Exceptional orthogonal polynomials and the Darboux transformation.

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Abstract. We adapt the notion of the Darboux transformation to the context of polynomial Sturm-Liouville problems. As an application, we characterize the recently described X_m Laguerre polynomials in terms of an isospectral Darboux transformation. We also show that the shape-invariance of these new polynomial families is a direct consequence of the permutability property of the Darboux-Crum transformation.

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1. Introduction

The Darboux-Crum transformation is a well-known and powerful technique in Quantum Mechanics for generating new exactly solvable potentials from known ones [12, 13]. A situation of particular interest arises when the Darboux transformation can also be applied to construct new families of orthogonal polynomials from known ones, since it is often the case that the bound states of an exactly solvable potential are polynomial after a change of independent variable and a rescaling of the wave function by a suitable non-vanishing weight function. Of course, care has to be exercised in order to characterize those cases in which the eigenfunctions obtained by this procedure do indeed give rise to orthogonal polynomial families which will in turn correspond to a well-defined Sturm-Liouville problem. In particular, once it is known that the transformed eigenfunctions are polynomial, one still has to show that they are complete in the underlying weighted L^2 space. Our purpose in this paper is to apply these ideas to generate novel families of complete orthogonal polynomial systems that are solutions of Sturm-Liouville problems, and in particular to show how the families of generalized Laguerre orthogonal polynomials recently constructed by Odake, Sasaki [23, 24, 25, 26], and Quesne [27, 28] fit into the classification principle for Darboux transformations first introduced in [12]. It is worth to stress that these new exceptional polynomial families, although solutions of a Sturm-Liouville problem, are outside the Askey-Wilson class [2].

To set our results in context, we first recall the foundational theorem of Bochner [5] which states that if an infinite sequence of polynomials $\{P_n(z)\}_{n=0}^\infty$ satisfies a second order eigenvalue equation of the form

$$p(x)P_n''(x) + q(x)P_n'(x) + r(x)P_n(x) = \lambda_n P_n(x), \qquad n = 0, 1, 2, \dots \tag{1}$$

then $p(x), q(x)$ and $r(x)$ must be polynomials of degree $2, 1$ and 0 respectively. In addition, if the $\{P_n(x)\}_{n=0}^\infty$ sequence is an orthogonal polynomial system, then it has to be (up to an affine transformation of z) one of the classical orthogonal polynomial systems of Jacobi, Laguerre or Hermite [1, 22, 9, 20, 19].

In a pair of recent papers [16, 17], we have shown that there exist complete orthogonal polynomial systems, defined by Sturm-Liouville problems, that extend beyond the classical families of orthogonal polynomials arising from Bochner's classical theorem on the characterization of Sturm-Liouville polynomial systems. What distinguishes our hypotheses from those made by Bochner is that the first eigenpolynomial of the sequence need not be of degree zero, even though the full set of eigenfunctions still forms a basis of the weighted L^2 space. The situation we considered in [17] is that of complete orthogonal polynomial systems starting in degree one. For this $m = 1$ case a full characterization of all Sturm-Liouville polynomial systems is available thanks to the classification of codimension one exceptional polynomial subspaces performed in [16]. The concept of an exceptional polynomial subspace was introduced in [14, 15]. For some recent applications of exceptional orthogonal polynomials see [21, 31].

In the present paper, we pursue this program in higher codimension by constructing explicit examples of complete systems of higher codimension using the Darboux transformation in a systematic fashion. In particular, we will construct the analogues in codimension m of the X_1 Laguerre polynomials. Our paper is organized as follows. In Section 2, we define the notion of an m-orthogonal polynomial system arising from modules of higher codimension and we introduce the notion of polynomial Sturm-Liouville problem, corresponding to the case in which the eigenfunctions of the Sturm-Liouville operator are polynomials. In Section 3, we carefully study the various cases in which the Darboux transformation preserves the polynomial character of the eigenfunctions (these are precisely the *algebraic Darboux transformations* introduced in [12]). We also show the role played by the property of shape invariance in the various factorizations that give rise to Darboux-Crum transformations. In Section 4, we apply these results to the case of the Sturm-Liouville problem defining the Laguerre polynomials and show precisely how the L1 and L2 families of codimension m Laguerre polynomials obtained in [25] fit into our general classification scheme. In Section 5, we show that these polynomials satisfy remarkable shape invariance properties that arise from the intertwining relations obtained between the partner second-order operators and iterations of the first-order operators defining the Darboux transformations. We will see that the shape invariance reported in [23] [27] follows from the shape invariance of the initial operator and the formal properties of the Darboux-Crum transformations.

