THE ROLE OF ADJUSTMENT COSTS IN INTEREST RATE DETERMINATION

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ABSTRACT.

We study in this paper the equilibrium influence of adjustment costs of capital on interest rates determination. Considering endogenous interest rates in optimal capital accumulation models introduces nonlinearities which together with expectations of future variables make the model hard to analyze. We use here a solution method that has recently been proposed in the literature, to show that the model is able to reproduce some of the correlations among output, consumption, interest rates and capital that can be observed in actual time series data.
INTRODUCTION.

The role of adjustment costs in the determination of the optimal capital accumulation path has been emphasized in the economic literature (see Treadway [9] and [10], Gould [1], Lucas [3] and Mortenson [6]). In a general equilibrium framework, fluctuations in the optimal stock of capital will affect as well as be influenced by interest rates. These, in turn, will be intertemporally correlated with output and consumption. Such a model could therefore be used to explain and interpret a number of dynamic statistical properties of real economies that have been observed in empirical work (see Litterman and Weiss [4], among others).

Interest rates have however been assumed in previous related work to be constant, or moving in an exogenously given deterministic fashion. A practical reason for such an assumption is that to make interest rates endogenous introduces nonlinearities in the model. Under uncertainty, the simultaneous presence of nonlinearities and expectations of future variables will preclude obtaining a closed form solution to the model, making impossible to analytically characterize the model's properties. An alternative is to generate sample realizations for the vector stochastic process of variables in the economy. We could then obtain some time series statistics: autocorrelation functions, cross-correlation functions for pairs of variables, impulse responses of the system to innovations in any one variable, decompositions of variance of forecast errors, and others. Functions like these have been used in previous empirical research to characterize the stochastic properties of univariate time series data as well as the intertemporal, dynamic interrelations between sets of variables. Therefore, the statistics obtained from the model could be compared with similar ones obtained from actual time series data, and small distances between these functions be used as criteria for goodness of fit of the model.

Unfortunately, the same analytical difficulties we have mentioned above
make this data generation process a non-trivial task. We utilize here a method suggested in Sims [8] and already utilized in Novales [7], to obtain explicit equilibrium solutions to nonlinear rational expectations models. The method does not need a closed form solution to the model, but requires some analysis so as to guarantee stability of the obtained solution.

The economy we consider is described in section 2, whereas in section 3 we propose the application of the solution method to this particular model. Section 3 contains a discussion of the calibration of the model, while in section 4 we present the numerical results that summarize the dynamic properties of the model. The paper closes with some conclusions and suggestions for further work.
II. THE ECONOMY.

Let us consider a single commodity economy with a continuum of identical agents. The commodity can either be consumed or used as capital. The stock of capital at time $t$ is an input in the production of time $t+1$ resources, according to the production function:

$$y_{t+1} = f(K_t, x_{t+1}) = y - \theta \cdot (K_t - x)^2 + \epsilon_{t+1}$$

where $y$ is a positive constant with $y > \max\{0; \theta \cdot x^2/2 - \inf\{\text{supp } \epsilon_t\}\}$, and $\epsilon_t$ is a stochastic process. The amount produced each period is therefore a random function of the previous period stock of capital.

At each time $t$, the cost of varying the stock of capital is given by:

$$\gamma(K_t, K_{t-1}) = \omega/2 \cdot (K_t - K_{t-1})^2$$

This quadratic specification for the adjustment costs of capital has been very popular in previous research (see references in the introduction). The convenience and interpretation of this assumption have been discussed in these references and will not be repeated here.

We assume the following single period utility function for the typical consumer:

$$U_t(C_t, C_{t-1}) = C_t - \frac{\mu}{2} \cdot C_t^2 - \frac{\beta}{2} \cdot (C_t - C_{t-1})^2$$

which is not time separable, for it makes the utility of current consumption a function of last period's consumption. From the functional form in (2.3), we can interpret the nonseparability as reflecting a distaste for rapid changes in the consumption values.

Consumers behave as price takers and solve the problem.
(2.4) \[ \text{Max } E_0 V(C_t, t = 0) = E_0 \sum_{t=0}^{\infty} \beta^t U_t(C_t, C_{t-1}) \]
subject to:
(2.5) \[ C_t + K_t - K_{t-1} = f(K_{t-1}, e_t) - y(K_t, K_{t-1}) \]

Together with (2.1)-(2.3) and \( C_t, K_t \geq 0 \) for all \( t \geq 0 \). Since there are random shocks to technology, this becomes an optimization problem under uncertainty. The information set \( \Omega_t \) based on which the conditional expectation \( E_t \) is calculated is assumed to contain: \( \{ Y_{t-s}, C_{t-s}, K_{t-s}, s \geq 0 \} \). In what follows, we reserve the notation \( U_t \) for the period \( t \) utility, and denote by \( V \) the lifetime discounted aggregate utility.

A change in consumption at time \( t \) changes not only the utility of the current period, but next period's utility as well, due to the nonseparability of preferences. The marginal lifetime utility of current consumption then is:

\[ \frac{3V}{3C_t} = \beta^t \cdot (1-nC_t-b \cdot (C_t-C_{t-1})) + \beta^{t+1} \cdot b \cdot (C_{t+1}-C_t) \]

which can be seen to be a random variable to be realized at time \( t+1 \). It is convenient to introduce the process of discounted marginal utility of current consumption:

(2.6) \[ W_t = \beta^{-t} \cdot \frac{3V}{3C_t} = 1 + \beta \cdot b \cdot C_{t+1} - (n+b \cdot \beta) \cdot C_t + b \cdot C_{t-1} \]

Which again, is a random variable at time \( t \).

It is shown in the appendix that the following is a necessary condition for optimality of an interior solution:

(2.7) \[ [1+\omega \cdot (K_t-K_{t-1})] \cdot E_t W_t = \beta \cdot E_t [W_{t+1} \cdot (1+\omega \cdot (K_{t+1}-K_t) - \theta \cdot (K_t-x))] \]

To interpret this condition, let us define the net revenue function at
time $t$ on last period's capital stock as the gross output produced at time $t$ minus the adjustment costs of capital between $t-1$ and $t$:

$$R_t(K_{t-1}, K_t) = \gamma + K_{t-1} - \frac{\theta}{2} (K_{t-1} - \alpha)^2 + \delta_t - \frac{\eta}{2} (K_t - K_{t-1})^2$$

which can be seen to depend on the current stock of capital as well. Consider the partial derivatives:

$$\frac{\partial R_t}{\partial K_t} = -\omega (K_t - K_{t-1}),$$

(2.8)

$$\frac{\partial R_{t+1}}{\partial K_t} = 1 - \theta (K_t - \alpha) + \omega (K_{t+1} - K_t)$$

If the current stock of capital is above last period's $K_{t-1}$ and we decrease $K_t$ by one unit, the return at time $t$ on $K_{t-1}$ then increases. The same result is obtained when $K_t$ is below $K_{t-1}$ and we increase $K_t$. In this economy, an increase of a unit in the stock of capital requires of a decrease in consumption of one unit plus the marginal cost of adjusting the stock of capital. That may add to more or less than a decrease of one unit of commodity in consumption, depending on whether the difference $K_t - K_{t-1}$ is positive or negative. From (2.8), the total change in current consumption needed to increase the stock of capital by one unit is therefore given by: $1 - \frac{\partial R_t}{\partial K_t}$.

