Equivalence of Fuzzy Rationality Measures

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Abstract
The notion of fuzzy rationality can naturally be extended to fuzzy preference relations which formulate the irrationality of individual preference over fixed sets of alternatives. Since fuzzy rationality measures have been systematically introduced, it is possible to investigate on the mathematical equivalent of two or more fuzzy rationality measures and to propose new characterization theorems for such an equivalence relation.

Key words: aggregation rules, fuzzy preferences, decision making.

1 Introduction and preliminaries

In some recent scientific investigations [2, 3], a set of axioms that characterize fuzzy rationality measures has been proposed. This mathematical tools is that rationality of individuals in a fuzzy environment. Therefore, it is possible to assign to each individual a value of rationality between 0 (absolute irrationality) and 1 (absolute rationality). This value, denoted by \( \alpha \), can be assigned in many different ways, and each of these different assignments corresponds to a different criterion for measuring the irrationality of an individual. Such criteria are called fuzzy rationality measures.

In particular, in [3] is analyzed the problem of defining rationality of individuals based only on their opinions expressed over a fixed set of alternatives and in [2] such a problem is given an intuitively sound solution by introducing a collection of conditions that are satisfied by any fuzzy rationality measure. Individuals expressing their opinions over pairs of elements of a finite set of alternatives are characterized by fuzzy preference relations. The latter were introduced by Zadeh [9] in order to capture degrees of preference. Given a finite set of alternatives \( X \) a fuzzy binary preference relation is defined on a fuzzy over all pairs of the elements of \( X \times X \) so that for each pair \( (x, y) \in X \times X \) represents the strength or intensity of preferences between \( x \) and \( y \), measured in the unit interval.

Throughout this paper we will assume that these preference relations are complete, i.e.

\[
\rho(x, y) + \rho(y, x) \leq 1 \quad \forall x, y \in X.
\]

2 Fuzzy rationality measures

Many measures for individual fuzzy rationality have been proposed in the past.

In the case in which the relations are crisp, i.e., \( \rho(x, y) \in \{0, 1\} \) for all \( x \neq y \), rationality has been characterized with the absence of cycles (see [9]).

In the context of fuzzy binary preference relations, a standard assumption is that rationality may not only be measured by the transitivity of the associated binary relation but also by the degree to which the transitivity is violated (see [9] and [10]). A fuzzy measure is said to be rational if and only if it is a crisp measure. However, the property of being transitive in crisp measures is not equivalent to the property of being transitive in fuzzy measures. Therefore, the fuzzy rationality of an individual does not allow for a simple classification.

The fuzzy classification of individual data obtained by means of the rationality measures given by Montero [11]. Such a measure is initially introduced in order to formulate the problem of rational agglomeration in group decision making (see [1] for a characterization of such rational aggregation rules).

In [2] fuzzy rationality measures are formally characterized as maps of type

\[
\rho : P \rightarrow [0, 1],
\]

where \( P = \bigcup X \times X \),

and a set of conditions that any fuzzy rationality measure must satisfy is introduced. Such conditions are the following ones:

(R1) For any finite set of alternatives \( X \), \( \rho(x, y) = 1 \) for any crisp binary preference \( \rho \) which is a strict preference on \( X \).
(II.5) Given \( \mu \in \mathcal{P} \) and a permutation \( \pi : X \rightarrow X \) then
\[
\sigma(\mu^{\pi}) = \sigma(\mu).
\]
where \( \sigma(\nu(x,y)) = \nu(\pi(x),\pi(y)) \) for all \( x,y \in X \).

(II.1) For all \( \pi \in \mathcal{P}, \), \( \sigma(\mu) = \sigma(\mu) \)
where \( \sigma(\nu(x,y)) = \nu(\pi(x),\pi(y)) \) for all \( x,y \in X \).

(II.11) Let \( Y \) be a non-empty finite set of alternatives and let \( \pi \) be an arbitrary permutation of \( Y \), then we can consider a fuzzy preference \( \sigma(\pi) = \sigma(\mu) \) such that \( \sigma(\nu(x,y)) = \nu(\pi(x),\pi(y)) \) for all \( x,y \in Y \).

(II.12) Let \( \pi \in \mathcal{P} \) be fixed. Given an arbitrary ordered pair of alternatives \( (a,b) \), an arbitrary point \( (\varepsilon, \delta) \in [0,1] \times [0,1] \), and two numbers \( \varepsilon, \delta \) such that \( \varepsilon + \delta \leq 1 \), we denote by \( \mu(\pi(a), \pi(b), \varepsilon, \delta) : \mathcal{P} \rightarrow [0,1] \) the fuzzy preference relation defined as
\[
\mu(\pi(a), \pi(b), \varepsilon, \delta) = \min(1 - \varepsilon, \delta).
\]

Let \( (\varepsilon, \delta) \in [0,1] \times [0,1] \) be fixed and let us consider the fuzzy preference relation \( \mu(\pi(a), \pi(b), \varepsilon, \delta) : \mathcal{P} \rightarrow [0,1] \) defined as
\[
\mu(\pi(a), \pi(b), \varepsilon, \delta) = \min(1 - \varepsilon, \delta).
\]

Then, one of the following three properties must be verified by \( \mu(\pi(a), \pi(b), \varepsilon, \delta) : \mathcal{P} \rightarrow [0,1] \).

