STABILITY OF CONIC BUNDLES (WITH AN APPENDIX BY MUNDET I RIERA)

T. GÓMEZ AND I. SOLS

Abstract. Roughly speaking, a conic bundle is a surface, fibered over a curve, such that the fibers are conics (not necessarily smooth). We define stability for conic bundles and construct a moduli space. We prove that (after fixing some invariants) these moduli spaces are irreducible (under some conditions). Conic bundles can be thought of as generalizations of orthogonal bundles on curves. We show that in this particular case our definition of stability agrees with the definition of stability for orthogonal bundles. Finally, in an appendix by I. Mundet i Riera, a Hitchin-Kobayashi correspondence is stated for conic bundles.

1. Introduction

In this paper we introduce the notion of stable conic bundle. This notion appears as the stability condition in the GIT construction of the moduli space of these objects.

Let X be a smooth complex curve of genus g. Let $r > 0$ and d be two integer numbers. Let $\mathscr{L}$ be a line bundle over X. These data will be fixed throughout the paper.

Definition 1.1. *A conic bundle on X of type $(r, d, \mathscr{L})$ is a pair $(\mathscr{E}, Q)$ where $\mathscr{E}$ is a vector bundle on X of rank r and degree d, and Q is a morphism*

$$Q : Sym^2\mathscr{E} \to \mathscr{L}.$$

A morphism between conic bundles $\varphi : (\mathscr{E}, Q) \to (\mathscr{E}', Q')$ is a morphism $f : \mathscr{E} \to \mathscr{E}'$ such that there is a commutative diagramme

$$\begin{tikzcd} \operatorname{Sym}^2 \mathscr{E} \arrow[r, "\operatorname{Sym}^2 f"] \arrow[d, "Q"'] & \operatorname{Sym}^2 \mathscr{E}' \arrow[d, "Q'"'] \\ \mathscr{L} \arrow[r, "g"] & \mathscr{L} \end{tikzcd}$$

where g is a scalar multiple of identity.

Then two conic bundles $(\mathscr{E}, Q)$ and $(\mathscr{E}', Q')$ will be isomorphic when there is an isomorphism $\mathscr{E} \cong \mathscr{E}'$ that takes Q into a scalar multiple of Q'. The name conic bundle comes from the case $r = 3$. We will be mostly interested in this case, and in fact we will only define stability for $r \leq 3$.

If $\mathscr{L} = \mathcal{O}_X$ and Q gives a nondegenerate quadratic form on each fiber, then the conic bundle is equivalent to an orthogonal bundle (see [6]). In this case there is

Date: July 2, 1999.
Mathematical Subject Classification: Primary 14D22, Secondary 14D20.

already a definition of stability, and we check in section 3.2 that it is a particular case of our definition.

Definition 1.2. *Consider a conic bundle $(\mathscr{E}, Q)$ and a subbundle $\mathscr{E}'$ of $\mathscr{E}$ of rank r'. Let x be a general point in X. If $\mathscr{F}_1$ and $\mathscr{F}_2$ are subbundles of $\mathscr{E}$, we denote by $\mathscr{F}_1\mathscr{F}_2$ the subbundle of $\operatorname{Sym}^2 \mathscr{E}$ generated by elements of the form $f_1 f_2$ where f_1 and f_2 are local sections of $\mathscr{F}_1$ and $\mathscr{F}_2$. We define a function $c_Q(\mathscr{E}')$ as follows:*

$$c_Q(\mathscr{E}') = \begin{cases} 2, & \textit{if} \quad Q|_{\mathscr{E}'\mathscr{E}'} \neq 0 \\ 1, & \textit{if} \quad Q|_{\mathscr{E}'\mathscr{E}} \neq 0 = Q|_{\mathscr{E}'\mathscr{E}'} \\ 0, & \textit{if} \quad Q|_{\mathscr{E}'\mathscr{E}} = 0. \end{cases}$$

Sometimes it will be convenient to write this type of conditions on Q in matrix form. Choosing a basis compatible with the filtration $\mathscr{E}' \subset \mathscr{E}$ these three cases can be expressed as follows

$$\begin{pmatrix} \times & \cdot \\ \cdot & \cdot \end{pmatrix}, \qquad \begin{pmatrix} 0 & \times \\ \times & \cdot \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 \\ 0 & \times \end{pmatrix},$$

where $\times$ means that that block is nonzero, 0 means that it is zero and $\cdot$ means that it can be anything.

Definition 1.3. *Let $(\mathscr{E}, Q)$ be a conic bundle. We say that two subbundles $\mathscr{E}_1 \subset \mathscr{E}_2 \subset \mathscr{E}$ give a critical filtration of $(\mathscr{E}, Q)$, if $\operatorname{rk}(\mathscr{E}_1) = 1$, $\operatorname{rk}(\mathscr{E}_2) = 2$, $\operatorname{rk}(\mathscr{E}) = 3$, $Q|_{\mathscr{E}_1\mathscr{E}_2} = 0$, and $Q|_{\mathscr{E}_1\mathscr{E}} \neq 0 \neq Q|_{\mathscr{E}_2\mathscr{E}_2}$.*

The fact that $\mathscr{E}_1 \subset \mathscr{E}_2 \subset \mathscr{E}$ is a critical filtration of $(\mathscr{E}, Q)$ means that for a generic point $x \in X$, the conic Q_x defined by Q on the fibre of $\mathbb{P}(\mathscr{E})$ over x is smooth, the point defined by $\mathscr{E}_1$ is in the conic and the line defined by $\mathscr{E}_2$ is tangent to the conic. In matrix form with a basis adapted to the filtration $\mathscr{E}_1 \subset \mathscr{E}_2 \subset \mathscr{E}$ this can be expressed as

$$Q = \begin{pmatrix} 0 & 0 & \times \\ 0 & \times & \cdot \\ \times & \cdot & \cdot \end{pmatrix}$$

Later on (definition 2.6) we will introduce a similar definition for filtrations of vector spaces.

Now we are ready to define the notion of stability. We will only define it for $r \leq 3$. As it is usual when one is working with vector bundles with extra structure, this notion will depend on a positive rational number τ. We could as well take τ to be a real number, but this wouldn't give anything new because when we vary τ the stability of a conic bundle can only change at rational values of τ.

We follow the notation of [1]: Whenever the word '(semi)stable' appears in a statement with the symbol '$(\leq)$', two statements should be read. The first with the word 'stable' and strict inequality, and the second with the word 'semistable' and the relation '$\leq$'.

Definition 1.4. *Let τ be a positive rational number. Let $(\mathscr{E}, Q)$ be a conic bundle with $r \leq 3$. We say that $(\mathscr{E}, Q)$ is (semi)stable with respect to τ if the following conditions hold*

(ss.1) If $\mathscr{E}'$ is a proper subbundle of $\mathscr{E}$, then

$$\frac{\deg(\mathscr{E}') - c_Q(\mathscr{E}')\tau}{\operatorname{rk}(\mathscr{E}')} (\leq) \frac{\deg(\mathscr{E}) - 2\tau}{r}.$$

(ss.2) If $\mathscr{E}_1 \subset \mathscr{E}_2 \subset \mathscr{E}$ is a critical filtration, then

$$\deg(\mathscr{E}_1) + \deg(\mathscr{E}_2)(\leq) \deg(\mathscr{E}).$$

Note that condition (ss.1) is reminiscent of the stability conditions for vector bundles with extra structure in the literature, but condition (ss.2) is new. It is due to the fact that in a conic bundle, Q is a nonlinear object. So far all objects that have been considered were linear, and this is why this kind of conditions didn't appear. This nonlinearity is responsible for the fact that the proof is more involved, and we have to consider only conic bundles with $r \leq 3$. For higher r we expect to have more conditions of the form (ss.2).

Lemma 1.5. *Let $(\mathscr{E}_1, Q_1)$ and $(\mathscr{E}_2, Q_2)$ be stable conic bundles of the same type $(r, d, \mathscr{L})$. Then any nontrivial morphism $\phi : (\mathscr{E}_1, Q_1) \to (\mathscr{E}_2, Q_2)$ is an isomorphism, and furthermore it is a scalar multiple of identity.*

Proof. Assume that ϕ is nontrivial. Let $f : \mathscr{E}_1 \to \mathscr{E}_2$ be the corresponding morphism of sheaves. Consider the subsheaves $\mathscr{E}' = \ker f$ of $\mathscr{E}_1$ and $\mathscr{E}'' = \operatorname{im} f$ of $\mathscr{E}_2$. Assume $\mathscr{E}' \neq 0$. By commutativity of the diagramme

$$\begin{tikzcd}
\operatorname{Sym}^2 \mathscr{E}' \arrow[r] \arrow[d, "Q_1"'] & \operatorname{Sym}^2 \mathscr{E}_1 \arrow[r, "\operatorname{Sym}^2 f"] \arrow[d, "Q_1"'] & \operatorname{Sym}^2 \mathscr{E}_2 \arrow[d, "Q_2"'] \\
\mathscr{L} \arrow[r, equal] & \mathscr{L} \arrow[r, "g"] & \mathscr{L}
\end{tikzcd}$$

we have that $c_{Q_1}(\mathscr{E}') = 0$, and then by stability

$$\frac{\deg(\mathscr{E}')}{\operatorname{rk}(\mathscr{E}')} < \frac{d - 2\tau}{r} < \frac{\deg(\mathscr{E}'') - 2\tau}{\operatorname{rk}(\mathscr{E}'')} \leq \frac{\deg(\mathscr{E}'') - c_{Q''}(\mathscr{E}'')\tau}{\operatorname{rk}(\mathscr{E}'')} < \frac{d - 2\tau}{r}$$

which is a contradiction. Then $\mathscr{E}' = 0$ and f is an isomorphism. Now let $x \in X$ be a point, and let λ be an eigenvalue of f at the fibre over x. Then $h = f - \lambda \operatorname{id}_{\mathscr{E}_1}$ is not surjective at x, hence h cannot be an isomorphism and then $h = 0$.

□

A flat family of (semi)stable conic bundles of type $(r, d, \mathscr{L})$ parametrized by a scheme T is a triple $(\mathscr{E}_T, Q_T, \mathscr{N})$ where $\mathscr{E}$ is a vector bundle on $X \times T$, flat over T, that restricts to a vector bundle of rank r and degree d on each fibre $X \times t$, and Q_T is a morphims $Q_T : \operatorname{Sym}^2 \mathscr{E}_T \to p_X^* \mathscr{L} \otimes p_T^* \mathscr{N}$ where $\mathscr{N}$ is a line bundle on T, and this morphim restricts to (semi)stable conic bundles on each fibre. Two families $(\mathscr{E}_T, Q_T, \mathscr{N})$ and $(\mathscr{E}'_T, Q'_T, \mathscr{N}')$ will be considered equivalent if there is a line bundle $\mathscr{M}$ on T, an isomorphism $f : \mathscr{E}_T \otimes p_T^* \mathscr{M} \to \mathscr{E}'_T$ and a commutative diagramme

$$\begin{tikzcd}
\operatorname{Sym}^2 \mathscr{E}_T \otimes p_T^* \mathscr{M}^2 \arrow[r, "\operatorname{Sym}^2 f"] \arrow[d, "Q"'] & \operatorname{Sym}^2 \mathscr{E}'_T \arrow[d, "Q'"'] \\
p_X^* \mathscr{L} \otimes p_T^* \mathscr{N} \otimes p_T^* \mathscr{M}^2 \arrow[r, "\cong"] & p_X^* \mathscr{L} \otimes p_T^* \mathscr{N}'
\end{tikzcd}$$

Let $\mathfrak{M}_\tau(r, d, \mathscr{L})^\natural$ (resp. $\overline{\mathfrak{M}}_\tau(r, d, \mathscr{L})^\natural$) be the functor that sends a scheme T to the set of flat families of stable (resp. semistable) conic bundles of type $(r, d, \mathscr{L})$ parametrized by T. The moduli space for this functor will be denoted by $\mathfrak{M}_\tau(r, d, \mathscr{L})$ (resp. $\overline{\mathfrak{M}}_\tau(r, d, \mathscr{L})$).

Theorem I. *Let X be a Riemann surface. Let $\tau > 0$ be a rational number. There exist a projective coarse moduli space $\overline{\mathfrak{M}}_\tau(r,d,\mathscr{L})$ of semistable conic bundles with respect to τ of fixed type $(r,d,\mathscr{L})$. The closed points of $\overline{\mathfrak{M}}_\tau(r,d,\mathscr{L})$ correspond to S-equivalence classes of conic bundles. There is an open set $\mathfrak{M}_\tau(r,d,\mathscr{L})$ corresponding to stable conic bundles. This open set is a fine moduli space of stable conic bundles. Points in this open set correspond to isomorphism classes of conic bundles.*

For a definition of S-equivalence, see subsection 2.3

At the same time we wrote this article, I. Mundet i Riera found the conditions for existence of solutions to a generalization of the vortex equation associated to Kaehler fibrations. As expected, the condition he finds is, in the case of conic bundles, the same as the condition we have found for stability. This is explained in the appendix.