The additional unit of capital at time $t$ has two real effects at time $t+1$: a) it increases production (that is, output at time $t+1$) by the marginal product: $1 - \theta (K_t - \alpha)$, and b) it contributes to the adjustment costs to be paid at $t+1$, since the stock of capital is now one unit larger. The aggregate effect is therefore given by $\frac{\partial R_{t+1}}{\partial K_t}$. Consumption at time $t+1$ can be increased by the net amount of these two effects. The implied utility gain at $t+1$ should exactly compensate the utility loss from having decreased consumption at time $t$ to gain one additional unit of capital. But that is exactly the message in (2.7), which can be written:
In equilibrium, the representative agent is indifferent between his allocation and a movement along his feasible set that involves giving up enough current consumption so as to increase the stock of capital by one unit and consume tomorrow the output increment produced by this change.

Recently, different models have obtained and tested implications of the type: "The marginal utility of consumption behaves as a first order Markov process" (see Hall [2], for example). The model in this paper shows that with adjustment costs of capital the condition is somewhat different. If we denote:

\[
\xi_t = \frac{3R_t}{3K_t} \frac{1}{\left(1 - \frac{3R_t}{3K_t}\right)}
\]

and define \( \xi_t = \prod_{s=1}^{t} \xi_s \), then, multiplying through (2.7) by \( \xi_t \) we get:

\[
E_t(\xi_t \cdot W_t) = E_t(\xi_{t+1} \cdot W_{t+1})
\]

With a time separable utility of consumption, the marginal utility discounted by the random factor \( \xi_t \) is a martingale process. When the utility function is not time separable, the discounted marginal utility \( \xi_t \cdot W_t \) then satisfies a condition which is weaker than a martingale: the current predictions of any two consecutive future values are the same. (Notice that with our specified utility function, \( \xi_t \cdot W_t \) is \( \Omega_{t+1} \)-measurable and \( \xi_{t+1} \cdot W_{t+1} \) is \( \Omega_{t+2} \)-measurable).

This result reduces to Hall's under his set of assumptions: Time separability of preferences implies that \( E_t(\xi_t W_t) = \xi_t W_t \). Furthermore, without costs of adjustment, then \( \xi_{t+1} / \xi_t \) is equal to the discount factor \( \beta \) times the marginal productivity of capital, equal to one plus the real rate of interest. Therefore, under these conditions, we have:
where \( W_t \) is the marginal utility of consumption, which is Hall's result. All of these are different degrees of generalization of Hall's result about the behavior of consumption as a first order Markov process when the utility function is time separable and approximately quadratic in a neighborhood of the equilibrium steady state value of consumption, and the interest rate is constant.

One way to introduce interest rates in the model is by defining the real rate at each time \( t \) to be equal to the marginal rate of time preference:

\[
\frac{E_t W_{t+1}}{\beta (1 + r_t)} = 1 + r_t = \frac{1}{\beta} \cdot \frac{E_t W_{t+1}}{E_t W_t}
\]

This condition arises from the utility maximization problem the consumer solves in the case when there are some investment opportunities (this is shown in appendix 2). Under uncertainty, the equilibrium rates of return on all assets in the economy would be equal to the gross real rate of interest. From the way uncertainty enters our model, we get here a weaker condition:

\[
\frac{3R_t(K_{t+1}, K_t)}{\beta K_t} \quad \frac{E_t W_{t+1}}{\beta \cdot E_t W_t} = 1
\]

which is just a rewriting of (2.9), and shows that in the conditional expectation sense, the equilibrium return on capital is equal to the rate of time preference. Unfortunately, we cannot conclude that the two rates are the same (see appendix 2).
SOLUTION OF THE MODEL.

The model in the previous section has been shown to produce interesting dynamic relations among consumption, capital, interest rates and output. The joint presence of nonlinearities and expectations of future variables in the equilibrium conditions prevent us from obtaining the closed form solution that would be needed to analytically characterize the model's properties regarding the interrelationships among these variables. An alternative way to analyze the model could be to generate equilibrium time series realizations that would be used to compute statistics (autocorrelation functions, cross-correlation functions, impulse response functions), that would summarize the dynamics of the economy under study. Unfortunately, this alternative approach is far from trivial, due to the presence of the conditional expectations in the equilibrium conditions.

The equilibrium in the economy described in the previous section is characterized by the set of optimality conditions:

\begin{align}
(3.1) \quad [1+\omega \cdot \Delta K_t] \cdot E_t W_t &= \beta \cdot E_t [W_{t+1} \cdot (1+\omega \cdot \Delta K_{t+1} - \theta \cdot (K_t - \alpha))] \\
(3.2) \quad 1 + r_t &= \frac{E_t W_{t+1}}{\beta} \\
the \ technology: \ \\
(3.3) \quad Y_t &= y - \frac{\theta}{2} \cdot (K_{t-1} - \alpha)^2 + \epsilon_t \\
and \ the \ budget \ constraint: \ \\
(3.4) \quad C_t + K_t &= K_{t-1} + \omega/2 \cdot (K_t - K_{t-1})^2 = Y_t \\
\end{align}

\begin{align} \text{together with the expression for the marginal utility of consumption:} \end{align}
The equilibrium conditions (3.1)-(3.5) are five equations in $(W_t, C_t, K_t, Y_t, r_t, \varepsilon_t)_{t=0}$. Actually, we are not one equation short, for we have not specified yet a stochastic representation for the technology shock $\varepsilon_t$ which could be used to obtain realizations for it. Equivalently, this means that the model needs of an additional condition to be closed. The difficulties we have mentioned suggest that arbitrary ways of closing the model will not in general allow us to solve the model. In particular, assuming that $\varepsilon_t$ follows a univariate ARIMA representation will let us generate realizations for $\varepsilon_t$, but will not allow us to use these realizations in (3.1)-(3.5) to generate data for all the other variables, again due to the simultaneous presence of conditional expectations and nonlinearities.

