arXiv:hep-ph/9401282v1 19 Jan 1994

ONE LOOP CALCULATIONS ON THE WESS-ZUMINO-WITTEN ANOMALOUS FUNCTIONAL AT FINITE TEMPERATURE

R.F. Alvarez-Estrada, **A. Dobado** and **A. Gómez Nicola**
Departamento de Física Teórica, Universidad Complutense
28040, Madrid, Spain

February 1, 2008

Abstract

We analyze the finite temperature (T) extension of the Wess-Zumino -Witten functional, discussed in a previous work, to one loop in chiral perturbation theory. As a phenomenological application, we calculate finite temperature corrections to the amplitude of π^0 decay into two photons. This calculation is performed in three limits : i) $T/M_\pi << 1$, ii) the chiral limit at finite T and iii) $T/M_\pi >> 1$ (M_π being the pion mass). The T-corrections tend to vanish in the chiral limit, where only the kaon contribution remains (although it is exponentially suppressed).

FT/UCM/15-93

The extension of the Wess-Zumino-Witten (WZW) anomalous lagrangian [1] for the finite temperature ($T \neq 0$) case has been analyzed recently [2]. Instead of the usual S^4 compactification of Euclidean space-time at $T = 0$ [1], a $S^1 \times S^3$ one was employed for $T \neq 0$. Our task is now to evaluate one-loop effects using the $S^1 \times S^3$ lagrangian, in the trivial (pionic) sector with baryon number $N_B = 0$. In particular, we are interested in the finite temperature corrections to the amplitude (or the correlation function) corresponding to the decay $\pi^0 \to \gamma\gamma$. For the trivial sector, the finite T WZW action gauged with the electromagnetic field A_μ, can be obtained directly from the one discussed in [2]. It reads:

$$
\begin{aligned}
\Gamma_{WZW}[U,A] &= \Gamma_0[U] - e\int_{S^1\times S^3} A_\mu J^\mu + \frac{e^2}{24\pi^2} N_c \int_{S^1\times S^3} \epsilon_{\mu\nu\alpha\rho}\partial_\mu A_\nu A_\alpha Tr[Q^2(\partial_\rho U)U^{-1} + \\
&+ Q^2U^{-1}(\partial_\rho U) + \frac{1}{2}Q\partial_\rho U Q U^{-1} - \frac{1}{2}QUQ\partial_\rho U^{-1}] \\
J_\mu &= \frac{N_c}{48\pi^2}\epsilon^{\mu\nu\alpha\rho} Tr[Q(\partial_\nu U U^{-1})(\partial_\alpha U U^{-1})(\partial_\rho U U^{-1}) + \\
&+ Q(U^{-1}\partial_\nu U)(U^{-1}\partial_\alpha U)(U^{-1}\partial_\rho U)] \\
\Gamma_0[U] &= \frac{N_c}{240\pi^2}\int_0^1 dt \int_{S^1\times S^3} \epsilon^{ijklm} U_t^{-1}\partial_i U_t U_t^{-1}\partial_j U_t U_t^{-1}\partial_k U_t U_t^{-1}\partial_l U_t U_t^{-1}\partial_m U_t \quad (1)
\end{aligned}
$$

where $i,j,k,l,m = 1,...5$, $t \in [0,1]$, and U_t being a mapping from $[0,1] \times S^1 \times S^3$ into $SU(3)$ with $U_0 = 1$, $U_1 = U(\vec{x},\tau)$. In turn, $U \equiv U(\vec{x},\tau)$ is a mapping from $S^1 \times S^3$ to $SU(3)$, which can be parametrized as $U(\vec{x},\tau) = \exp(i\pi^a(\vec{x},\tau)T_a/F_\pi)$ with $a = 1...8$ and $\pi^a(\vec{x},\tau)$ being the Goldstone boson fields (π, K, η). T_a are the $SU(3)$ generators and $F_\pi \simeq 93$ MeV is the pion decay constant. Notice that $\Gamma_0[U]$ is the direct restriction of eq.(14) in [2] to the trivial ($j = 0$) sector. The trace in (1) is over $SU(3)$, $\beta = 1/T$ is the radius of S^1, N_c is the number of colours, and Q is the quark charge matrix:

$$Q = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix} \tag{2}$$

At low energies, the total action of the system is obtained by adding to the WZW term (1) the non-linear sigma model piece in the Euclidean $S^1 \times S^3$ space:

$$\begin{aligned} \Gamma &= \frac{1}{4} F_\pi^2 \int_{S^1 \times S^3} Tr(D_\mu U^{-1} D^\mu U - M^2(U + U^{-1})) + \Gamma_{WZW} \\ D_\mu &= \partial_\mu + ie[Q, U] A_\mu \end{aligned} \tag{3}$$

where M is the quark mass matrix. Using standard techniques, one generates the Chiral Perturbation Theory (χ PT) expansion. Higher energy or temperature contributions can be included by adding higher derivative terms to the above action. Since we are interested only in the low temperature domain those terms will not be considered here. Note also that working at finite T the χPT expansion is done in powers of the parameter $T^2/8F_\pi^2$ [3].

Now we concentrate on the $\pi^0 \to \gamma\gamma$ process. In general, the π^0 decay amplitude at any T can be expressed as:

$$A_{\mu\nu} = \frac{iN_c e^2}{12\pi^2 F_\pi} C(T) \epsilon_{\mu\nu\alpha\rho} k_2^\alpha k_3^\rho \tag{4}$$

where k_2 and k_3 are the fourmomenta of the outgoing photons and $C(T)$ is some suitable function. At $T = 0$ the lowest order approximation can be obtained from the WZW action at the tree level and corresponds to $C = 1$. Next order contributions come from the one-loop diagrams a) and b) in Fig.1 (plus some renormalization graphs to be discussed later). Those diagrams have been studied at $T = 0$ in [4] [5]. Let us now consider those graphs together with the usual renormalization diagrams for F_π (graph c) in Fig.1) and for the pion mass and wave function (graph d) in Fig.1). Then, the final result for the one-loop $\pi^0 \to \gamma\gamma$ amplitude is the same as the tree level one, provided that F_π be replaced by its physical value, for on-shell photons. In other words it is given by (4) also with $C(0) = 1$. Let us sketch its justification. The contributions of the graphs a), c) and d) in Fig.1 are proportional to $G_\pi(0)$ and $G_K(0)$, $G_\pi(x)$ and $G_K(x)$ being the pion and kaon propagators ($x \in S^4$). At $T = 0$ it can be shown that (in dimensional regularization) the only surviving piece of the graph b) that contributes to the decay amplitude is a sum of terms proportional to $G_\pi(0)$ and $G_K(0)$. The corresponding coefficients of every contribution cancel with one other, for pions and kaons separately. Notice that $G_\pi(0)$ and $G_K(0)$ are divergent quantities, so that the above cancellations imply that there are no one-loop divergent contributions to the anomaly that could renormalize the N_c coefficient of the WZW action [4].

Now we turn to analyze the $T \neq 0$ case. Physically, it would correspond to the effect of a medium at constant temperature T, in which the π^0 could live. Such a situation could take place in a neutron star, or in a cosmological hadronic medium below the chiral phase transition point [3] [6] [7]. In agreement with previous studies of the $\pi^0 \to \gamma\gamma$ decay at finite temperature [8], one expects to find non-vanishing T-corrections. In addition such corrections should be ultraviolet finite and no renormalization of the anomaly at finite T should be needed.