2. GIT Construction

In this section we will construct the moduli space of semistable conic bundles. This construction is based on the ideas of Simpson for the construction of the moduli space of semistable sheaves ([7]). We will follow closely the paper [2] of King and Newstead and the paper [1] of Huybrechts and Lehn. In 2.1 we prove some boundness theorems that are needed later, and in 2.2 we give the construction of the moduli space and prove the semistability condition. The base field k can be any algebraically closed field of characteristic zero, but we are mainly interested in $\mathbb{C}$.

2.1. **Boundness theorems.**

Proposition 2.1. *Let X be a genus g curve. Let $\mathcal{S}$ be a set of vector bundles on X with degree d and rank r. Assume that there is a constant b such that if $\mathscr{E} \in \mathcal{S}$ and $\mathscr{E}'$ is a nonzero subsheaf of $\mathscr{E}$, then*

$$\mu(\mathscr{E}') = \frac{\deg(\mathscr{E}')}{\mathrm{rk}(\mathscr{E}')} \le b.$$

Then there is a constant m_0 such that if $m \ge m_0$, for all $\mathscr{E} \in \mathcal{S}$, we have $h^1(\mathscr{E}(m)) = 0$ and $\mathscr{E}(m)$ is generated by global sections. Hence $\mathcal{S}$ is bounded.

Proof. Let x be a point of the curve X and $\mathscr{E} \in \mathcal{S}$. The exact sequence

$$0 \to \mathscr{E}(m) \otimes \mathcal{O}_X(-x) \to \mathscr{E}(m) \to \mathscr{E}(m)|_x \to 0$$

gives that if $h^1(\mathscr{E}(m) \otimes \mathcal{O}_X(-x)) = 0$ for all $x \in X$, then $\mathscr{E}(m)$ is generated by global sections and $h^1(\mathscr{E}(m)) = 0$.

Assume that $h^1(\mathscr{E}(m) \otimes \mathcal{O}_X(-x)) \neq 0$. Then by Serre duality there is a nonzero morphism $\mathscr{E}(m) \otimes \mathcal{O}_X(-x) \to \mathscr{K}_X$, where $\mathscr{K}_X$ is the canonical divisor. This gives an effective divisor D on X and an exact sequence

$$0 \to \mathscr{E}'(m) \to \mathscr{E}(m) \to \mathscr{K}_X(x - D) \to 0.$$

Let $d' = \deg(\mathscr{E}')$. We have $\mathrm{rk}(\mathscr{E}') = r - 1$. Then

$$d' = (1-r)m + d + rm - (2g - 1 - \deg(D)) \ge d - 2g + 1 + m$$

On the other hand, by hypothesis $d' \le (r-1)b$, and combining both inequalities we get

$$m \le (r-1)b - d + 2g - 1.$$

Then if we take $m_0 > (r-1)b - d + 2g - 1$, for any $m \geq m_0$ and $x \in X$ we will have $h^1(\mathscr{E}(m) \otimes \mathcal{O}_X(-x)) = 0$, thus $\mathscr{E}(m)$ is generated by global sections and $h^1(\mathscr{E}(m)) = 0$. By standard methods using the Quot scheme, this implies that $\mathcal{S}$ is bounded.

□

Corollary 2.2. *The same conclusion is true for the set of vector bundles $\mathscr{E}$ occurring in semistable conic bundles $(\mathscr{E}, Q)$ of fixed type. The constant m_0 depends on X, τ, r and d, but not on $\mathscr{L}$.*

Proof. By condition (ss.1) we have that for every subsheaf $\mathscr{E}'$ of $\mathscr{E}$

$$\frac{\deg(\mathscr{E}')}{\operatorname{rk}(\mathscr{E}')} \leq \frac{d-2\tau}{r} + \frac{c_Q(\mathscr{E}')\tau}{\operatorname{rk}(\mathscr{E}')} \leq \frac{d-2\tau}{r} + 2\tau.$$

Take $b = \frac{d-2\tau}{r} + 2\tau$ and apply the proposition.

□

Corollary 2.3. *Let $\mathcal{S}$ be the set of semistabilizing sheaves, i.e. sheaves $\mathscr{E}'$, $\mathscr{E}_1$, $\mathscr{E}_2$ that give equality in condition (ss.1) or (ss.2). Then the conclusions of proposition 2.1 are also true for $\mathcal{S}$.*

Proof. By semistability, the slope of a subsheaf of a sheaf in $\mathcal{S}$ is bounded. On the other hand there are only a finite number of possibilities for rhe rank and degree of a sheaf in $\mathcal{S}$, then we can apply proposition 2.1.

□

Now we will state two lemmas of King and Newstead ([2, lemma 2.2] and [2, corollary 2.6.2]).

Lemma 2.4. *Let $\mathscr{E}$ be a torsion free sheaf such that for all subsheaf $\mathscr{F}$ of $\mathscr{E}$, $\mu(\mathscr{F}) \leq b$. If $b < 0$, then $h^0(\mathscr{E}) = 0$. If $b \geq 0$ then $h^0(\mathscr{E}) \leq \operatorname{rk}(\mathscr{E})(b+1)$.*

□

Lemma 2.5. *Fix R, b, k. Then there exists an n_0 such that if $\mathcal{S}$ is a set of torsion free sheaves with*
(i) $\operatorname{rk}(\mathscr{E}) \leq R$
(ii) $\mu(\mathscr{F}) \leq b$ *for all nonzero subsheaves* $\mathscr{F}$ *of* $\mathscr{E}$
(iii) For some $n \geq n_0$

$$h^0(\mathscr{E}(n)) \geq \operatorname{rk}(\mathscr{E})(\chi(\mathcal{O}_X(n)) + k)$$

Then the set $\mathcal{S}$ is bounded.

□

2.2. **Construction and proof of main theorem.**

Now we will give the GIT construction of the moduli space. We will assign a point in a projective scheme Z to a conic bundle $(\mathscr{E}, Q)$ of fixed type $(r, d, \mathscr{L})$. Let P be the Hilbert polynomial of $\mathscr{E}$, i.e. $P(m) = rm + d + r(1-g)$. We will assume that m is large enough so that corollaries 2.2 and 2.3 are satisfied. Let V be a vector space of dimension $p = P(m)$. Let $\mathcal{H}$ be the Hilbert scheme $\operatorname{Hilb}(V \otimes \mathcal{O}_X(-m), P)$ parametrizing quotients of $V \otimes \mathcal{O}_X(-m)$ with Hilbert polynomial P. Let $l > m$ be an integer, $W = H^0(\mathcal{O}_X(l-m))$, and G be the Grassmannian $\operatorname{Grass}(V \otimes W, p)$ of quotients of $V \otimes W$ of dimension p. For l large enough we have embeddings

$$\mathcal{H} \to G \to \mathbb{P}(\Lambda^{P(l)}(V \otimes W))$$

Let $B = H^0(\mathscr{L})$ and $\mathcal{P} = \mathbb{P}(\operatorname{Sym}^2(V^\vee \otimes B))$. Given $(\mathscr{E}, Q)$ and an isomorphism $V \cong H^0(\mathscr{E}(m))$ we get a point $(\tilde{q}, \tilde{Q})$ in $\mathcal{H} \times \mathcal{P}$ as follows:

The vector bundle $\mathscr{E}(m)$ is generated by global sections (corollary 2.2), then we have a quotient

$$q : V \otimes \mathcal{O}_X(-m) \cong H^0(\mathscr{E}(m)) \otimes \mathcal{O}_X(-m) \twoheadrightarrow \mathscr{E}.$$

Denote by $\tilde{q}$ the point in $\mathcal{H}$ corresponding to this quotient. On the other hand, we get a point $\tilde{Q}$ in $\mathcal{P}$ by the composition

$$\operatorname{Sym}^2 V \cong \operatorname{Sym}^2 H^0(\mathscr{E}(m)) \to H^0(\operatorname{Sym}^2 \mathscr{E}(m)) \to H^0(\mathscr{L}(2m)) = B$$

Let $\tilde{Q}$ be a point in $\mathcal{P}$. We will denote by Q' a representative of $\tilde{Q}$, i.e. $Q' : \operatorname{Sym}^2 V \to B$. This gives (up to multiplication by a scalar) an evaluation

$$\mathrm{ev} : \operatorname{Sym}^2 V \otimes \mathcal{O}_X(-2m) \to B \otimes \mathcal{O}_X(-2m) \to \mathscr{L}.$$

Let Z be the closed subset of $\mathcal{H} \times \mathcal{P}$ of points $(\tilde{q}, \tilde{Q})$ such that (some multiple) of this evaluation map factors through $\operatorname{Sym}^2 \mathscr{E}$

$$\operatorname{Sym}^2 V \otimes \mathcal{O}_X(-2m) \to \operatorname{Sym}^2 \mathscr{E} \to \mathscr{L}.$$

The group $\mathrm{SL}(V)$ acts in a natural way on $\mathcal{H} \times \mathcal{P}$. A point in Z will be called "good" if the quotient

$$q : V \otimes \mathcal{O}_X(-m) \twoheadrightarrow \mathscr{E}$$

induces an isomorphism $V \xrightarrow{\cong} H^0(\mathscr{E}(m))$, and $\mathscr{E}$ is torsion free. Note that a conic bundle $(\mathscr{E}, Q)$ gives a "good" point in Z and conversely we can recover the conic bundle from the point, and two "good" points correspond to the same conic bundle iff they are in the same orbit of the action of $\mathrm{SL}(V)$. This action on $\mathcal{H} \times \mathcal{P}$ preserves the subscheme Z and the subset of "good" points.

Let $\mathscr{M}$ be the line bundle on $\mathcal{H}$ given by the embedding $\mathcal{H} \to \mathbb{P}(\Lambda^{P(l)}(V \otimes W))$. Embedd Z in projective space with $\mathcal{O}_Z(n_1, n_2) = p_{\mathcal{H}}^* \mathscr{M}^{\otimes n_1} \otimes p_{\mathcal{P}}^* \mathcal{O}_{\mathcal{P}}(n_2)$

$$Z \hookrightarrow \mathbb{P}(\operatorname{Sym}^{n_1}[\Lambda^{P(l)}(V \otimes W)] \otimes \operatorname{Sym}^{n_2}[\operatorname{Sym}^2 V^\vee \otimes B])$$

The group $\mathrm{SL}(V)$ acts naturally on $\operatorname{Sym}^{n_1}[\Lambda^{P(l)}(V \otimes W)] \otimes \operatorname{Sym}^{n_2}[\operatorname{Sym}^2 V^\vee \otimes B]$, and this gives a linearization for the action of $\mathrm{SL}(V)$ on Z.

Now we will characterize the (semi)stable points of Z under the action of $\mathrm{SL}(V)$ with the linearization induced by $\mathcal{O}_X(n_1, n_2)$. We will take

$$\frac{n_2}{n_1} = \frac{P(l) - P(m)}{P(m) - 2\tau} \tau.$$

Notation. Given a point $(\tilde{q}, \tilde{Q}) \in Z$ and a subspace $V' \subset V$ we denote by $\mathscr{E}_{V'}$ the image of $V' \otimes \mathcal{O}_X(-m)$ under the quotient $q : V \otimes \mathcal{O}_X(-m) \to \mathscr{E}$. Note that $V' \subset H^0(\mathscr{E}_{V'}(m))$, but in general they are not equal. If $\mathscr{E}' \subset \mathscr{E}$ is a subsheaf of $\mathscr{E}$ we have $\mathscr{E}_{H^0(\mathscr{E}'(m))} \subset \mathscr{E}'$, with equality if $\mathscr{E}'(m)$ is generated by global sections. Given a sheaf $\mathscr{F}$, we will denote by $P_{\mathscr{F}}$ its Hilbert polynomial.

The following definition is analogous to definition 1.3.

Definition 2.6. *Let $(\tilde{q},\tilde{Q})$ be a point in Z. Let $V_1 \subset V_2 \subset V_3 = V$ be a filtration of V. Let Q'_{ab} be the restriction of $Q' : \mathrm{Sym}^2 V \to B$ to $V_a \otimes V_b$. We say that V_1, V_2 give a critical filtration of $(\tilde{q},\tilde{Q})$, if $\mathrm{rk}(\mathscr{E}_{V_1}) = 1$, $\mathrm{rk}(\mathscr{E}_{V_2}) = 2$, $Q'_{12} = 0$, and $Q'_{13} \neq 0 \neq Q'_{22}$.*

Proposition 2.7. *For l large enough the point $(\tilde{q},\tilde{Q}) \in Z$ is (semi)stable by the action of $\mathrm{SL}(V)$ with respect to the linearization by $\mathcal{O}_Z(n_1,n_2)$ iff:*

(.1) If $V' \subsetneq V$ is a subspace of V, then*

$$\dim V'(n_1P(l)+2n_2)(\leq)\dim V(n_1P_{\mathscr{E}_{V'}}(l)+c_Q(\mathscr{E}_{V'})n_2).$$

(.2) If $V_1 \subset V_2 \subset V$ is a critical filtration, then*

$$(\dim V_1+\dim V_2)(n_1P(l)+2n_2)(\leq)\dim V(n_1(P_{\mathscr{E}_{V_1}}(l)+P_{\mathscr{E}_{V_2}}(l))+2n_2).$$

Proof. We will apply the Hilbert-Mumford criterion: a point $(\tilde{q},\tilde{Q})$ is (semi)stable iff for all one-parameter subgroup (1-PS) λ of $\mathrm{SL}(V)$ we have $\mu((\tilde{q},\tilde{Q}),\lambda)(\leq)0$, where $\mu((\tilde{q},\tilde{Q}),\lambda)$ is the minimum weight of the action of λ on $(\tilde{q},\tilde{Q})$.

