^2g|_0(\cdot)),)$$

where the product law in $GL(n,\mathbb{R}) \ltimes (S^2(n) \times S^3(n), \circ)$ is given by

$$(a, t, T) \cdot (a', t', T') := (aa', (t_{a'-1}, T_{a'-1}) \circ (t', T')) = (aa', t_{a'-1} + t', T_{a'-1} + \frac{1}{2}[t_{a'-1}, t']_0^3 + T')$$

(what implies $(a, t, T)^{-1} = (a^{-1}, -t_{a}, -T_{a})$).

**Remark.** Note that the term $\frac{1}{2}[t_{a'-1}, t']_0^3$ prevents this product to be a semidirect product of the form $G^2(n) \ltimes S^3(n)$; the group $G^2(n)$ has no natural structure as a subgroup of $G^3(n)$ (see the first remark in section 3).

As the following lemma establishes, the above chart $(u_0, \bar{u}_1, \bar{u}_2)$ provides, by restriction, an isomorphic image of $G^3$ in terms of $G$ and the first two prolongations of $g$, and this is a very interesting fact for computations.

**Lemma.** The group $G^3$ is isomorphic —via $(u_0, \bar{u}_1, \bar{u}_2)$— to the group $G \ltimes (g_1 \times g_2, \circ)$.

**Proof.** As we said above, it holds: $j^3_0(g) \in G^3 \iff \exists \kappa \in \mathcal{F}(\mathbb{R}^n, G)$ such that

1. $Dg|_0 = \kappa(0) \in G$,
2. $D^2g|_0 = D\kappa|_0(\cdot) \in S^2(n)$,
3. $D^3g|_0 = D^2\kappa|_0(\cdot, \cdot) \in S^3(n)$.
As for $G^2$, conditions (1) and (2) lead to
\[ \tilde{u}_1(j_3^0(g)) \in g_1. \]
Now we consider condition (3). Let us call $\kappa_1(\cdot) \equiv \kappa^{-1}D\kappa \in \mathcal{F}(\mathbb{R}^n, L(\mathbb{R}^n, g))$ and $\kappa_2(\cdot, \cdot) \equiv \kappa^{-1}D^2\kappa \in \mathcal{F}(\mathbb{R}^n, L_{\text{sim}}(\mathbb{R}^n \times \mathbb{R}^n, gl(n, \mathbb{R}))).$ Note that although $\kappa_1|_0(\cdot) \in L(\mathbb{R}^n, g)$, in general $\kappa_2|_0(\cdot, \cdot)$ does not belong to $L(\mathbb{R}^n \times \mathbb{R}^n, g)$; so conditions (1) and (3) do not lead to $Dg|_0^{-1}D^3g|_0 \in g_2$.

From $D(\kappa^{-1})(\cdot) = -\kappa^{-1}D\kappa(\cdot)\kappa^{-1}$, it immediately follows
\[ \kappa_2(\cdot, \cdot) = D\kappa_1(\cdot, \cdot) + \kappa_1(\cdot)\kappa_1(\cdot); \]
then, at the origin of $\mathbb{R}^n$,
\[ \kappa_2|_0(\cdot, \cdot) \in \kappa_1|_0(\cdot)\kappa_1|_0(\cdot) + L(\mathbb{R}^n \times \mathbb{R}^n, g). \]
Thus conditions (1) and (3) lead to
\[ (3') \quad \kappa_2|_0(\cdot, \cdot) \in (\kappa_1|_0(\cdot)\kappa_1|_0(\cdot) + L(\mathbb{R}^n \times \mathbb{R}^n, g)) \cap S^3(n). \]
Now observe that, given any $s \in g_1 := L(\mathbb{R}^n, g) \cap S^2(n)$, the identity
\[ s(\cdot)s(\cdot, \cdot) = \frac{3}{2}S(s(\cdot)s(\cdot, \cdot)) + \frac{1}{2}s^s(\cdot, \cdot) \]
holds with $s^s(\cdot) \in L(\mathbb{R}^n, g) \subset L(\mathbb{R}^n \times \mathbb{R}^n, g)$; applying this identity to $s = \kappa_1|_0$, we get from $(3')$
\[ \kappa_2|_0(\cdot, \cdot) = \frac{3}{2}S(\kappa_1|_0(\cdot)\kappa_1|_0(\cdot, \cdot)) \in L(\mathbb{R}^n \times \mathbb{R}^n, g) \cap S^3(n) = g_2; \]
thus again (1) and (3) lead to
\[ Dg|_0^{-1}D^3g|_0 - \frac{3}{2}S(Dg|_0^{-1}D^2g|_0(\cdot)Dg|_0^{-1}D^2g|_0(\cdot, \cdot)) \in g_2, \]
or, in other words, $\tilde{u}_2(j_3^0(g)) \in g_2$.

