On Besov spaces of logarithmic smoothness and Lipschitz spaces

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We compare Besov spaces $B^{s,b}_{p,q}$ with zero classical smoothness and logarithmic smoothness $b$ defined by using the Fourier transform with the corresponding spaces $B^{0,b}_{p,q}$ defined by means of the modulus of smoothness. In particular, we show that $B^{0,b+1/2}_{2,2} = B^{b}_{2,2}$ for $b > -1/2$. We also determine the dual of $B^{0,b}_{p,q}$ with the help of logarithmic Lipschitz spaces $\text{Lip}^{(1,−s)}_{p,q}$. Finally, we show embeddings between spaces $\text{Lip}^{(1,−s)}_{p,q}$ and $B^{b}_{p,q}$ which complement and improve embeddings established by Haroske (2000) [28].

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1. Introduction

Besov spaces $B^{s}_{p,q}$ play a central role in the theory of function spaces as can be seen in the monographs by Triebel [39–41]. For the complete solution of some natural questions as compactness in limiting embeddings [33,11] or spaces on fractals [20,21], more general spaces have been introduced where smoothness of functions is considered in a more delicate manner than in $B^{s}_{p,q}$. These spaces of generalized smoothness have been studied for long and from different points of view. See, for example, the papers by DeVore, Riemenschneider and Sharpley [16], Brézis and Wainger [5], Gol’dman [26], Merucci [34], Kalyabin and Lizorkin [32], Cobos and Fernandez [9], Edmunds and Haroske [18], Haroske and Moura [30], Farkas and Leopold [24], Triebel [41, pp. 52–55] and the references given there.

As in the case of $B^{s}_{p,q}$, spaces of generalized smoothness on $\mathbb{R}^n$ can be introduced by following the Fourier analytic approach or by means of the modulus of smoothness. If we take classical smoothness $s$ and additional logarithmic smoothness with exponent $b$, the first way leads to spaces $B^{s,b}_{p,q}$ and the second to spaces $B^{s,b}_{p,q}$ (precise definitions are given in Section 2). If $1 \leq p \leq \infty$ and $s > 0$, it turns out that $B^{s,b}_{p,q} = B^{s,b}_{p,q}$ with equivalence of norms (see [30, Theorem 2.5] and [39, 2.5.12]); but if $0 < p < 1$ and $0 < q \leq 1$ then

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$B^{n(1/p-1)}_{p,q} \neq B^{n(1/p-1)}_{p,q}$ as it is shown in [37, Corollary 3.10]). However, the relation between these two kinds of spaces when $s = 0$ has not been described yet. This problem is stated in the report of Triebel [42, p. 6], where first results on this question have been shown: Working with spaces on the unit cube $Q^n$ in $\mathbb{R}^n$ and using the Haar basis, Triebel established in [42, Proposition 9] that $B^{0,b}_{p,p}(Q^n) \hookrightarrow B^{0,b}_{p,p}(Q^n)$ provided that $1 < p \leq 2$ and $b \geq 0$ or $2 < p < \infty$ and $b > 1/2 - 1/p$. See also the paper by Besov [4], where spaces $B^{0}_{p,q}$, $1 \leq p, q \leq \infty$, are compared with certain spaces defined by first differences.

In this paper we compare spaces $B^{s,b}_{p,q}$ and $B^{0,b}_{p,q}$ with the help of the limiting real method

$$(A_0, A_1)_{(\theta, \eta), q} = \left\{ a \in A_0 : \|a\|_{A_0(\theta, \eta), q} = \left( \int_0^1 \frac{K(t, a)}{t^{\eta} (1 - \log t)^\theta} \frac{q \, dt}{t} \right)^{1/q} < \infty \right\}.$$  

Here $\theta = 0$ or $1$, $A_1 \hookrightarrow A_0$ and $K(t, a)$ is the $K$-functional of Peetre (see Section 2). Among other things, for $b > -1/p$ we show that

$$B^{0,b+1/p}_{p,p} \hookrightarrow B^{0,b}_{p,p} \hookrightarrow B^{0,b+1/2}_{p,p} \quad \text{if } 1 < p \leq 2,$$

$$B^{0,b+1/2}_{p,p} \hookrightarrow B^{0,b}_{p,p} \hookrightarrow B^{0,b+1/p}_{p,p} \quad \text{if } 2 \leq p < \infty.$$  

Therefore, $B^{0,b+1/2}_{2,2} = B^{0,b}_{2,2}$. In particular, this implies that the classical space $B^{0}_{2,2}$, defined by the modulus of continuity, coincides with the space $B^{0,1/2}_{2,2}$, defined by the Fourier transform, with zero classical smoothness and logarithmic smoothness with exponent $1/2$.

We also consider embeddings between spaces $B^{s,b}_{p,q}$. According to [38, Theorem 2.8.1] or [2, Corollary 5.4.21], if $1 \leq p \leq r < \infty$, $1 \leq q \leq \infty$ and $s > 0$, then $B^{n(1/p-1)+s}_{p,q} \hookrightarrow B^{s}_{r,q}$. Note that in the embedding the two spaces have the same differential dimension. The limit case where $s = 0$ has been studied by DeVore, Riemenschneider and Sharpley [16, Corollary 5.3(ii)], where they showed that the embedding holds with a loss of a unit in the exponent of the logarithmic smoothness. To be more precise, if $1 \leq p \leq r \leq \infty$, $1 \leq q \leq \infty$ and $b > -1/q$ then $B^{n(1/p-1)+s}_{p,q} \hookrightarrow B^{s}_{r,q}$. This result has been improved recently by Gogatishvili, Opic, Tikhonov and Trebels [25, Corollary 2.8] by showing that the embedding holds with the loss of only $1/\min(q, r)$ in the exponent of the logarithmic smoothness. In this paper, we use limiting interpolation to derive the embedding $B^{n(1/p-1)+s,1/\min(q, r)}_{p,q} \hookrightarrow B^{s}_{r,q}$ following a more simple approach than in [25].