Let $p = P(m)$. A 1-PS λ of $\mathrm{SL}(V)$ is equivalent to a basis $\{v_1,\dots,v_p\}$ of V and a weight vector $(\gamma_1,\dots,\gamma_p)$ with $\gamma_i \in \mathbb{Z}$, $\gamma_1 \leq \cdots \leq \gamma_p$, and $\sum \gamma_i = 0$. The set $\mathcal{C}$ of all weight vectors is a cone in $\mathbb{Z}^p$. If a basis of V has been chosen, then by a slight abuse of notation we will denote $\mu((\tilde{q},\tilde{Q}),\lambda)$ by $\mu((\tilde{q},\tilde{Q}),\gamma)$, where $\gamma \in \mathcal{C}$.

We will choose a set of one-parameter subgroups, calculate $\mu((\tilde{q},\tilde{Q}),\lambda)$, and then imposing $\mu((\tilde{q},\tilde{Q}),\lambda)(\leq)0$ we will obtain necessary conditions for $(\tilde{q},\tilde{Q})$ to be (semi)stable.

Then we will show that the chosen set of one-parameter subgroups is sufficient, in the sense that if we check that $\mu((\tilde{q},\tilde{Q}),\lambda)(\leq)0$ for all one-parameter subgroups in this set, then the same will hold for any arbitrary one-parameter subgroup in $\mathcal{C}$.

We have $\mu((\tilde{q},\tilde{Q}),\lambda) = n_1\mu(\tilde{q},\lambda)+n_2\mu(\tilde{Q},\lambda)$, where $\mu(\tilde{q},\lambda)$ (resp. $\mu(\tilde{Q},\lambda)$) is the minimum weight of the action of λ on $\tilde{q} \in \mathcal{H}$ (resp. $\tilde{Q} \in \mathcal{P}$). Fix a basis $\{v_1,\dots,v_p\}$ of V. Define $\varphi(i) = \dim q'(\langle v_1,\dots,v_i\rangle \otimes W)$, where $q' : V \otimes W \twoheadrightarrow k^{P(l)}$ is the quotient corresponding to the point $\tilde{q} \in \mathcal{H}$. We have (see [7])

$$\mu(\tilde{q},\gamma) = \sum_{i=1}^{p}\gamma_i(\varphi(i)-\varphi(i-1)).$$

On the other hand

$$\mu(\tilde{Q},\gamma) = \min_{i,j\in\{1,\dots,p\}}\{\gamma_i+\gamma_j : Q'(v_i,v_j) \neq 0\}.$$

Note that $\mu(\tilde{q},\gamma)$ is linear on $\gamma \in \mathcal{C}$, but $\mu(\tilde{Q},\gamma)$ is not.

GIT (semi)stable implies conditions (*)

Let $(\tilde{q},\tilde{Q})$ be a (semi)stable point in Z. Let $\{v_1,\dots,v_p\}$ be a basis of V. Define

$$i_k = \min\{i : \mathrm{rk}(\mathscr{E}_{\langle v_1,\dots,v_i\rangle}) \geq k\}.$$

Note that if $(\tilde{q},\tilde{Q})$ is "good", then the map $V \to H^0(\mathscr{E}(m))$ is an isomorphism (in particular injective), and then $i_1 = 1$. Later on we will see that for sufficiently large m, a semistable point is "good", but now we won't assume that $(\tilde{q},\tilde{Q})$ is "good". Define a filtration of V

$$V_1 = \langle v_1,\dots,v_{i_1}\rangle \subset V_2 = \langle v_1,\dots,v_{i_2}\rangle \subset V_3 = V.$$

Let Q'_{ab} be the restriction of $Q' : \operatorname{Sym}^2 V \to B$ to $V_a \otimes V_b$. To calculate $\mu(\tilde{Q},\lambda)$ we distinguish seven cases.

$$\begin{array}{lll}
1) & Q'_{11} \neq 0 & \mu(\tilde{Q},\lambda) = 2\gamma_{i_1} \\
2) & Q'_{11} = 0,\ Q'_{12} \neq 0 & \mu(\tilde{Q},\lambda) = \gamma_{i_1} + \gamma_{i_2} \\
3) & Q'_{12} = 0,\ Q'_{13} \neq 0 \neq Q'_{22} & \mu(\tilde{Q},\lambda) = \min(2\gamma_{i_2}, \gamma_{i_1} + \gamma_{i_3}) \\
4) & Q'_{13} = 0,\ Q'_{22} \neq 0 & \mu(\tilde{Q},\lambda) = 2\gamma_{i_2} \\
5) & Q'_{22} = 0,\ Q'_{13} \neq 0 & \mu(\tilde{Q},\lambda) = \gamma_{i_1} + \gamma_{i_3} \\
6) & Q'_{13} = Q'_{22} = 0,\ Q'_{23} \neq 0 & \mu(\tilde{Q},\lambda) = \gamma_{i_2} + \gamma_{i_3} \\
7) & Q'_{23} = 0,\ Q'_{33} \neq 0 & \mu(\tilde{Q},\lambda) = 2\gamma_{i_3}
\end{array}$$

Note that in all cases, except case 3, $\mu(\tilde{Q},\lambda)$ is a linear function of $\gamma \in \mathcal{C}$.

First we will consider weight vectors of the form

$$\gamma^{(i)} = (\overbrace{i-p,\ldots,i-p}^{i},\overbrace{i,\ldots,i}^{p-i}) \quad (1 \leq i < p). \tag{1}$$

and define $V' = \langle v_1,\ldots,v_i\rangle$ (it is clear that any subspace of V can be written in this form, after choosing an appropriate bases for V).

We have $\mu(\tilde{q},\gamma^{(i)}) = -p\varphi(i) + i\varphi(p)$. To obtain a formula for $\mu(\tilde{Q},\gamma^{(i)})$ we have to analyze each of the seven cases. We will only work out the details for cases 2 and 3, the remaining cases being similar to case 2.

In case 2 we have $\mu(\tilde{Q},\gamma^{(i)}) = \gamma^{(i)}_{i_1} + \gamma^{(i)}_{i_2}$. Then, according to the value of i we have

$$\mu(\tilde{Q},\gamma^{(i)}) = \begin{cases} 2i & , i < i_1 \qquad (\operatorname{rk}(\mathscr{E}_{V'}) = 0) \\ 2i - p & , i_1 \leq i < i_2 \quad (\operatorname{rk}(\mathscr{E}_{V'}) = 1) \\ 2i - 2p & , i_2 \leq i \qquad (\operatorname{rk}(\mathscr{E}_{V'}) \geq 2) \end{cases}$$

In case 3 we have $\mu(\tilde{Q},\gamma^{(i)}) = \min(2\gamma^{(i)}, \gamma^{(i)}_{i_1} + \gamma^{(i)}_{i_3})$, hence

$$\mu(\tilde{Q},\gamma^{(i)}) = \begin{cases} 2i & , i < i_1 \qquad (\operatorname{rk}(\mathscr{E}_{V'}) = 0) \\ 2i - p & , i_1 \leq i < i_2 \quad (\operatorname{rk}(\mathscr{E}_{V'}) = 1) \\ 2i - 2p & , i_2 \leq i \qquad (\operatorname{rk}(\mathscr{E}_{V'}) \geq 2) \end{cases}$$

Doing the calculation for the seven cases we check that in every case we have

$$\mu(\tilde{Q},\gamma^{(i)}) = 2i - c_Q(\mathscr{E}_{V_i})p$$

Then $\mu((\tilde{q},\tilde{Q}),\gamma^{(i)})(\leq)0$ gives

$$n_1(-p\varphi(i) + i\varphi(p)) + n_2(2i - c_Q(\mathscr{E}_{V'}))(\leq)0$$

If we vary V' (allowing $V' = V$), the submodules $\mathscr{E}_{V'}$ are bounded, so we can take l large enough such that $\varphi(i) = P_{\mathscr{E}_{V'}}(l)$. We have $i = \dim V'$, and $\varphi(p) = P(l)$, and then we obtain condition (*.1).

To obtain condition (*.2), assume that we have subspaces $V_1 \subset V_2 \subset V$ giving a critical filtration. Let $i = \dim V_1$ and $j = \dim V_2 - \dim V_1$. Take a bases $\{v_1,\ldots,v_p\}$ of V adapted to this filtration, i.e. such that $V_1 = \langle v_1,\ldots,v_i\rangle$ and $V_2 = \langle v_1,\ldots v_{i+j}\rangle$. Consider the weight vector

$$\gamma^{(i)} + \gamma^{(i+j)} =$$
$$(\overbrace{2i+j-2p,\ldots,2i+j-2p}^{i},\overbrace{2i+j-p,\ldots,2i+j-p}^{j},\overbrace{2i+j,\ldots,2i+j}^{p-i-j}). \tag{2}$$

An easy computation then shows $\mu(\tilde{Q},\gamma^{(i)}+\gamma^{(i+j)})=2(\dim V_1+\dim V_2-\dim V)$. On the other hand $\mu(\tilde{q},\gamma^{(i)}+\gamma^{(i+j)})=(\dim V_1+\dim V_2)P(l)-\dim V(P_{V_1}(l)+P_{V_2}(l))$, and then $\mu((\tilde{q},\tilde{Q}),\gamma^{(i)}+\gamma^{(i+j)})(\leq)0$ gives condition (*.2).

Conditions (*) imply GIT (semi)stable

Now we have to show that the one-parameter subgroups that we have used are sufficient. As we did before, we will fix an arbitrary base V, and we consider the seven different cases. In all cases except 3, $\mu(\tilde{Q},\gamma)$, and hence $\mu((\tilde{q},\tilde{Q}),\gamma)$, is a linear function of $\gamma\in\mathcal{C}$, and then to prove that $\mu((\tilde{q},\tilde{Q}),\gamma)(\leq)0$ for all γ it is enough to check it on the generators $\gamma^{(i)}$ defined above (1).

In case 3 we have $\mu(\tilde{Q},\gamma)=\min(2\gamma_{i_2},\gamma_{i_1}+\gamma_{i_3})$, hence it is no longer linear on γ, and it is not enough to check the condition on the generators $\gamma^{(i)}$. But it is a piecewise linear function. The cone $\mathcal{C}$ of weights is divided in two cones

$$\mathcal{C}^{>}=\{(\gamma_1,\dots,\gamma_p)\in\mathcal{C}:2\gamma_{i_2}\geq\gamma_{i_1}+\gamma_{i_3}\}$$
$$\mathcal{C}^{<}=\{(\gamma_1,\dots,\gamma_p)\in\mathcal{C}:2\gamma_{i_2}\leq\gamma_{i_1}+\gamma_{i_3}\}$$

Observe that $\mu(\tilde{Q},\gamma)$ is linear on each of these cones. We will use the following lemma.

Lemma 2.8. *Let $\mathcal{C}$ be a cone in $\mathbb{Z}^p$, let $\gamma^{(i)}$ be a set of generators of $\mathcal{C}$, i.e. $\mathcal{C}=(\oplus_i\mathbb{Q}^+\gamma^{(i)})\cap\mathbb{Z}^p$. Let $A:\mathbb{Z}^p\to\mathbb{Q}$ be a linear function such that $A(\gamma^{(i)})\in\{1,0,-1\}$. Let $\mathcal{C}^>$ be the subcone $\{v\in\mathcal{C}:A(v)\geq 0\}$. Then the set of vectors*

$$v_{i,j}=\begin{cases}\gamma^{(i)} & ,\ A(e_i)\geq 0\\ \gamma^{(i)}+\gamma^{(i+j)} & ,\ A(e_i)=-1,\ A(e_{i+j})=1\\ 0 & ,\ \textit{otherwise.}\end{cases}$$

generate $\mathcal{C}^>$.

□

We apply this lemma with $A(\gamma)=(2\gamma_{i_2}-\gamma_{i_1}-\gamma_{i_3})/p$ (and then with the negative of this, for $\mathcal{C}^<$), and we obtain a set of generators for $\mathcal{C}^>$ and $\mathcal{C}^<$. But all these vectors are either of the form $\gamma^{(i)}$ with $1\leq i<p$, or of the form $\gamma^{(i)}+\gamma^{(i+j)}$ with $\mathrm{rk}(\mathscr{E}_{\langle v_1,\dots,v_i\rangle})=1$ and $\mathrm{rk}(\mathscr{E}_{\langle v_1,\dots,v_{i+j}\rangle})=2$, and we have already considered them.

□

Remark 2.9. In the following propositions we will prove that conditions (*) are equivalent to the stability conditions (s). Recall that $\dim V=p=P(m)$, $\varphi(i)=P_{E_{V'}}(l)$, $\varphi(p)=P(l)$ and $\dim V'=i$. The idea is to show that for $l\gg m\gg 0$, we can replace $P(l)$ by $\mathrm{rk}(E)l$, $P_{E_{V'}}(l)$ by $\mathrm{rk}(E')l$, $P(m)$ by $\deg(E)+rm$ and $\dim V'$ by $P_{E_{V'}}(m)$, and this by $\deg(E')+rm$.