Once we know that $(u_0, \tilde{u}_1, \tilde{u}_2)(G^3) \subset G \times g_1 \times g_2$, the assert of the lemma follows immediately.

\[ \square \]

**Corollary.** If $g_2 = 0$, every $j_3^0(g) \in G^3$ satisfies
\[ Dg|_0^{-1}D^3g|_0 - \frac{3}{2}S(Dg|_0^{-1}D^2g|_0(\cdot)Dg|_0^{-1}D^2g|_0(\cdot, \cdot)) = 0. \]

**Remark.**

Thus the *algebraic* condition for a matrix group $G$ of being of finite type 2 *analytic* property: every $G\mathbb{R}^n$-preserving diffeomorphism $\varphi \in D_{0,v}(\mathbb{R}^n)$, $v \in \mathbb{R}^n$, must satisfy
\[ D\varphi|_0^{-1}D^3\varphi|_0 - \frac{3}{2}S(D\varphi|_0^{-1}D^2\varphi|_0(\cdot)D\varphi|_0^{-1}D^2\varphi|_0(\cdot, \cdot)) = 0. \]
In particular, let $CO(n, \mathbf{R})$ (with $n \geq 3$) be the linear conformal group with respect to some (pseudo) euclidean scalar product $\eta$ in $\mathbf{R}^n$; as it is well known, $CO(n, \mathbf{R})$ is of finite type 2. It thus follows that every conformal (local) diffeomorphism in $(\mathbf{R}^n, \eta)$ satisfies this condition, as one immediately checks.

Using the above isomorphism $(u_0, \tilde{u}_1, \tilde{u}_2)$, we get the following

**Theorem.** The representations $Ad^{G^3}$ (adjoint) and $A^{G^3}$ —and the corresponding representations $ad^{G^3}$ and $a^{G^3}$— are given by

\[
Ad^{G^3} : G^3 \simeq G \otimes (g_1 \times g_2, \circ) \rightarrow \text{Aut}(g + g_1 + g_2, \mathbf{R}^n) \\
(a, t, T) \mapsto \begin{pmatrix}
\text{Ad}_a^G & 0 \\
-(T^{(\cdot)}_a) & (\cdot)_a
\end{pmatrix},
\]

or $Ad^{G^3}_{(a,t,T)}(\beta, s, S) = (a\beta a^{-1}, s_a - (t^\beta)_a, (S - T^\beta + [t, s]_3 + \frac{1}{2}[t^\beta, t]_3)_a)$;

\[
ad^{G^3} : g^3 \simeq G \otimes (g_1 \times g_2, \circ) \rightarrow \text{Der}(g + g_1 + g_2, \mathbf{R}^n) \\
(\alpha, t, T) \mapsto \begin{pmatrix}
ad_{\alpha}^G & 0 \\
-t(\cdot) & (\cdot)_\alpha
\end{pmatrix},
\]

or $ad^{G^3}_{(\alpha,t,T)}(\beta, s, S) = ([\alpha, \beta], s^\alpha - t^\beta, S^\alpha - T^\beta + [t, s]_3) = [(\alpha, t, T), (\beta, s, S)]_3$;