In addition we determine the dual of $B^{0,b}_{p,q}$ for $1 < p < \infty$, $1 \leq q < \infty$ and $b > -1/q$. This is done with the help of logarithmic Lipschitz spaces $\text{Lip}^{(1, -\alpha)}_{p,q}$. This problem was considered by Haroske [28,29] and Neves [35] among other authors. Our approach allows us to cover some critical cases which come up for the techniques used in [28]. As a consequence, we complement and improve several results of Haroske [28].

2. Preliminaries

Subsequently, given two quasi-Banach spaces $X$, $Y$, we put $X \hookrightarrow Y$ to mean that $X$ is continuously embedded in $Y$.

If $U$, $V$ are non-negative quantities depending on certain parameters, we write $U \lesssim V$ if there is a constant $c > 0$ independent of the parameters in $U$ and $V$ such that $U \leq cV$. We put $U \sim V$ if $U \lesssim V$ and $V \lesssim U$.

Let $\hat{A} = (A_0, A_1)$ be a quasi-Banach couple, that is to say, two quasi-Banach spaces $A_0$, $A_1$ which are continuously embedded in some Hausdorff topological vector space. The Peetre’s $K$-functional is given by
where the infimum is taken over all representations $a = a_0 + a_1$ with $a_0 \in A_0$ and $a_1 \in A_1$.

For $0 < \theta < 1$ and $0 < q \leq \infty$, the real interpolation space $\tilde{A}_{\theta,q} = (A_0,A_1)_{\theta,q}$ is formed by all $a \in A_0 + A_1$ having a finite quasi-norm

$$\|a\|_{\tilde{A}_{\theta,q}} = \left( \int_0^\infty \frac{t^{-\theta}K(t,a)^q}{t} \frac{dt}{t} \right)^{1/q}$$

(as usual, when $q = \infty$ the integral should be replaced by the suprema). See [3,6] or [38].

For $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$, let $\ell(t) = 1 + |\log t|$ and

$$\ell^\theta(t) = \begin{cases}
\ell^{\alpha_0}(t) & \text{for } t \in (0,1], \\
\ell^{\alpha_\infty}(t) & \text{for } t \in (1,\infty).
\end{cases}$$

Replacing $t^\theta$ by $t^\theta/\ell^\theta(t)$ we obtain the spaces

$$\tilde{A}_{\theta,q,\mathbb{A}} = (A_0,A_1)_{\theta,q,\mathbb{A}} = \left\{ a \in A_0 + A_1 : \|a\|_{\tilde{A}_{\theta,q,\mathbb{A}}} = \left( \int_0^\infty \frac{t^{-\theta}K(t,a)^q}{t} \frac{dt}{t} \right)^{1/q} < \infty \right\}$$

(see [22,23]). Under suitable assumptions on $\mathbb{A}$ and $q$, spaces $(A_0,A_1)_{\theta,q,\mathbb{A}}$ are well-defined even if $\theta = 0$ or $\theta = 1$. In the special case $\alpha_0 = \alpha_\infty = \alpha$, we simply write $(A_0,A_1)_{\theta,q}$ instead of $(A_0,A_1)_{\theta,q,(\alpha,\alpha)}$.

We shall also need the following limiting real spaces. Let $A_1 \hookrightarrow A_0$, $0 < q \leq \infty$ and $-\infty < \eta < \infty$. For $\theta = 1$ or $\theta = 0$, the space $A_{(\theta,\eta),q} = (A_0,A_1)_{(\theta,\eta),q}$ consists of all those $a \in A_0$ with

$$\|a\|_{A_{(\theta,\eta),q}} = \left( \int_0^1 \left( \frac{K(t,a)^q}{t^{\theta}(1-\log t)^{\eta}} \right) \frac{dt}{t} \right)^{1/q} < \infty$$

(see [27,10,12]). To avoid that $\tilde{A}_{(1,\eta),q} = \{0\}$, if $\theta = 1$, we assume that $\eta > 1/q$ if $q < \infty$, and $\eta \geq 0$ if $q = \infty$.

The following result is established in [8, Lemma 2.5] by using the connection between limiting real spaces $A_{(\theta,\eta),q}$ and logarithmic spaces $A_{\theta,q,\mathbb{A}}$ [19, Proposition 1], and reiteration results for logarithmic spaces [22, Theorems 5.9*, 4.7*, 5.7 and 4.7]. It will be important in our later considerations (note that our notation here follows [22] and so it is slightly different from [8]).