(II.1.1) \( \mu(\pi(a), \pi(b), \varepsilon, \delta) \) is monotone, i.e., either
\[
\mu(\pi(a), \pi(b), \varepsilon, \delta) \leq \mu(\pi(a), \pi(b), \varepsilon, \delta)
\]
for any \( \lambda \leq \mu \) or
\[
\mu(\pi(a), \pi(b), \varepsilon, \delta) \geq \mu(\pi(a), \pi(b), \varepsilon, \delta)
\]
for any \( \lambda \geq \mu \).

(II.1.2) There exists a constant \( K \) such that
\[
\mu(\pi(a), \pi(b), \varepsilon, \delta) \geq K \mu(\pi(a), \pi(b), \varepsilon, \delta)
\]
for all \( \lambda \leq \mu \) such that either \( \lambda \leq K \) or \( \lambda > K \).

(II.1.3) There exists a constant \( K \) such that
\[
\mu(\pi(a), \pi(b), \varepsilon, \delta) \geq K \mu(\pi(a), \pi(b), \varepsilon, \delta)
\]
for all \( \lambda \leq \mu \) such that either \( \lambda < K \) or \( \lambda > K \).

We then give some examples of fuzzy rationality measures.

3.1 Normal fuzzy rationality measures

(II.1) Let \( \mu \) be a binary fuzzy preference relation. We consider all values of type
\[
C = (a, b, c, d, e) \in \mathbb{R}^3
\]

that we call \( B_{\mu(a,b)} \). To a given cycle \( C = (a, b, c, d, e) \), we associate the weight
\[
W(C) = B_{\mu(a,b)}(a, b, c, d, e)
\]

where for convenience \( a_{(a,b)} = a \). We then put
\[
\mu'(a) = 1 - \max(W(C) \in C_{\mu(a,b)}).
\]
2.1 Possibilistic fuzzy rationality measures
The first example of possibilistic fuzzy rationality measure is given by fuzzy transitivity. (Eq. 23)

\[ R_{p} \left( a, b \right) = \begin{cases} 1 & \text{if } R_{p} \text{ is maximally transitive} \\ 0 & \text{otherwise} \end{cases} \]

An example of possibilistic fuzzy rationality measure which allows a fuzzy classification is given by the measure introduced by Montesinos in [8, 1].

Such a measure is computed by looking at all possible chains \( C = (a_1 \rightarrow a_2 \rightarrow \ldots \rightarrow a_n) \) of distinct alternatives and considering all cycles of type

\[ a_1 \rightarrow a_2 \rightarrow \ldots \rightarrow a_n \rightarrow a_1 \]

where \( R_{p}(a_1, a_2), R_{p}(a_2, a_3), \ldots, R_{p}(a_{n-1}, a_n), R_{p}(a_n, a_1) \) is transitive if either

- \( R_{p}(a_1, a_2) \) for all \( a_1, a_2, \ldots, a_n \) and \( A \in (a_1, a_2) \) for all \( a_1, a_2, \ldots, a_n \) and \( W \in (a_1, a_2) \) for all \( a_1, a_2, \ldots, a_n \);

or

- \( R_{p}(a_1, a_2) \) for all \( a_1, a_2, \ldots, a_n \) and \( W \in (a_1, a_2) \) for all \( a_1, a_2, \ldots, a_n \);

where a cycle is rational if it is not transitive.

Given a chain \( C = (a_1 \rightarrow a_2 \rightarrow \ldots \rightarrow a_n) \), the weight associated to \( C \) and denoted by \( A_{p}(C) \) is defined as

\[ A_{p}(C) = \sum_{i=1}^{n} d(C) \]

In particular (see [5]), \( A_{p}(C) \) verifies

\[ A_{p}(C) = 1 - \left( R_{p}(a_1, a_2) \oplus R_{p}(a_2, a_3) \oplus \ldots \oplus R_{p}(a_{n-1}, a_n) \right) \]

(31)

In view of (23), Montesinos's rationality is defined as the fuzzy property \( R_{p} : P \rightarrow \left[ 0, 1 \right] \) with

\[ A_{p}(a) > 0.5 \text{ for all } a \in A \]

2.3 Optimistic fuzzy rationality measure
We now give an example of an optimistic fuzzy rationality measure \( p_{o} \), which is introduced (Eq. 24).

We define \( p_{o} \) as

\[ p_{o}(x) = \frac{\left( \sum_{y \in X} p(x, y) \right)}{\left| X \right|} \]

where \( p(x, y) \rightarrow 0 \) for simplicity the restrictions of the fuzzy preferences \( p : X \rightarrow [0, 1] \) to \( Y \times Y \).
References