\[
A^{G^3} : G^3 \simeq G \otimes (g_1 \times g_2, \circ) \rightarrow GL(\mathbf{R}^n + g + g_1) \\
(a, t, T) \mapsto \begin{pmatrix}
\text{Ad}_a^G t(\cdot) & 0 \\
0 & \text{Ad}_a^G
\end{pmatrix},
\]

or $A^{G^3}_{(\alpha,t,T)}(v, \beta, s) = (av, \alpha (t(v) + \beta) a^{-1}, (T(v) - \frac{1}{2}[t^\beta, t^\alpha])_a)$;

\[
a^{G^3} : g^3 \simeq G \otimes (g_1 \times g_2, \circ) \rightarrow GL(\mathbf{R}^n + g + g_1) \\
(\alpha, t, T) \mapsto \begin{pmatrix}
\alpha & 0 \\
0 & \alpha + t(\cdot) & (\cdot)_\alpha
\end{pmatrix},
\]

or $a^{G^3}_{(\alpha,t,T)}(v, \beta, s) = (av, t(v) + [\alpha, \beta], T(v) + s^\alpha - t^\beta)$.

**Proof.** A careful computation is needed; for more details see [9]. The expressions for $Ad^{G^3}_{(a,t,T)} := I_{Ad^{G^3}_{(a,t,T)}}$ and $A^{G^3}_{(\alpha,t,T)} := T^{G^3}_{(\alpha,t,T)}$ follow from the
following expressions of the inner automorphisms of $G^3$

\[
\tilde{\mathcal{A}}_{\langle a, t \rangle}^G : \text{Aut}(\mathbb{R}^n + g + g_1, [\cdot, \cdot]_2^2) \to \mathbb{R}^n + g + g_1 \times [\cdot, \cdot]_2^2 \\
(a, t) \mapsto \left( \begin{array}{ccc} a & 0 & 0 \\ Ad_a^G t(\cdot) & Ad_a^G 0 \\ \frac{-1}{2}(t^{(2)})_a & -(t^{(3)})_a & (\cdot)_a \end{array} \right),
\]

or \( \tilde{\mathcal{A}}_{\langle a, t \rangle}^G(v, \beta, s) = (av, a(t(v) + \beta)a^{-1}, (-\frac{1}{2}t^{(v)} - t^{(3)} + s)_a). \)

**Remarks.**

(i) In the case \( g_2 = 0 \), observe that: \( \tilde{\mathcal{A}}_{\langle a, t \rangle}^{g_1} = Ad_{\langle a, t \rangle}^{g_1} \) and \( \tilde{\pi}_{\langle a, t \rangle}^{g_1} = ad_{\langle a, t \rangle}^{\text{Def} + \mathbb{R}^2}. \)

(ii) When \( G = CO(n, \mathbb{R}) \), the linear conformal group, it is well known [6] that \( CO(n, \mathbb{R})^2 \) is isomorphic to a Lie subgroup \( L_0 \) of the full conformal group \( L \equiv O(n + 2)/\mathbb{Z}_2 \), as follows:

\[
CO(n, \mathbb{R})^2 \simeq CO(n, \mathbb{R}) \times co(n, \mathbb{R})_1 \xrightarrow{\simeq} L_0
\]

\[
(kb, -\tilde{\mu}) \mapsto A_{\langle k, b, \mu \rangle} \equiv \left( \begin{array}{ccc} k & 0 & 0 \\ b\mu & \mu & 0 \\ (2k)^{-1}(\mu, \mu') & k^{-1}\mu & k^{-1} \end{array} \right),
\]

with \( k \in \mathbb{R}^+, \ b \in O(n, \mathbb{R}), \ \mu \in \mathbb{R}^n, \ \tilde{\mu} \in co(n, \mathbb{R})_1, \ \mu' \in \mathbb{R}^n \) (we
have used the isomorphisms: $\mathbb{R}^{n*} \rightarrow \mathbb{R}^n, \, \mu \mapsto \mu^i$, determined by $\eta(\mu^i, \cdot) = \mu$, with $\eta$ a standard scalar product of arbitrary signature, and $\mathbb{R}^{n*} \rightarrow \text{co}(n, \mathbb{R})_1, \, \mu \mapsto \tilde{\mu}$, where 
\[ \tilde{\mu}(v, w) := \mu(v)w + \mu(w)v - \eta(v, w)\mu^i. \]