Proposition 2.10. *For m and l large enough we have that conditions (*) are equivalent to:*

*(**.1) If $V'\subsetneq V$ is a subspace of V, then*

$$r(\dim V'-c_Q(\mathscr{E}_{V'}))\leq\mathrm{rk}(\mathscr{E}_{V'})(\dim V-2\tau),$$

and in case of equality we also require $\dim V'(\leq)P_{\mathscr{E}_{V'}}(m)$.

*(**.2) If $V_1\subset V_2\subset V$ is a critical filtration, then*

$$\dim V_1+\dim V_2\leq\dim V$$

and in case of equality we also require $\dim V_1 + \dim V_2 (\leq) P_{\mathscr{E}_{V_1}}(m) + P_{\mathscr{E}_{V_2}}(m)$.

Proof. We rewrite (*.1) using

$$\frac{n_2}{n_1} = \frac{P(l) - P(m)}{P(m) - 2\tau}\tau.$$

We obtain

$$[(\dim V_1 - c_Q(\mathscr{E}_{V'})\tau)r - \mathrm{rk}(\mathscr{E}_{V'})(\dim V - 2\tau)](l-m) + \\ +(\dim V - 2\tau)\dim V(\dim V' - P_{\mathscr{E}_{V'}})(m)(\leq)0.$$

We have $(l-m) \gg 0$ and $m \gg 0$, hence $\dim V > 2\tau$ and the result follows. Now we rewrite (*.2), using $r = 3$, $\mathrm{rk}(\mathscr{E}_{V_1}) = 1$ and $\mathrm{rk}(\mathscr{E}_{V_2}) = 2$.

$$3(\dim V_1 + \dim V_2 - \dim V)(l-m) + \\ +(\dim V - 2\tau)\dim V(\dim V_1 + \dim V_2 - P_{\mathscr{E}_{V_1}}(m) - P_{\mathscr{E}_{V_2}}(m))(\leq)0,$$

and the result follows.

□

Proposition 2.11. *For m and l large enough, we have*

(i) If $(\mathscr{E}, Q)$ is a (semi)stable conic bundle, then the corresponding point $(\tilde{q}, \tilde{Q})$ in Z is GIT (semi)stable under the action of $\mathrm{SL}(V)$.

(ii) If $(\tilde{q}, \tilde{Q}) \in Z$ is a GIT semistable point, then $\tilde{q}$ is "good" and $h^1(\mathscr{E}(m)) = 0$.

(iii) If $(\tilde{q}, \tilde{Q}) \in Z$ is a GIT (semi)stable point, then the corresponding conic bundle $(\mathscr{E}, Q)$ is (semi)stable.

Note that thanks to (ii), in (iii) we know that $\mathscr{E}$ is torsion free.

Proof. We will proof the three items in three steps

Step 1. (Semi)stable conic bundle ⇒ GIT (semi)stable $(\tilde{q}, \tilde{Q})$

We will use proposition 2.10. We will start checking (**.1). Let $\mathcal{S}$ be the set of vector bundles $\mathscr{E}'$ that are subsheaves of bundles $\mathscr{E}$ occurring in semistable conic bundles. It satisfies hypothesis (i) and (ii) of lemma 2.5 with $R = 3$ and $b = \frac{d-2\tau}{r} + 2\tau$. Let $k = \frac{d-2\tau}{r}$, n large enough, so that propositions 2.5, 2.2 and 2.3 hold, and let $\mathcal{S}_n$ be the subset of $\mathcal{S}$ consisting of bundles $\mathscr{E}'$ that satisfy hypothesis (iii) of lemma 2.5. Then the set $\mathcal{S}_n$ is bounded. Taking $m > n$ large enough we then have $h^1(\mathscr{E}'(m)) = 0$ for $\mathscr{E}' \in \mathcal{S}_n$. In other words,

$$h^0(\mathscr{E}'(m)) = \mathrm{rk}(\mathscr{E}')(\chi(\mathcal{O}_X(m)) + \frac{\det(\mathscr{E}')}{\mathrm{rk}(\mathscr{E}')}), \quad \text{for } \mathscr{E}' \in \mathcal{S}_n \tag{3}$$

On the other hand, we still have

$$h^0(\mathscr{E}'(m)) < \mathrm{rk}(\mathscr{E}')(\chi(\mathcal{O}_X(m)) + \frac{d-2\tau}{r}), \quad \text{for } \mathscr{E}' \in \mathcal{S} \setminus \mathcal{S}_n \tag{4}$$

Let V' be a subspace of V, and $\mathscr{E}_{V'}$ the corresponding sheaf. If $\mathscr{E}_{V'}$ belongs to $\mathcal{S}_n$, we get that condition (ss.1) implies (**.1), because

$$\frac{\dim V' - c_Q(\mathscr{E}_{V'})\tau}{\mathrm{rk}(\mathscr{E}_{V'})} \leq \frac{h^0(\mathscr{E}_{V'}(m)) - c_Q(\mathscr{E}_{V'})\tau}{\mathrm{rk}(\mathscr{E}_{V'})} =$$
$$= \frac{\deg(\mathscr{E}_{V'}) - c_Q(\mathscr{E}_{V'})\tau}{\mathrm{rk}(\mathscr{E}_{V'})} + \chi(\mathcal{O}_X(m))(\leq)$$
$$(\leq)\frac{d-2\tau}{r} + \chi(\mathcal{O}_X(m)) = \frac{\dim V - 2\tau}{r}$$

and $\dim V' \leq h^0(\mathscr{E}_{V'}(m)) = P_{\mathscr{E}_{V'}}(m)$, because $h^1(\mathscr{E}'(m)) = 0$. On the other hand, if $\mathscr{E}_{V'}$ belongs to $\mathcal{S} \setminus \mathcal{S}_n$, inequality (4) implies (**.1)

$$\frac{\dim V' - c_Q(\mathscr{E}_{V'})}{\mathrm{rk}(\mathscr{E}_{V'})} \leq \frac{h^0(\mathscr{E}_{V'}(m))}{\mathrm{rk}(\mathscr{E}_{V'})} < \frac{d-2\tau}{r} + \chi(\mathcal{O}_X(m)) = \frac{\dim V - 2\tau}{r}$$

In both cases, if inequality (ss.1) is strict, then inequality (**.1) is also strict. But assume that there is a semistabilizing subsheaf $\mathscr{E}'$ of $\mathscr{E}$ (i.e. giving equality in (ss.1)). By corollary 2.3, $\mathscr{E}'(m)$ is generated by global sections. Let $V' = H^0(\mathscr{E}'(m)) \subset H^0(\mathscr{E}(m)) = V$. Then $\mathscr{E}' = \mathscr{E}_{V'}$, and we have

$$\frac{\dim V' - c_Q(\mathscr{E}_{V'})\tau}{\mathrm{rk}(\mathscr{E}_{V'})} = \frac{\deg(\mathscr{E}_{V'}) - c_Q(\mathscr{E}_{V'})}{\mathrm{rk}(\mathscr{E}_{V'})} + \chi(\mathcal{O}_X(m)) =$$
$$\frac{d-2\tau}{r} + \chi(\mathcal{O}_X(m)) = \frac{\dim V - 2\tau}{r}$$

and $\dim V' = P_{\mathscr{E}_{V'}}(m)$.

Now we will check condition (**.2). Let $\mathcal{T}$ be the set of vector bundles of the form $\mathscr{E}_1 \oplus \mathscr{E}_2$ such that $\mathscr{E}_1 \subset \mathscr{E}_2 \subset \mathscr{E}$ gives a critical filtration of a (semi)stable conic bundle $(\mathscr{E}, Q)$. Hypothesis (i) and (ii) of lemma 2.5 are satisfied with $R = 3$ and $b = \frac{d-2\tau}{r} + 2\tau$. Let $k = d/3$, and n large enough. Let $\mathcal{T}_n$ be the subset of $\mathcal{T}$ consisting of vector bundles $\mathscr{E}_1 \oplus \mathscr{E}_2$ satisfying hypothesis (iii). Then $\mathcal{T}_n$ is bounded, and taking m large enough we have $0 = h^1((\mathscr{E}_1 \oplus \mathscr{E}_2)(m)) = h^1(\mathscr{E}_1(m)) + h^1(\mathscr{E}_2(m))$ for $\mathscr{E}_1 \oplus \mathscr{E}_2 \in \mathcal{T}_n$. Hence for $\mathscr{E}_1 \oplus \mathscr{E}_2 \in \mathcal{T}_n$,

$$h^0(\mathscr{E}_1(m)) + h^0(\mathscr{E}_2(m)) = 3\chi(\mathcal{O}_X(m)) + \deg(\mathscr{E}_1) + \deg(\mathscr{E}_2) \tag{5}$$

On the other hand, for $\mathscr{E}_1 \oplus \mathscr{E}_2 \in \mathcal{T} \setminus \mathcal{T}_n$ we still have

$$h^0(\mathscr{E}_1(m)) + h^0(\mathscr{E}_2(m)) < 3\chi(\mathcal{O}_X(m)) + d \tag{6}$$

Let $V_1 \subset V_2 \subset V$ be a critical filtration of V. If $\mathscr{E}_{V_1} \oplus \mathscr{E}_{V_2} \in \mathcal{T}_n$, we get that (ss.2) implies (**.2), because

$$\dim V_1 + \dim V_2 \leq h^0(\mathscr{E}_{V_1}(m)) + h^0(\mathscr{E}_{V_2}(m)) =$$
$$= 3\chi(\mathcal{O}_X(m)) + \deg(\mathscr{E}_{V_1}) + \deg(\mathscr{E}_{V_2})(\leq)3\chi(\mathcal{O}_X(m)) + \deg(\mathscr{E}) = \dim V$$

and also $\dim V_1 + \dim V_2 \leq h^0(\mathscr{E}_{V_1}(m)) + h^0(\mathscr{E}_{V_2}(m)) = P_{\mathscr{E}_{V_1}}(m) + P_{\mathscr{E}_{V_2}}(m)$.

On the other hand, if $\mathscr{E}_{V_1} \oplus \mathscr{E}_{V_2} \in \mathcal{T} \setminus \mathcal{T}_n$, inequality (6) implies (**.2)

$$\dim V_1 + \dim V_2 \leq h^0(\mathscr{E}_{V_1}(m)) + h^0(\mathscr{E}_{V_2}(m)) < 3\chi(\mathcal{O}_X(m)) + d = \dim V$$

In both cases, if inequality (ss.2) is strict, also (**.2) is strict. But assume that we have subsheaves $\mathscr{E}_1 \subset \mathscr{E}_2 \subset \mathscr{E}$ giving a critical filtration of a semistable conic bundle $(\mathscr{E}, Q)$. By lemma 2.3 $\mathscr{E}_1(m)$ and $\mathscr{E}_2(m)$ are generated by global sections and $h^1(\mathscr{E}_1(m))=h^1(\mathscr{E}_2(m))=0$. Taking $V_1=H^0(\mathscr{E}_1(m))$ and $V_2=H^0(\mathscr{E}_2(m))$ we have $\mathscr{E}_{V_1}=\mathscr{E}_1$ and $\mathscr{E}_{V_2}=\mathscr{E}_2$, hence

$$\begin{aligned}\dim V_1+\dim V_2 &= \deg(\mathscr{E}_1)+\deg(\mathscr{E}_2)+3\chi(\mathcal{O}_X(m))=\\ &= \deg(\mathscr{E})+3\chi(\mathcal{O}_X(m))=\dim V\end{aligned}$$

and also $\dim V_1+\dim V_2=P_{\mathscr{E}_{V_1}}(m)+P_{\mathscr{E}_{V_2}}(m)$

Step 2. $(\tilde{q},\tilde{Q})$ ***GIT (semi)stable*** $\Rightarrow$ $h^1(\mathscr{E}(m))=0$ ***and*** $\tilde{q}$ ***good***

If $h^1(\mathscr{E}(m))\neq 0$, then by Serre duality $\operatorname{Hom}(\mathscr{E},\mathscr{K}_X)\neq 0$. Take $\psi\in\operatorname{Hom}(\mathscr{E},\mathscr{K}_X)$. The composition $V\otimes\mathcal{O}_X\to\mathscr{E}(m)\to\mathscr{K}_X$ gives a linear map

$$f: V\to H^0(\mathscr{K}_X).$$

Let U be the kernel of f. We have $\dim U\geq \dim V-\dim H^0(\mathscr{K}_X)=p-g$. Then by (semi)stability of $(\tilde{q},\tilde{Q})$ we have

$$r(p-g-c_Q(\mathscr{E}_U)\tau)\leq r(\dim U-c_Q(\mathscr{E}_U)\tau)(\leq)\operatorname{rk}(E_U)(p-2\tau),$$

$$(r-\operatorname{rk}(\mathscr{E}_U))p\leq r(g+c_Q(\mathscr{E}_U)\tau)-\operatorname{rk}(\mathscr{E}_U)2\tau$$

We have $\operatorname{rk}(\mathscr{E}_U)\leq r$. Then if m is large enough the inequality forces $r=\operatorname{rk}(\mathscr{E}_U)$. By definition of U we have $\mathscr{E}_U(m)\subset\ker\psi$, then $\operatorname{rk}(\ker\psi)=r$, $\operatorname{rk}(\operatorname{im}\psi)=0$, and then $\psi=0$ because $\mathscr{K}_X$ is torsion free. We conclude that (for m large enough) $h^1(\mathscr{E}(m))=0$.