**Lemma 2.1.** Let $A_0, A_1$ be quasi-Banach spaces with $A_1 \hookrightarrow A_0$. Assume that $0 < \theta < 1$, $0 < p,q \leq \infty$ and $\gamma < -1/q < \eta$. The following continuous embeddings hold:

(a) $(A_0,A_1)_{\theta,q,\gamma+1/\min\{p,q\}} \hookrightarrow (A_0,(A_0,A_1)_{\theta,\eta})_{(1,-\gamma),q} \hookrightarrow (A_0,A_1)_{\theta,q,\gamma+1/\max\{p,q\}},$

(b) $(A_0,A_1)_{\theta,q,\eta+1/\min\{p,q\}} \hookrightarrow ((A_0,A_1)_{\theta,\eta},A_1)_{(0,-\eta),q} \hookrightarrow (A_0,A_1)_{\theta,q,\eta+1/\max\{p,q\}}.$

**Remark 2.2.** Note that if $\gamma = -1/q$ then $(A_0,(A_0,A_1)_{\theta,\eta})_{(1,1/q),q} = \{0\}$, so none of the embeddings in statement (a) of Lemma 2.1 hold in this case. As for statement (b) when $\eta = -1/q$, if $p = q$ we can determine explicitly $((A_0,A_1)_{\theta,\eta},A_1)_{(0,1/p),p}$. Indeed, by Holmstedt’s formula [31, Remark 2.1]

$$K(t,a;A_0,A_1) \sim \left( \int_0^{1/(1-\theta)} K(s,a;A_0,A_1)^p \frac{ds}{s} \right)^{1/p}.$$
Hence, we obtain

\[
\|a\|_{(A_0,A_1)_{\theta,q}(0,1/p),p} \sim \left( \int_0^1 \frac{1}{1 - \log t} \int_0^1 \left( \frac{K(s,a;A_0,A_1)}{s^\theta} \right)^p \frac{ds}{1 - \log t} \frac{dt}{s} \right)^{1/p} = \left( \int_0^1 \left( \frac{K(s,a;A_0,A_1)}{s^\theta} \right)^p \int_0^1 \frac{1}{1 - \log t} \frac{dt}{s} ds \right)^{1/p} \sim \left( \int_0^1 \left( \frac{K(s,a;A_0,A_1)}{s^\theta} \right)(\log(1 - \log s))^{1/p} \right)^{1/p}.
\]

Therefore, if \( \eta = -1/q \) and \( p = q \) we still have the embedding of the right-hand side in statement (b) of Lemma 2.1 because \( \eta + 1/\max\{p,q\} = 0 \) and so \((A_0,A_1)_{\theta,q,\eta+1/\max\{p,q\}} = (A_0,A_1)_{\theta,q}\). But the embedding of the left-hand side in (b) fails.

Other kind of limiting reiteration formulae can be seen in [13].

Let \( S \) and \( S' \) be the Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions on \( \mathbb{R}^n \), and the space of tempered distributions on \( \mathbb{R}^n \), respectively. By \( \mathcal{F} \) we denote the Fourier transform on \( S \) and by \( \mathcal{F}^{-1} \) the inverse Fourier transform.

Take \( \varphi_0 \in S \) such that

\[
\text{supp} \varphi_0 \subset \{ x \in \mathbb{R}^n : |x| \leq 2 \} \quad \text{and} \quad \varphi_0(x) = 1 \quad \text{if} \quad |x| \leq 1.
\]

For \( j \in \mathbb{N} \) and \( x \in \mathbb{R}^n \) let \( \varphi_j(x) = \varphi_0(2^{-j}x) - \varphi_0(2^{-j+1}x) \). Then the sequence \( (\varphi_j)_{j=0}^\infty \) forms a dyadic resolution of unity, \( \sum_{j=0}^\infty \varphi_j(x) = 1 \) for all \( x \in \mathbb{R}^n \).

For \( 1 \leq p \leq \infty, 0 < q \leq \infty \) and \( s, b \in \mathbb{R} \), the space \( B_{s,b}^{s,b} \) consists of all \( f \in S' \) having a finite quasi-norm

\[
\|f\|_{B_{s,b}^{s,b}} = \left( \sum_{j=0}^\infty (2^{js}(1 + j)^b \|\mathcal{F}^{-1}(\varphi_j \mathcal{F}f)\|_{L_p})^q \right)^{1/q}
\]

(with the usual modification if \( q = \infty \)). See [34,9,33,30]. Note that if \( b = 0 \) then \( B_{s,0}^{s,0} \) coincides with the usual Besov space \( B_{s,q}^{s,q} \).

Besov spaces of generalized smoothness can be also introduced by using the modulus of smoothness as we recall next. Let \( f \) be a function on \( \mathbb{R}^n \), let \( h \in \mathbb{R}^n \) and \( k \in \mathbb{N} \). We put

\[
(\Delta_h^k f)(x) = f(x + h) - f(x) \quad \text{and} \quad (\Delta_{h+1}^k f)(x) = \Delta_h^k(\Delta_h^k f)(x).
\]

The \( k \)-th order modulus of smoothness of a function \( f \in L_p \) is defined by

\[
\omega_k(f,t)_p = \sup_{|h| \leq t} \|\Delta_h^k f\|_{L_p}, \quad t > 0.
\]

If \( k = 1 \) we simply write \( \omega(f,t)_p \) instead of \( \omega_1(f,t)_p \).

For \( 1 \leq p \leq \infty \), the following connection holds between the \( K \)-functional for the couple \((L_p,W_p^k)\) and the \( k \)-th order modulus of smoothness: There are positive constants \( c_1 \) and \( c_2 \) such that

\[
c_1 K(t^k,f;L_p,W_p^k) \leq \min(1,t^k)\|f\|_{L_p} + \omega_k(f,t)_p \leq c_2 K(t^k,f;L_p,W_p^k)
\]

(2.1)

for all \( f \in L_p \) and \( t > 0 \) (see [2, Theorem 5.4.12]).
For $1 \leq p \leq \infty$, $0 < q \leq \infty$, $-\infty < b < \infty$, $s \geq 0$ and $k \in \mathbb{N}$ with $k > s$, the space $B_{p,q}^{s,b}$ consists of all $f \in L_p$ such that

$$
\|f\|_{B_{p,q}^{s,b}} = \|f\|_{L_p} + \left( \int_0^1 \left( (1 - \log t)^b \omega_k(f,t) \right)_t^q \frac{dt}{t} \right)^{1/q} < \infty.
$$

See [16,30]. Note that if $s = 0$ and $b < -1/q$, then $B_{p,q}^{0,b} = L_p$.