The perspective taken in this paper is largely that of the formal calculus of differential operators. Previously, analytic aspects of the Darboux-Crum method for Sturm-Liouville systems were considered in [10]. Our approach is different in that we focus on algebraic properties and exact solutions. In a subsequent paper [18] we shall study the codimension m Laguerre polynomial system in a functional analytic setting by giving a spectral theoretic characterization of the codimension m Laguerre polynomial system in the context of Sturm-Liouville theory. A detailed analysis of the asymptotic properties of the zeros of these polynomials will be given. We shall also further develop some of the key formal properties of these polynomials that result from the factorization and shape invariance of their defining operators, namely their orthogonality properties, Rodrigues-type formulas and generating functions. Finally we mention that this entire analysis can also be carried out in the case of Jacobi polynomials.

2. Preliminaries

We will say that a differential operator

$$T(y) = p(x)y'' + q(x)y' + r(x)y, \quad y = y(x) \tag{2}$$

is exactly solvable by polynomials (PES) if it admits infinitely many real, polynomial eigenfunctions $y_j(x)$:

$$T(y_j) = \lambda_j y_j, \quad \deg y_j < \deg y_{j+1}, \quad \lambda_j \in \mathbb{R},\ j = 1, 2, \ldots. \tag{3}$$

Moreover, we say that a sequence of polynomials $\{y_j\}_{j=1}^{\infty}$ has codimension m if

$$\deg y_j = m + j - 1, \quad j = 1, 2, \dots \tag{4}$$

We also say that $T(y)$ is an m-PES operator if the eigenpolynomials satisfy the above condition. We say that $T(y)$ is primitive, if the eigenpolynomials do not possess a common root (real or complex).

Note: if $T(y)$ has at least 3 linearly independent polynomial eigenfunctions, then necessarily, $p(x), q(x), r(x)$ must be rational functions.

Let $I = (x_1, x_2)$ be an open interval (bounded, unbounded, or semi-bounded) and let $W dx$ be a positive measure on I with finite moments of all orders. We say that a sequence of real polynomials $\{y_j\}_{j=1}^{\infty}$ forms an *orthogonal polynomial system* (OPS for short) if the polynomials constitute an orthogonal basis of the Hilbert space $\mathrm{L}^2(I, W dx)$. If (4) holds, we speak of an m-OPS.

The following definition encapsulates the notion of a system of orthogonal polynomials defined by a second-order differential equation. Consider a boundary value problem

$$-(Py')' + Ry = \lambda W y \tag{5}$$

$$\lim_{x \to x_i^{\pm}} (Py'u - Pu'y)(x) = 0, \quad i = 1, 2, \tag{6}$$

where $P(x), W(x) > 0$ on the interval $I = (x_1, x_2)$, and where $u(x)$ is a fixed polynomial solution of (5). We speak of a polynomial Sturm-Liouville problem (PSLP) if the resulting spectral problem is self-adjoint, pure-point and if all eigenfunctions are polynomial. We speak of an m-PSLP if the eigenpolynomials satisfy (4). If $m = 0$, then we recover the classical orthogonal polynomials, the totality of which is delineated by Bochner's theorem. For $m > 0$, Bochner's theorem no longer applies and we encounter a generalized class of polynomials; we name these exceptional, or X_m polynomials.

Given a PSLP, the operator

$$T(y) = W^{-1}(Py')' - W^{-1}Ry$$

is PES. Letting $p(x), q(x), r(x)$ be the rational coefficients of $T(y)$ as in (2), we have

$$P(x) = \exp\left(\int^x q/p\right), \tag{7}$$

$$W(x) = (P/p)(x), \tag{8}$$

$$R(x) = -(rW)(x), \tag{9}$$

Hence, for a PSLP, $P(x), R(x), W(x)$ belong to the quasi-rational class[11], meaning that their logarithmic derivative is a rational function.

Conversely, given a PES operator $T(y)$ and an interval $I = (x_1, x_2)$ we formulate a PLSP (5) by employing (7)–(9) as definitions, and by adjoining the following assumptions:

(i) $P(x), W(x)$ are continuous and positive on I

(ii) $W dx$ has finite moments, i.e. $\int_I x^n W(x) dx < \infty, \quad n = 0, 1, 2, \dots$

(iii) $\lim_{x\to x_i} P(x)x^n = 0, \quad i = 1,2, \quad n = 0,1,2\dots$

(iv) the eigenpolynomials of $T(y)$ are dense in the Hilbert space $\mathrm{L}^2(I, W dx)$.

These definitions and assumptions (i) and (ii) imply Green's formula:

$$\int_{x_1}^{x_2} T(f)g\, W dx - \int_{x_1}^{x_1} T(g) f\, W dx = P(f'g - fg')\Big|_{x_1}^{x_2} \tag{10}$$

By (iii) if $f(x), g(x)$ are polynomials, then the right-hand side is zero. If f and g are *eigenpolynomials* of $T(y)$ with unequal eigenvalues, then necessarily, they are orthogonal in $\mathrm{L}^2(I, W dx)$. Finally, by (iv) the eigenpolynomials of $T(y)$ are complete in $\mathrm{L}^2(I, W dx)$, and hence satisfy the definition of an OPS.