Then $\dim V=p=h^0(\mathscr{E}(m))$, and to show that $\tilde{q}$ is "good" it is enough to show that the induced linear map

$$V\to H^0(\mathscr{E}(m))$$

is injective. Let V' be the kernel. Then we have $\operatorname{rk}(\mathscr{E}_{V'})=0$. By semistability we have (**.1)

$$r(\dim V'-c_Q(\mathscr{E}_{V'}))\leq 0,$$

but $c_Q(\mathscr{E}_{V'})=0$, and then $\dim V'$ must be zero.

To show that $\mathscr{E}$ is torsion free, let $\mathscr{T}\subset\mathscr{E}$ be the torsion subsheaf. We have $V\cong H^0(\mathscr{E}(m))$, and then $U=H^0(\mathscr{T}(m))$ is a subspace of V. The associated sheaf $\mathscr{E}_U$ has rank equal to zero, and arguing as above we get $U=0$.

Step 3. GIT (Semi)stable $(\tilde{q},\tilde{Q})$ $\Rightarrow$ ***(semi)stable conic bundle***

By the previous step we know that we can choose m large enough so that $\tilde{q}$ is "good". We will check first (ss.1). Let $\mathscr{E}'$ be a subsheaf of $\mathscr{E}$. Define $V'=H^0(\mathscr{E}'(m))$. We have $\mathscr{E}_{V'}\subset\mathscr{E}'$, $\operatorname{rk}(\mathscr{E}_{V'})\leq\operatorname{rk}(\mathscr{E}')$, $\dim V'\geq P_{\mathscr{E}'}(m)$, and $c_Q(\mathscr{E}')\geq c_Q(\mathscr{E}_{V'})$. Then

$$\begin{aligned}r(P_{\mathscr{E}'}(m)-c_Q(\mathscr{E}')\tau)&\leq r(\dim V'-c_Q(\mathscr{E}_{V'})\tau)(\leq)\\ (\leq)\operatorname{rk}(\mathscr{E}_{V'})(\dim V-2\tau)&\leq\operatorname{rk}(\mathscr{E}')(\dim V-2\tau).\end{aligned}$$

Note that if (**.1) is strict, then also (ss.1) is strict. But assume that there is a subspace $V'\subset V$ that is semistabilizing, i.e. both conditions in (**.1) are equalities. Then

$$\begin{aligned}\frac{\deg(\mathscr{E}_{V'})-c_Q(\mathscr{E}_{V'})\tau}{\operatorname{rk}(\mathscr{E}_{V'})}&=\frac{\dim V'-c_Q(\mathscr{E}_{V'})\tau}{\operatorname{rk}(\mathscr{E}_{V'})}-\chi(\mathcal{O}_X(m))=\\ &=\frac{\dim V-2\tau}{r}-\chi(\mathcal{O}_X(m))=\frac{\deg(\mathscr{E})-2\tau}{r},\end{aligned}$$

and we get that (ss.1) for $\mathscr{E}_{V'}$ also gives equality.

Now we are going to check (ss.2). As in step 1, consider the set $\mathcal{T}$ of vector bundles of the form $\mathscr{E}_1 \oplus \mathscr{E}_2$ such that $\mathscr{E}_1 \subset \mathscr{E}_2 \subset \mathscr{E}$ gives a critical filtration. We have already proved condition (ss.1), thus hypothesis (i) and (ii) of lemma 2.5 are again satisfied. Then, as in step 1, we define the subset $\mathcal{T}_n \subset \mathcal{T}$, and taking m large enough we can assume that the vector bundles $\mathscr{E}_1$ and $\mathscr{E}_2$ are generated by global sections if $\mathscr{E}_1 \oplus \mathscr{E}_2 \in \mathcal{T}$.

Let $\mathscr{E}_1 \subset \mathscr{E}_2 \subset \mathscr{E}$ be a critical filtration of $(\mathscr{E}, Q)$. Let $V_1 = H^0(\mathscr{E}_1(m))$ and $V_2 = H^0(\mathscr{E}_2(m))$. If $\mathscr{E}_{V_1} \oplus \mathscr{E}_{V_2} \in \mathcal{T}_n$, then $\mathscr{E}_{V_1}$ and $\mathscr{E}_{V_2}$ are generated by global sections and then $\mathscr{E}_{V_1} = \mathscr{E}_1$, $\mathscr{E}_{V_2} = \mathscr{E}_2$, and $V_1 \subset V_2 \subset V$ is a critical filtration of V and (**.2) holds

$$\begin{aligned} \deg(\mathscr{E}_1) + \deg(\mathscr{E}_2) = \dim V_1 + \dim V_2 - 3\chi(\mathcal{O}_X(m))(\leq) \\ (\leq) \dim V - 3\chi(\mathcal{O}_X(m)) = \deg(\mathscr{E}). \end{aligned}$$

On the other hand, if $\mathscr{E}_{V_1} \oplus \mathscr{E}_{V_2} \in \mathcal{T} \setminus \mathcal{T}_n$, inequality (6) implies (**.2)

$$\deg(\mathscr{E}_1) + \deg(\mathscr{E}_2) \leq h^0(\mathscr{E}_1(m)) + h^0(\mathscr{E}_2(m)) - 3\chi(\mathcal{O}_X(m)) < \deg(\mathscr{E}).$$

Note that if (**.2) is strict then (ss.2) is also strict. But assume that there is a semistabilizing critical sequence $V_1 \subset V_2 \subset V$, i.e. a critical sequence giving equality in both conditions of (**.2). Then

$$\begin{aligned} \deg(\mathscr{E}_{V_1}) + \deg(\mathscr{E}_{V_2}) = \dim V_1 + \dim V_2 - 3\chi(\mathcal{O}_X(m)) = \\ = \dim V - 3\chi(\mathcal{O}_X(m)) = \deg \mathscr{E}, \end{aligned}$$

and we also get an equality in (ss.2).

□

Once we have established proposition 2.11 and lemma 1.5 we can prove theorem I using standard techniques. We follow closely [1].

Proof of theorem I. Let $\overline{\mathfrak{M}}_\tau(r,d,\mathscr{L})$ (resp. $\mathfrak{M}_\tau(r,d,\mathscr{L})$) be the GIT quotient of Z (resp. Z^s) by $\mathrm{SL}(V)$.

First we construct a universal family on Z^{ss} using the universal families of the Quot scheme $\mathcal{Q}$ and on $\mathcal{P} = \mathbb{P}(\mathrm{Sym}^2 V^\vee \otimes B)$ (we think of $\mathcal{P}$ as the Grassmannian of one dimensional subspaces of $\mathrm{Sym}^2 V^\vee \otimes B$, and hence the universal subbundle of subspaces is $\mathcal{O}_{\mathcal{P}}(-1)$).

Recall that Z^{ss} is in $\mathcal{H} \times \mathcal{P}$. The universal quotient $\mathscr{E}_{\mathcal{H}}$ on $\mathcal{H} \times X$ pulls back to a vector bundle $\mathscr{E}_{Z^{ss}} = p^*_{\mathcal{H}\times\mathcal{Q}}\mathscr{E}_{\mathcal{H}}$ on Z^{ss}. On the other hand the universal subbundle on $\mathcal{P} \times X$ gives a morphism $\mathrm{Sym}^2 V \otimes p_X^*\mathcal{O}_X(-2m) \to p_X^*\mathscr{L} \otimes \mathcal{O}_{\mathcal{P}}(1)$ of sheaves over $Z^{ss} \times X$. By the definition of Z, there is a line bundle $\mathscr{N}$ on Z^{ss} such that this last morphism factors and gives $Q_{Z^{ss}} : \mathrm{Sym}^2 \mathscr{E}_{Z^{ss}} \to p_X^*\mathscr{L} \otimes p^*_{Z^{ss}}\mathscr{N}$. Note that the line bundle $\mathscr{N}$ is needed because the factorization on Z is only up to scalar multiplication. The triple $(\mathscr{E}_{Z^{ss}}, Q_{Z^{ss}}, \mathscr{N})$ is a universal conic bundle.

Given a family $(\mathscr{E}_T, Q_T)$ of conic bundles parametrized by T, and using the universal family on Z^{ss}, we obtain a morphism $T \to \overline{\mathfrak{M}}_\tau(r,d,\mathscr{L})$. This is done in the following way: Let m be large enough so that proposition 2.11 holds. Given a family $(\mathscr{E}_T, Q_T, \mathscr{N})$ of conic bundles parametrized by T, consider the locally free sheaf $\mathcal{V} = p_{T*}(\mathscr{E}_T \otimes p_X^*\mathcal{O}_X(m))$, and note that $p_T^*\mathcal{V} \otimes p_X^*\mathcal{O}_X(-m) \to \mathscr{E}_T$ is a surjection. Cover T with open sets U_i such that there are isomorphisms $\phi_i : V \otimes \mathcal{O}_{U_i} \to \mathcal{V}$. Then we have quotients $q_i : V \otimes p_X^*\mathcal{O}_X(-m) \twoheadrightarrow \mathscr{E}_{U_i}$ and families of subspaces $\mathcal{O}_{U_i} \hookrightarrow \mathrm{Sym}^2 V^\vee \otimes B \otimes \mathcal{O}_{U_i}$, and these give maps $U_i \to Z^{ss}$. On the intersections $U_i \cap U_j$ this maps in general will differ by the action of $\mathrm{SL}(V)$, then they combine to give a well defined morphism $T \to \overline{\mathfrak{M}}_\tau(r,d,\mathscr{L})$.

It is straightforward to check the universal property for $\overline{\mathfrak{M}}_\tau(r,d,\mathscr{L})$, and then $\overline{\mathfrak{M}}_\tau(r,d,\mathscr{L})$ is a coarse moduli space.

Now we will show that the universal family restricted to Z^s descends to $\mathfrak{M}_\tau(r,d,\mathscr{L})$, making it a fine moduli space. Applying Luna's étale slice theorem [3], we can find an étale cover U' of $\mathfrak{M}_\tau(r,d,\mathscr{L})$ over which there is a universal family $(\mathscr{E}'_{U'}, Q'_{U'})$. Consider $U'' = U' \times_{\mathfrak{M}_\tau(r,d,\mathscr{L})} U'$ and take an isomorphism $\Phi : p_1^*(\mathscr{E}'_{U'}, Q'_{U'}) \to p_2^*(\mathscr{E}'_{U'}, Q'_{U'})$ with the condition $p_1^* Q'_{U'} = p_2^* Q'_{U'} \circ \mathrm{Sym}^2 \Phi$. This isomorphism exists and is unique by lemma 1.5, and then it satisfies the cocycle condition of descend theory [5, Chap. VII], and hence the family $(\mathscr{E}'_{U'}, Q'_{U'})$ descends to $\mathfrak{M}_\tau(r,d,\mathscr{L})$.

$\square$

2.3. **S-equivalence.**

Let $(\mathscr{E},Q)$ and $(\mathscr{E}',Q')$ be two nonisomorphic conic bundles. If they are strictly semistable, it could still happen that the corresponding points in the moduli space $\overline{\mathfrak{M}}_\tau(r,d,\mathscr{L})$ coincide. In this case we say that they are S-equivalent (note that this is not the usual definition. Usually one defines two bundles as S-equivalent if the graded objects of their Jordan-Hölder filtrations coincide, and then proves that S-equivalence classes corresponds to points of the moduli space). In this section, given a strictly semistable conic bundle $(\mathscr{E},Q)$, we will show how to obtain a canonical representative $(\mathscr{E}^S,Q^S)$ of its S-equivalent class. In other words, given two semistable conic bundles $(\mathscr{E},Q)$ and $(\mathscr{E}',Q')$, they will be S-equivalent iff $(\mathscr{E}^S,Q^S)$ is isomorphic to $({\mathscr{E}'}^S,{Q'}^S)$.