3. Besov spaces with logarithmic smoothness

If $s > 0$ it is well-known that the definition of $B_{p,q}^{s,b}$ does not depend on the choice of $k > s$ (see [30, Theorem 2.5]). Next we show that the same property holds for $B_{p,q}^{0,b}$. We also characterize spaces $B_{p,q}^{0,b}$ by interpolation.

**Theorem 3.1.** Let $1 \leq p \leq \infty$, $0 < q \leq \infty$, $-\infty < b < \infty$ and $k \in \mathbb{N}$.

(a) The space $B_{p,q}^{0,b}$ does not depend on the choice of $k \in \mathbb{N}$.

(b) We have $B_{p,q}^{0,b} = (L_p, W_{p}^{k})_{(0, -b), q}$ with equivalence of quasi-norms.

**Proof.** Let $k \in \mathbb{N}, k > 1$. Put

$$
\|f\|_{B_{p,q}^{k}} = \|f\|_{L_p} + \left( \int_0^1 \left( (1 - \log t)^b \omega_k(f,t) \right)_t^q \frac{dt}{t} \right)^{1/q} \tag{3.1}
$$

and

$$
\|f\|_{B_{p,q}^{(k)}} = \|f\|_{L_p} + \left( \int_0^1 \left( (1 - \log t)^b \omega_k(f,t) \right)_t^q \frac{dt}{t} \right)^{1/q}.
$$

Our aim is to show the equivalence between the quasi-norms $\| \cdot \|_{B_{p,q}^{k,b}}$ and $\| \cdot \|_{B_{p,q}^{(k)}}^k$. Since $\omega_k(f,t)_t \leq 2^{k-1} \omega(f,t)_t$, it is clear that $\|f\|_{B_{p,q}^{k,b}} \lesssim \|f\|_{B_{p,q}^{(k)}}$. Let us check the converse inequality. Using Marchaud’s inequality [2, Theorem 5.4.4], for $0 < t \leq 1$ we obtain

$$
\frac{\omega(f,t)_t}{t} \lesssim \int_t^\infty \frac{\omega_k(f,s)_s}{s} \frac{ds}{s}
$$

$$
\lesssim \int_t^1 \frac{\omega_k(f,s)_s}{s} \frac{ds}{s} + \|f\|_{L_p} \int_1^\infty s^{-1} \frac{ds}{s}
$$

$$
\sim \int_t^1 \frac{\omega_k(f,s)_s}{s} \frac{ds}{s} + \|f\|_{L_p}.
$$

Therefore, since $\omega_k(f,s)_s/s^k$ is equivalent to a decreasing function, applying Hardy’s inequality [1, Theorem 6.4], we get
\[
\left( \int_0^1 \left( (1 - \log t)^b \omega(f, t) \frac{dt}{t} \right)^{\frac{1}{q}} \right)^{1/q} \lesssim \left( \int_0^1 \left( t(1 - \log t)^b \frac{dt}{t} \right)^{\frac{1}{q}} \right)^{1/q} \|f\|_{L_p} \\
+ \left( \int_0^1 \left[ \int t^2(1 - \log t)^b \frac{dt}{t^2} \left( \omega(f, t) \frac{dt}{t} \right)^{\frac{1}{q}} \right]^{\frac{1}{q}} \right)^{1/q} \lesssim \|f\|_{L_p} + \left( \int_0^1 \left[ \int \left( 1 - \log t \right)^b \omega_k(f, t) \frac{dt}{t} \right]^{\frac{1}{q}} \right)^{1/q} = \|f\|_{B^{0,b}_{p,q}}.
\]

This proves statement (a).

As for (b), using (2.1), we obtain

\[
\|f\|_{(L_p, W^k_p)_{0, -b, q}} = \left( \int_0^1 \left[ (1 - \log t)^b K(t, f; L_p, W^k_p) \right]^{\frac{1}{q}} \frac{dt}{t} \right)^{1/q} \\
\lesssim \left( \int_0^1 \left[ (1 - \log t)^b K(t^k, f; L_p, W^k_p) \right]^{\frac{1}{q}} \frac{dt}{t} \right)^{1/q} \\
\lesssim \left( \int_0^1 \left[ (1 - \log t)^b \left( \|f\|_{L_p} + \omega_k(f, t) \right) \right]^{\frac{1}{q}} \frac{dt}{t} \right)^{1/q} \\
\lesssim \|f\|_{L_p} + \left( \int_0^1 \left[ (1 - \log t)^b \omega_k(f, t) \right]^{\frac{1}{q}} \frac{dt}{t} \right)^{1/q} = \|f\|_{B^{k,b}_{p,q}}
\]

where we have used (a) in the last equivalence. This completes the proof. \(\square\)

**Remark 3.2.** For Besov spaces defined over \(T^n\), the fact that the definition of \(B^{0,b}_{p,q}(T^n)\) is independent of \(k \in \mathbb{N}\) has been proved in [25, pp. 1041–1043] by using Jackson inequality and Bernstein inequality.

In what follows we assume that \(B^{0,b}_{p,q}\) is quasi-normed by (3.1).

Next we compare \(B^{0,b}_{p,q}\) and \(B^{k,b}_{p,q}\).

