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UNIVERSIDAD COMPLUTENSE

FACULTAD DE ECONOMICAS

Campus de Somosaguas

28223 MADRID

Teléfono 91 394 26 11 - FAX 91 294 26 13

Internet: http://www.ucm.es/info/icae/

E-mail: icaesec@ccee.ucm.es

Documento de trabajo

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José Casals
Miguel Jerez
Sonia Sotoca

No. 0006 Junio 2000

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AN EXACT MULTIVARIATE MODEL-BASED STRUCTURAL DECOMPOSITION

José Casals
Miguel Jerez†
Sonia Sotoca

Departamento de Fundamentos del Análisis Económico II
Facultad de CC. Económicas y Empresariales
Universidad Complutense de Madrid

ABSTRACT

We describe a simple procedure for decomposing a vector of time series into trend, cycle, seasonal and irregular components. Contrary to common practice, we do not assume these components to be orthogonal conditional on their past. However, the state-space representation employed assures that their smoothed estimates converge to exact values, with null variances and covariances. Among other implications, this means that the components are not revised when the sample increases. The practical application of the method is illustrated both with simulated and real data.

Keywords: State-space models, seasonal adjustment, trends, unobserved components.

JEL Classification: C32, C53, E27, E37

† Corresponding author. Departamento de Fundamentos del Análisis Económico II. Universidad Complutense de Madrid. Campus de Somosaguas. 28223 Madrid (Spain). Phone: (+34) 913 94 23 61. Fax: (+34) 913 94 26 13. E-mail: mjerez@ccee.ucm.es.

1. Introduction.

Many works define the additive structural decomposition of a time series, z_t, as:

$$z_t = t_t + c_t + s_t + \varepsilon_t \quad (1)$$

where:

- t_t is the *trend component*, representing the long-term behavior of the series,
- c_t is the *cyclical component*, describing autocorrelated transitory fluctuations,
- s_t is the *seasonal component*, associated with persistent patterns over seasons, and
- ε_t is an *irregular component*.

As in other fields, the decomposition (1) has been of long-standing interest to economists. Early research emphasized the extraction of the seasonal component to obtain a seasonally-adjusted series, useful in assessing the importance of fluctuations in economic variables without the "contamination" of seasonal variability. In Macroeconomics, interest soon extended to the trend component, associated with long-run growth, and to the transitory components, interpretable as short-term fluctuations.

There are two basic approaches to obtain the unobserved components (UC) on the right-hand-side of (1): *ad-hoc methods* and *model-based methods*.

Ad-hoc methods consist of filtering the series by means of a differential equation, designed to extract the components generating peaks of spectral power at previously chosen frequencies. The most famous examples of this approximation are in the Census X-11 saga, see Shiskin *et al.* (1967) and Findley *et al.* (1998). An influential proposal specialized in trend extraction is the HP filter, due to Hodrick and Prescott (1980). The simplest versions of these methods have a clear advantage in simplicity, as they do not require any previous analysis. However, their mechanical application often yields spurious decompositions, as they implicitly assume that different time series follow the same stochastic process.

Potential inadequacy of *ad-hoc* methods inspired the *model-based methods*, which emphasize the coherence between the properties of the observed series and those of the structural components. This principle is at the basis of three methodologies, the *ARIMA-model-based* (AMB), the *Forecast Decomposition* (FD) and the *Structural Time Series Models* (STSM) approaches.

AMB techniques were originally developed by Box *et al.* (1978), Burman (1980), Hillmer and Tiao (1982) and Bell and Hillmer (1984). For a self-contained survey, see Planas (1997). A recent software implementation is due to Gómez and Maravall (1996). These methods start from a reduced-form ARIMA model for z_t, and afterwards obtain a structural representation defined by individual

ARIMA processes for each UC, constrained that their sum is observationally equivalent to the reduced-form model.

The FD approach is due to Box *et al.* (1987), for a recent paper on FD see Espasa and Peña (1995). It consists of decomposing the *h*-steps-ahead forecast function of a given univariate model, generally belonging to the ARIMA family, into persistent and transitory components, which can also be broken down into seasonal and nonseasonal terms.

Last, STSM are directly set up in terms of the components in (1), which are represented by state-space (from now on SS) models specified according to the statistical properties of the time series, see Engle (1978), Harvey (1989), Harvey and Shephard (1993) and Young *et al.* (1999). Whereas AMB and FD techniques are essentially univariate, the simpler structure of SS models makes it easy to define STSM for vectors of time series and allows extensions to nonlinear and non-gaussian systems or models with stochastic variances. This approach is implemented with some differences in three main software packages: MICRO-CAPTAIN, see Young and Brenner (1991), BATS, see Pole *et al.* (1994), and STAMP, see and Koopman *et al.* (1995).

Once the models for the components have been specified and estimated using any of these methodologies, the final step in the analysis consists of estimating the components. To this purpose most approaches use a class of algorithms known in general as "symmetric filters" such as, *e.g.*, the Wiener-Kolmogorov filter, see Burman (1980) and Bell and Hillmer (1984), and the fixed-interval smoother, see Anderson and Moore (1979). The word "symmetric" alludes to the fact that current estimates of the components depend on past and future values of the time series. FD methods are an important exception to this general approach, because the components implied by a forecast function depend only on past sample values and, therefore, one-side asymmetric filters are natural choices for extracting the empirical components.

The literature about structural decomposition is so rich that any generalization is deemed to be incorrect in specific cases. However, we can say that most *ad-hoc*, AMB and STSM methods share three main shortcomings: first, they need arbitrary restrictions; second, theoretical and empirical components have different properties; and third, estimates of the components change when the sample increases.

Arbitrary restrictions are necessary because, in general, Eq. (1) and the models for the components make up an underidentified econometric model in structural form. To improve identificability, most methods assume that the components are uncorrelated. This restriction is not justified on economic grounds but has a compelling practical motivation because structural components are often analyzed separately and, therefore, some kind of independence among them is desirable. When independence of the components is not enough to achieve exact identification, the literature suggests other restrictions such as canonical constraints, see Hillmer and Tiao (1982).

Second, the statistical properties of theoretical and empirical components are different because, whereas the theoretical components depend only on past information, their symmetric filtered estimates depend on both, past and future information, see *e.g.* Planas (1997, pp. 113). As a consequence, the empirical components: a) violate the independence assumption, see García-Ferrer and del Hoyo (1992), and b) are generated by stochastic processes which do not derive from the model of the time series.

Finally, the use of symmetric filters also implies that the empirical components change when the sample increases. These "revisions" are clearly a major problem when the decomposition is applied to generate public macroeconomic information.

The empirical components obtained by FD methods avoid these general shortcomings because they are obtained from a reduced-form model, so they do not require arbitrary identification assumptions, and are extracted by filtering only past information, thus avoiding revisions and distortions in their stochastic structure. However, using only past information implies a certain inefficiency and heterogeneity of the within-the-sample structural components, as the values computed in t are based in a different information set than those computed in any other instant. Also there is some ambiguity about the choice of the lead-time for the forecast function.

In this paper we use SS techniques to obtain the structural decomposition of a vector of time series generated by a linear stochastic process, addressing the shortcomings mentioned above. Section 2 defines the basic notation, summarizes some previous results about the SS representation of the data generating process and characterizes the structural components. To do this, we state a one-to-one correspondence between the eigenvalues of the transition matrix in the SS representation and the corresponding frequencies displaying peaks of spectral power. Using consensus ideas about the properties of the components, the state variables with unit eigenvalues are associated with the trend, those with peaks at seasonal frequencies are assigned to the seasonal component and the rest of the states are included in the cycle term. The structural components are then defined by unique linear combinations of the state variables, characterized by the coefficients in the observation equation. Section 3 discusses the estimation of the structural components. As the SS representation employed is observable, the symmetric and asymmetric filtered estimates of the states (and therefore, those of the structural components) converge to conditionally orthogonal values with null variances. In this sense, the decomposition obtained is exact. Section 4 organizes the results in Sections 2 and 3 into a five-stage methodology. Practical application of our proposal is illustrated by the examples in Sections 5 and 6. Finally, Section 7 summarizes the main conclusions and suggests some additional applications.