We now describe a construction that systematically generates polynomial Sturm-Liouville systems of arbitrarily large codimension $m \geq 0$.

3. The Darboux transformation

Let $T(y)$ be a differential operator (2) with rational coefficients. We speak of a rational factorization if

$$T - \lambda_0 = BA \tag{11}$$

where $A(y), B(y)$ are first order operators with rational coefficients and where λ_0 is a constant. Let us write

$$A(y) = b(y' - wy), \tag{12}$$

$$B(y) = \hat{b}(y' - \hat{w}y), \tag{13}$$

where $w(x), \hat{w}(x), b(x), \hat{b}(x)$ are all rational functions. Given a rational factorization we introduce the partner operator

$$\hat{T} = AB + \lambda_0 \tag{14}$$

whose explicit form is

$$\hat{T}(y) = py'' + \hat{q}y' + \hat{r}y \tag{15}$$

where

$$\hat{q} = q + p' - 2pb'/b \tag{16}$$

$$\hat{r} = -p(\hat{w}' + \hat{w}^2) - \hat{q}\hat{w} + \lambda_0. \tag{17}$$

We will refer to

$$\phi(x) = \exp\left(\int^x w\right),$$

as a factorization eigenfunction (quasi-rational) and to $b(x)$ as the factorization gauge (rational). The former satisfies

$$T(\phi) = \lambda_0 \phi. \tag{18}$$

Equivalently,

$$w(x) = \phi'(x)/\phi(x) \tag{19}$$

is a rational solution of the following Ricatti-like equation:

$$p(w' + w^2) + qw + r = \lambda_0. \tag{20}$$

For a fixed $T(y)$, a rational factorization is fully determined by a quasi-rational factorization eigenfunction and a rational factorization gauge. Indeed, given p, q, r, w, b relation (11) gives us

$$\hat{b} = p/b, \tag{21}$$

$$\hat{w} = -w - q/p + b'/b. \tag{22}$$

The choice of $b(x)$ determines the gauge of the partner operator. Consider two factorization gauges $b_1(x), b_2(x)$ and let $\hat{T}_1(y), \hat{T}_2(y)$ be the corresponding partner operators. Then,

$$\hat{T}_2 = \mu^{-1}\hat{T}_1\mu,$$

where

$$\mu(x) = b_1(x)/b_2(x).$$

The above construction of the partner operator is symmetric with respect to the interchange of the hatted and unhatted variables. Letting $P(x)$ be as in (7), and setting

$$\hat{\phi}(x) = \exp\left(\int^x \hat{w}\right) = \frac{1}{(P/b)(x)\phi(x)}, \tag{23}$$

we have

$$\hat{T}(\hat{\phi}) = \lambda_0\hat{\phi}.$$

We also have

$$P/b = \hat{P}/\hat{b}, \tag{24}$$

$$\hat{b}b = p, \tag{25}$$

$$\hat{q}/p - \hat{b}'/\hat{b} = q/p - b'/b \tag{26}$$

Thus, starting with $\hat{T}$ and taking $\hat{\phi}$ as the factorization function and $\hat{b}(x)$ as the factorization gauge, we recover $T(y)$.

Next, suppose that T is a PES operator with eigenpolynomials $\{y_j\}$. If $\mu(x)$ is a polynomial, then $\mu T \mu^{-1}$ is also a PES operator, with eigenpolynomials $\{\mu y_j\}$. Therefore, we can fix the gauge of a PES operator by requiring that the eigenpolynomials are primitive (no common roots).

By construction, partner operators obey the following intertwining relations:

$$\hat{T}A = AT, \qquad B\hat{T} = TB. \tag{27}$$

Hence, if T is a PES operator with eigenpolynomials $\{y_j\}$, then $\{A(y_j)\}$ are eigenfunctions of the partner operator $\hat{T}$ with the same eigenvalues. By inspection of (12), with the appropriate choice of $b(x)$, the $A(y_j)$ are polynomials. Hence, if T is PES, then so is $\hat{T}$. Furthermore, the requirement that the eigenpolynomials of $\hat{T}$ be primitive fixes $b(x)$ up to a choice of scalar multiple. In many cases, such as the factorization shown in (67)-(69), it will suffice to take $b(x)$ to be the denominator of $w(x)$. However, there are other cases, such as the factorization shown in (128)-(131), where $b(x)$ must be a rational function.