In order to follow that approach, the expectations in (3.1)-(3.4) raise an important difficulty, because that increases the number of variables to solve for, given that we then have not only realized current and past values of variables, but the values taken by the conditional expectations of future variables as well.

We close the model by considering the process:

\begin{equation}
Z_{t+1} = W_t \cdot \left(1 + w \cdot sK_t \right)
\end{equation}

The reason to date the process $Z_t$ as in (3.6) is that with the assumed utility function (2.5), the discounted marginal utility is a variable to be realized at time $t+1$. The process $Z_t$ can be used to write the optimality condition (3.1) as:

\begin{equation}
\beta^{-1} \cdot E_t Z_{t+1} = E_t Z_{t+2} - \theta \cdot (K_t - \alpha) \cdot E_t W_{t+1}
\end{equation}

Assumption 1.- $Z_t$ is a stationary process that admits a first order autoregressive representation:

\begin{equation}
(3.5) \quad W_t = 1 + a_0 \cdot C_{t+1} + a_1 \cdot C_t + a_2 \cdot C_{t-1}
\end{equation}

with: $a_0 = \beta \cdot b$, $a_1 = -(n+b+\beta \cdot b)$; $a_2 = b$, so that $W_t$ is not realized until time $t+1$.
This amounts to assuming Granger causal priority of \( Z_t \) with respect to all other variables in the economy. However, the Granger causal priority concept, as we usually think of it applies to linear models, and it is unclear what the assumption made above implies about the Granger causal properties of consumption and capital, the two components in the process \( Z_t \). One would guess that the Granger causal priority of \( Z_t \) can be compatible with almost any possible causal ordering among \( C_t \), \( K_t \) and all the other variables in the economy.

**Stability Analysis:**

Closing the model must be done in a way that achieves stability of the implied solution. We can analyze the stability properties of the model by constructing a linear approximation around its steady state. Equations (2.5) and (2.7) can be approximated around the steady state values \( (W^*, K^*, \epsilon^*) \) by:

\[
(2.5) \quad \begin{bmatrix}
-(1-L) & W^* \cdot [\mu \cdot (1-L) \cdot (L-\beta) + \beta \cdot \theta \cdot L] & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

\[
(2.7) \quad \begin{bmatrix}
Q^{-1}(L) & (1-L) \cdot (1-\omega \cdot K^*) + \theta \cdot (K^* - \alpha) \cdot L & -1
\end{bmatrix}
\]

where \( Q(.) \) is the lag polynomial (3.5) that defines \( W \) as a function of current, future and past consumption values. Suppose we now complete the system with a third row of the form: \( [E \ F \ 0 \ 1] \), in which the last entry is taken to be zero because it simplifies the computations of the eigenvalues of the resulting matrix. It is not hard to show that if \( E \) is zero, the resulting matrix has then a determinant with a unit root. On the other hand, if \( F \) is chosen to be zero, then the determinant is a quadratic, with at least one of the roots inside the unit circle, and consequently, the system is not stable.

Therefore, we need to close the system with an assumption on the
stochastic distribution of a function of both, \( W_t \) and \( K_t \). Suppose that we assume the autoregression: \( B(L) \cdot [W_t \cdot (1+\omega \cdot (K_t-K_{t-1}))] = \nu_{t+1} \). The third row of the matrix then becomes:

\[
\begin{bmatrix}
B(L) & B(L) \cdot W^* \cdot \omega \cdot (1-L) & 0
\end{bmatrix}
\]

and the lag polynomial determinant has as roots those of \( B(L) \) together with:

\[
L = \frac{-\omega \cdot (1-\beta)}{-\omega \cdot (1-\beta) + \beta \cdot \theta}
\]

which needs of the relation: \( \omega > \theta \cdot \beta / (1-\beta) \). This inequality is of course a constraint on the chosen parameter values when numerically solving the model. The solution will be stable so long as this condition among the parameter values is satisfied and the autoregression for \( Z_t \) is chosen to be stationary. Our equation (3.8) is just one such stationary autoregression.

Recursive solution of the model:

With assumption 1, equation (3.8) then becomes:

\[
\beta^{-1} \cdot (AZ_t + \mu) = A^2 
Z_t + (A+1) \cdot \mu - \theta \cdot (K_t-\alpha) \cdot E_t W_{t+1}
\]

which gives us the conditional expectation formula:

\[
E_t W_{t+1} = \frac{A(A-\beta^{-1})Z_t-(\beta^{-1}-A-1) \cdot \mu}{\theta \cdot (K_t-\alpha)}
\]  

and suggests that:

\[
W_{t+1} = \frac{A(\beta^{-1}-A) \cdot Z_t+(\beta^{-1}-A-1) \cdot \mu}{-\theta \cdot (K_t-\alpha)} + \eta_{t+2}
\]
with $E_{t} \eta_{t+2} = 0$. Again, the date in $\eta_{t+2}$ comes about because $W_{t+1}$ is realized at time $t+2$. As a consequence of (3.10), $\eta_{t}$ is a first order moving average random shock:

(3.11) $\eta_{t} = \xi_{t} - \rho \cdot \eta_{t+1}$, where $\xi_{t}$ is a white noise.

Equations (3.6) and (3.10) together give us:

(3.12) $K_{t} = K_{t-1} + \omega \left( 1 - \frac{2_{t+1}}{\omega} \right) - \frac{A(\beta^{-1}-A)Z_{t-1} + (\beta^{-1}-A-1) \cdot \mu}{-\theta \cdot (K_{t-1} - \alpha)}$.

which is a recursive formula to determine the equilibrium path of capital accumulation as a function of the process $Z_{t}$. The equilibrium value of the interest rate is given by:

(3.13) $r_{t} = \frac{1}{\beta} \cdot \frac{E_{t} W_{t} - \theta \cdot (K_{t-1} - \alpha)}{E_{t} W_{t+1}} - \frac{A(\beta^{-1}-A) \cdot Z_{t-1} + (\beta^{-1}-A-1) \cdot \mu}{\beta \cdot \frac{-\theta \cdot (K_{t-1} - \alpha)}{E_{t} W_{t+1}}}$.