Let $(\mathscr{E},Q)$ be a strictly semistable conic bundle. Then there exists at least one "semistabilizing object", i.e. there exists either a subbundle $\mathscr{E}' \subset \mathscr{E}$ that gives equality on (ss.1)(and then we say that $\mathscr{E}'$ is a semistabilizing object of type I), or there is a critical filtration $\mathscr{E}_1 \subset \mathscr{E}_2 \subset \mathscr{E}$ giving equality on (ss.2)(and then we say that the filtration is a semistabilizing object of type II). Choose one semistabilizing object. We define a new conic bundle $(\mathscr{E}_0,Q_0)$ as follows (it will depend on which semistabilizing object we choose):

In the first case (corresponding to (ss.1)), the vector bundle is defined to be $\mathscr{E}_0 = \mathscr{E}' \oplus \mathscr{E}/\mathscr{E}'$ (note that if $\mathscr{E}$ is semistable and $\mathscr{E}'$ gives equality on (ss.1), then $\mathscr{E}/\mathscr{E}'$ is torsion free). To define Q_0, let v and w be local sections of $\mathscr{E}_0$ on an open set U. We distinguish three cases:

$$\text{If } \ c_Q(\mathscr{E}') = \left\{ \begin{array}{ll} 2, \text{ then } Q_0(v,w) = & \left\{ \begin{array}{ll} Q(v,w) & v,w \in \mathscr{E}'(U) \\ 0 & \text{otherwise} \end{array} \right. \\ 1, \text{ then } Q_0(v,w) = & \left\{ \begin{array}{ll} Q(v,w) & v \in \mathscr{E}'(U) \text{ or } w \in \mathscr{E}'(U) \\ 0 & \text{otherwise} \end{array} \right. \\ 0, \text{ then } Q_0(v,w) = & Q(v,w) \end{array} \right.$$

In matrix form this can be written as follows

$$\text{If } Q = \begin{pmatrix} \times & \cdot \\ \cdot & \cdot \end{pmatrix}, \quad \text{then } Q_0 = \begin{pmatrix} \times & 0 \\ 0 & 0 \end{pmatrix}$$

$$\text{If } Q = \begin{pmatrix} 0 & \times \\ \times & \cdot \end{pmatrix}, \quad \text{then } Q_0 = \begin{pmatrix} 0 & \times \\ \times & 0 \end{pmatrix}$$

$$\text{If } Q = \begin{pmatrix} 0 & 0 \\ 0 & \times \end{pmatrix}, \quad \text{then } Q_0 = \begin{pmatrix} 0 & 0 \\ 0 & \times \end{pmatrix}$$

It is easy to see that this is well defined.

In the second case (corresponding to (ss.2)) we define the vector bundle to be $\mathscr{E}_0 = \mathscr{E}_1 \oplus \mathscr{E}_2/\mathscr{E}_1 \oplus \mathscr{E}/\mathscr{E}_2$. Again let v and w be local sections of $\mathscr{E}_0$ on an open set U. Then we set

$$Q_0(v,w) = \begin{cases} Q(v,w) & v \text{ and } w \in \mathscr{E}_2(U) \\ Q(v,w) & v \text{ or } w \in \mathscr{E}_1(U) \\ 0 & \text{otherwise,} \end{cases}$$

and in matrix form

$$Q = \begin{pmatrix} 0 & 0 & \times \\ 0 & \times & \cdot \\ \times & \cdot & \cdot \end{pmatrix} \quad Q_0 = \begin{pmatrix} 0 & 0 & \times \\ 0 & \times & 0 \\ \times & 0 & 0 \end{pmatrix}.$$

Again it is easy to see that this is well defined.

Proposition 2.12. *The conic bundle $(\mathscr{E}_0, Q_0)$ is also semistable. Furthermore, if we repeat this process, eventually we will get a conic bundle that we will call $(\mathscr{E}^S, Q^S)$ with the following properties*

(i) $(\mathscr{E}^S, Q^S)$ is semistable, and if we apply this process to it with any object we obtain an isomorphic conic bundle (i.e. this process stops).

(ii) $(\mathscr{E}^S, Q^S)$ only depends on the isomorphism class of $(\mathscr{E}, Q)$.

(iii) Two conic bundles $(\mathscr{E}, Q)$ and $(\mathscr{E}', Q')$ are S-equivalent if and only if $(\mathscr{E}^S, Q^S)$ is isomorphic to $({\mathscr{E}'}^S, {Q'}^S)$.

Remark 2.13. The conic bundle $(\mathscr{E}^S, Q^S)$ is the analogue of the graded object $\mathrm{gr}(\mathscr{E})$ of the Jordan-Hölder filtration of a semistable torsion-free sheaf. Note that $\mathrm{gr}(\mathscr{E})$ can also be obtained by a process similar to this.

Proof. We start with a general observation about GIT quotients. Let Z be a projective variety with a linearized action by a group G. Two points in the open subset Z^{ss} of semistable points are S-equivalent (they are mapped to the same point in the moduli space) if there is a common closed orbit in the closures (in Z^{ss}) of their orbits. Let $z \in Z^{ss}$. Let B be the unique closed orbit in the closure $\overline{G\cdot z}$ in Z^{ss} of its orbit $G\cdot z$. Assume that z is not in B. Then there exists a one-parameter subgroup λ such that the limit $z_0 = \lim_{t\to 0}\lambda(t)\cdot z$ is in $\overline{G\cdot z}\setminus G\cdot z$. Note that we must have $\mu(z,\lambda)=0$ (otherwise z_0 would be unstable). Note that $G\cdot z_0 \subset \overline{G\cdot z}\setminus G\cdot z$, and then $\dim G\cdot z_0 < \dim G\cdot z$. Repeating this process with z_0 we then get a sequence of points that eventually stops and gives $\tilde z \in B$. Two points z_1 and z_2 will then be S-equivalent if and only if after applying this procedure to both of them the orbits of $\tilde z_1$ and $\tilde z_2$ are the same.

We will use the notation introduced in subsection 2.2. We will prove the proposition using the previous observation. The fact that choosing a "semistabilizing object" of $(\mathscr{E}, Q)$ induces a one parameter subgroup with $\mu(z,\lambda) = 0$ (where z is the corresponding point on Z^{ss}) follows from proposition 2.11 and the proof of proposition 2.7. The fact that the limit point z_0 corresponds to $(\mathscr{E}_0, Q_0)$ is an easy calculation (see [7, lemma 1.26]). The conic bundle $(\mathscr{E}_0, Q_0)$ is semistable by proposition 2.14.

It is easy to check that $\tilde z$ corresponds to $(\mathscr{E}^S, Q^S)$, and then items (ii) and (iii) follow from the fact that $\tilde z$ is in B.

$\square$

Proposition 2.14. *Let λ be a 1-PS of* $\mathrm{SL}(V)$*. Let* $\mathrm{SL}(V)$ *act on Z. Asume this action is linearized with respect to an ample line bundle $\mathscr{H}$. Let $z \in Z^{ss}$. Let $z_0 = \lim_{t\to 0} \lambda(t) \cdot z$. If $\mu(z,\lambda) = 0$, then $z_0 \in Z^{ss}$.*

Proof. This proof was given to us by A. King. We can assume, without loss of generality, that the polarization $\mathscr{H}$ of Z is very ample, and then Z embedds in $\mathbb{P}(H^0(\mathscr{H})^\vee)$ and $\mathrm{SL}(V)$ acts on $H^0(\mathscr{H})^\vee$. A point $x \in Z$ is (semi)stable iff its image in $\mathbb{P}(H^0(\mathscr{H})^\vee)$ is (semi)stable, and then we can assume $(Z, \mathscr{H}) = (\mathbb{P}(\mathbb{C}^n), \mathcal{O}(1))$, with $\mathrm{SL}(V)$ acting on $\mathbb{C}^n$.

Let $\pi : \mathbb{C}^n \setminus \{0\} \to \mathbb{P}(\mathbb{C}^n)$ be the projection. Let $z \in \mathbb{P}(\mathbb{C}^n)$ be a semistable point and λ a 1-PS with $\mu(z,\lambda) = 0$. Let $\tilde{z} \in \mathbb{C}^n$ be a point in the fibre $\pi^{-1}(z)$, and let

$$\tilde{z}_0 = \lim_{t\to 0} \lambda(t) \cdot \tilde{z}.$$

This limit exists and it is not the origin because $\mu(z,\lambda) = 0$. We have $z_0 := \lim_{t\to 0} \lambda(t) \cdot z = \pi(\tilde{z}_0)$ (by continuity of π). Assume that the point z_0 is unstable. Then the closure of the orbit of $\tilde{z}_0$ contains the origin, but this closure is included in the closure of the orbit of $\tilde{z}$, and this doesn't contain the origin because z is semistable. Then z_0 is semistable. Furthermore, z_0 cannot be stable because $\mu(z_0,\lambda) = \mu(z,\lambda) = 0$, then z_0 is strictly semistable.

□

3. Properties of conic bundles

3.1. Irreducibility of moduli space.

First we will show that the semistability and stability of conic bundles are open conditions.

Proposition 3.1. *Let $(\mathscr{E}_T, Q_T, \mathscr{N})$ be a flat family of conic bundles parametrized by T. The subset T^s (resp. T^{ss}) corresponding to stable (resp. semistable) conic bundles is open.*

Proof. Let m and l be large enough so that $\mathscr{V} = p_{T*}(\mathscr{E}_T \otimes p_X^*\mathcal{O}_X(m))$ is locally free, $p_T^*\mathscr{V} \otimes p_X^*\mathcal{O}_X(-m) \to \mathscr{E}_T$ is a surjection and proposition 2.11 holds.

Note that the universal family that was constructed on Z^{ss} in the proof of theorem I can be extended to the set Z^{good} of "good" points. Arguing as in the proof of theorem I, there is a finite open cover $\{U_i\}_{i\in I}$ of T and morphisms $f_i : U_i \to Z^{good} \subset Z$. These morphisms depend on the choices made (the choice of local trivializations of $\mathscr{V}$), but the $\mathrm{SL}(V)$ orbit of $f_i(t)$ are independent of the choices. In particular, the property of $f_i(t)$ belonging to Z^{ss} only depends on t. By proposition 2.11, $f_i(t)$ lies in Z^s (resp. Z^{ss}) iff the conic bundle $(\mathscr{E}_t, Q_t)$ is stable (resp. semistable). Then

$$T^s = \bigcup_{i\in I} f_i^{-1}(Z^s)$$

and the openness of Z^s in Z proves that T^s is open (the same argument works for Z^{ss}).

□

Theorem 3.2. *Let X be a Riemann surface. Fix r, d and τ. Then there exists an integer l_0 such that if $\deg \mathscr{L} > l_0$, then $\overline{\mathfrak{M}}_\tau(r,d,\mathscr{L})$ and $\mathfrak{M}_\tau(r,d,\mathscr{L})$ are irreducible or empty.*

Proof. We will construct a flat family of conic bundles parametrized by an irreducible scheme $\widetilde{Y}$ with the property that every semistable conic bundle of type $(r,d,\mathscr{L})$ belongs to the family. Then there is a surjective morphism $\widetilde{Y}^{ss} \to \overline{\mathfrak{M}}_\tau(r,d,\mathscr{L})$, where $\widetilde{Y}^{ss}$ is the open subset representing semistable points, and this proves that $\overline{\mathfrak{M}}_\tau(r,d,\mathscr{L})$ is irreducible. Repeating this with the open subset $\widetilde{Y}^s$ corresponding to stable points, we prove that $\mathfrak{M}_\tau(r,d,\mathscr{L})$ is also irreducible.

Let m be large enough so that for any semistable conic bundle $(\mathscr{E},Q)$ in $\overline{\mathfrak{M}}_\tau(r,d,\mathscr{L})$, the vector bundle $\mathscr{E}(m)$ is generated by global sections (corollary 2.2), and such that

$$2g-2-d-rm<0. \tag{7}$$

Note that m only depends on X, r, d and τ, but not on $\mathscr{L}$. If we choose $r-1$ generic sections of $\mathscr{E}(m)$, we have an exact sequence

$$0 \to \mathcal{O}_X^{\oplus r-1}(-m) \to \mathscr{E} \to \mathscr{M}(-m) \to 0$$

where $\mathscr{M}$ is a line bundle of degree $d+rm$.

By standard methods we can construct a universal family $\mathscr{F}_Y$ of extensions of line bundles of degree $d+(r-1)m$ by $\mathcal{O}_X^{\oplus r-1}(-m)$. This will be parametrized by a scheme Y that has a morphism to $\operatorname{Pic}^{d+rm}(X)$, and the fibre over a line bundle $\mathscr{M}$ is naturally isomorphic to $\operatorname{Ext}^1(\mathscr{M}(-m),\mathcal{O}_X^{\oplus r-1}(-m))$. Note that to construct this family we need that the dimension of this Ext^1 group is constant when we vary $\mathscr{M}$, but this is true thanks to (7). Each point $y\in Y$ corresponds to an extension of the form

$$0 \to \mathcal{O}_X^{\oplus r-1}(-m) \to \mathscr{F}_y \to \mathscr{M}(-m) \to 0.$$

It follows from the argument in the previous paragraph that all vector bundles in semistable conic bundles do occur in this family.

Note that, if $(\mathscr{E},Q)$ is a conic bundle, Q can be thought of as an element of $H^0(\operatorname{Sym}^2 \mathscr{F}_y^\vee \otimes \mathscr{L})$. Now choose l_0 large enough so that for any line bundle $\mathscr{L}$ of degree $\deg(\mathscr{L})>l_0$ the following holds

$$H^1(\operatorname{Sym}^2 \mathscr{F}_y^\vee \otimes \mathscr{L})=0$$

for any $y\in Y$. Then $H^0(\operatorname{Sym}^2 \mathscr{F}_y^\vee \otimes \mathscr{L})$ is constant when we vary y, and we can construct a (flat) family of conic bundles parametrized by $\widetilde{Y}=\mathbb{V}(\operatorname{Sym}^2 \mathscr{F}^\vee \otimes p_X^*\mathscr{L})$, and every semistable conic bundle of type $(r,d,\mathscr{L})$ belongs to this family.