**Theorem 3.3.** Let \(1 < p < \infty\), \(0 < q \leq \infty\) and \(b > -1/q\). Then

\[
B^{0,b+1/\min\{2,p,q\}}_{p,q} \hookrightarrow B^{0,b}_{p,q} \hookrightarrow B^{0,b+1/\max\{2,p,q\}}_{p,q}.
\]

**Proof.** Recall that

\[
B^s_{p,\min\{p,q\}} \hookrightarrow F^s_{p,q} \hookrightarrow B^s_{p,\max\{p,q\}}
\]
where $F_{p,q}^s$ stands for the Triebel–Lizorkin space (see [39, Proposition 2.3.2/2(iii)]). Moreover, $F_{p,2}^s = H_{p}^s$ [39, Theorem 2.5.6(i)] and so $F_{p,2}^0 = L_p$. According to Theorem 3.1(b), Lemma 2.1 and [9, Theorem 5.3 and Remark 5.4], we derive

$$B_{p,q}^{0,b} = (L_p, W_p^1)_{(0,-b)} \leftrightarrow (B_{p,\max\{2,p\}}^{0}, H_p^1)_{(0,-b)},$$

$$= \left((H_p^{-1}, H_p^1)_{1/2, \max\{2,p\}}, H_p^1\right)_{(0,-b)}$$

$$\leftrightarrow (H_p^{-1}, H_p^1)_{1/2, q, b+1/ \max\{2,p,q\}}$$

$$= B_{p,q}^{0,b+1/ \max\{2,p,q\}}.$$

Similarly, we have

$$B_{p,q}^{0,b+1/ \min\{2,p,q\}} = \left((H_p^{-1}, H_p^1)_{1/2, q, b+1/ \min\{2,p,q\}}\right)$$

$$\leftrightarrow \left((H_p^{-1}, H_p^1)_{1/2, \min\{2,p\}}, H_p^1\right)_{(0,-b)}$$

$$= (B_{p,\min\{2,p\}}^{0}, H_p^1)_{(0,-b)}$$

$$\leftrightarrow (L_p, W_p^1)_{(0,-b)},$$

$$= B_{p,q}^{0,b}.$$  \hfill \Box

**Remark 3.4.** In general $B_{p,q}^{0,b} \neq B_{p,q}^{0,b+1/ \max\{2,p,q\}}$ because in any of the cases

$$\begin{cases} 1 < p < \infty, & 0 < q \leq \min\{2, p\}, & b + \frac{1}{\max\{2, p\}} < 0 < b + \frac{1}{q}, \\ 1 < p \leq 2, & p < q \leq \infty, & 0 < b + \frac{1}{q} \leq \frac{1}{p} - \frac{1}{\max\{2, q\}}, \\ 2 < q < \infty, & 2 < q \leq \infty, & 0 < b + \frac{1}{q} \leq \frac{1}{2} - \frac{1}{\max\{p, q\}}. \end{cases}$$

the space $B_{p,q}^{0,b+1/ \max\{2,p,q\}}$ does not contain only regular distributions (see [7, Theorem 4.3]).

**Corollary 3.5.** Let $1 < p < \infty$ and $b > -1/p$.

(a) If $1 < p \leq 2$ then $B_{p,p}^{0,b+1/p} \hookrightarrow B_{p,p}^{0,b} \hookrightarrow B_{p,p}^{0,b+1/2}$.

(b) If $2 \leq p < \infty$ then $B_{p,p}^{0,b+1/2} \hookrightarrow B_{p,p}^{0,b} \hookrightarrow B_{p,p}^{0,b+1/p}$.

In particular, for $b > -1/2$ we obtain with equivalence of norms

$$B_{2,2}^{0,b+1/2} = B_{2,2}^{0,b}.$$

**Remark 3.6.** As in Remark 3.4, note that

$$B_{p,p}^{0,b} \neq B_{p,p}^{0,b+1/2} \quad \text{if} \quad 1 < p < 2 \quad \text{and} \quad b + \frac{1}{2} < 0 < b + \frac{1}{p},$$

and

$$B_{p,p}^{0,b} \neq B_{p,p}^{0,b+1/p} \quad \text{if} \quad 2 < p < \infty \quad \text{and} \quad 0 < b + \frac{1}{p} \leq \frac{1}{2} - \frac{1}{p}.$$
We finish this section with an embedding result into spaces $B_{p,q}^{0,b}$. This problem has been considered in [16, Corollary 5.3(ii)] and [25, Corollary 2.8]. We follow a more simple approach than in [25] based on limiting interpolation.

**Theorem 3.7.** Let $1 \leq p < r < \infty$, $0 < q \leq \infty$, $b > -1/q$ and $\alpha = n(1/p - 1/r)$. Then

$$B_{p,q}^{\alpha,b+1/\min\{q,r\}} \hookrightarrow B_{r,q}^{0,b}.$$  

**Proof.** According to [2, Corollary 5.4.20], we have

$$B_{p,r}^{\alpha} \hookrightarrow B_{r}.$$  

(3.2)

On the other hand, let $k \in \mathbb{N}$ such that $k > \alpha$ and $0 < \theta < 1$ such that $\theta k > \alpha$. By [2, Corollaries 5.4.13 and 5.4.21], we derive

$$W_p^{k} \hookrightarrow (L_p, W_p^{k})_{\theta,p} = B_{p,p}^{\theta k} \hookrightarrow B_{r,p}^{\theta k - \alpha}.$$  

(3.3)