The imaginary time formalism of Finite Temperature Field Theory (FTFT) [9] can be implemented by using the $S^1 \times S^3$ compactification of the WZW action discussed in [2]. As usual the energies are discretized and energy integrals must be replaced by sums over Matsubara frequencies. Let us concentrate on the contribution of graph b) in Fig.1. It splits into a pionic and a kaonic part, depending on the particle inside the loop. From the lagrangian in (1) and (3), the pionic part reads in Euclidean momentum space :

$$
\begin{aligned}
A^{(b)}_{\mu\nu}(T) &= \frac{iN_c e^2}{3\pi^2 F_\pi}\frac{1}{F_\pi^2}\epsilon_{\nu\alpha\rho\gamma}k^{2\gamma}k^{3\alpha}I^\rho_\mu(T) \\
I^\rho_\mu(T) &= \frac{1}{\beta}\sum_{n=-\infty}^{+\infty}\int\frac{d^{d-1}\vec{q}}{(2\pi)^{d-1}}\frac{q_\mu q^\rho}{(\omega_n^2+\vec{q}^2+M_\pi^2)((\omega_n-k_{20})^2+(\vec{q}-\vec{k_2})^2+M_\pi^2)}
\end{aligned}
\tag{5}
$$

where $\omega_n = 2\pi n/\beta$, $d = 4-\epsilon$ (see [3]) and M_π is the pion mass. The kaon term has the same structure as (5) with M_π replaced by M_K. Now, by using the well known formulae:

$$
\frac{1}{AB} = \int_0^1 dx \frac{1}{[Ax+B(1-x)]^2} \tag{6}
$$

and

$$
\frac{i}{\beta}\sum_{n=-\infty}^{+\infty} f(\frac{2n\pi i}{\beta}) = \frac{1}{2\pi}\sum_{s=\pm 1}\int_{-i\infty+s\epsilon}^{i\infty+s\epsilon}\frac{d\omega f(\omega)}{\exp[s\beta\omega]-1}+\frac{1}{2\pi}\int_{-i\infty}^{i\infty}d\omega f(\omega) \quad ;\epsilon\to 0^+ \tag{7}
$$

in (5) we can separate the $T = 0$ part and arrive to an integral over the ω complex plane for the $T \neq 0$ piece. This integral can be performed by using the residue theorem inside the contours C_1 and C_2 showed in Fig.2. In fact, the resulting $f(\omega)$ has always two poles at the points $\omega = ik_0 \pm B(\vec{q};x)$

with $B(\vec{q};x) = \sqrt{A(\vec{q};x) - k_0^2}$, $A(\vec{q};x) = \vec{q}^2 - 2\vec{q}\cdot\vec{k} + M^2(x)$, $M^2(x) = M_\pi^2 + (1-x)(k_{20}^2 + \vec{k}_2^{\,2})$ and $k_\mu = (1-x)k_{2\mu}$. One of those poles lies always inside C_1 and the other one inside C_2.

In the following, in order to make the discussion simpler we will consider three different limiting cases, namely the low temperature limit $T/M_\pi << 1$, the chiral limit $M_\pi = 0$ and finally the limit $M_\pi/T << 1$. Note that this last limit can be understood as the one caracterizing the leading corrections to the previously mentioned chiral limit $M_\pi = 0$.

We shall consider first the low temperature limit $\frac{T}{M_\pi} << 1$. In this regime, the $T \neq 0$ piece in $I_\mu^\rho(T)$ can be written , as $d \to 4$, as a linear combination of the one-dimensional integrals:

$$I_k(n;T,x) = \frac{2}{(4\pi)^{\frac{k-1}{2}}\Gamma(\frac{k-1}{2})}\int_0^\infty dy \frac{y^{k-2}}{(y^2+M^2(x))^{n/2}} \exp\left(\frac{-\sqrt{y^2+M^2(x)}}{T}\right) \tag{8}$$

The integrals in (8) are always finite for $M_\pi \neq 0$. Then we see that at low temperatures there are no divergent temperature corrections for $d \to 4$, in agreement with the idea that renormalization at $T = 0$ ensures that no extra divergent terms appear at $T \neq 0$. In particular, that is also true for off-shell photons with $k_{20}^2 + \vec{k}_2^{\,2} \neq 0$ (after analytic continuation of k_{20} to continuous values), as studied in [4] for $T = 0$.

