GROUPS OF TRANSFORMATIONS OF A G-STRUCTURE
WHICH LEAVE INARIANT A SUBSTRUCTURE

by

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1. INTRODUCTION.

Throughout this paper our manifolds will be Hausdorff, infinitely differentiable, and second countable. We fix the following objects: A m-dimensional manifold M, two closed subgroups G, H of GL (m; R), with $G \supset H$, and a G-structure $p: A \rightarrow M$. Let $\Sigma$ be a Lie group of transformations of A. We ask: Is there an $H$-structure $B \subset A$ such that $\Sigma \subset \text{Aut}(B)$? If the answer is affirmative we say that $\Sigma$ is inessential. Thus, inessential groups are groups of transformations of a substructure of A. In order to avoid trivial cases we will assume that there are $H$-structures contained in A. The aim of this paper is to impose conditions on $M$, $G$, $H$, $A$, and $\Sigma$ so that $\Sigma$ be inessential.

2. AUXILIARY RESULTS.

We will write $G/H = L$. The group $G$ acts canonically on $L$ on the left, and we denote by $E$ the bundle associated to $A$ with fibre $L$. There is a projection $A \times L \rightarrow E$ which we write $(a, z) \mapsto az$. The group $\Sigma$ acts on the left on $E$ by $(\sigma_x, az) \mapsto (\sigma_x(a))z$, where $\sigma_x$ is the bundle isomorphism of $A$ induced by $\sigma$. There are canonical bijections between the set of $H$-structures contained in $A$, the set of sections of $E$, and the set of maps $\Phi: A \rightarrow L$ such that $\Phi(ga) = g^{-1}\Phi(a)$ for all $a \in A$, $g \in G$. In fact, the $H$-structure $B$ corresponds to $\Phi$ if and only if $B = \Phi^{-1}(H)$, and $\Phi$ corresponds to the section $s$ if and only if for all $a \in A$, $s(p(a)) = a \Phi(a)$.

The following lemma is easy.
2.1. **Lemma.** Let $\sigma \in \text{Aut} \,(A)$, and let $B$ be an $\Omega$-structure contained in $A$ determined by the map $\Phi: \, A \to L$, or by a section $s$ of $E$. Then, the following statements are equivalent

a) $\sigma \in \text{Aut} \,(B)$

b) $\Phi \circ \sigma = \Phi$

c) For all $x \in M$, we have $\sigma \,(s\,(x)) = s\,(\sigma \,(x))$.

2.2. **Lemma.** Let $\Sigma$ be a Lie group acting on the manifolds $M$ and $N$, and $f: N \to M$ an equivariant map; i.e. $f\,(\sigma \,(x)) = \sigma \,(f \,(x))$ for all $x \in M$ and $\sigma \in \Sigma$. Suppose there is a submanifold $P$ of $M$ such that

a) The map $\Sigma \times P \to M$, $(\sigma, x) \to \sigma \,(x)$ is a surjective submersion.

b) There is a map $s': P \to N$ such that $f \circ s' = \text{id}_P$ and for all $\sigma \in \Sigma$ and $x \in P$ with $\sigma \,(x) \in P$ we have $s' \,(\sigma \,(x)) = \sigma \,(s' \,(x))$.

Then the map $s'$ can be extended to a unique map $s: M \to N$ such that $f \circ s = \text{id}_M$, and $s\,(\sigma \,(x)) = \sigma \,(s\,(x))$ for all $\sigma \in \Sigma$ and $x \in M$.

**Proof:** Define $h: \Sigma \times P \to N$ and $g: \Sigma \times P \to M$ by $h\,(\sigma, x) = \sigma \,(s' \,(x))$ and $g\,(\sigma, x) = \sigma \,(x)$. It is clear from the hypothesis that $h$ is constant on the fibres of $g$. Hence there is a $s: M \to N$ such that $s \circ g = h$, and one checks easily that it is the required extensión.

We will give without proofs some results about a Lie group acting properly on a manifold.

2.3. **Lemma.** If $\Sigma$ acts freely and properly on $M$, the orbit space is a quotient manifold of $M$. In fact, $M$ is a principal $\Sigma$-bundle with base the orbit space.