2. Model representations and characterization of the structural components.

Assume that a structural decomposition analogous to (1) is to be computed for the $m \times 1$ random vector z_t. For reasons that will be discussed in Section 3, our method requires z_t to be the output of a *steady-state innovations* SS model (hereafter "innovations model") defined by:

$$x_{t+1} = \Phi x_t + \Gamma u_t + E a_t \tag{2}$$

$$z_t = H x_t + D u_t + a_t \tag{3}$$

where:

- x_t is an $n \times 1$ vector of *state variables*,
- u_t is an $r \times 1$ vector of exogenous variables,
- a_t is an $m \times 1$ vector of errors, such that $a_t \sim \text{iid}(\mathbf{0}, B)$.

The *transition equation* (2) characterizes all the dynamic structure of z_t. On the other hand, the *observation equation* (3) describes how z_t is generated by the sum of: a) a linear combination of the dynamic components, given by Hx_t, b) the instantaneous effect of the exogenous variables, given by Du_t, and c) the error a_t.

In Subsection 2.1 we will discuss how to: a) obtain a representation as (2)-(3) from a standard econometric model, and b) redefine the state variables into independent and clearly defined dynamic components. Subsection 2.2 discusses the spectral properties of the state variables and, using consensus ideas, associates them to the structural components. Finally, Subsection 3.3 defines a generalized structural decomposition and relates its components to linear combinations of the state variables in the innovations model.

2.1. Previous results about the innovations model.

The representation (2)-(3) can be obtained in two ways, corresponding to Results 1.1 and 1.2.

Result 1.1. If the model for z_t is (or can be expressed as) a general time-invariant SS model:

$$x^*_{t+1} = \Phi x^*_t + \Gamma u_t + E^* w_t \tag{4}$$

$$z_t = H x^*_t + D u_t + C v_t \tag{5}$$

where the errors w_t, v_t are independent of the initial state, x^*_1, and such that: a) $w_t \sim \text{iid}(\mathbf{0}, Q)$, $v_t \sim \text{iid}(\mathbf{0}, R)$, $\text{cov}(w_t v_t) = S$ for all $t = 1, 2, \ldots, N$ and b) $x^*_1 \sim (\mu_1, \Sigma_1)$.

Then under weak assumptions, see Casals *et al.* (1999, Theorem 1), z_t is also the output of the innovations model (2)-(3) with:

$$x_1 \sim (\mu_1, \Sigma_1 - P) \tag{6}$$

$$P = \Phi P \Phi^T + E^* Q E^{*T} - E^* B E^{*T} \tag{7}$$

$$E = (\Phi P H^T + E^* S C^T) B^{-1} \tag{8}$$

$$B = H P H^T + C R C^T \tag{9}$$

Note that model (4)-(5) includes two error terms, w_t and v_t, whereas the innovations model has only one, a_t. In some sense, this means that the relationship between both formulations is analogous to that existing between the structural and the reduced-form of an econometric model, *i.e.*, both are observationally equivalent but some information that is explicit in the covariance matrices of the "structural model" (4)-(5), Q, R and S, is lost when they are combined in the "reduced-form model" covariance, B. As we will see later, this is a very relevant fact for some applications.

Result 1.1 has the advantage of generality, as it supports any model that can be written as a standard linear fixed-coefficients SS model. However, it requires to solve the nonlinear equations (7)-(9). Ionescu *et al.* (1997) describe efficient and stable procedures to do this, but this can be an undesirable complication for many users. Therefore, if the model for the time series is (or can be written as) a VARMAX process, then Result 1.2 is more convenient:

Result 1.2. Assume that z_t follows the VARMAX(p,s,q) process:

$$F(B) z_t = G(B) u_t + \Xi(B) a_t \tag{10}$$

where a_t is a $m \times 1$ vector of white noise errors, u_t is a $r \times 1$ vector of exogenous variables; the polynomial matrices $F(B)$, $G(B)$ and $\Xi(B)$, are defined by:

$$F(B) = I + \sum_{i=1}^{p} F_i B^i, \quad G(B) = \sum_{i=0}^{s} G_i B^i, \quad \Xi(B) = I + \sum_{i=1}^{q} \Xi_i B^i \tag{11}$$

and may contain roots in the unit circle; finally, B denotes the backward-shift operator, such that for any sequence x_t: $B^{\pm k} x_t = x_{t \mp k}$.

Under these conditions, (10)-(11) can be expressed in the equivalent innovations form defining:

$$\Phi = \begin{bmatrix} -F_1 & I & 0 & \cdots & 0 \\ -F_2 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -F_{k-1} & 0 & 0 & \cdots & I \\ -F_k & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} G_1 - F_1 G_0 \\ G_2 - F_2 G_0 \\ \vdots \\ G_{k-1} - F_{k-1} G_0 \\ G_k - F_k G_0 \end{bmatrix}, \quad E = \begin{bmatrix} \Xi_1 - F_1 \\ \Xi_2 - F_2 \\ \vdots \\ \Xi_{k-1} - F_{k-1} \\ \Xi_k - F_k \end{bmatrix} \tag{12}$$

$$H = [I \quad 0 \quad \cdots \quad 0], \quad D = G_0 \tag{13}$$

where the *state vector* x_t has $n = m \cdot k$ rows, and $k = \max\{p, s, q\}$, see Aoki (1990) and Terceiro (1990), and the error term in the innovations model is the same as in model (10).

After obtaining a innovations model for the data, the problem consists of restructuring its dynamics in an equivalent and more meaningful representation. This can be done using Result 2.

Result 2. If z_t is realized by (2)-(3), then it is also realized by the model:

$$\bar{x}_{t+1} = \bar{\Phi} \bar{x}_t + \bar{\Gamma} u_t + \bar{E} a_t \tag{14}$$

$$z_t = \bar{H} \bar{x}_t + D u_t + a_t \tag{15}$$

where the states in (14)-(15) are related to those in (2)-(3) by a linear transformation $\bar{x}_t = U x_t$, such that $|U| \neq 0$. Accordingly, the matrices in (14)-(15) are related to those in (2)-(3) by the expressions: $\bar{\Phi} = U \Phi U^{-1}$, $\bar{\Gamma} = U \Gamma$, $\bar{E} = U E$, $\bar{H} = H U^{-1}$, and the transformed transition matrix $\bar{\Phi}$ is block-diagonal. The dimensions of these blocks can be: 1×1, representing a unique real eigenvalue, 2×2, representing a pair of imaginary eigenvalues, and $v \times v$, representing v repeated eigenvalues.

This factorization of a transition matrix can be obtained by different procedures, being the Jordan decomposition the most popular algorithm. For the purposes of this paper, we use a simpler and more stable method, which consists of applying the Schur decomposition to Φ and diagonalizing the resulting real Schur matrix by solving a system of Sylvester equations. For a complete description of this procedure, see Petkov *et al* (1991, pp. 103-106).

2.2. Frequency-domain properties of the state variables.

The block-diagonal representation (14)-(15) decomposes the dynamic response of the system to shocks in the inputs, a_t and u_t, into several basic movements corresponding to different eigenvalues of the transition matrix. These basic movements, which are termed *latent components* by West (1997), represent independent reactions of the state variables.

In the domain of time it is difficult to associate these latent movements with the structural components. In the frequency domain, however, all the structural components can be clearly defined.

Following Burman (1980), the trend is represented by a peak at low frequencies, the irregular component should display no significant peak at any frequency, the seasonal component comprises the spectral peaks at the basic seasonal frequency and its multiples. By exclusion, any other component should be assigned to the cyclic term. The implementation of these ideas in the block-diagonal innovations model (14)-(15) is easy, as the eigenvalues of the transition matrix characterize unambiguously the properties of the different states in both time and frequency domains.

Assume that $\lambda_{j,k} = a_j \pm b_j i$ is a pair of conjugate eigenvalues of $\bar{\Phi}$, associated with the states j and k. Under these conditions, it is easy to show that both states generate a peaks in the pseudo-spectrum in the frequency $f_{j,k} = (2\pi)^{-1} \arctan(b_j/a_j)$, where $f_{j,k}$ is in cycles per unit time. There are several particular cases worth considering:

1) If the real part of $\lambda_{j,k}$ is zero, then $b_j/a_j = \infty$, implying that $f_{j,k} = 1/4$ if $b_j > 0$ or $f_{j,k} = 3/4$ if $b_j < 0$.

2) If an eigenvalue is a real number, then $b_j/a_j = 0$, implying that $f_{j,k} = 0$ if $a_j > 0$ or $f_{j,k} = 1/2$ if $a_j < 0$.