In order to study the finite T corrections to the π^0 decay amplitude, we consider: i) the usual analytic continuation from discretized to continuous frequencies of the external photons, ii) on-shell photons ($M^2(x) = M_\pi^2$ and $I_k(n;T,x)$ being, then, independent of x) and iii) the π^0 rest frame, in which

$k_{2j} = -k_{3j}$, $k_{20} = k_{30} = iM_\pi/2$. On the other hand, the contributions of diagrams a),c) and d) of Fig.1 are now proportional to $G_\pi^T(0)$. In turn $G_\pi^T(x)$ is the thermal pionic propagator (now $x \in S^1 \times S^3$). One has [3] :

$$
\begin{aligned}
G_\pi^T(0) &= G_\pi(0) + g_1(M_\pi, T) \\
g_1(M_\pi, T) &= 2\sum_{n=1}^{\infty}\int_0^{\infty} \frac{d\lambda}{(4\pi\lambda)^{d/2}} \exp(-\lambda M_\pi^2 - \frac{n^2}{4\lambda T^2})
\end{aligned} \tag{9}
$$

In the π^0 rest frame, the only surviving contribution of I_μ^ρ in (5) is the I_i^j part ($i, j = 1...d-1$). This part is splitted into two pieces, one of them proportional to δ_i^j and the other one to $k_{2i}k_2^j$. The later vanishes in $A_{\mu\nu}^{(b)}(T)$ in (5). The δ_i^j piece is written in terms of the I_k integrals in (8). The final result for the π^0 decay amplitude at low temperature (having accomplished the steps i),ii) and iii) above and letting $d \to 4$) is given in (4), with:

$$
\begin{aligned}
C^{RF}(T) &= 1 + \frac{2}{F_\pi^2}(2f(M_\pi, T) - g_1(M_\pi, T)) + (kaon \quad terms) \\
f(M_\pi, T) &= \frac{2\pi}{M_\pi}\sinh(\frac{M_\pi}{2T})(I_6(2; T) + TI_6(3; T))
\end{aligned} \tag{10}
$$

$I_6(2; T)$ and $I_6(3; T)$ are given by the right-hand-side of (8) with $M^2(x) = M_\pi^2$. The superindex RF refers to the pion rest frame. From (8-10) we obtain the following low temperature expansion for $C^{RF}(T)$:

$$
\begin{aligned}
C^{RF}(T) &= 1 + 2\frac{T^2}{F_\pi^2}\frac{1}{(2\pi)^{3/2}}\tau^{-1/2}e^{-1/\tau}(2\tau\sinh\frac{1}{2\tau} - 1)\{1 + \frac{3}{8}\tau - \frac{15}{128}\tau^2 + \\
&+ O(\tau^3)\} + O(e^{-2/\tau}) + (kaon \quad terms)
\end{aligned} \tag{11}
$$

where $\tau = T/M_\pi$. As the exponentials in the above equation clearly suggest, the contribution of other families (the kaon terms), is exponentially small at low temperatures. The same occurs for other thermodynamical quantities [3].