(It follows from proposition (1.2.3), th (1.1.3), and th. (4.1) in [4]).

2.4. **Lemma.** Let $\Sigma$ be a Lie group acting properly on a manifold $M$. There exists an open set $U$ and a closed set $C$ such that $C \subset U$, $\Sigma \, C = M$, and for each compact $K$, $\{ \sigma \in \Sigma \, | \, U \cap \sigma \,(K) \neq \varnothing \}$ is relatively compact (See [3] l. lemma 2).
3. Case of a Free Proper Action.

3.1. Theorem. If \( \Sigma \subset \text{Aut} (A) \) is diffeomorphic to \( \mathbb{R}^k \) for some \( k \) and acts freely and properly on \( M \), then \( \Sigma \) is inessential.

Proof: If \( Q \) is the orbit space of \( \Sigma \), we know by (2.3) that \( M \to Q \) is a principal bundle with group \( \Sigma \). The fibre of this bundle is diffeomorphic to \( \mathbb{R}^k \). Hence, this bundle admits a global section. Being a principal bundle it must be trivial. Therefore there is a submanifold \( P \subset M \) such that the map \( \Sigma \times P \to M, (\sigma, x) \to \sigma(x) \) is a diffeomorphism. Let \( s' \) be the associated section to an \( H \)-structure contained in \( A \). We still denote by \( s' \) the restriction to \( P \). If \( x \in P \) and \( \sigma(x) \in P \), then \( \sigma = \text{id}_M \). Therefore we may apply (2.2) getting a section \( s \) of \( E \). If \( B \) is the \( H \)-structure associated to \( s \), we get from (2.1) and (2.2) that \( \Sigma \subset \text{Aut} (B) \).

Example: We take \( G = \text{GL} (m; \mathbb{R}) \) and \( H = \text{O} (m; \mathbb{R}) \). Thus, any free proper action of \( \mathbb{R} \) on \( M \) induces a group \( \Sigma \) to which (3.1) can be applied, and we get that \( \Sigma \) can be considered as a group of isometries of a certain Riemannian metric on \( M \). The vector field induced by the \( R \)-action is a Killing vector field.


We will denote by \( \Sigma_x \) the isotropy group of \( \Sigma \) at \( x \in M \). Consider the property: Any Lie homomorphism \( h : \Sigma_x \to G \) has its image contained in a conjugate of \( H \).

4.1. Theorem: If \( \Sigma \subset \text{Aut} (A) \) acts transitively and the property above holds, then \( \Sigma \) is inessential.

Proof: Choose a frame \( a \in p^{-1} (x) \). For each \( \sigma \in \Sigma_x \), \( \sigma_a (a) \in p^{-1} (x) \). Therefore there is a unique element \( h (\sigma) \) of \( G \) such that \( \sigma_a (a) = a h (\sigma) \). It is clear that \( h : \Sigma_x \to G \) is a Lie group homomorphism, and that if \( a' \in p^{-1} (x) \) is written \( a' = a g \), the corresponding \( h' \) is related to \( h \) by \( h' (\sigma) = g^{-1} h (\sigma) g \). We may assume then that \( a \) has been chosen with the condition \( h (\Sigma_x) \subset H \). We take in (2.2) \( P = \{ x \} \). Hypothesis (a) holds clearly because the action is transitive. We define \( s' : P \to P \) by \( s'(x) = a H \). Then \( \sigma (x) \in P \) if and only if \( \sigma \in \Sigma_x \), and we have

\[
\sigma (s'(x)) = \sigma (a H) = (a h (\sigma)) H = a (h (\sigma) H) = a H = s'(x) = s' (\sigma (x)).
\]
If B is the H-structure associated to the extension s of s', it follows from (2.1) that Σ ⊂ Aut (B).

We point out some cases in which the property of the isotropy group holds.