□

3.2. **Orthogonal bundles.**

An orthogonal bundle is a vector bundle associated to a principal bundle with (complex) orthogonal structure group. Equivalently, it is a conic bundle $(\mathscr{E},Q)$ with $\mathscr{L}=\mathcal{O}_X$, such that the bilinear form $Q:\operatorname{Sym}^2\mathscr{E}\to\mathcal{O}_X$ induces an isomorphism $Q:\mathscr{E}\to\mathscr{E}^\vee$. We will call such a conic bundle a smooth conic bundle. In this case the conic bundle gives a smooth conic $\mathscr{C}_x$ for each point $x\in X$. Note that the isomorphism $Q:\mathscr{E}\to\mathscr{E}^\vee$ induces an isomorphism $\det Q:\det\mathscr{E}\to\det\mathscr{E}^\vee$, and then $\deg(\mathscr{E})=0$ (in fact $(\det(\mathscr{E}))^{\otimes 2}=\mathcal{O}_X$).

There is a notion of stability for orthogonal bundles (see [6]): a bundle $\mathscr{E}$ is orthogonal (semi)stable iff for every proper isotropic subbundle $\mathscr{F}$, $\deg(\mathscr{F})(\leq)0$. The notion of stability that we have defined for conic bundles depends in principle on a parameter τ, but we will show that in the case of a smooth conic bundle, the notion of stability doesn't depend on the particular value of the parameter. In fact we will prove that a smooth conic bundle is τ-(semi)stable iff it is (semi)stable as an orthogonal bundle.

Lemma 3.3. *Let $(\mathscr{E},Q)$ be a smooth conic bundle, and let $\mathscr{F}$ be a proper vector subbundle of $\mathscr{E}$. Then*

(i) There is an exact sequence

$$0 \to \mathscr{F}^{\perp} \to \mathscr{E} \to \mathscr{F}^{\vee} \to 0,$$

and $\deg(\mathscr{F}) = \deg(\mathscr{F}^{\perp})$.

(ii) If $\mathscr{F}$ is isotropic ($c_Q(\mathscr{F}) \leq 1$), then $\mathrm{rk}(\mathscr{F}) = 1$.

(iii) If $\mathrm{rk}(\mathscr{F}) = 1$, *then* $c_Q(\mathscr{F}) \geq 1$

Proof. (i) Follows from the exact sequence

$$0 \to \mathscr{F}^{\perp} \to \mathscr{E} \cong \mathscr{E}^{\vee} \to \mathscr{F}^{\vee} \to 0,$$

and the fact that $\deg(\mathscr{E}) = 0$.

(ii) Assume that $\mathrm{rk}(\mathscr{F}) = 2$. Then, in a basis adapted to $\mathscr{F} \subset \mathscr{E}$

$$Q = \begin{pmatrix} 0 & 0 & \cdot \\ 0 & 0 & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

and then $\det Q = 0$, contradicting the fact that the conic bundle is smooth.

(iii) If $c_Q(\mathscr{F}) = 0$, then

$$Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \cdot & \cdot \\ 0 & \cdot & \cdot \end{pmatrix}$$

and then $\det Q = 0$, again contradicting the fact that the conic bundle is smooth. □

Proposition 3.4. *A smooth conic bundle $(\mathscr{E},Q)$ is τ-semistable iff the vector bundle $\mathscr{E}$ is semistable as an orthogonal bundle. Furthermore, it is τ-stable iff it is stable as an orthogonal bundle.*

Proof. Let $(\mathscr{E},Q)$ be a smooth τ-semistable conic bundle. Let $\mathscr{F}$ be an isotropic vector subbundle. By lemma 3.3 (ii), $\mathrm{rk}(\mathscr{F}) = 1$. We have $\mathscr{F} \subset \mathscr{F}^{\perp}$, $\mathrm{rk}(\mathscr{F}^{\perp}) = 2$ (by lemma 3.3 (i)), and we check that $\mathscr{F} \subset \mathscr{F}^{\perp} \subset \mathscr{E}$ is a critical filtration. Then $\deg(\mathscr{F}) + \deg(\mathscr{F}^{\perp}) \leq 0$, but $\deg(\mathscr{F}) = \deg(\mathscr{F}^{\perp})$ (lemma 3.3 (i)), and then $\deg(\mathscr{F}) \leq 0$, which proves that $\mathscr{E}$ is semistable as an orthogonal bundle. Furthermore, if $(\mathscr{E},Q)$ is τ-stable, then $\deg(\mathscr{F}) + \deg(\mathscr{F}^{\perp}) < 0$, $\deg(\mathscr{F}) < 0$ and $\mathscr{E}$ is stable as an orthogonal bundle.

Conversely, let $\mathscr{E}$ be an orthogonal semistable bundle. Let $\mathscr{F}$ be any vector subbundle. Following [6] let $\mathscr{N} = \mathscr{F} \cap \mathscr{F}^{\perp}$, and let $\mathscr{N}'$ be the vector subbundle generated by $\mathscr{N}$. We have an exact sequence

$$0 \to \mathscr{N}' \to \mathscr{F} \oplus \mathscr{F}^{\perp} \to \mathscr{M}' \to 0 \tag{8}$$

where $\mathscr{M}'$ is the subbundle of $\mathscr{E}$ generated by $\mathscr{F} + \mathscr{F}^{\perp}$. We have $\mathscr{M}' = (\mathscr{N}')^{\perp}$.

If $\mathscr{N}' = 0$, then $\mathscr{E} = \mathscr{F} \oplus \mathscr{F}^{\perp}$, $c_Q(\mathscr{F}) = 2$, and $\deg(\mathscr{F}) = 0$ (lemma 3.3 (i)). Then

$$\frac{\deg(\mathscr{F}) - c_Q(\mathscr{F})\tau}{\mathrm{rk}(\mathscr{F})} = \frac{-2\tau}{\mathrm{rk}(\mathscr{F})} < \frac{-2\tau}{3} = \frac{\deg(\mathscr{E}) - 2\tau}{3}.$$

If $\mathscr{N}' \neq 0$, then $\deg(\mathscr{F}) = \deg(\mathscr{N}')$ (by lemma 3.3 (i) and the exact sequence (8)), and then $\deg(\mathscr{F}) \leq 0$ (because $\mathscr{E}$ is orthogonal semistable and $\mathscr{N}'$ is isotropic). If $\mathrm{rk}(\mathscr{F}) = 2$, then $c_Q(\mathscr{F}) = 2$ (by lemma 3.3 (ii)), and if $\mathrm{rk}(\mathscr{F}) = 1$, then $c_Q(\mathscr{F}) \geq 1$ (by lemma 3.3 (iii)). In any case

$$\frac{\deg(\mathscr{F}) - c_Q(\mathscr{F})\tau}{\mathrm{rk}(\mathscr{F})} \leq \frac{-c_Q(\mathscr{F})\tau}{\mathrm{rk}(\mathscr{F})} < \frac{-2\tau}{3} = \frac{\deg(\mathscr{E}) - 2\tau}{3}$$

Now let $\mathscr{E}_1 \subset \mathscr{E}_2 \subset \mathscr{E}$ be a critical filtration. Then $\mathscr{E}_1$ is isotropic, $\mathscr{E}_2 = \mathscr{E}_1^{\perp}$, and then

$$\deg(\mathscr{E}_1) + \deg(\mathscr{E}_2) = 2\deg(\mathscr{E}_1) \leq 0,$$

because $\mathscr{E}_1$ is isotropic and $\mathscr{E}$ is orthogonal semistable. This finishes the proof that $(\mathscr{E}, Q)$ is τ-semistable. Furthermore, if $\mathscr{E}$ is orthogonal stable, the last inequality is strict, and we obtain that $(\mathscr{E}, Q)$ is τ-stable.

□

Appendix: Hitchin-Kobayashi correspondence for conic bundles

(By I. Mundet i Riera)

In this appendix I use the result in [4] to relate the notion of stability for conic bundles to the existence of solutions to a certain PDE. This is similar to the well known relation between stability of vector bundles and existence of Hermite–Einstein metrics, or between stability of holomorphic pairs and solutions to the vortex equations. As usual in the literature, I call such a relation a Hitchin–Kobayashi correspondence (see [4] and the references therein).

Take a non-degenerate conic bundle $Q : \mathrm{Sym}^2 \mathscr{E} \to \mathscr{L}$ on a Riemann surface X. Let E be the smooth bundle underlying $\mathscr{E}$. We denote $\overline{\partial}_{\mathscr{E}}$ the $\overline{\partial}$ operator on E given by $\mathscr{E}$. Fix a metric (In this appendix *metric* will always mean *Hermitian metric*). on $\mathscr{L}$ and consider the following equation on a metric h on E:

$$i\Lambda F_{\overline{\partial}_{\mathscr{E}},h} + \frac{\tau}{2}\frac{Q \otimes Q^{*_h}}{\|Q\|_h^2} = c\mathrm{Id}, \tag{9}$$

where $F_{\overline{\partial}_{\mathscr{E}},h}$ is the curvature of the Chern connection of $\overline{\partial}_{\mathscr{E}}$ with respect to h, $\Lambda : \Omega^2(X) \to \Omega^0(X)$ is the adjoint of wedging with the Kaehler form of X and the subscript h in $*$ and $\|\cdot\|$ is to recall that both depend on h. Finally, $\tau > 0$ and c are real numbers.

We will take a (rather standard) point of view putting equation (9) inside the setting considered in [4]. Then we will study the existence criterion to solutions of the equation given in [4] applied to this particular case, thus arriving again at the notion of stability for conic bundles.

Fix a metric h_0 in E. Let $\mathcal{G}^c$ be the complex gauge group of E, i.e., the group of its smooth automorphisms covering the identity on X. The group $\mathcal{G}^c$ acts on the space of $\overline{\partial}$ operators on E by pullback. So $g \in \mathcal{G}^c$ sends $\overline{\partial}_{\mathscr{E}}$ to $g^*\overline{\partial}_{\mathscr{E}} = g \circ \overline{\partial}_{\mathscr{E}} \circ g^{-1}$. Any metric h on $\mathscr{E}$ is the pullback $h = g^*h_0$ by some $g \in \mathcal{G}^c$. Furthermore, for any metric h and gauge transformation g

$$g\left(F_{(g^{-1})^*\overline{\partial}_{\mathscr{E}},h}\right)g^{-1} = F_{\overline{\partial}_{\mathscr{E}},g^*h} \qquad \text{and} \qquad g\left(\frac{g^{-1}Q \otimes (g^{-1}Q)^{*_h}}{\|g^{-1}Q\|_h^2}\right)g^{-1} = \frac{Q \otimes Q^{*_{(g^*h)}}}{\|Q\|_{g^*h}^2}.$$

So if $h = g^*h_0$ solves (9) then, conjugating by g, we get

$$i\Lambda F_{(g^{-1})^*\overline{\partial}_{\mathscr{E}},h_0} + \frac{\tau}{2}\frac{g^{-1}Q \otimes g^{-1}Q^{*_{h_0}}}{\|g^{-1}Q\|^2_{h_0}} = c\mathrm{Id}. \tag{10}$$

We will now see that equation (10) on $g \in \mathcal{G}^c$ is a particular case of the equations considered in [4].

Let $F = \mathbb{P}(\mathrm{Sym}^2(\mathbb{C}^3)^*)$. Take on F the symplectic structure $\tau\omega$, where ω is the symplectic structure on F obtained from the canonical metric on $\mathrm{Sym}^2(\mathbb{C}^3)^*$. (For that we view F as the symplectic quotient $F = \mu_0^{-1}(-i)/S^1$, where $\mu_0(z) = -i|z|^2$ is the moment map of the action of S^1 on $\mathrm{Sym}^2(\mathbb{C}^3)^*$.) Consider on F the action of $U(3,\mathbb{C})$ induced by the canonical action on $\mathbb{C}^3$. This action is Hamiltonian, and the moment map evaluated at $x \in F$ is

$$\mu(x) = -i\frac{\tau}{2}\left(\frac{\hat{x} \otimes \hat{x}^*}{\|\hat{x}\|^2}\right), \tag{11}$$

where $\hat{x} \in \mathrm{Sym}^2(\mathbb{C}^3)^*$ is any lift of x.

Let P be the $U(3,\mathbb{C})$ principal bundle of h_0-unitary frames of E, and let $\mathcal{F} = P \times_{U(3,\mathbb{C})} F$. The conic bundle Q gives a section $\Phi \in \Gamma(\mathcal{F})$, and by formula (11) the term in (10) involving Q is $i\mu(\Phi)$. Let $\mathcal{A}$ be the set of connections on P. Let $A = A_{\overline{\partial}_{\mathscr{E}},h_0}$ be the Chern connection. The action of $\mathcal{G}^c$ on $\mathcal{A}$ considered in [4] is as follows: $g \in \mathcal{G}^c$ sends A to $g(A) = A_{g^*\overline{\partial}_{\mathscr{E}},h_0}$. Finally, since the conic bundle $Q : \mathrm{Sym}^2\mathscr{E} \to \mathscr{L}$ is non-degenerate, the pair (A,Φ) is simple. So by the theorem in [4] there is a solution $g \in \mathcal{G}^c$ to equation (10) if and only if (A,Φ) is c-stable. Furthermore, the metric g^*h_0 is unique.

The previous discussion applies also to bundles of quadrics on projective bundles of arbitrary dimension. In the next section we will study the c-stability condition on any rank and in the next one we will give a more precise description of c-stability for conic bundles.

Stability for bundles of quadrics.