Starting from initial conditions $(Z_{0}, K_{0}, \xi_{0})$ and a parameter vector $(A, \mu, \beta, \omega, \alpha, \theta, \sigma_{u}^{2}, \sigma_{\epsilon}^{2})$, and drawing a random realization for the vector process $(\nu_{t}, \xi_{t})$, we can then use (3.8) and (3.11) to generate time series for $Z_{t}$ and $\eta_{t}$, and then together with $K_{0}$ in (3.12) to generate a time series for $K_{t}$. After that, (3.10) can be used to get a time series for $W_{t}$. The final step is to recover the equilibrium path of consumption from (3.5), the expression for the marginal utility of consumption $W_{t}$, which can be written:

$W_{t} = 1 + \beta \cdot b \cdot D(L) \cdot C_{t+1}$

with:
\[ Q(L) = 1 - \frac{(n+b+\beta.b)}{\beta.b}L + L^2 \]

and therefore, we have for each time \( t \) the identity:

\[ W_t - 1 \]

(3.14) \[ Q(L).C_{t+1} = ------- \]

\[ \beta.b \]

Let \( \lambda_1 \) and \( \lambda_2 \) denote the two roots of the lag polynomial \( Q(L) \).

They can be obtained from:

\[ n+b+\beta.b \]

\[ \lambda_1 + \lambda_2 = ------- \]

\[ \beta.b \]

\[ \lambda_1 \lambda_2 = 1/\beta \]

Given a value for the discount factor \( \beta \), then a choice of \( \lambda_1 \) determines the value of \( \lambda_2 \) and finally, a value of \( n \) gives us the value of \( b \) by:

\[ b = n / [\beta(\lambda_1+\lambda_2-1)-1] \]

Let us assume that the roots are chosen so that \( |\lambda_1| > 1 \), \( |\lambda_2| < 1 \). We can then expand (3.14) forward:

(3.15) \[ C_{t+1} = \lambda_2^* C_t + [b, \beta, (\lambda_1-1)]^{-1} - [\beta.b]^{-1} \cdot \sum_{i=1}^{\infty} (\beta \lambda_2)^i W_{t+i} \]

We want to use (3.15), together with a realization of the process of marginal utility \( W_t \), to compute the corresponding realization for consumption. Clearly, in order to do so, we will have to truncate the infinite sum in (3.15) at some \( t+T \), and use the approximation:
(3.16) \[ C_{t+1} = \lambda_2 C_t + \frac{1}{\beta \cdot b} \left( \frac{1}{\lambda_1 - 1} - \sum_{i=1}^{\infty} \lambda_1^{-i} W_{t+i} \right) \]

Hence, to generate a consumption series, we choose a \( \lambda_2 \) value, which given a value of \( \alpha \) determines the values of \( b \) and \( \lambda_1 \). Equation (3.16) can then be used to start from an initial condition \( C_{-1} \) to get a consumption series. If we assume that \( W_t \) is roughly constant (and equal to \( W^\alpha \)) on the time interval \([t+1, t]\), the approximate error due to truncation at (3.16) is then given by:

\[
W^\alpha = \frac{1}{\beta \cdot b} \frac{1}{\lambda_1 T (\lambda_1 - 1)}
\]

This error will clearly be smaller the bigger the absolute value of \( \lambda_1 \) and the larger the number \( T \) of terms included in the approximation. Finally, we get a time series for output from (3.4), and a realization for \( \xi_t \) from (3.3). It is therefore clear that the autoregression (3.8) does not introduce a new shock \( \nu_t \) into the model, for \( \xi_t \) and \( \nu_t \) are one an exact function of the other. However, the nonlinear nature of this relationship implies that stochastic assumptions made on \( \nu_t \) will not translate into similar properties for \( \xi_t \). In particular we will show that even when \( \nu_t \) is independently and identically distributed over time, the technology shock \( \xi_t \) may be autocorrelated.

Nonuniqueness of the solution:

Linear rational expectations models are characterized by not having, in general, a unique solution. The analytically more complex nonlinear models are not different in that respect. An advantage of our solution method is to clearly point out the source of nonuniqueness in the model. For the one in this paper, we have just shown that any stationary autoregression assumed for the process \( Z_t \) will produce a stationary, and hence acceptable, solution.
EMPIRICAL RESULTS.

The deterministic steady state of the model provides some guidance in choosing parameter values for the simulations. Those values are chosen so that the implied steady state values of the variables are close to the ones in actual detrended time series data. In general, it is not possible to match the steady state values for all the variables for the model imposes restrictions among those values which do not hold in actual data. For example, the budget constraint (2.5) implies that the steady state values of consumption and output are the same. This observation, while being important because it introduces a constraint the parameter values to be used when solving the model is not however a result that emerges from actual time series data.

To obtain the deterministic steady state of the economy we set the values of the random shocks equal to their unconditional mean (zero) at all times $t$, to get:

$$(z^*, k^*, w^*, l^*, r^*, y^*) = \left( \begin{array}{ccccc} \frac{\mu}{1-H^{-1}} & \gamma + \ldots; \frac{1}{1-A}; \alpha & \gamma + \ldots; \frac{1}{1-A}; \beta^{-1} & \ldots; \frac{(1-H^{-1})^2}{2\theta} \end{array} \right).$$

which as we can see, is independent of the values of $\lambda_1$ and $\lambda_2$.

The monthly discount factor was chosen $\theta = .997$, which implies annual interest rate values fluctuating around 3.6%. The parameters in the $z_t$-autoregression were chosen $A = .90$, $\mu = .03$, and $\sigma_y = 3 \times 10^{-4}$. Values of $A$ close to 1.0 are plausible because one would expect $z_t$ to be a smooth process. Variations in the value of $A$ in that range did not produce any noticeable change in the results. The value of $\theta$ just affects the mean values of the series but not the model's stochastic properties.

The values of $n$ and $\alpha$ affect just the average consumption and capital stock values. They were chosen to be $n = .00165$ and $\alpha = 380.0$, which imply steady state values of consumption and the stock of capital equal to 424.24 and 374.0, respectively.

The value of $w$ was chosen to be 1.0 which implies that $w \cdot (k^* - \alpha) = -6.0$, well below -1.0, the necessary condition to guarantee
stability of the solution. The value of \( \theta \) affects the size of the fluctuations of the generated time series. A value of \( \theta = .0005 \) produced plausible size fluctuations. As a consequence of the restriction we mentioned above, the value of \( \gamma \) must be 47.24. We chose the nonstationary root of \( g(L) \) to be \( \lambda_1 = 1.20 \), which implies values \( \lambda_2 = .835 \) and \( b = .0289 \). Experimentation with different values of \( \lambda_1 \) proved that choice not to produce any important change in the results. The value of \( \rho \) in the moving average representation (3.11) for \( \eta_t \) was chosen to be .25, and the standard deviation of \( \xi_t \) was \( \sigma_\xi = 10^{-4} \). While the value of \( \rho \) is mainly irrelevant for the results, the value of the variance is very important to obtain appropriate fluctuations in the solutions.