(4.2) Suppose Σ_X is compact and H is a normal subgroup such that G/H is isomorphic to R^k for some k. If h: Σ_X → G is a continuous homomorphism, then h(Σ_X) ⊂ H. If this were not the homomorphism h': Σ_X → R^k, composition of h, the projection G → G/H, and the isomorphism G/H → R^k would not be constant. Then h(Σ_X) cannot be bounded and Σ_X is not compact.

(4.3) Let G have a finite number of connected components. There is a compact subgroup H of G having the following property: If K is a compact subgroup of G, then K is contained in a conjugate of H. (See [2] (XV.3.1)). Clearly the property holds for G and H. It is well known that if G = GL(m; R), we can take H = O(m, R). Analogously, if G = GL(n; C), we can take H to be the unitary group. We get then for example, that if Σ is an isotropy compact Lie group of transformations acting transitively on M, there is a Riemannian metric for which Σ is a subgroup of its group of isometries.

5. CASE OF A PROPER ACTION.

(5.1) Theorem: Suppose there is a vector space V and a linear action of G on V such that: (a) For a certain v_o ∈ V, H is the isotropy group at v_o. (b) The orbit W of v_o is an open cone; i.e. if w ∈ W and r > 0, r w ∈ W. If Σ ⊂ Aut (A) acts properly, then Σ is inessential.

Proof: Let i: G/H → W be the canonical diffeomorphism, and j its inverse. Let Φ: A → G/H be a map corresponding to an M-structure B ⊂ A. Take C and U as in (2.4) and let f: M → R be a map which is 1 on C and 0 outside U. For each a ∈ A define h_a: Σ → V by h_a(σ) = f(p(σ_a(a))) (iΦ(σ_a(a))). Now, define

Φ': A → G/H , \quad \Phi'(a) = \int h_a(σ) \, dσ,

where the integral is the left invariant Haar integral on Σ. Since one-point sets are compact, Φ'(a) is defined and Φ'(σ(σ)) = Φ'(σ), for the integral is invariant under left translations. Given g ∈ G one checks that h_ag = g^{-1}h_a. Then, since the integral commutes with linear maps we obtain Φ'(ag) = g^{-1}Φ'(a). If B' is the H-structure associated to Φ' we have Σ ⊂ Aut (B').

Compact groups act properly. Therefore.
(5.2) **Corollary:** If $G$ and $H$ verify the hypothesis of (5.1), any compact group $\Sigma \subset \text{Aut} (\Lambda)$ is inessential.

(5.3) **Example:** Let $G$ be the group of matrices with positive determinant, and $H = \text{SL} (m; \mathbb{R})$. We take as $V$ the space of alternating $m$-multilinear maps on $\mathbb{R}^m$, and as $\nu_0$ the determinant. We get that if $M$ is an oriented manifold and $\Sigma$ a Lie group of transformations preserving the orientation, there is a volume element inducing the orientation such that all elements of $\Sigma$ are volume preserving.

(5.4) **Example:** Let $G$ be the conformal group and $H$ the orthogonal group of $\mathbb{R}^m$. We take as $V$ the space of multiples of the standard inner product $\nu_0$. The conclusion of (5.1) is that for any Lie group of conformal transformations of a metric $g$, there is a metric $g'$ of the form $g' = hg$, with $h$ a positive map, such that $\Sigma$ is a group of isometries of $g$.

(5.5) **Example:** Let $G = \text{GL} (m; \mathbb{R})$, and $H = \text{O} (m; \mathbb{R})$. We take as $V$ the space of bilinear symmetric forms on $\mathbb{R}^m$ on which $G$ acts by pull-back, and $\nu_0$ is the standard inner product. The conclusion of (5.1) is that for any Lie group of transformations $\Sigma$ acting properly on $M$ there is a Riemannian metric $g$ such that $\Sigma$ is contained in its group of isometries.

There is a similar example for $G = \text{GL} (n; \mathbb{C})$ and $H = \text{U} (n; \mathbb{C})$, where $m = 2n$.

Examples (5.4) and (5.5) are known, although the result is proved by a different method (See [1] ths. 1 and 4, and [4] (4.3.1)). Besides showing other examples, our theorem allows us to give a simpler treatment with the help of (2.4).
BIBLIOGRAPHY