3) If $b_j = 0$ and a_j tends to one, the spectral power tends to infinity at $f_{j,k} = 0$.

According to these results, and assuming that the system is not explosive, the states in the block-diagonal model can be naturally assigned to the structural components with the rules summarized in Table 1.

[Insert Table 1]

where F_S is the set of seasonal frequencies defined as: $F_S = \{ f_j = k_j/s \ ; \ k_j = 1, 2, ..., [s/2] \}$, being s the seasonal period and $[s/2] = s/2$ if s is even, or $[s/2] = (s-1)/2$ if s is odd.

Therefore, all the states with a unit eigenvalue are assigned to the trend component, all the states generating peaks at nonseasonal frequencies are assigned to the cycle component, and all the states generating peaks at seasonal frequencies are assigned to the seasonal component. Note that in a SS model the eigenvalues of the transition matrix are the reciprocals of the roots of the AR characteristic polynomial. Then, an assignation analogous to that in Table 1 can be made in terms of the AR roots.

2.3. Characterization of the structural components.

Assume that z_t is generated by the block-diagonal innovations model (14)-(15). Consider also the multivariate extension of (1):

$$z_t = t_t + c_t + s_t + d_t + \varepsilon_t \tag{16}$$

where t_t, c_t, s_t and ε_t are $m \times 1$ vectors of trend, cycle, seasonal and irregular components, respectively. A new term, d_t, represents the instantaneous effects of exogenous variables on z_t. In a framework of structural decomposition, this term is often used to model calendar effects or outliers.

Under such conditions, the components in (16) can be characterized by restructuring the block-diagonal model as:

$$\begin{bmatrix} \bar{x}_{t+1}^t \\ \bar{x}_{t+1}^c \\ \bar{x}_{t+1}^s \end{bmatrix} = \begin{bmatrix} \bar{\Phi}^t & 0 & 0 \\ 0 & \bar{\Phi}^c & 0 \\ 0 & 0 & \bar{\Phi}^s \end{bmatrix} \begin{bmatrix} \bar{x}_t^t \\ \bar{x}_t^c \\ \bar{x}_t^s \end{bmatrix} + \begin{bmatrix} \bar{\Gamma}^t \\ \bar{\Gamma}^c \\ \bar{\Gamma}^s \end{bmatrix} u_t + \begin{bmatrix} \bar{E}^t \\ \bar{E}^c \\ \bar{E}^s \end{bmatrix} a_t \tag{17}$$

$$z_t = \begin{bmatrix} \bar{H}^t & \bar{H}^c & \bar{H}^s \end{bmatrix} \begin{bmatrix} \bar{x}_t^t \\ \bar{x}_t^c \\ \bar{x}_t^s \end{bmatrix} + D u_t + a_t \tag{18}$$

where $\bar{x}_t^t$ is the vector of nonstationary states, $\bar{x}_t^c$ is the vector of stationary (nonseasonal) states and $\bar{x}_t^s$ is the vector of seasonal states. Accordingly, the structural components are defined as:

$$t_t = \bar{H}^t \bar{x}_t^t \tag{19}$$

$$c_t = \bar{H}^c \bar{x}_t^c \tag{20}$$

$$s_t = \bar{H}^s \bar{x}_t^s \tag{21}$$

$$d_t = D u_t \tag{22}$$

$$\varepsilon_t = a_t \tag{23}$$

Note that there are infinite block-diagonalizing matrices, U, but the components (19)-(23) are unique, because block-diagonalization is a similar transformation of the original transition matrix.

Expressions (16) and (19)-(23) can be interpreted as a decomposition of the one-step-ahead forecast function. To see this, note that (23) defines the irregular component to be the error term of the econometric model. Then the within-the-sample components in (19)-(22) add up to the corresponding fitted value, whereas the component in (24) coincides with the residual. When computed out of the sample, (19)-(22) add up to the corresponding forecast.

3. Estimation of the structural components.

Given the characterization of the structural components made in Section 2, the problem reduces to obtaining estimates of the state variables and combining them according to (19)-(23). The SS literature provides two basic algorithms to do this: the Kalman filter and the fixed-interval smoother (FIS), see Anderson and Moore (1979). These methods differ mainly in the information considered. While the former is a one-sided asymmetric filter, providing efficient estimates of the first and second-order moments of the states conditional on past information, the latter is a two-sided symmetric filter, which uses all the information in the sample. Therefore, FIS estimates are more precise in general. For the purposes of this paper, we use an efficient implementation of the FIS algorithm which allows for both, stationary and nonstationary roots, see Casals *et al.* (2000).

Denoting the information set up to time i as $\Omega^i = \{z_1, z_2, \ldots, z_i, u_1, u_2, \ldots, u_i\}$, and the FIS estimates by $x_{t|N} = E(x_t \mid \Omega^N)$ and $P_{t|N} = E[(x_t - x_{t|N})(x_t - x_{t|N})^T \mid \Omega^N]$, N being the sample size, the following result holds:

Result 3. In an innovations model, the FIS covariance, $P_{t|N}$ converges to zero as t increases.

This important property can be derived as a Corollary of Theorem 4.2. in De Souza *et. al* (1986), taking into account that: a) an innovations model is detectable and b) the strong solution of the corresponding Riccati equation is zero. For our purposes, the following argument is simpler and more direct.

Proof. Eq. (3) implies that $a_t = z_t - Hx_t - Du_t$. Substituting this expression in (2) yields:

$$x_{t+1} = (\Phi - EH)x_t + (\Gamma - ED)u_t + Ez_t \tag{24}$$

If the initial states x_1 were known, all the sequence of FIS states would be exactly determined by (24) because the inputs in the right-hand-side of this expression are contained in Ω^N. In most cases x_1 is unknown and treated as a random variable. Then the covariance of smoothed estimates, conditional on all the sample, would be:

$$P_{t|N} = (\Phi - EH)^{t-1} P_{1|N} [(\Phi - EH)^{t-1}]^T \tag{25}$$

and the eigenvalues of $\Phi - EH$ coincide with the reciprocal roots of the MA terms, see *e.g.* (11) and (12)-(13). Then if the model is invertible, *i.e.*, if the eigenvalues of $\Phi - EH$ are inside the unit circle, $P_{t|N}$ converges to zero at an exponential rate as t increases. If the MA polynomial has roots on the unit circle, the convergence of $\mathrm{tr}(P_{t|N})$ to zero is more complex. ■

Result 3 implies that the states in an innovations model are observable, *i.e.*, its FIS estimates

converge to exact values. Therefore, the FIS states and the empirical structural components, which are linear combinations of these states, can be treated as actual data and interpreted separately, as they are mutually independent. Also Result 3 assures that, after convergence to null covariances has been achieved, the empirical structural components will not be revised, as FIS revisions are proportional to the uncertainty affecting the estimates, see Casals *et al.* (2000).

The following example illustrates the different modes of convergence implied by Result 3.

Example. Consider the models:

$$(1 - 1.5B + .5B^2)z_t = a_t\,,\ \sigma_a^2 = .1 \tag{26}$$

$$(1 - 1.5B + .5B^2)z_t = (1 - .8B)a_t\,,\ \sigma_a^2 = .1 \tag{27}$$

Figure 1.a. shows the trace of FIS covariance of the states in the SS representation of (26)-(27), computed with 100 simulated observations. Note that:

1) When the model has a pure autoregressive structure, as in (26), the trace becomes zero after processing a number of observations equal to the order of the process.

2) If the model has invertible moving average terms, as in (27), $\mathrm{tr}(P_{t|N})$ converges exponentially to zero as $t \to N$. The rate of convergence is governed by the decay of the coefficients in the corresponding autoregressive representation. Then, if the moving average factor is close to noninvertibility, convergence will be slower, and vice versa.

The fact that models (26) and (27) have an AR unit root does not affect the result, but unit roots in the MA factors imply a weaker convergence property. For example, if the model were:

$$(1 + .3B - .5B^2)z_t = (1 - B)a_t\,,\ \sigma_a^2 = .1 \tag{28}$$

the trace of the FIS covariance would be as shown in Figure 1.b, for 100 and 200 observations. Then, when the model is noninvertible, the trace of $P_{t|N}$ converges to a small value for each sample size, and this value tends to zero when the sample increases. Then, convergence of $\mathrm{tr}(P_{t|N})$ to zero is assured when $t \to \infty$.