Let us now analyze what happens in the chiral limit $M_\pi \to 0$ at finite T. First it is important to notice that one should take the infinite volume limit, $R \to \infty$, R being the radius of the S^3 sphere, before the $M_\pi \to 0$ limit is taken. This is so because the chiral limit would lead to inconsistencies if R would have remained finite, since there are not massless Goldstone bosons at finite volume (see [10]). From current algebra theorems at $T = 0$, one expects the π^0 decay amplitude to be entirely dominated by the anomaly in the chiral limit, and then, the low energy corrections are expected to vanish. As we have derived the finite T corrections displayed in (10)-(11) from a low energy action, they are expected to be vanishingly small in the chiral limit by the previous argument, as the anomaly does not depend on temperature. Indeed, this is the case, which constitutes an interesting check of consistency of our calculations. In fact, it is possible to calculate this limit for $I_\mu^\rho(T)$ in (5) by using again (6) and (7). Then we obtain for $d \to 4$ and on-shell photons:

$$I_\mu^\rho(T, M_\pi = 0) \quad = \quad I_\mu^\rho(0, M_\pi = 0) - g_\mu^\rho \frac{T^2}{12} + k_{2\mu} k_2^\rho - terms \qquad (12)$$

The contribution of the $k_{2\mu} k_2^\rho$ terms in (12) to $A_{\mu\nu}^{(b)}(T)$ in (5) vanishes. The g_μ^ρ piece cancels with the contribution coming from $g_1(M_\pi, T)$, which is the massless Bose gas self-energy (see [3]). So we find that the only T

contribution in the chiral limit is the one coming from the kaonic part. The latter is exponentially diminished as long as the temperature lies below the kaon mass:

$$C(T, M_\pi = 0) = 1 + O(\frac{T^2}{F_\pi^2}\frac{T^3}{M_K^3}e^{-\frac{M_K}{T}}) \tag{13}$$

Let us now come to the $T/M_\pi >> 1$ regime. The chiral limit contribution previously analyzed can also be viewed as the leading one in this limit. In the $M_\pi \neq 0$ case, it is possible to find higher order contributions in this limit, taking into account that the parameter $T^2/8F_\pi^2$ must remain small (in order to keep the one-loop approximation in χPT valid). For this calculation, we leave the sum over frequencies (over n) as it appears in (5), without going to the complex plane. Only (6) is then used in (5). The integral over the $d-1$ momentum space can then be reduced to a one-dimensional integral by using standard dimensional regularization formulas. By expanding in terms of $\sigma = M_\pi/(2\pi T)$, it is found that the n-sum in every term of the expansion is of the form $\sum_{n=1}^{\infty} 1/n^{p+\epsilon} = \zeta(p+\epsilon)$ with ζ the Riemann's zeta function and $\epsilon = 4-d$. Notice that in this treatment, the $T=0$ part has not been separated from the $T \neq 0$ one up to this stage. The $p=1$ contribution in ζ carries a $1/\epsilon$ infinite term which is nothing but the $T=0$ one. Once this $T=0$ part is separated, we again find that the remaining $T \neq 0$ part is finite. In particular, as the first order term in the expansion we find the $M_\pi = 0$ term discussed before. Collecting the terms coming from $g_1(M_\pi, T)$ that have already been calculated for $\sigma << 1$ in [3], we have for $C^{RF}(T)$ in this regime:

$$
\begin{aligned}
C^{RF}(T) &= 1+\frac{T^2}{F_\pi^2}(c_1\sigma + c_2\sigma^2 + O(\sigma^4) + O(e^{-\frac{1}{T}(M_K-\frac{M_\pi}{2})})) \\
c_1 &= 1-\frac{\sqrt{3}}{4}-\frac{\pi}{6} \qquad c_2=\frac{-1}{12}
\end{aligned}
\tag{14}
$$

Notice that the $O(\sigma^2\log\sigma)$ terms appearing in the expansion of $g_1(M_\pi,T)$ [3] cancel exactly with the ones coming from the graph b).