Interpolating embeddings (3.2) and (3.3) by the limiting real method we get

$$(B_{p,r}^{\alpha}, W_p^{k})_{(0,-b),q} \hookrightarrow (L_r, B_{r,p}^{\theta k - \alpha})_{(0,-b),q}.$$  

The target space in this embedding can be determined by using [8, Lemma 2.2(b)] and Theorem 3.1(b). Indeed,

$$(L_r, B_{r,p}^{\theta k - \alpha})_{(0,-b),q} = (L_r, (L_r, W_p^{k})_{\frac{\theta k - \alpha}{r,p}})_{(0,-b),q}$$  

$$= (L_r, W_p^{k})_{(0,-b),q} = B_{r,q}^{0,b}.$$  

As for the domain space, Lemma 2.1(b) yields

$$B_{p,q}^{\alpha,b+1/\min\{q,r\}} = (L_p, W_p^{k})_{\alpha/k,q,b+1/\min\{q,r\}}$$  

$$\hookrightarrow ((L_p, W_p^{k})_{\alpha/k,r}, W_p^{k})_{(0,-b),q}$$  

$$= (B_{p,r}^{\alpha}, W_p^{k})_{(0,-b),q}.$$  

This completes the proof. □

**4. Duality**

Let $1 < p < \infty$, $1 \leq q < \infty$ and $-\infty < b < \infty$. Since $B_{p,q}^{0,b} = (H_{p}^{-1}, H_{p}^{1})_{1/2,q,b}$, using the duality formula for spaces $(A_0, A_1)_{\theta,q,A}$ (see [15, Theorem 3.1] or [36, Theorem 2.4]) and that $(H_{p}^{s})' = H_{p'}^{-s}$ [38, Theorem 2.6.1], it follows that

$$(B_{p,q}^{0,b})' = B_{p',q'}^{-0,-b} \quad \text{where } 1/p + 1/p' = 1 = 1/q + 1/q'.$$  

(see also [24, Theorem 3.1.10]).

In order to determine the dual space of $B_{p,q}^{0,b}$, we first establish an auxiliary result and recall the definition of logarithmic Lipschitz spaces (see [28] and [29]).
Lemma 4.1. Let $A_0$, $A_1$ be Banach spaces with $A_1$ continuously and densely embedded in $A_0$. Assume that $1 \leq q < \infty$, $1/q + 1/q' = 1$, and $\eta > -1/q$. Then we have with equivalence of norms

$$(A_0, A_1)_{(0, -\eta), q} = (A'_1, A'_0)_{(1, \eta+1), q'}.$$

Proof. Since $A_1 \hookrightarrow A_0$, we have that $K(t, a; A_0, A_1) \sim \|a\|_{A_0}$ for $t \geq 1$. Take any $\tau < -1/q$. It follows that

$$\left( \int_1^\infty \frac{(1 + \log t)\tau K(t, a; A_0, A_1)}{t} \right)^{1/q} \sim \left( \int_1^\infty \frac{(1 + \log t)\tau q}{t} \right)^{1/q} \|a\|_{A_0}.$$

This yields that

$$(A_0, A_1)_{(0, -\eta), q} = (A_0, A_1)_{0, q, (\eta, \tau)} = (A_1, A_0)_{1, q, (\tau, \eta)}.$$

Since $\tau + 1/q < 0 < \eta + 1/q$, we can apply the duality formula established in [14, Theorem 5.6] to derive

$$(A_0, A_1)_{(0, -\eta), q} = (A_1, A_0)_{1, q, (\tau, \eta)} = (A'_1, A'_0)_{1, q', (-\eta-1, -\tau-1)}.$$

Density of the embedding $A_1 \hookrightarrow A_0$ implies that $A'_0 \hookrightarrow A'_1$. So $K(t, g; A'_1, A'_0) \sim \|g\|_{A'_1}$ for $t \geq 1$. Now, using that $K(t, g)/t$ is a decreasing function we get

$$\left( \int_1^\infty \frac{t^{-\frac{1}{q'}}(1 + \log t)^{-\frac{1}{q'}-1}K(t, g; A'_1, A'_0)}{t} \right)^{1/q'} \sim \frac{\|g\|_{A'_1}}{K(1, g; A'_1, A'_0)} \left( \int_0^1 \frac{(1 - \log t)^{-\frac{1}{q'}}(1 - \log t)^{-\frac{1}{q'}}}{t} \right)^{1/q'}$$

$$\leq \frac{\|g\|_{(A'_1, A'_0)_{(1, \eta+1), q'}}}{(A'_1, A'_0)_{(1, \eta+1), q'}}.$$

Consequently, $$(A_0, A_1)_{(0, -\eta), q} = (A'_1, A'_0)_{(1, \eta+1), q'}.$$

□

Definition 4.2. Let $1 \leq p \leq \infty$, $0 < q \leq \infty$ and $\alpha > 1/q$ ($\alpha \geq 0$ if $q = \infty$). The space $\text{Lip}_{p, q}^{(1, -\alpha)}$ is formed by all functions $f \in L_p$ having a finite quasi-norm

$$\|f\|_{\text{Lip}_{p, q}^{(1, -\alpha)}} = \|f\|_{L_p} + \left( \int_0^1 \frac{\omega(f, t)_p}{t(1 + \log t)^{-\frac{1}{q'}}} \right)^{1/q'}.$$