[Insert Figure 1]

The following corollaries provide further insight into the implications of Result 3.

Corollary 3.1. The Kalman filter covariance of an innovations model, $P_{t|t-1}$, converges to zero.

Proof. The proof is identical to that of Result 3, replacing the subindex N in (25) by t-1. ■

Corollary 3.2. In an innovations model the FIS and Kalman filter estimates of the states converge to the same values, so that $\|x_{t|N} - x_{t|t-1}\| \underset{t\to\infty}{\to} 0$, being $\| \;\|$ a vector norm.

Proof. Immediate from Result 3 and Corollary 3.1. ■

Therefore, the one-sided Kalman filter estimates also converge to exact values (Corollary 3.1) and, consequently, to the FIS estimates of the states (Corollary 3.2). These properties allow for computationally efficient implementations of the structural decomposition, as the cost of Kalman filtering is much smaller than the cost of FIS.

4. Structure of the method.

Building on the ideas discussed in Sections 2 and 3, the structural decomposition of a vector of time series, z_t, can be organized in the following steps:

Step 1) Obtain an adequate representation for z_t and the equivalent innovations model, using Results 1.a or 1.b.

Step 2) According to Result 2, obtain the equivalent block-diagonal innovations model.

Step 3) Compute FIS estimates of the states, $\bar{x}_{t|N}$, and the corresponding covariances, $P_{t|N}$, which by Result 3 converge to zero.

Step 4) Classify the different states according to their eigenvalues and compute estimates of the trend, cycle and seasonal components by combining the FIS estimates of the states with the corresponding coefficients in the observation equation, see (19)-(21).

Step 5) Compute the instantaneous effect of the exogenous variables as indicated in (22).

Step 6) Compute FIS estimates of the irregular component as $a_{t|N} = z_t - \bar{H}\bar{x}_{t|N} - Du_t$, see (15) and (23).

To illustrate the practical application of this methodology, Sections 5 and 6 present two examples.

The first example consists of a worst-case comparison of our method with STSM and AMB decompositions, using simulated data. To this end, we simulate an stochastic process which can be interpreted both, as a STSM and an AMB decomposition. The components are then extracted using the direct SS representation and the equivalent innovations model and compared in different aspects. In this comparison, our method is in deliberate disadvantage, as the data generating process makes explicit certain variance proportionality constraints which get lost in the innovations representation. Despite this fact, the components obtained by both methodologies result very similar overall and our components have some advantages derived from their convergence to exact values. Also, it is shown that the informational disadvantage can be compensated by means of an *ex-post* smoothing of some components.

In the second example we illustrate the flexibility and multivariate capacities of our method by modeling two real time series. The empirical analysis suggests the existence of a common trend and a common harmonic cycle. In our framework, the extraction and separate analysis of the common and specific components in both series results straightforward. Also, the presence of some missing data in the sample results only a minor inconvenient.

5. An example with simulated data: comparison between the components implied by a structural time series model and the corresponding innovations model.

Consider the data generating process defined in Table 2:

[Insert Table 2]

Following the terminology of Harvey (1989), the trend component follows a particular case of the *stochastic trend model*, implying that the trend in $t+1$ is equal to the trend in t plus a random walk derivative of the trend, Δ_t. The seasonal component follows a quarterly *dummy variable seasonality model*, where the sum of the seasonal components over a year is a random disturbance. The error terms η_t, ω_t and e_t are gaussian white noise processes, with an instantaneous covariance matrix:

$$\mathrm{COV}\begin{bmatrix}\eta_t\\ \omega_t\\ e_t\end{bmatrix}=\begin{bmatrix}1/1600 & 0 & 0\\ 0 & .1 & 0\\ 0 & 0 & 1\end{bmatrix} \tag{29}$$

note that the noise-variance-ratio, $\sigma_\eta^2/\sigma_e^2 = 1/1600$, coincides with the value assumed in the HP filter for quarterly data.

In the second column of Table 2 the model is interpreted as an AMB decomposition, where the observed time series follows an $\mathrm{ARIMA}(0,1,5)\times(0,1,0)_4$ model (very close to a standard airline model), the trend follows an integrated random-walk and the seasonal component follows an AR(3) process, with all its roots on the unit circle.

Table 3 shows the SS representation of the data generating process. Note that this model is not in innovations form, as the shocks affecting the state variables are different from the observation error.

[Insert Table 3]

Using Result 1.1, this model can be written in an equivalent innovations form, and then block-diagonalized using Result 2. The representation obtained in this way is shown in Table 4.

[Insert Table 4]

Note that: a) the states are now clearly separated in different (1×1) and (2×2) blocks, and b) the eigenvalues of the transition matrix are identical to those in Table 3. According to their frequencies, the first and second states correspond to the trend and the states third to fifth correspond to the seasonal

component. Taking into account this classification of the states and the coefficients in the last row of Table 4, the structural components are given by:

$$t_t = \bar{x}_{1t} \tag{30}$$

$$s_t = .619\bar{x}_{3t} - .342\bar{x}_{4t} - .577\bar{x}_{5t} \tag{31}$$

$$\varepsilon_t = z_t - t_t - s_t \tag{32}$$

Combining the SS model in Table 4 with (30)-(32), we can obtain the ARIMA models for these components. Table 5 compares them with the corresponding ARIMA representations of the data generating process.

[Insert Table 5]

Note that both sets of models have the same autoregressive structures, but different error terms. While the components of the data generating process follow pure autoregressive processes, the components implied by the innovations model receive shocks from the delayed irregular component, affected by different moving average polynomials and scale parameters.

5.1. Comparison between the properties of the empirical components.

We now generate 200 random draws from the data generating process, see Appendix A, and compute the FIS estimates of the components using both, the direct SS formulation in Table 3 and the equivalent innovations model in Table 4. Figure 2 shows that the trend and seasonal components obtained in both cases are very similar. The irregular components have different volatilities as expected, see Table 5.

[Insert Figure 2]

Despite their overall similarity, the statistical properties of these components are very different. Figure 3 compares the FIS variances of the trend and seasonal components. Note that the variances of the STSM components display the "U-shape" characteristic of symmetric filters, meaning that the estimates at the center of the sample are more precise than those at the extremes. When the decomposition is applied to obtain seasonally adjusted data this fact is very important, because an increase in the sample generates a revision effect proportional to the uncertainty of the components. On the other hand, the components implied by the innovations model converge to deterministic values within the sample, as was stated in Result 3. Therefore, its most recent values are not affected by future values of the time series.

[Insert Figure 3]

Figure 4 shows the sample autocorrelation functions of the second-order differences of the FIS trends. Note that the autocorrelation function corresponding to the STSM trend is not white noise, despite the theoretical model for this component, see Table 5. In comparison, the autocorrelations of the trend implied by the innovations model are coherent with its ARIMA(0,2,1) model.

[Insert Figure 4]

The second difference of the empirical STSM trend displays a (perhaps nonstantionary) second-order autoregressive structure. Therefore, the level of this variable has at least a fourth-order AR structure, with two or even three roots on the unit circle. This fact suggests some mutual contamination between the trend and quarterly components. As a matter of fact this was to be expected because, despite the independence constraint in (29), the FIS estimates of the components in the STSM are correlated. For example, the instantaneous correlations between the trend and seasonal states in the STSM, computed from the FIS covariance matrices, vary between ±.20 at the beginning and at the end of the sample, being almost null at the middle. In comparison, the empirical components resulting from innovations model converge to independent values and, therefore, its properties correspond neatly with those of the theoretical models.

5.2. Comparison between the trend derivatives.

Structural decompositions are often used to measure the turning points in the trend, see *e.g.*, García-Ferrer and Queralt (1998). Figure 5.a compares the derivatives of the trend resulting from the STSM model and the innovations model. The latter is more volatile because the "smoothness" of the trend results from the ratios between the variances in (29), which are explicit in the data generating process but not in the innovations model. Therefore, the trend derivative obtained directly from the innovations model is useless for this purpose.

If smoothness of the trend is relevant for the analysis, it can be achieved by applying the HP filter to the seasonally adjusted data obtained from the innovations model. In this case this is coherent with the theoretical model, because the variances in (29) were chosen according to the assumptions of this filter. Figure 5.b shows that the derivatives of the trend resulting from the STSM and the *ex-post* smoothed trend are practically undistinguishable. In fact, the absolute differences between both derivatives range between 10^{-1} and 10^{-4}.