We now compare the different behaviours of $C^{RF}(T)$ for low and high temperatures by looking at eqs. (11) and (14). Let us define $s(\tau)$ through $C^{RF}(T)=1+\frac{T^2}{F_\pi^2}s(\tau)$ ($s(\tau)$ being dimensionless). In Fig.3 we have plotted $s(\tau)$ versus T (for M_π= 135 MeV), up to temperatures of 200 MeV (which is an interval where our approximations are expected to be valid). The kaon contribution has been plotted in the $T/M_K << 1$ limit. In order to check the consistency of the several approximations considered, we have also evaluated numerically the $T\neq 0$ part in $I_\mu^\rho(T)$ in (5), through the exact expressions of the residues, valid for any T. The results for some values of T are also showed in Fig.3. We see that the analytical results are in good agreement with the numerical computations in the two limits considered.

It is interesting to observe that for low temperatures $s(\tau)$ is an increasing function of τ, while, in the large T regime, it decreases with τ. Whether or not this change of behaviour of the decay amplitude could be due to the presence of the chiral phase transition, near the temperature range in which $s(\tau)$ decreases, stands as an open question (see other studies of the chiral phase transition for $M_\pi\neq 0$ like [3][6]).

Summarizing, the finite-T WZW functional [2], gauged with the electromagnetic field in the trivial sector, has been explored up to one-loop in χPT. The resulting finite-T corrections to the $\pi^0 \rightarrow \gamma\gamma$ decay amplitude have been analyzed in three regimes : i)$T << M_\pi$, ii)$M_\pi = 0$ at finite T and iii)$T >> M_\pi$. As a check of consistency, we have found that the T-corrections to the decay amplitude tend to vanish in the chiral limit, where only a exponentially supressed kaon contribution remains.

The financial support of CICYT (Proyecto AEN93-0776) is acknowledged.

References

[1] J. Wess and B. Zumino, *Phys.Lett* **37B** (1971) 95
E. Witten, *Nucl.Phys.* **B223** (1983) 422; **B223** (1983) 433
L. Alvarez-Gaumé and P. Ginsparg, *Ann.Phys.* **161** (1985) 423

[2] R.F. Alvarez-Estrada, A. Dobado and A. Gómez Nicola, *Phys.Lett.* **B319** (1993) 238.

[3] P. Gerber and H. Leutwyler, *Nucl.Phys.* **B321** (1989) 387-429.

[4] J.F. Donoghue and D. Wyler, *Nucl.Phys.* **B316** (1989) 289-306

[5] J.F. Donoghue, B.R. Holstein and Y.-C.R.Lin, *Phys.Rev.Lett.* **55** (1985) 2766

[6] A. Barducci, R. Casalbuoni, S. de Curtis, R. Gatto and G. Pettini, *Phys.Rev* **D46** (1992) 2203

[7] A. Schenk, *Phys.Rev.* **D47** (1993) 5138

[8] C. Contreras and M. Loewe, *Zeit. Phys.* **C40** (1988) 253
A. Gómez Nicola and R.F. Alvarez-Estrada, *Zeit.Phys.* **C60** (1993), 711

[9] N.P. Landsman and Ch.G. Van Weert, *Phys.Rep.* 145 (1987) 1941

[10] J. Gasser and H. Leutwyler, *Phys.Lett.* **B188** (1987) 477
H. Leutwyler, *Phys.Lett.* **B189** (1987) 197.

FIGURE CAPTIONS

*Fig.1.*Diagrams contributing to the π^0 decay amplitude. Graph c) gives rise to F_π renormalization and d) is the pion mass and wave function renormalization graph . The dashed line in graph c) represents the axial current (see [4][5]).

*Fig.2.*Contours C_1 and C_2 in the ω complex plane.

Fig.3 $s(\tau)$ as a function of T for $M_\pi = 135$ MeV. The two regimes (high T and low T) analyzed in the text are plotted, together with the numerical results and the kaon contribution.

This figure "fig1-1.png" is available in "png" format from:

http://arXiv.org/ps/hep-ph/9401282v1

This figure "fig1-2.png" is available in "png" format from:

http://arXiv.org/ps/hep-ph/9401282v1

This figure "fig1-3.png" is available in "png" format from:

http://arXiv.org/ps/hep-ph/9401282v1