Now we are ready to describe the dual space of $B^{a,b}_{p,q}$. Recall that the usual lift operator $I_s$ is defined by

$$I_s f = F^{-1} (1 + |x|^2)^{s/2} F f, \quad -\infty < s < \infty.$$
Theorem 4.3. Let $1 < p < \infty$, $1 \leq q < \infty$ and $b > -1/q$. The space $(B^{0,b}_{p,q})'$ consists of all $f \in H^{-1}_{p'}$ such that $I_{-1} f \in \text{Lip}^{(1,-b-1)}_{p',q'}$ with $1/p + 1/p' = 1 = 1/q + 1/q'$. Moreover,
\[
\|f\|_{(B^{0,b}_{p,q})'} \sim \|I_{-1} f\|_{\text{Lip}^{(1,-b-1)}_{p',q'}}.
\]

Proof. By Theorem 3.1(b) and Lemma 4.1, we derive
\[
(B^{0,b}_{p,q})' = ((L_p, W^1_p)_{(0,-b),q})' = (H^{-1}_{p'}, L_{p'})_{(1,b+1),q'}.
\]
On the other hand, lift operators

\[ I_{-1} : H^{-1}_{p'} \rightarrow L_{p'}, \quad I_{-1} : L_{p'} \rightarrow W^1_{p'} \]

are bijective and bounded. Hence

\[
K(t,f; H^{-1}_{p'}, L_{p'}) \sim K(t, I_{-1} f; L_{p'}, W^1_{p'}) \sim \min(1,t)\|I_{-1} f\|_{L_{p'}} + \omega(I_{-1} f, t)_{p'}
\]

where we have used (2.1) for the last equivalence. Consequently

\[
\|f\|_{(B^{0,b}_{p,q})'} \sim \left( \int_0^1 (1 - \log t)^{-b-1} q' \frac{dt}{t} \right)^{1/q'} \|I_{-1} f\|_{L_{p'}}
\]

\[
+ \left( \int_0^1 \left[ \frac{\omega(I_{-1} f, t)_{p'}}{t(1 - \log t)^{b+1}} \right] q' \frac{dt}{t} \right)^{1/q'}
\]

\[
\sim \|I_{-1} f\|_{\text{Lip}^{(1,-b-1)}_{p',q'}}. \quad \square
\]

5. Embeddings between Besov and Lipschitz spaces

We start by showing that Lipschitz spaces can be generated by interpolation from the couple $(L_p, W^1_p)$.

Lemma 5.1. Let $1 \leq p \leq \infty$, $0 < q \leq \infty$ and $\alpha > 1/q$ ($\alpha \geq 0$ if $q = \infty$). Then

\[
(L_p, W^1_p)_{(1,\alpha),q} = \text{Lip}^{(1,-\alpha)}_{p,q}
\]

with equivalent quasi-norms.

Proof. Using (2.1) we derive

\[
\|f\|_{(L_p, W^1_p)_{(1,\alpha),q}} = \left( \int_0^1 \left[ \frac{K(t,f; L_p, W^1_p)}{t(1 - \log t)^{\alpha}} \right] q \frac{dt}{t} \right)^{1/q}
\]

\[
\sim \left( \int_0^1 (1 - \log t)^{-\alpha q} \frac{dt}{t} \right)^{1/q} \|f\|_{L_p} + \left( \int_0^1 \left[ \frac{\omega(f,t)_{p'}}{t(1 - \log t)^{\alpha}} \right] q \frac{dt}{t} \right)^{1/q}
\]

\[
\sim \|f\|_{\text{Lip}^{(1,-\alpha)}_{p,q}}. \quad \square
The next result describes the position of Lipschitz spaces between Besov spaces with classical smoothness 1 and additional logarithmic smoothness.

**Theorem 5.2.** Let $1 < p < \infty$, $0 < q \leq \infty$ and $\alpha > 1/q$. Then

\[ B_{p,q}^{1,-\alpha+1/\min\{2,p,q\}} \hookrightarrow \text{Lip}_{p,q}^{(1,-\alpha)} \hookrightarrow B_{p,q}^{1,-\alpha+1/\max\{2,p,q\}}. \]

**Proof.** By Lemmata 5.1, 2.1(a) and [9, Theorem 5.3 and Remark 5.4], we obtain

\[
\text{Lip}_{p,q}^{(1,-\alpha)} = (L_p, W_p^1)_{(1,\alpha),q} \hookrightarrow (L_p, B_{p,\max\{2,p\}}^1)_{(1,\alpha),q}
\]

\[
= (L_p, (L_p, W_p^2)_{1/2,\max\{2,p\}})_{(1,\alpha),q}
\]

\[
\hookrightarrow (L_p, W_p^2)_{1/2,\alpha+1/\max\{2,p,q\}}
\]

\[
= B_{p,q}^{1,-\alpha+1/\max\{2,p,q\}}.
\]

Similarly, we derive

\[
B_{p,q}^{1,-\alpha+1/\min\{2,p,q\}} = (L_p, W_p^2)_{1/2,\alpha-\alpha+1/\min\{2,p,q\}}
\]

\[
\hookrightarrow (L_p, (L_p, W_p^2)_{1/2,\min\{2,p\}})_{(1,\alpha),q}
\]

\[
= (L_p, B_{p,\min\{2,p\}}^1)_{(1,\alpha),q}
\]

\[
\hookrightarrow (L_p, W_p^1)_{(1,\alpha),q}
\]

\[
= \text{Lip}_{p,q}^{(1,-\alpha)}. \quad \square
\]

**Corollary 5.3.** Let $1 < p < \infty$ and $\alpha > 1/p$.

(a) If $1 < p \leq 2$ then $B_{p,p}^{1,-\alpha+1/p} \hookrightarrow \text{Lip}_{p,p}^{(1,-\alpha)} \hookrightarrow B_{p,p}^{1,-\alpha+1/2}$.