[Insert Figures 5.a and 5.b]

6. An example with real data: common features in wheat prices.

The degree of market integration in different periods has always attracted the interest of quantitative economic historians. Many of their works share two common features: a) they focus in grain markets, due to their importance in the pre-industrial economies, and b) are based in the statistical analysis of time series of prices for different grains, measured in several locations.

In this example we analyze the yearly series of prices of the wheat sold in the *Monasterio de Sandoval* (S_t) and in the *Fábrica de la iglesia de Alaraz* (A_t), between 1691 and 1788 (N=98), in reales per fanega. The second series has six missing values due to discontinuities in the source, see Appendix B. The goals of the analysis are: a) testing if the prices in these locations display common statistical features, see Engle and Kozicki (1993) and, if found, b) apply our method to estimate them.

After a standard analysis, the following univariate models were fitted to the log-transformed series and to the difference of (log) prices:

$$\underset{(.098)}{(1-.049B}\underset{(.098)}{+.289B^2)}\nabla\log S_t=\hat{a}_t^S\ ;\ \hat{\sigma}_S^2=.077\ ;\ Q(10)=5.55\ ; p=4.12,\ d=.54 \tag{33}$$

$$\underset{(.100)}{(1-.004B}\underset{(.100)}{+.343B^2)}\nabla\log A_t=\hat{a}_t^A\ ;\ \hat{\sigma}_A^2=.113\ ;\ Q(10)=12.25\ ; p=3.99,\ d=.59 \tag{34}$$

$$\underset{(.102)}{(1-.280B)}(\log A_t-\log S_t)=\underset{(.031)}{.107}+\hat{a}_t\ ;\ \hat{\sigma}^2=.067\ ;\ Q(10)=6.81 \tag{35}$$

where the figures in parentheses are the standard deviations of the estimates; $Q(10)$ is the Ljung-Box Q statistic, computed with the first 10 lags of the sample autocorrelation function; and the values p, d are, respectively, the period and damping factor of the pseudo-cycle implied by the corresponding AR(2) factor. Due to the presence of missing values, the estimates in (34) and (35) were obtained using SS techniques, see Kohn and Ansley (1986) and Terceiro (1990, Chapter 5).

Then, single log prices are adequately represented by ARIMA(2,1,0) processes, implying that the series are nonstationary and have harmonic pseudo-cycles with a period of roughly four years. Also, the model for the difference of log prices is a stationary AR(1) with a constant term, meaning that both series share a common trend and a common harmonic cycle. The presence of these cofeatures supports the idea that grain markets were substantially integrated in this historical period.

Further evidence can be found by building a multivariate model for the series, constrained to represent these cofeatures. A bivariate analysis according to the methodology of Jenkins and Alavi (1981) provided the following model:

$$\begin{bmatrix} 1-.231B & 0 \\ (.089) & \\ 0 & (1-.026B+.247B^2)(1-B) \\ & (.086)\quad (.082) \end{bmatrix} \begin{bmatrix} z_{1t} \\ z_{2t} \end{bmatrix} = \begin{bmatrix} .110 \\ (.026) \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ -.619B & 1 \\ (.090) & \end{bmatrix} \begin{bmatrix} \hat{a}_{1t} \\ \hat{a}_{2t} \end{bmatrix} \quad (36)$$

$$\hat{\Sigma}_a = \begin{bmatrix} .068 & -.003 \\ = & .052 \end{bmatrix} \;;\; Q(10) = \begin{bmatrix} 6.99 & 11.06 \\ 9.18 & 9.93 \end{bmatrix} \quad (37)$$

where z_t is defined as:

$$\begin{bmatrix} z_{1t} \\ z_{2t} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \log A_t \\ \log S_t \end{bmatrix} \quad (38)$$

and $Q(10)$ is a matrix of Ljung-Box statistics for the residual sample autocorrelations and cross-correlations. Table 6 summarizes the structure of model (36)-(37) in block-diagonal innovations form. According to the eigenvalues of the transition matrix, the first state correspond to the trend, the second state corresponds to a cyclical movement and the third and fourth states correspond to a harmonic pseudo-cycle.

[Insert Table 6]

The next step consists of obtaining the FIS estimates for the states. Figure 6 represents the trace of their FIS covariance matrices. Note that the trace drops quickly to zero, with transitory peaks of uncertainty due to the effect of the missing observations.

[Insert Figure 6]

Obtaining the structural decomposition of z_t is now trivial, taking into account the observation equation in Table 6. However, the FIS results can be combined in a more expressive form. By (38) the variables in z_t are related with the log-prices by:

$$\begin{bmatrix} \log A_t \\ \log S_t \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} z_{1t} \\ z_{2t} \end{bmatrix} \quad (39)$$

therefore, pre-multiplying the observation equation by:

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad (40)$$

we obtain the following decomposition of the log prices:

$$\begin{bmatrix} \log A_t \\ \log S_t \end{bmatrix} = \begin{bmatrix} .971 & 1 & 1.106 & -.391 \\ .971 & 0 & 1.106 & -.391 \end{bmatrix} \begin{bmatrix} \bar{x}_{1t} \\ \bar{x}_{2t} \\ \bar{x}_{3t} \\ \bar{x}_{4t} \end{bmatrix} + \begin{bmatrix} .110 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{a}_{1t} \\ \hat{a}_{2t} \end{bmatrix} \quad (41)$$

which yields the structural components:

$$t_t = .971\bar{x}_{1t} \quad (42)$$

$$c_t = 1.106\bar{x}_{3t} - .391\bar{x}_{4t} \quad (43)$$

$$c_t^A = \bar{x}_{2t} + .110 \quad (44)$$

$$\varepsilon_t^A = \hat{a}_{1t} + \hat{a}_{2t} \quad (45)$$

$$\varepsilon_t^S = \hat{a}_{2t} \quad (46)$$

According to (41), t_t is the common trend, c_t is the common cycle and c_t^A is a stationary component including all the individual features of the log price in Alaraz. A strict application of the decomposition defined in Section 2 would imply computing a separate component for the constant term, $d_t = .110$. Obviously this would be a meaningless complication, so we decided to add a constant to the cyclical component in (44). Last, the white noise terms, ε_t^A and ε_t^S are, respectively, the irregular components affecting the series of Alaraz and Sandoval, respectively. The FIS estimates of all these components are shown in Figures 7 and 8.

[Insert Figures 7 and 8]

Finally, according to (45)-(46), the covariance matrix of the irregular components is:

$$cov\begin{pmatrix} \varepsilon_t^A \\ \varepsilon_t^S \end{pmatrix} = \begin{bmatrix} .114 & .049 \\ .049 & .052 \end{bmatrix} \quad (47)$$

implying an instantaneous cross-correlation of .636. This high value further confirms that there could be a substantial degree of market integration.

After decomposing the series, the exercise could continue by looking for exogenous variables causing the cofeatures. For example, a climatological indicator could explain part of the common trend and/or the common cycle.

7. Concluding remarks.

The structural decomposition proposed in this paper improves existing methods in several aspects.

First, it enforces consistency between the properties of a time series and those of the structural components, as model-based methods do, but avoids the *ad-hoc* identification constraints required by the AMB or the STSM approaches, as it only requires a reduced-form econometric model for the data.

Second, the signal extraction algorithm employed is a symmetric FIS. Therefore, the past and future sample information is efficiently processed when computing within-the-sample empirical components.

Third, the method guarantees convergence of the empirical components to certain values, assuring coherence between the properties of the theoretical and empirical components, providing a rigorous statistical foundation for using the empirical components as observed time series and avoiding revisions of the empirical components when the sample increases. Obviously, these properties are in practice conditional on the model for the observed time series. Changes in the formulation of the model or in its parameters affect both the theoretical and empirical components.

Fourth, it easily accommodates multivariate time series and constraints upon the structural components, such as cointegration or other cofeatures, see the example in Section 6.

Last, our method is independent of particular model-building techniques or model specifications. The only requirement is that the model for the data should have an equivalent linear and time-invariant SS representation. Therefore it can be applied to many common stochastic processes such as ARIMA, VARMAX, univariate transfer functions or periodic VARMAX, using Result 1.2, and also to STSM or models with errors-in-variables, using Result 1.1. The example in Section 5 illustrates clearly this point, as the data generating process is an STSM, which has equivalent innovations and ARMA representations.