This induces the following isomorphism between the Lie algebras
\[
\text{co}(n, \mathbb{R})^2 \simeq \text{co}(n, \mathbb{R}) \oplus \text{co}(n, \mathbb{R})_1 \xrightarrow{\sim} \mathfrak{l}_0
\]
\[
(\kappa I + \beta, -\tilde{\mu}) \mapsto a_{(\kappa, b, \mu)} \equiv \begin{pmatrix}
\kappa & 0 & 0 \\
\mu^i & \beta & 0 \\
0 & \mu & -\kappa
\end{pmatrix}
\]
with $\kappa \in \mathbb{R}, \, \beta \in \text{co}(n, \mathbb{R}), \, \mu \in \mathbb{R}^{n*}$.

Moreover one sees very easily the isomorphism
\[
(\mathbb{R}^n + \text{co}(n, \mathbb{R}) + \text{co}(n, \mathbb{R})_1, \, [ \cdot, \cdot ]_{2,1}^2) \xrightarrow{\sim} \mathfrak{l}
\]
\[
(w, \kappa I + \beta, -\tilde{\mu}) \mapsto \begin{pmatrix}
\kappa & w^i & 0 \\
\mu^i & \beta & w \\
0 & \mu & -\kappa
\end{pmatrix}
\]
(being $w^i = \eta(w, \cdot)$) with
\[ [w, w]_{2,1}^2 = 0, \]
\[ [\beta, w]_{2,1}^2 = \beta w', \]
\[ [	ilde{\mu}, w]_{2,1}^2 = \tilde{\mu}(w'), \]
\[ [\beta, \beta]_{2,1}^2 = \beta \beta' - \beta' \beta, \]
\[ [\tilde{\mu}, \beta]_{2,1}^2 = -\tilde{\mu} \beta' - \mu \beta, \]
\[ [\tilde{\mu}, \tilde{\mu}]_{2,1}^2 = 0. \]

The last isomorphism yields (after a straightforward computation) the correspondence
\[ \mathcal{A}^2_{(k, b, -\tilde{\mu})}(w, \kappa I + \beta, -\tilde{\mu}) \mapsto \text{Ad}^2_{A_{(\kappa, b, \mu)}} a_{(\kappa, \beta, \nu)}. \]

Thus we conclude that
\[ \mathcal{A}^2_{\mathfrak{j}_0(g)} = \text{Ad}^2_{\mathfrak{j}_0(g)}, \quad \forall \mathfrak{j}_0^2(g) \in CO(n, \mathbb{R})^2; \]
for details see [9].

6 Final comment

As we mentioned in the introduction we are interested in the relationships between true connections and Cartan connections on 2-jet lifts $P^2(M, G^2)$ of 1-integrable
$G$-structures when $P$ is of finite type 2. The behaviour of such geometrical objects under the principal action is controlled by the above representations $Ad^{G^2}$ and $\mathcal{A}^{G^2}$, respectively. Using such representations we have been able to prove that, given a Cartan connection on $P^2$ and a symmetric connection on $P$, a distinguished connection on $P^2$ arises which is not the trivial prolongation of the latter one; in the conformal case, the geodesics of that second order connection are closely related to the so called “conformal circles”.

References


Eduardo Aguirre-Dabán
Departamento de Geometría y Topología
Facultad de Matemáticas
Universidad Complutense
28040 Madrid
Spain
Email: Eduardo_Aguirre@Mat.ucm.es

Ignacio Sánchez-Rodríguez
Departamento de Geometría y Topología
Facultad de Ciencias
Universidad de Granada
18071 Granada
Spain
Email: ignacios@goliat.ugr.es