With these parameter values we generated monthly time series for (1960,1)-(1984,12). We then tried to replicate actual data collection by accumulating the flows of output and consumption over a quarter and averaging the interest rate values. These aggregated/averaged series were used in all the computations. A total of 50 simulations were run and we calculated the sample means and standard deviations for all the statistics we obtained. All the results we present are these sample means, whereas the numbers in brackets are the standard deviations obtained from the empirical distribution. Since we omitted the first 10 quarters from all the computations to minimize sampling error, a total of 90 observations were available. A number of them had to be skipped in each case depending on the number of lags involved in the calculus.

To characterize the stochastic properties of the univariate time series produced by the model we computed the autocorrelation and partial autocorrelation functions for output, interest rates and consumption, which can be seen in table 1. There is evidence in these functions that consumption and output could each be represented by an AR(2) model, and interest rates by an ARIMA(0,1,1) model. These short autoregressions for output and consumption match actual data well. Another interesting observation is that the simulated technology shock shows important serial correlation, and seems to be well represented by an AR(1) model with coefficient .94. Autocorrelation in the technology shock is a major reason to produce the serial correlation in output that we observe in table 1 as well as in actual output data.
In Table 2 we see that there is a weak positive effect from interest rates lagged 1 to 4 quarters to output, as well as a weak negative effect from output to interest rates 1 to 3 quarters ahead. These series are contemporaneously uncorrelated. The contemporaneous correlation between consumption and output is very important and extends up to 3 quarters in each direction. This property matches a characteristic of actual time series data, except that in actual data the correlation extends to a longer period. It is not surprising after these two observations to see that the correlations between interest rates and consumption have the same characteristics as those between interest rates and output.

The impulse responses of bivariate systems to shocks in one variable are presented in Table 3. Output has a smooth negative response to an interest rate innovation, recovering after more than two years. This is the type of response of output to nominal rates in actual data. The interest rates in our model are real, so these two similar results would just be comparable with no time uncertainty.

The response of consumption to an interest rate innovation is positive for two quarters and then becomes negative until it recovers to zero. The responses of interest rates to output or consumption innovations are initially positive to then become negative. In either case they just last for about 6 quarters.

**Remark 1.** The Hessian of the utility function (2.3) is a tridiagonal matrix:

\[
\begin{bmatrix}
    a_{11} & a_{12} & 0 & 0 & \ldots & 0 & 0 \\
    a_{21} & a_{22} & a_{23} & 0 & \ldots & 0 & 0 \\
    0 & a_{32} & a_{33} & a_{34} & \ldots & 0 & 0 \\
    \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    0 & 0 & 0 & 0 & \ldots & a_{T-1,T} & a_{T,T}
\end{bmatrix}
\]
This matrix can be decomposed (see Johnson and Riess [3]) as the product $LU$ of the matrices:

$$
L = \begin{pmatrix}
1 & 0 & 0 & \ldots & 0 \\
x_{21} & 1 & 0 & \ldots & 0 \\
oop & x_{32} & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
o & 0 & 0 & \ldots & x_{T-1,T} 1
\end{pmatrix},
$$

$$
U = \begin{pmatrix}
u_{11} & a_{12} & 0 & 0 & \ldots & 0 \\
0 & u_{22} & a_{23} & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & a_{T-1,T} \\
0 & 0 & 0 & \ldots & u_{T,T}
\end{pmatrix},
$$

so that: $|H| = |L| |U| = |U| = \prod_{i=1}^T u_{ii}$

Under our parameter choice, the values of the $u_{ii}$ elements alternate in sign starting with negative, and consequently, for these parameter values, the lifetime utility (2.5) is a concave function.

**Remark 2.** Equations (3.5) and (3.9) imply the forecasting formula:

$$
\mathbb{E}_t \begin{pmatrix} Z_{t+2} \\ 1+w\cdot\Delta K_{t+1} \end{pmatrix} = \frac{A. (\beta^{-1}-A). Z_t + w. (\beta^{-1}-A-1)}{1+w\cdot\Delta K_{t+1}} - \theta.(K_t-\alpha)
$$

for a nonlinear function of $Z_{t+2}$ and $K_{t+1}$, given $K_t$. That conditional expectation is a nonlinear function of $Z_t$ and $K_t$. However, that formula does not help us to identify any particular linear stochastic structure for $Z_{t+2}/(1+w\cdot K_{t+1})$ and hence, does not produce any inconsistency with the set of assumptions made above. In particular, it is consistent with the first order autoregression for $Z_t$ assumed in (3.8).
Remark 3. An alternative estimation strategy would start by postulating a finite order autoregression for the marginal utility of consumption. For simplicity, we assume a first order autoregression:

\[ W_{t+1} = B W_t + b + \epsilon_{t+1} \]

Along a stationary equilibrium path, consumption will fluctuate around its steady state and consequently, the discounted marginal utility of consumption would go to zero. In order not to impose any restriction on the parameters of (3.18), we chose to make our assumption on the undiscounted, rather than the discounted marginal utility. Then, from (3.7):

\[ E_t(I_{t+2} - \beta^{-1} Z_{t+1}) = \theta \cdot (K_t - \alpha) \cdot (B W_t + b) \]

which suggests the specification:

\[ Z_{t+1} = \beta^{-1} Z_t + \theta \cdot (K_{t-1} - \alpha) \cdot (B W_{t-1} + b) + \epsilon_{t+1} \]

with \( E_{t-1} \epsilon_{t+1} = 0 \) but \( E_t \epsilon_{t+1} \) may be different from zero. If we now use our definition of the process \( Z_t \):

\[ Z_{t+1} = -W_t \cdot [1+w(K_t - K_{t-1})] \]

then we get:

\[ (1+w(K_t - K_{t-1})) = \beta^{-1} W_{t-1} [1+w(K_{t-1} - K_{t-2})] + \theta (K_{t-1} - \alpha) (B W_{t-1} + b) + \epsilon_{t+1}, \]

\[ W_t \]

\[ (wW_t) - (wW_t + \beta^{-1} wW_{t-1} + \theta (B W_{t-1} + b)) K_{t-1} + \beta^{-1} W_{t-1} [1+w(K_t - K_{t-1})] + \theta (K_{t-1} - \alpha) (B W_{t-1} + b) - \epsilon_{t+1} = 0 \]

which is a non linear second order difference equation on \( (W_t, K_t) \) with nonlinear coefficients, which makes impossible to analyze the equilibrium of the model in a way like the one we suggested in section 3.
CONCLUSIONS

We have analyzed in this paper the role that costs of adjustment of the optimal capital stock play on the equilibrium determination of interest rates. Considering endogenous rates of interest in optimal capital accumulation models creates difficulties, since the implied decision rules are nonlinear. Under uncertainty, these decision rules will also include expectations of future variables, and that combination of things makes impossible the use of standard methods for the analysis of dynamic economic models.