In applications not reported here, we found our decomposition to be useful in testing for common dynamic components between seasonal time series and forecasting a time series with long-run constraints on its components.

Acknowledgments.

We thank Enrique Llopis for providing the time series listed in Appendix B. Arthur B. Treadway, Jaime Terceiro, José Luis Gallego and Marcos Bujosa read early drafts of this paper and made many useful suggestions. All errors are our responsibility.

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Table 1. Correspondence between the state variables and the structural components.

eig($\bar{\Phi}$)		**Spectral peaks at**	**Structural component**
Real	$\lambda_j = 1$	$f_j = 0$	Trend
	$\lambda_j = a_j,\ -1 \le a_j < 1$	$f_j = 0$ if $0 < a_j < 1$	Cycle
		If $a_j = 0$, then x_j is a redundant state, which has no effect on the structural components	
		$f_j = 1/2$ if $-1 \le a_j < 0$	Seasonal if $1/2 \in F_S$, Cycle otherwise
Complex	$\lambda_{j,k} = a_j \pm b_j i$	$f_{j,k} = (2\pi)^{-1} \arctan(b_j / a_j)$	Seasonal if $f_{j,k} \in F_S$, Cycle otherwise

Table 2: Definition of the data generating process.

Component	**STSM representation**	**ARIMA representation**
Trend	$T_{t+1} = T_t + \Delta_t$ $\Delta_{t+1} = \Delta_t + \eta_t$	$(1 - B)^2 T_{t+1} = \eta_t$
Seasonal	$(1 + B + B^2 + B^3) S_{t+1} = \omega_t$	$(1 + B + B^2 + B^3) S_{t+1} = \omega_t$
Time series	$z_t = T_t + S_t + e_t$	$(1 - B)(1 - B^4) z_t$ $= (1 - .933B + .091B^2 - .047B^3 - .585B^4 + .548B^5) a_t$ $\approx (1 - .933B)(1 - .585B^4) a_t$; $a_t \sim \text{nid}(0, 1.824)$

Table 3. Structure of data generating process in SS form.

	Inputs								
	T_t	Δ_t	S_t	S_{t-1}	S_{t-2}	η_t	ω_t	ε_t	eig(Φ)
Outputs	$\boldsymbol{\Phi}$					$\boldsymbol{E}$		—	
T_{t+1}	1	1	0	0	0	0	0	—	1
Δ_{t+1}	0	1	0	0	0	1	0	—	1
S_{t+1}	0	0	-1	-1	-1	0	1	—	$\pm i$
S_t	0	0	1	0	0	0	0	—	
S_{t-1}	0	0	0	1	0	0	0	—	-1
	$\boldsymbol{H}$					—		$\boldsymbol{C}$	
z_t	1	0	1	0	0	—	—	1	

Table 4. Structure of the data generating process in block-diagonal innovations form.

	Inputs								
	$\bar{x}_{1t}$	$\bar{x}_{2t}$	$\bar{x}_{3t}$	$\bar{x}_{4t}$	$\bar{x}_{5t}$	a_t	eig($\bar{\Phi}$)	Freq. (f_j)	Component
Outputs			$\bar{\Phi}$			$\bar{E}$			
$\bar{x}_{1t+1}$	1.000	1.000	0	0	0	.188	1	0	Trend
$\bar{x}_{2t+1}$	.000	1.000	0	0	0	.019	1	0	
$\bar{x}_{3t+1}$	0	0	.489	1.461	0	-.070	$\pm i$	.25	Seasonal
$\bar{x}_{4t+1}$	0	0	-.848	-.489	0	-.116			
$\bar{x}_{5t+1}$	0	0	0	0	-1	.203	-1	.5	
			$\bar{H}$			—			
z_t	1.000	.000	.619	-.342	-.577	1			

Table 5: Comparison between the models for the structural components in ARIMA notation.

Component	**Data generating process**	**Innovations model**
Trend	$(1-B)^2 T_t = \eta_{t-1}$ $\eta_{t-1} \sim \text{nid}(0, 1/1600)$	$(1-B)^2 t_t = (1-.899B).188\,\varepsilon_{t-1}$
Seasonal	$(1+B+B^2+B^3)S_t = \omega_{t-1}$ $\omega_{t-1} \sim \text{nid}(0, .1)$	$(1+B+B^2+B^3)s_t$ $= (1+1.402B+2.347B^2)(-.120)\varepsilon_{t-1}$
Irregular	$e_t \sim \text{nid}(0, 1)$	$\varepsilon_t \sim \text{nid}(0, 1.824)$

Table 6. Structure of model (36) in block-diagonal innovations form.

Outputs	Inputs							$\mathrm{eig}(\bar{\Phi})$	Component
	$\bar{x}_{1t}$	$\bar{x}_{2t}$	$\bar{x}_{3t}$	$\bar{x}_{4t}$	$u_t = 1$	$\hat{a}_{1t}$	$\hat{a}_{2t}$		
	$\bar{\Phi}$				$\bar{\Gamma}$	$\bar{E}$			
$\bar{x}_{1t+1}$	1	0	0	0	0	.522	.844	1	Trend
$\bar{x}_{2t+1}$	0	.231	0	0	.026	.231	0	.231	Cycle
$\bar{x}_{3t+1}$	0	0	.333	-.866	0	.052	.223	$.013 \pm .496i$	Cycle (harmonic)
$\bar{x}_{4t+1}$	0	0	.403	-.307	0	-.140	.103		
	$\bar{H}$				D	—			
z_{1t}	0	1	0	0	.110	1	0		
z_{2t}	.971	0	1.106	-.391	0	0	1		

Figure 1.a. Trace of $P_{t|N}$ for models (26)-(27).

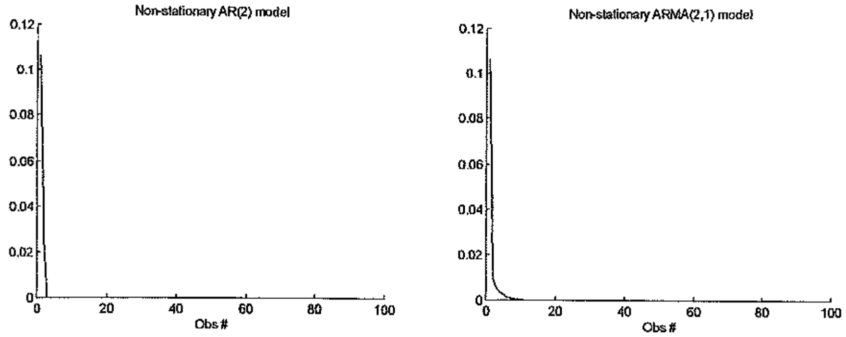


Figure 1.b. Trace of $P_{t|N}$ for model (28).

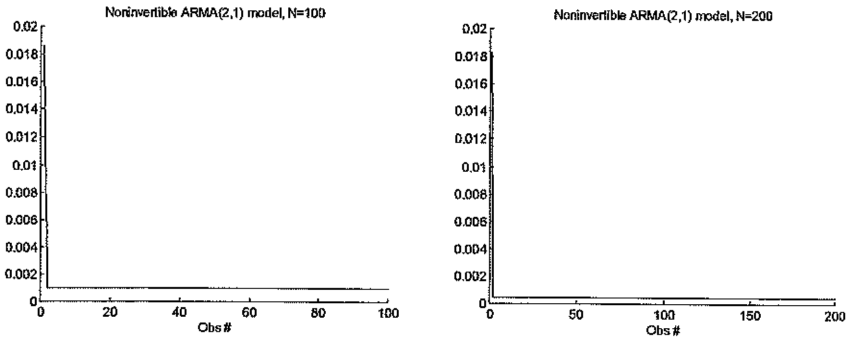


Figure 2. Comparison between the components obtained from the STSM and the innovations model.