(b) If $2 \leq p < \infty$ then $B_{p,p}^{1,-\alpha+1/2} \hookrightarrow \text{Lip}_{p,p}^{(1,-\alpha)} \hookrightarrow B_{p,p}^{1,-\alpha+1/p}$.

In particular, if $\alpha > 1/2$ we have

\[ B_{2,2}^{1,-\alpha+1/2} = \text{Lip}_{2,2}^{(1,-\alpha)}. \]

Next we recall a result of Haroske [28, Proposition 16].

**Proposition 5.4.** Let $1 \leq p \leq \infty$, $0 < q, v \leq \infty$, $\alpha > 1/q$ and $\beta > 1/v$. Then

\[
\text{Lip}_{p,q}^{(1,-\alpha)} \hookrightarrow \text{Lip}_{p,v}^{(1,-\beta)} \text{ if, and only if, }
\]

\[
\left\{\begin{array}{l}
\beta - \frac{1}{v} \geq \alpha - \frac{1}{q} \quad \text{and} \quad v \geq q, \\
\beta - \frac{1}{v} > \alpha - \frac{1}{q} \quad \text{and} \quad v < q.
\end{array}\right.
\]

In the remaining part of this section we show that combining Proposition 5.4 with the previous results we can derive some complements and improvements of the results of [28].
Corollary 5.5. Let $1 < p < \infty$, $0 < q, v \leq \infty$ and $\alpha > 1/v$. Then

$$B_{p,q}^1 \hookrightarrow \text{Lip}_{p,v}^{(1,-\alpha)} \quad \text{if} \quad \begin{cases} 0 < q \leq \min\{2,p\}, \\
\min\{2,p\} < q, v < q & \text{and} \quad \alpha > \frac{1}{v} + \frac{1}{\min\{2,p\}} - \frac{1}{q}, \\
\min\{2,p\} < q \leq v & \text{and} \quad \alpha \geq \frac{1}{v} + \frac{1}{\min\{2,p\}} - \frac{1}{q}. \end{cases}$$

Proof. If $0 < q \leq \min\{2,p\}$, we obtain

$$B_{p,q}^1 \hookrightarrow B_{p,\min\{2,p\}}^1 \hookrightarrow (L_p, B_{p,\min\{2,p\}}^1)_{(1,\alpha),v} \hookrightarrow (L_p, W_p^1)_{(1,\alpha),v} = \text{Lip}_{p,v}^{(1,-\alpha)}.$$ 

If $\min\{2,p\} < q$, let $\beta = 1/\min\{2,p\}$. By Theorem 5.2 and Proposition 5.4, we derive

$$B_{p,q}^1 = B_{p,q}^{1, -\beta + \frac{1}{\min\{2,p\} - \alpha}} \hookrightarrow \text{Lip}_{p,q}^{(1,-\beta)} \hookrightarrow \text{Lip}_{p,v}^{(1,-\alpha)}. \quad \Box$$

Remark 5.6. Theorem 5.5 confirms a conjecture of Haroske in [28, Remark 12] and closes a problem also mentioned in [29, p. 115] by showing that if $1 < p < \infty$ the embedding $B_{p,q}^1 \hookrightarrow \text{Lip}_{p,v}^{(1,-\alpha)}$ holds not only for $\alpha = 1/q' + 1/v$ but even for smaller values of $\alpha$.

Remark 5.7. Note that the argument given in Theorem 5.5 when $0 < q \leq \min\{2,p\}$ works for any $\alpha \geq 0$ if $v = \infty$. So when $q = \min\{2,p\}$ we recover a result proved by Neves [35, Proposition 5.6] using different techniques.

Next for $1 \leq q < \infty$ we cover a limit case left open in [28, Corollary 23(i)] (see also [29, Corollary 7.20(i)]).

Corollary 5.8. Let $1 < p < \infty$, $1 \leq q \leq \infty$ and $0 < \beta = \alpha - 1/q$. Then

$$B_{p,1}^{1, -\beta} \hookrightarrow \text{Lip}_{p,q}^{(1,-\alpha)}.$$ 

Proof. Using again Theorem 5.2 and Proposition 5.4 we obtain

$$B_{p,1}^{1, -\beta} \hookrightarrow \text{Lip}_{p,1}^{(1,-\beta-1)} \hookrightarrow \text{Lip}_{p,q}^{(1,-\alpha)}. \quad \Box$$

Proceeding as in Corollary 5.8, we can also derive the embedding

$$B_{p,\min\{2,p\}}^{1, -\alpha+1/q} \hookrightarrow \text{Lip}_{p,q}^{(1,-\alpha)}$$

provided that $1 < p < \infty$, $0 < q \leq \infty$ and $\alpha > 1/q$. This improves [28, embedding (29), p. 793] because

$$B_{p,\min\{1,q\}}^{1, -\alpha+1/q} \hookrightarrow B_{p,\min\{2,p,q\}}^{1, -\alpha+1/q}.$$ 

Note also that from Theorem 5.2 we can recover [28, embeddings (41), p. 796] for $1 < p < \infty$. Besides, Theorem 5.2 also yields that if $\alpha > 1/q$ then

$$\text{Lip}_{p,q}^{(1,-\alpha)} \hookrightarrow B_{p,q}^{1, -\alpha+1/q} \quad \text{if} \max\{2,p\} \leq q,$$

and

$$B_{p,q}^{1, -\alpha+1/q} \hookrightarrow \text{Lip}_{p,q}^{(1,-\alpha)} \quad \text{if} \ q \leq \min\{2,p\}.$$
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