A general method that has recently been introduced is utilized to solve the nonlinear rational expectations model in this paper. The method allows for generation of equilibrium time series data that can be used to characterize the model's properties concerning the interrelations between output, consumption, capital and interest rates. It is shown in the paper that our simple general equilibrium model of capital accumulation is able to explain some autocorrelation properties as well as interesting cross-correlations that are observed in actual time series data.
### TABLE 1

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<td>.20 (.05)</td>
<td>.62 (.06)</td>
<td>-.22 (.06)</td>
</tr>
<tr>
<td>3</td>
<td>.17 (.05)</td>
<td>.31 (.11)</td>
<td>-.16 (.07)</td>
</tr>
<tr>
<td>4</td>
<td>.15 (.04)</td>
<td>.04 (.13)</td>
<td>-.09 (.06)</td>
</tr>
<tr>
<td>5</td>
<td>.09 (.03)</td>
<td>-.16 (.14)</td>
<td>-.04 (.05)</td>
</tr>
<tr>
<td>6</td>
<td>.05 (.04)</td>
<td>-.28 (.14)</td>
<td>.0 (.03)</td>
</tr>
<tr>
<td>7</td>
<td>.0 (.04)</td>
<td>-.33 (.14)</td>
<td>.03 (.06)</td>
</tr>
<tr>
<td>8</td>
<td>-.02 (.04)</td>
<td>-.33 (.13)</td>
<td>.04 (.07)</td>
</tr>
</tbody>
</table>
# TABLE 3

**IMPULSE RESPONSES**

I. **System: Output - Interest Rates.**

<table>
<thead>
<tr>
<th>Quarters ahead</th>
<th>Output Responses</th>
<th>Interest Rates Responses</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Shock in Output</td>
<td>Shock in Interest Rates</td>
</tr>
<tr>
<td>1</td>
<td>1.0 (.0)</td>
<td>.0 (.0)</td>
</tr>
<tr>
<td>2</td>
<td>2.09 (.12)</td>
<td>-.40 (.13)</td>
</tr>
<tr>
<td>3</td>
<td>2.62 (.33)</td>
<td>-1.47 (.33)</td>
</tr>
<tr>
<td>4</td>
<td>2.49 (.33)</td>
<td>-1.79 (.44)</td>
</tr>
<tr>
<td>5</td>
<td>1.82 (.67)</td>
<td>-1.58 (.50)</td>
</tr>
<tr>
<td>6</td>
<td>1.08 (.69)</td>
<td>-.87 (.47)</td>
</tr>
<tr>
<td>7</td>
<td>.42 (.58)</td>
<td>-.50 (.40)</td>
</tr>
<tr>
<td>8</td>
<td>-.13 (.39)</td>
<td>-.16 (.30)</td>
</tr>
<tr>
<td>12</td>
<td>-.76 (.52)</td>
<td>.50 (.30)</td>
</tr>
<tr>
<td>16</td>
<td>.0 (.29)</td>
<td>.07 (.22)</td>
</tr>
<tr>
<td>20</td>
<td>.30 (.22)</td>
<td>-.18 (.10)</td>
</tr>
<tr>
<td>24</td>
<td>.04 (.22)</td>
<td>-.05 (.15)</td>
</tr>
</tbody>
</table>

II. **System: Interest Rates - Consumption.**

<table>
<thead>
<tr>
<th>Quarters ahead</th>
<th>Interest Rates Responses</th>
<th>Consumption Responses</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Shock in Interest Rates</td>
<td>Shock in Consumption</td>
</tr>
<tr>
<td>1</td>
<td>1.0 (.0)</td>
<td>.0 (.0)</td>
</tr>
<tr>
<td>2</td>
<td>-.95 (.13)</td>
<td>.56 (.06)</td>
</tr>
<tr>
<td>3</td>
<td>-.09 (.12)</td>
<td>.19 (.12)</td>
</tr>
<tr>
<td>4</td>
<td>.19 (.07)</td>
<td>-.42 (.13)</td>
</tr>
<tr>
<td>5</td>
<td>.08 (.11)</td>
<td>-.26 (.13)</td>
</tr>
<tr>
<td>6</td>
<td>-.05 (.12)</td>
<td>-.05 (.10)</td>
</tr>
<tr>
<td>7</td>
<td>-.04 (.05)</td>
<td>-.04 (.07)</td>
</tr>
<tr>
<td>8</td>
<td>.06 (.07)</td>
<td>-.10 (.08)</td>
</tr>
<tr>
<td>12</td>
<td>.0 (.0)</td>
<td>.02 (.04)</td>
</tr>
<tr>
<td>16</td>
<td>.0 (.0)</td>
<td>.02 (.02)</td>
</tr>
<tr>
<td>20</td>
<td>.0 (.0)</td>
<td>.0 (.02)</td>
</tr>
<tr>
<td>24</td>
<td>.0 (.0)</td>
<td>-.01 (.01)</td>
</tr>
</tbody>
</table>
### Table 4

**Autoregressive Systems**

#### I. Output - Interest Rates.

<table>
<thead>
<tr>
<th>$Y_{t-1}$</th>
<th>$Y_{t-2}$</th>
<th>$Y_{t-3}$</th>
<th>$Y_{t-4}$</th>
<th>$r_{t-1}$</th>
<th>$r_{t-2}$</th>
<th>$r_{t-3}$</th>
<th>$r_{t-4}$</th>
<th>Const.</th>
<th>$R^2$</th>
<th>Q</th>
<th>SEE</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.31</td>
<td>-1.62</td>
<td>.15</td>
<td>.06</td>
<td>-.02</td>
<td>-.06</td>
<td>-.02</td>
<td>.0</td>
<td>43.5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(.16)</td>
<td>(.28)</td>
<td>(.18)</td>
<td>(.06)</td>
<td>(.0)</td>
<td>(.0)</td>
<td>(.0)</td>
<td>(.0)</td>
<td>(11.3)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$r_{t}$</td>
<td>14.1</td>
<td>-11.0</td>
<td>-7.9</td>
<td>4.71</td>
<td>-1.25</td>
<td>-1.45</td>
<td>-.74</td>
<td>-.14</td>
<td>81.1</td>
<td>.62</td>
<td>27.3</td>
</tr>
<tr>
<td>(1.85)</td>
<td>(4.7)</td>
<td>(5.2)</td>
<td>(2.0)</td>
<td>(1.12)</td>
<td>(.11)</td>
<td>(.10)</td>
<td>(.07)</td>
<td>(204.5)</td>
<td>(.04)</td>
<td>(8.7)</td>
<td></td>
</tr>
</tbody>
</table>