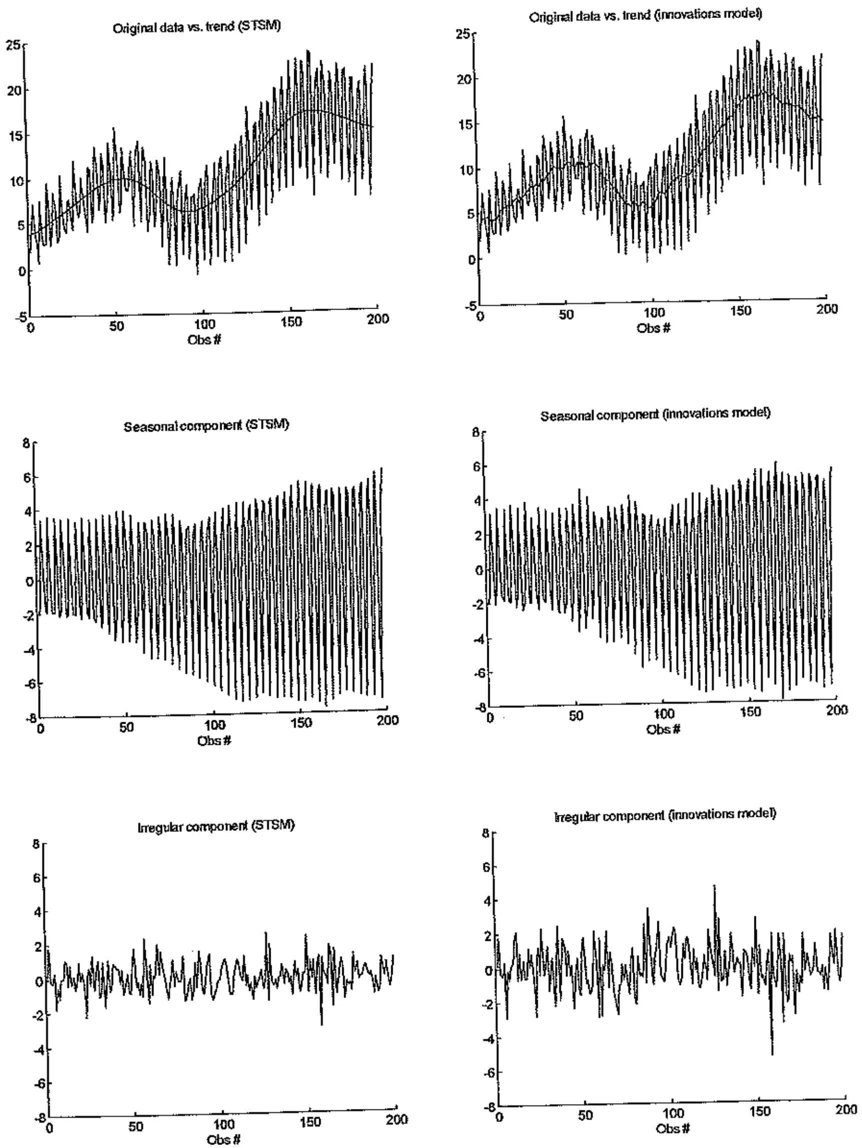


Figure 3. Comparison between the variances of the components implied by the STSM ("+") and the innovations model (continuous line).

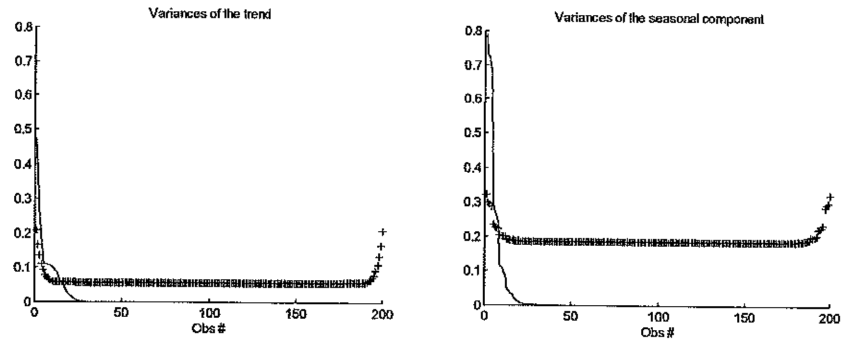
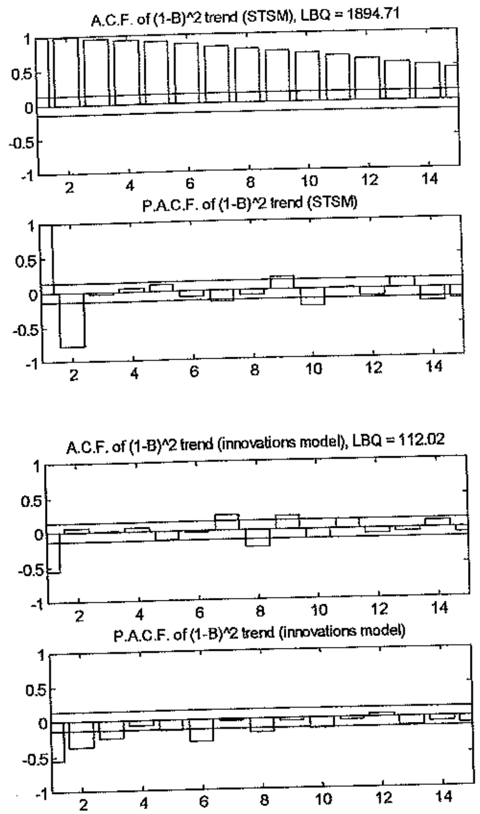


Figure 4. Autocorrelation analysis of the second-order differences of the trends implied by the STSM and the innovations model†.



† The value "LBQ" appearing at the header of each plot is the Ljung-Box Q statistic computed with the first 15 lags of the sample autocorrelation function.

Figure 5.a. Comparison between the derivatives of the trend implied by the STSM ("+") and the innovations model (continuous line).

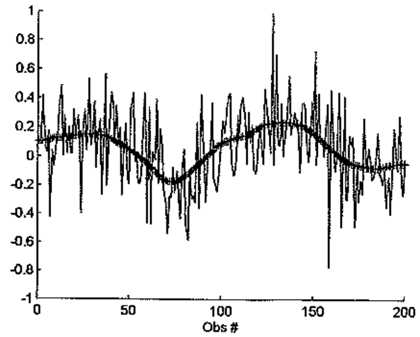


Figure 5.b Comparison between the derivatives of the trend after the *ex-post* smoothing.

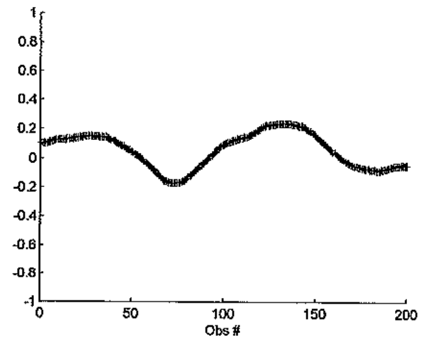


Figure 6. Trace of the FIS covariance of the states.

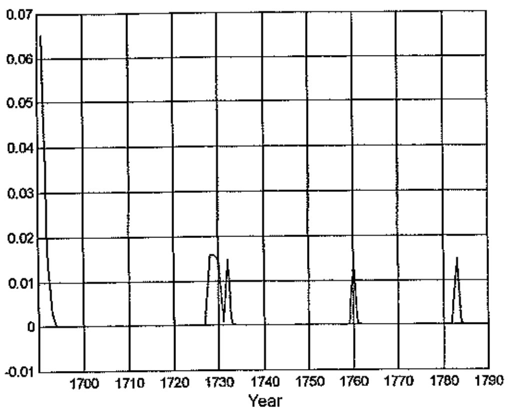


Figure 7. Common features of wheat prices (in logs).