We will suppose from now on that $\mathrm{Vol}(X) = 1$. The stability condition stated in [4] refers to reductions of the structure group of our bundle to parabolic subgroups plus antidominant characters of those parabolic subgroups. In our case the structure group is $GL(n,\mathbb{C})$, so a parabolic reduction is equivalent to a filtration by subbundles:

$$0 \subset E_1 \subset \cdots \subset E_r = E,$$

where the ranks strictly increase. The action on E of any antidominant character for this reduction is given by a matrix of this form (written using any splitting $E = E_1 \oplus E_2/E_1 \oplus \cdots \oplus E_r/E_{r-1}$)

(12)

$$\chi = \begin{pmatrix} z + m_1 + \cdots + m_{r-1} & 0 & \dots & 0 \\ 0 & z + m_2 + \cdots + m_{r-1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & z \end{pmatrix} - \sum_{k=1}^{r-1} m_k \frac{\mathrm{rk}(E_k)}{\mathrm{rk}(E)}\mathrm{Id},$$

where z is any real number and $m_j \leq 0$ are negative real numbers (strictly speaking this is the action of i times an antidominant character; however, we will ignore this in the sequel. Using the notation of [4] this matrix is $ig_{\sigma,\chi}$, where χ is an antidominant character).

Stability of a quadric. To write the stability notion for (A,Φ) we need to compute the maximal weight of the action of χ on the section Φ. So fix a point $x \in X$ and write

$$0 \subset W_1 \subset \cdots \subset W_r = W$$

the induced filtration in the fibre $W = E_x$ over x. Take a basis $e_1,\ldots,e_n$ of W such that for any $1 \le k \le r$, $\{e_1,\ldots,e_{\mathrm{rk}(W_k)}\}$ is also a basis of W_k. Write $e_1^*,\ldots,e_n^*$ the dual basis, so that Q gives on W the quadratic form

$$Q = Q(x) = \sum_{i\le j} \alpha_{ij}(e_i^* e_j^*).$$

The action of χ on $\mathrm{Sym}^2 W^*$ diagonalizes in the basis $\{e_i^* e_j^*\}_{i\le j}$, and one has

$$\chi(e_i^* e_j^*) = \\ \left(-(2z + 2m_I + \cdots + 2m_{J-1} + m_J + \cdots + m_{r-1}) + 2\sum_{k=1}^{r-1} m_k \frac{\dim(W_k)}{\dim(W)}\right)(e_i^* e_j^*).$$

Here and in the sequel we follow this convention: the index I (resp. J) is the minimum one such that e_i belongs to W_I (resp. e_j belongs to W_J). From this one deduces that

$$\mu(Q(x);\chi) = \max_{\alpha_{ij}\neq 0}\{-(2z+2m_I+\cdots+2m_{J-1}+m_J+\cdots+m_{r-1})\}+2\sum_{k=1}^{r-1} m_k \frac{\dim(W_k)}{\dim(W)}.$$

Define $M_I = -(m_I + \cdots + m_{r-1})$. Given two subspaces $W', W'' \subset W$, we will write $Q(W',W'') = 0$ if for any $w' \in W'$ and $w'' \in W''$, $Q(w',w'') = 0$. Otherwise we will write $Q(W',W'') \neq 0$. Then,

$$\mu(Q(x);\chi) = \max_{Q(W_I,W_J)\neq 0}\{M_I + M_J - 2z\} + 2\sum_{k=1}^{r-1} m_k \frac{\dim(W_k)}{\dim(W)}\mathrm{Id}. \tag{13}$$

Stability for the bundle of quadrics. The pair (A,Φ) is by definition c-stable if for any filtration σ of E by subbundles

$$0 \subset E_1 \subset \cdots \subset E_r = E$$

and any antidominant character χ as in (12) one has

$$\deg(\sigma,\chi) + \tau\int_{x\in X} \mu(Q(x);\chi) - \langle \chi, c\mathrm{Id}\rangle > 0. \tag{14}$$

Here the degree of the pair (σ,χ) is

$$\deg(\sigma,\chi) = z\deg(E) + \sum_{j=1}^{r-1} m_j\left(\deg(E_j) - \frac{\mathrm{rk}(E_j)}{\mathrm{rk}(E)}\deg(E)\right),$$

and, on the other hand, $\langle\chi, c\mathrm{Id}\rangle = zc\,\mathrm{rk}(E) = zcn$. The map Q is holomorphic, and the function $\mu(Q(x);\chi)$ is lower semicontinuous and takes a finite number of values as x moves on X. Hence, $\mu(Q(x);\chi)$ takes its maximal value in a Zariski open dense subset of X, and so

$$\int_{x\in X}\mu(Q(x);\chi) = \mathrm{Vol}(X)\max_{x\in X}\mu(Q(x);\chi) = \max_{x\in X}\mu(Q(x);\chi).$$

For any pair of subbundles $E', E'' \subset E$, define $Q(E', E'') = \max_{x\in X} Q(E'_x, E''_x)$. Then

$$\max_{x\in X} \mu(Q(x);\chi) = \max_{Q(E_i,E_j)\neq 0} \{M_i + M_j - 2z\} + \sum_{k=1}^{r-1} 2m_k \frac{\mathrm{rk}(E_k)}{\mathrm{rk}(E)}. \tag{15}$$

Putting everything together (14) becomes

$$0 < z\deg(E) - 2\tau z - zc\,\mathrm{rk}(E) + \sum_{k=1}^{r-1} m_k \left(\deg(E_k) - \frac{\mathrm{rk}(E_k)}{\mathrm{rk}(E)}\deg(E) + 2\tau\frac{\mathrm{rk}(E_k)}{\mathrm{rk}(E)}\right)$$
$$+ \tau \max_{Q(E_i,E_j)\neq 0}\{M_i + M_j\}.$$

This must be true for any real number z, so the pair (A, Φ) can only be stable if

$$c = \frac{\deg(E) - 2\tau}{\mathrm{rk}(E)}.$$

Define now

$$d_k = \deg(E_k) - \frac{\mathrm{rk}(E_k)}{\mathrm{rk}(E)}\deg(E) + 2\tau\frac{\mathrm{rk}(E_k)}{\mathrm{rk}(E)}.$$

Then the stability condition reduces to

$$\sum_{k=1}^{r-1} m_k d_k + \tau \max_{Q(E_i,E_j)\neq 0}\{M_i + M_j\} > 0. \tag{16}$$

And this must hold for any choice of (not all zero) negative numbers $m_1, \ldots, m_{r-1}$.

4. The case $\mathrm{rk}(E) = 3$

In the sequel we will use the following notation. If E' is a vector bundle and α is any real number,

$$\mu_\alpha(E') = \frac{\deg(E') - \alpha}{\mathrm{rk}(E')}.$$

In this section we assume that $\mathrm{rk}(E) = 3$. Hence, Q describes a bundle of conics in a bundle of projective planes $\mathbb{P}(E)$ on X. Recall that we assume that Q is (generically) non-degenerate. We have seen above that the pair (A, Φ) cannot be c-stable unless

$$c = \mu_{2\tau}(E).$$

Suppose this holds. Now, according to formula (16), (A, Φ) is stable if and only if for any filtration $0 \subset E_1 \subset E_2 \subset E$ and for any pair of (not all zero) real numbers $m_1, m_2 \leq 0$,

$$m_1 d_1 + m_2 d_2 + \tau \max_{Q(E_i,E_j)\neq 0}\{M_i + M_j\} > 0, \tag{17}$$

where, as before, $d_k = \deg(E_k) - \frac{\mathrm{rk}(E_k)}{\mathrm{rk}(E)}\deg(E) + 2\tau\frac{\mathrm{rk}(E_k)}{\mathrm{rk}(E)}$. There are three cases to consider:

- $Q(E_1, E_1) = Q(E_1, E_2) = 0$, $Q(E_2, E_2) \neq 0$. Geometrically, E_1 gives fibrewise a point on the conic and E_2 a tangent line to the conic at the point given by E_1. In this case,

$$\max_{Q(E_i,E_j)\neq 0}\{M_i + M_j\} = \max\{-2m_2, -m_1, -m_2\}.$$

Hence,

$$
\begin{aligned}
&0 > d_1 - \tau = \deg(E_1) - \frac{\mathrm{rk}(E_1)}{\mathrm{rk}(E)}\deg(E) + 2\tau\frac{\mathrm{rk}(E_1)}{\mathrm{rk}(E)} - \tau,\\
&0 > d_2 - 2\tau = \deg(E_2) - \frac{\mathrm{rk}(E_2)}{\mathrm{rk}(E)}\deg(E) + 2\tau\frac{\mathrm{rk}(E_2)}{\mathrm{rk}(E)} - 2\tau,\\
&0 > d_1 + d_2 - 2\tau.
\end{aligned}
$$

Simplifying, we obtain the following conditions:

$$
\mu_\tau(E_1) < \mu_{2\tau}(E), \qquad \mu_{2\tau}(E_2) < \mu_{2\tau}(E), \qquad \deg(E_1) + \deg(E_2) < \deg(E).
$$

- $Q(E_1, E_1) = 0$, $Q(E_1, E_2) \neq 0$ ($\Rightarrow$ $Q(E_2, E_2) \neq 0$). Geometrically, E_1 is a point on the conic and E_2 a line passing through E_1 but generically not tangent to the conic. In this case,

$$
\max_{Q(E_i,E_j)\neq 0}\{M_i + M_j\} = -m_1 - 2m_2.
$$

Hence,

$$
\begin{aligned}
&0 > d_1 - \tau = \deg(E_1) - \frac{\mathrm{rk}(E_1)}{\mathrm{rk}(E)}\deg(E) + 2\tau\frac{\mathrm{rk}(E_1)}{\mathrm{rk}(E)} - \tau,\\
&0 > d_2 - 2\tau = \deg(E_2) - \frac{\mathrm{rk}(E_2)}{\mathrm{rk}(E)}\deg(E) + 2\tau\frac{\mathrm{rk}(E_2)}{\mathrm{rk}(E)} - 2\tau.
\end{aligned}
$$

Simplifying, we obtain the following two conditions:

$$
\mu_\tau(E_1) < \mu_{2\tau}(E), \qquad \mu_{2\tau}(E_2) < \mu_{2\tau}(E).
$$

- $Q(E_1, E_1) \neq 0$ ($\Rightarrow$ $Q(E_1, E_2) \neq 0$ and $Q(E_2, E_2) \neq 0$). Geometrically, E_1 gives a point generically not on the conic and E_2 any line through E_1. In this case,

$$
\max_{Q(E_i,E_j)\neq 0}\{M_i + M_j\} = -2m_1 - 2m_2.
$$

Hence,

$$
\begin{aligned}
&0 > d_1 - 2\tau = \deg(E_1) - \frac{\mathrm{rk}(E_1)}{\mathrm{rk}(E)}\deg(E) + 2\tau\frac{\mathrm{rk}(E_1)}{\mathrm{rk}(E)} - 2\tau,\\
&0 > d_2 - 2\tau = \deg(E_2) - \frac{\mathrm{rk}(E_2)}{\mathrm{rk}(E)}\deg(E) + 2\tau\frac{\mathrm{rk}(E_2)}{\mathrm{rk}(E)} - 2\tau.
\end{aligned}
$$

Simplifying, we obtain the following two conditions:

$$
\mu_{2\tau}(E_1) < \mu_{2\tau}(E), \qquad \mu_{2\tau}(E_2) < \mu_{2\tau}(E).
$$

In conclusion, and as claimed at the beginning, the condition of c-stability obtained from studying equation (9) coincides with that of stability obtained from the GIT construction of the moduli space of conic bundles.

Acknowledgements. We would like to thank R. Hernández, A. King, S. Ramanan and C.S. Seshadri for discussions and comments.

References

[1] D. Huybrechts and M. Lehn, *Framed modules and their moduli,* Internat. J. Math., **6** (1995), 297–324.

[2] A. D. King and P. E. Newstead, *Moduli of Brill-Noether pairs on algebraic curves,* Internat. J. Math. **6** (1995), 733–748.

[3] D. Luna, *Slices étales,* Bull. Soc. Math. France, Mémoire **33** (1973), 81–105.

[4] I. Mundet i Riera, *A Hitchin-Kobayashi correspondence for Kaehler fibrations*, `math.DG/9901076`

[5] J. Murre, *Lectures on an Introduction to Grothendieck's Theory of the Fundamental Group,* Lecture Notes, Tata Institute of Fundamental Research, Bombay, 1967.

[6] S. Ramanan, *Orthogonal and spin bundles over hyperelliptic curves,* Proc. Indian Acad. Sci., Math. Sci., **90** (1981), 151–166.

[7] C. Simpson, *Moduli of representations of the fundamental group of a smooth projective variety I,* Publ. Math. I.H.E.S. **79** (1994), 47–129.

T. Gómez
tomas@math.tifr.res.in
School of Mathematics
Tata Institute of Fundamental Research
Homi Bhabha Road, 400 005 Mumbai (India)

I. Sols
sols@eucmax.sim.ucm.es
Departamento de Algebra
Facultad de Ciencias Matemáticas
Universidad Complutense de Madrid, 28040 Madrid (Spain)

I. Mundet i Riera
ignasi.mundet@uam.es
Dep. Matemáticas, Facultad de Ciencias
Universidad Autónoma de Madrid, Madrid (Spain)