$\Sigma = \begin{pmatrix} .0039 & .04 \\ (.0007) & (.01) \\ .49 & 1.72 \\ (.08) & (.31) \end{pmatrix}$ \quad \ln |\Sigma| = -5.3 \quad (30)

#### II. Interest Rates - Consumption.

<table>
<thead>
<tr>
<th>$r_{t-1}$</th>
<th>$r_{t-2}$</th>
<th>$r_{t-3}$</th>
<th>$r_{t-4}$</th>
<th>$C_{t-1}$</th>
<th>$C_{t-2}$</th>
<th>$C_{t-3}$</th>
<th>$C_{t-4}$</th>
<th>Const.</th>
<th>$R^2$</th>
<th>Q</th>
<th>SEE</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1.26</td>
<td>-1.45</td>
<td>-.73</td>
<td>-.14</td>
<td>14.63</td>
<td>-11.8</td>
<td>-7.74</td>
<td>4.83</td>
<td>49.4</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(.12)</td>
<td>(.11)</td>
<td>(.11)</td>
<td>(.07)</td>
<td>(1.91)</td>
<td>(5.09)</td>
<td>(5.6)</td>
<td>(2.14)</td>
<td>(196.7)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$C_{t}$</td>
<td>-0.02</td>
<td>-0.05</td>
<td>-0.02</td>
<td>.0</td>
<td>2.38</td>
<td>-1.78</td>
<td>.28</td>
<td>.03</td>
<td>41.0</td>
<td>.97</td>
<td>21.7</td>
</tr>
<tr>
<td>(.0)</td>
<td>(.0)</td>
<td>(.0)</td>
<td>(.0)</td>
<td>(.16)</td>
<td>(.29)</td>
<td>(.18)</td>
<td>(.06)</td>
<td>(10.5)</td>
<td>(.01)</td>
<td>(7.8)</td>
<td></td>
</tr>
</tbody>
</table>

$\Sigma = \begin{pmatrix} 1.69 & .036 \\ (.31) & (.009) \\ .47 & .0034 \\ (.08) & (.0006) \end{pmatrix}$ \quad \ln |\Sigma| = -5.45 \quad (30)
Consider the consumer's optimization problem:

\[
\max E_0 V(C_t; \Delta_t) = E_0 \sum_{t=0}^\infty \beta^t U(C_t) = E_0 \sum_{t=0}^\infty \beta^t (C_t - \frac{\beta}{2} C_t^2 - \frac{\beta}{2} (C_t - C_{t-1})^2)
\]

subject to the constraints:

\[
C_t + K_t - K_{t-1} + \frac{\gamma}{2} (K_t - K_{t-1})^2 = Y_t
\]

\[
Y_t = \gamma - \frac{\beta}{2} (K_{t-1} - \alpha)^2 + \epsilon_t
\]

\[
C_t, K_t \geq 0
\]

Following the approach in Kushner [..], there exists a sequence of random Lagrange multipliers \((\lambda_t)\) such that:

\[
E_t V_{C_t} - E_t \lambda_t \leq 0 \quad \text{and} \quad 0 \quad \text{if} \quad C_t > 0
\]

\[
-E_t \lambda_t \{1 + \epsilon t \Delta K_t\} + E_t \{1 - \epsilon (K_t - \alpha) + \epsilon (K_{t+1} - K_t)\} \leq 0 \quad \text{and} \quad 0 \quad \text{if} \quad K_t > 0
\]

and the transversality condition:

\[
\lim_{T \to \infty} \beta^t K_t \lambda_t = 0
\]

where \(E_t\) denotes the operator expectation conditional on the sigma algebra \(\mathcal{G}_t\) of subsets of \(\mathcal{G}_t\), a set which includes current and past decision and states. Hence, the conditional expectation of any current decision variable is that same variable.

Assuming an interior solution, i.e., \(C_t, K_t > 0 \ \forall t\) and using standard properties of the conditional expectation operator, we get:

\[
E_t V_{C_t} = E_t \lambda_t
\]

\[
E_{t+1} V_{C_{t+1}} = E_{t+1} \lambda_{t+1} \quad \Rightarrow \quad E_t V_{C_{t+1}} = E_t \lambda_{t+1}
\]

and:

\[
E_t V_{C_t} (1 + \epsilon t \Delta K_t) = E_t \{\lambda_t + 1 - \epsilon (K_t - \alpha) + \epsilon \Delta K_{t+1}\} =
\]

\[
E_t \{E_{t+1} (1 - \epsilon (K_t - \alpha) + \epsilon \Delta K_{t+1})\} = E_t \{1 - \epsilon (K_t - \alpha) + \epsilon \Delta K_{t+1} \cdot E_{t+1} \lambda_{t+1}\} =
\]

\[
E_t \{1 - \epsilon (K_t - \alpha) + \epsilon \Delta K_{t+1} \cdot E_{t+1} V_{C_{t+1}}\} = E_t \{E_{t+1} (1 - \epsilon (K_t - \alpha) + \epsilon \Delta K_{t+1} \cdot V_{C_{t+1}})\} =
\]

\[
E_t (1 - \epsilon (K_t - \alpha) + \epsilon \Delta K_{t+1} \cdot V_{C_{t+1}})
\]
APPENDIX 2.- In order to clarify the equilibrium relationships among real interest rates, the marginal rate of return on capital, and the marginal rate of time preference, we split now the optimization problem in appendix 1 into two problems, that of a representative consumer who lends to the single firm in the economy, consuming each period his endowment net of this loans, and the problem of a firm which takes care of production, borrows from consumers (that is, issues some one period bonds $B_t$ which are sold to consumers), and distributes some dividends $Y^*_t$ to the owners, in this case, the representative consumer in the economy.

a) Consider now the optimization problem of a consumer whose only possibility of transferring resources over time is by buying bonds (lending) from the production firm in the economy:

$$\max_{(C_t, B_t)} \sum_{t=0}^{T-1} \beta^t \cdot (C_t - \frac{B_t}{2} - Y^* - \frac{1}{2}(C_t - C_{t-1})^2)$$

subject to:

$$C_t + B_t = Y^*_t + (1+r_{t-1})B_{t-1}$$

given $r_{-1}$, $C_{-1}$ and $B_{-1}$ and taking the sequences $(r_t, Y^*_t)$ as given.