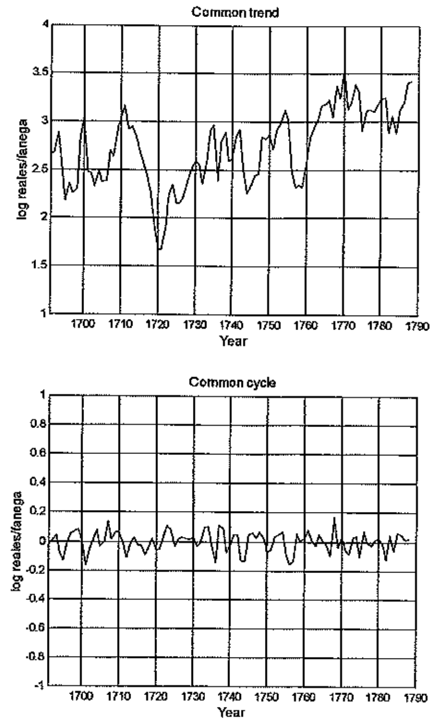
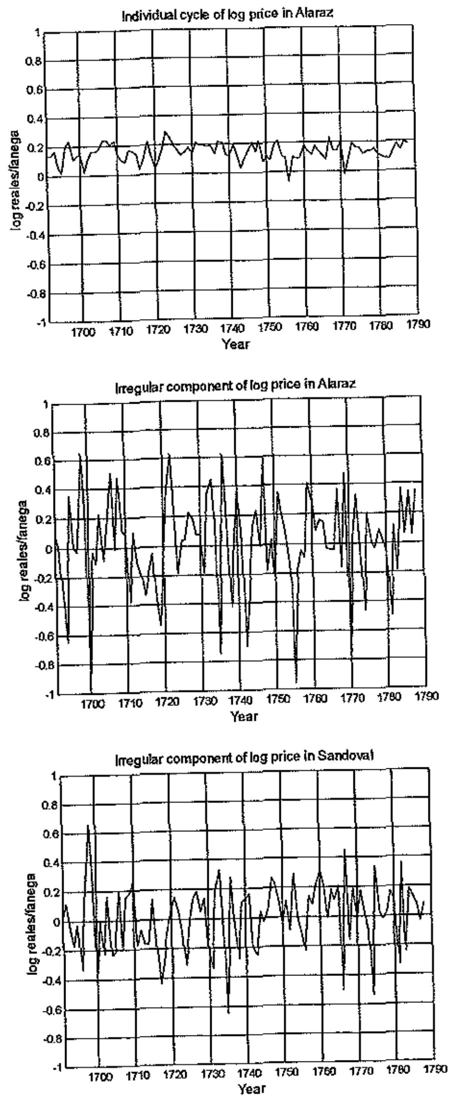


Figure 8. Individual features of (log) wheat prices.



Appendix A: Simulated data used in Section 5.

Obs.	z_t	Obs.	z_t	Obs.	z_t	Obs.	z_t	Obs.	z_t
1	1.9839	41	6.5252	81	0.4915	121	1.5334	161	9.5232
2	4.1365	42	7.8095	82	6.0958	122	8.5688	162	13.8117
3	7.2564	43	12.7802	83	10.0224	123	13.5509	163	23.6179
4	4.0816	44	11.6161	84	7.8136	124	14.4775	164	23.3728
5	2.7511	45	5.1148	85	0.2390	125	2.4660	165	7.5703
6	0.6828	46	7.4685	86	7.2630	126	9.1358	166	16.3808
7	7.5726	47	13.8770	87	8.5484	127	17.6577	167	20.6114
8	3.6843	48	10.4788	88	10.7379	128	13.3015	168	21.5621
9	2.7747	49	5.2899	89	1.4390	129	6.2545	169	10.0252
10	2.8758	50	9.1778	90	5.4900	130	9.6068	170	15.5322
11	9.5626	51	15.5739	91	7.8018	131	15.4648	171	22.4384
12	6.5804	52	10.9532	92	9.1682	132	15.8465	172	20.0040
13	2.6921	53	6.1788	93	0.9772	133	4.3906	173	9.4084
14	4.2375	54	7.3731	94	7.7581	134	10.6376	174	13.5416
15	8.5775	55	13.3941	95	7.7703	135	16.9306	175	21.2057
16	6.3674	56	10.6307	96	8.1144	136	18.0164	176	20.1215
17	2.9625	57	8.6376	97	-0.6658	137	5.5506	177	10.9987
18	3.6781	58	8.6398	98	5.7581	138	11.8807	178	13.7982
19	10.4649	59	11.7467	99	8.8322	139	18.1150	179	22.0740
20	6.6928	60	12.9966	100	9.8072	140	17.9973	180	20.7452
21	4.6681	61	4.6345	101	0.8696	141	6.9070	181	9.2546
22	3.9816	62	8.7030	102	7.6816	142	11.0437	182	13.3336
23	7.6172	63	13.0195	103	10.5368	143	19.6207	183	21.2908
24	7.6901	64	14.1113	104	11.3036	144	19.4881	184	21.5541
25	5.5549	65	5.1189	105	0.2456	145	8.2034	185	9.0118
26	4.3479	66	9.7714	106	6.1984	146	12.0909	186	13.0161
27	12.0189	67	13.4173	107	8.9129	147	18.7055	187	20.4928
28	7.4748	68	11.7304	108	11.5264	148	20.7599	188	20.4763
29	5.7602	69	4.0712	109	1.1415	149	8.5824	189	7.3662
30	6.4299	70	6.3484	110	8.7756	150	16.3149	190	12.6438
31	10.0989	71	11.7760	111	11.4379	151	18.8405	191	18.7619
32	8.1723	72	10.8252	112	12.1508	152	22.4995	192	20.0436
33	6.8097	73	4.6275	113	0.6590	153	8.8153	193	9.2562
34	4.0742	74	6.5335	114	9.5043	154	15.1631	194	12.8691
35	11.2922	75	11.9646	115	10.7862	155	19.4532	195	19.3712
36	11.0120	76	9.8638	116	12.8343	156	22.7587	196	21.7131
37	5.0782	77	2.3814	117	0.5977	157	10.7683	197	7.4288
38	6.0249	78	6.6286	118	8.2299	158	11.0380	198	11.4007
39	13.4450	79	12.2037	119	10.9629	159	22.5336	199	19.2632
40	11.3456	80	8.0208	120	13.4723	160	22.4050	200	21.9119

Appendix B: Average price (in reales per fanega) of the wheat sold in the *Monasterio de Sandoval* (S_t) and the *Fábrica de la iglesia de Alaraz* (A_t)†.

Year	S_t	A_t	Year	S_t	A_t	Year	S_t	A_t
1691	14.88	16.33	1727	11.56	NA	1763	24.14	28.00
1692	17.67	22.32	1728	11.86	NA	1764	28.00	28.00
1693	15.38	12.69	1729	14.33	NA	1765	29.87	26.00
1694	10.17	6.76	1730	10.81	17.00	1766	15.00	25.56
1695	7.50	11.26	1731	8.87	NA	1767	30.10	35.00
1696	7.55	13.00	1732	12.42	18.00	1768	29.30	34.00
1697	12.33	12.30	1733	20.42	28.00	1769	30.00	46.00
1698	20.81	23.00	1734	19.06	28.00	1770	35.25	19.92
1699	23.63	28.00	1735	10.00	11.00	1771	25.75	22.14
1700	14.66	10.00	1736	12.37	20.00	1772	23.58	33.92
1701	10.00	10.00	1737	18.00	28.00	1773	27.58	34.00
1702	8.75	11.00	1738	15.00	16.00	1774	16.75	21.00
1703	12.10	15.18	1739	13.83	14.00	1775	23.25	23.58
1704	10.37	14.08	1740	15.25	22.00	1776	25.00	27.41
1705	8.39	14.52	1741	21.40	23.00	1777	22.50	25.09
1706	13.22	22.92	1742	15.81	11.00	1778	22.50	27.30
1707	14.08	21.37	1743	8.25	8.00	1779	28.37	28.70
1708	16.50	28.00	1744	8.83	10.58	1780	29.25	27.91
1709	24.10	28.00	1745	10.55	16.00	1781	18.00	16.76
1710	29.00	28.00	1746	13.00	15.00	1782	22.75	NA
1711	20.00	18.00	1747	15.68	25.00	1783	17.37	19.78
1712	15.45	20.00	1748	23.04	19.00	1784	20.25	28.00
1713	15.95	20.00	1749	18.91	19.00	1785	28.00	31.18
1714	14.90	17.00	1750	15.84	14.00	1786	28.00	41.87
1715	16.50	12.00	1751	15.96	22.00	1787	29.37	39.42
1716	10.87	11.00	1752	17.70	28.00	1788	33.75	52.75
1717	6.66	11.00	1753	28.00	28.00			
1718	6.96	8.00	1754	23.50	22.40			
1719	6.75	5.00	1755	17.16	8.00			
1720	5.43	5.00	1756	8.25	8.00			
1721	5.85	8.00	1757	10.25	9.50			
1722	7.40	16.00	1758	12.00	11.00			
1723	9.69	18.00	1759	12.62	NA			
1724	8.25	12.00	1760	18.93	22.00			
1725	8.25	10.50	1761	23.00	23.50			
1726	10.12	10.89	1762	19.16	25.50			

† NA: Not available.