The optimality conditions, as a function of the random Lagrange multipliers $(\delta_t)_{t=0}^T$ are:

$$E_t V_C = E_t \delta_t \quad \text{and} \quad 0 \quad \text{if} \quad C_t < 0$$

$$E_t \delta_t = (1+r_t) E_t \delta_{t+1} \quad \text{and} \quad 0 \quad \text{if} \quad B_t > 0$$

and the transversality condition:

$$\lim_{T \to \infty} \delta_T B_T = 0$$

which imposes a bound on the rate of growth of savings. But $B_t$ is a flow variable, not a stock, and one would expect it to converge to a finite steady state value. If consumption behaves in a similar fashion, then the transversality condition will be satisfied.

Assuming an interior solution, then:

$$E_t V_C = E_t \delta_t \quad \text{and}$$

$$E_{t+1} V_{C_{t+1}} = E_{t+1} \delta_{t+1} \quad \text{which implies:}$$

$$E_t V_{C_{t+1}} = E_t \delta_{t+1}$$

Since:

$$E_t \delta_t = (1+r_t) E_t \delta_{t+1}$$

then:

$$1+r_t = \frac{E_t V_{C_t}}{E_{t+1} V_{C_{t+1}}}$$
b) If we now consider the optimization problem of a firm which maximizes the expected present value of the stream of dividends distributed among the owners each period:

\[
\max_{Y^*, B_t, K_t} E_t \sum_{t=0}^{T_0} D_t . Y^*_t
\]

where \( D_t \) is a generic sequence of discount factors used by the firm, and the problem is solved subject to the sequence of constraints:

\[
Y^*_t + B_t = Y^*_t + (1 + r_{t-1}) B_{t-1} + K_t - K_{t-1} + (w/2) (K_t - K_{t-1})^2
\]

\[
Y_t = \gamma - \theta_a (K_{t-1} - \bar{x})^2 + \epsilon_t
\]

and given \( K_{-1}, B_{-1}, r_{-1} \).

Then, the following optimality conditions must hold at each time \( t \):

\[
E_t [ \eta_t^* - (1 + r_t) \cdot E_t \eta_{t+1} ] \leq 0 \quad \text{and} \quad = 0 \quad \text{if} \quad B_t > 0
\]

\[
E_t \left[ I - \Theta (K_t - \bar{x}) + w (K_{t+1} - K_t) + 1 \right] \cdot \eta_{t+1} - E_t \left[ w (K_t - K_{t-1}) + 1 \right] \eta_t \leq 0
\]

\[
E_t \Delta t - E_t \eta_t \leq 0 \quad \text{and} \quad = 0 \quad \text{if} \quad Y^*_t > 0
\]

as well as the transversality condition:

\[
\lim_{T \to \infty} D_T K_T = 0
\]

which imposes a bound on the rate of growth of the stock of capital.

Assuming an interior solution, i.e., \( Y^*_t, B_t, K_t > 0 \) \( \forall t \), then:

\[
1 + r_t = \frac{E_t \eta_t}{E_t \eta_{t+1}} = \frac{E_t \eta_{t+1}}{E_t \eta_t}
\]

(3)

\[
E_t \eta_t = E_t D_t
\]

(4)

\[
E_t [ \eta_{t+1} \cdot (1 - \Theta (K_t - \bar{x}) + w . \Delta K_{t+1}) ] = E_t [ \eta_t \cdot (1 + w . \Delta K_t) ]
\]

(5)

Suppose that the firm uses as a (random) discount factor the marginal utility of the representative shareholder, \( D_t = V_{C_t} \). Then (4) becomes identical to (2) and (5) becomes:

\[
E_t [ \eta_{t+1} \cdot (1 - \Theta (K_t - \bar{x}) + w . \Delta K_{t+1}) ] = (1 + w . \Delta K_t) . E_t \eta_t
\]

From (4) we have that the difference between \( \eta_t \) and \( V_{C_t} \) is a random variable \( \eta_{t+1} \) unpredictable as of time \( t \):

\[
\eta_t = V_{C_t} + \eta_{t+1}
\]
with \( E_t \gamma_{t+1} = 0 \). We then have:

\[
E_t \left[ \eta_{t+1} \cdot [1-\theta(K_t-\alpha)+\omega \cdot \Delta K_{t+1}] \right] = E_t \left[ (V_{C_{t+1}} + \gamma_{t+2}) \cdot [1-\theta(K_t-\alpha)+\omega \cdot \Delta K_{t+1}] \right] =
\]

\[
= E_t \left[ V_{C_{t+1}} \cdot [1-\theta(K_t-\alpha)+\omega \cdot \Delta K_{t+1}] \right] + E_t \left[ \gamma_{t+2} \cdot [1-\theta(K_t-\alpha)+\omega \cdot \Delta K_{t+1}] \right]
\]

and the second term in this sum is equal to:

\[
E_t \left[ \gamma_{t+2} \cdot [1-\theta(K_t-\alpha)+\omega \cdot \Delta K_{t+1}] \right] = E_t \left[ (1-\theta(K_t-\alpha)+\omega \cdot \Delta K_{t+1} \cdot E_{t+1} \gamma_{t+2} \right] = 0
\]

Analogously,

\[
E_t \left[ \eta_{t} \cdot [1+\omega \cdot \Delta K_{t}] \right] = E_t \left[ V_{C_{t}} \cdot [1+\omega \cdot \Delta K_{t}] \right] + E_t \left[ \gamma_{t+1} \cdot [1+\omega \cdot \Delta K_{t}] \right] =
\]

and the second term in this sum is equal to:

\[
(1+\omega \cdot \Delta K_{t}) \cdot E_t \gamma_{t+1} = 0
\]

and therefore, equation (5) becomes identical to (1), the first order condition in the first of our optimization problems.

Actually, all that is needed to get this result is that the firm uses a discount factor \( D_t \) such that \( E_t D_t = E_t V_{C_t} \). What this means is that the firm discounts by the current expected value of the marginal utility of current consumption.
BIBLIOGRAPHY