The duality between T and $\hat{T}$ has another aspect. Let $W(x)$ be as in (8) and let $\hat{W}(x)$ be analogously defined. Hence, by equations (24) (25),

$$\hat{W} = P/b^2 = pW/b^2 \tag{28}$$

Consequently, A and $-B$ are formally adjoint relative to these measures:

$$\int_{x_1}^{x_2} A(f)g\,\hat{W}dx + \int_{x_1}^{x_2} B(g)f\,W dx = (P/b)fg\Big|_{x_1}^{x_2} \tag{29}$$

If the above RHS vanishes for polynomial f, g, then A and $-B$, with suitably defined domains, give rise to adjoint operators in the rigorous sense of densely defined linear operators on Hilbert spaces $\mathrm{L}^2(I, Wdx)$ and $\mathrm{L}^2(I, \hat{W}dx)$, respectively.

Darboux transformations can be classified into three types as far as their spectral properties are concerned [6, 30]: state-deleting, state-adding, or isospectral.

(i) **state-deleting transformation:** In this case the factorizing function $\phi(x)$ satisfies $\phi \in L^2(I, Wdx)$ and the formal factorizing eigenvalue λ_0 is the maximum‡ of the spectrum of T.

(ii) **state-adding transformation:** In this case the partner factorizing function $\hat{\phi}(x)$ satisfies $\hat{\phi} \in L^2(I, \hat{W}dx)$ and the formal factorizing eigenvalue λ_0 must be above the maximum of the spectrum of T. Equivalently, from (23) and (28) it follows that

$$\hat{\phi} \in L^2(I, \hat{W}dx) \Leftrightarrow \frac{p^{1/2}}{P}\phi^{-1} \in L^2(I, Wdx)$$

so it is clear that the spectral properties of the transformation only depend on the choice of ϕ, not on the choice of gauge $b(x)$.

(iii) **isospectral transformation:** In this case $\phi \notin L^2(I, Wdx)$, $\hat{\phi} \notin L^2(I, \hat{W}dx)$ and the formal factorizing eigenvalue λ_0 must be above the maximum of the spectrum of T.

In the context of algebraic Darboux transformations discussed in this paper, if we assume that both T and $\hat{T}$ are PSLPs, the above spectral characterization can be particularized to a purely algebraic one.

(i) A state-deleting transformation corresponds to $\phi = y_1$, the first eigenpolynomial of T.

(ii) A state-adding transformation corresponds to $\hat{\phi}$ (as defined by (23)) being a polynomial.

‡ Note that, as opposed to the usual convention in Schrödinger operators where the spectrum is bounded from below, in this paper the spectrum of all Sturm-Liouville problems is bounded from above. The eigenfunction corresponding to the maximum of the spectrum corresponds therefore to the *ground state*.

(iii) Isospectral transformations correspond to neither ϕ nor $\hat{\phi}$ being polynomials.

The above conditions can be explicitly verified on the factorizations performed in Sections 4 and 5. For example, equations (67)-(69) show an isospectral factorization; neither of the factorizing eigenfunctions is a polynomial. By contrast, equations (128)-(131) show a state-deleting/state-adding factorization; one of the factorizing eigenfunctions is a polynomial, while its partner eigenfunction is not.

State-adding and state-deleting factorizations are dual notions, in the sense that if the factorization of T is state-deleting, then the factorization of $\hat{T}$ is state-adding, and vice versa.

As we already pointed out, the eigenpolynomials $\{y_j\}$ and $\{\hat{y}_j\}$ constitute orthogonal polynomial systems relative to $\mathrm{L}^2(I, W dx)$ and $\mathrm{L}^2(I, \hat{W} dx)$, respectively. The adjoint relation between A and B allows us to compare the L^2 norms of the two families. Indeed, by (11) (3) (29),

$$\int_I (A(y_j))^2 \, \hat{W} dx = -\int_I B(A(y_j)) y_j \, W dx = (\lambda_0 - \lambda_j) \int_I y_j^2 \, W dx \tag{30}$$

3.1. Shape-invariance

Suppose that

$$T_k(y) = p(x) y'' + q_k(x) y' + r_k(x) y, \quad k \in K, \tag{31}$$

is a family of PES operators, where K is some parameter index set. If this family is closed with respect to the state-deleting Darboux transformation, we speak of *shape-invariant* operators and polynomials. To be more precise, let $\pi_k(x) = y_{k,1}(x)$ be be the corresponding ground-state eigenpolynomial. Without loss of generality, we assume that that the ground-state energy is zero. and let

$$T_k = B_k A_k, \qquad A_k(\pi_k) = 0 \tag{32}$$

be the corresponding factorization. Shape-invariance means that there exists a one-to-one map $h : K \to K$ and real constants λ_k such that

$$T_{h(k)} = A_k B_k + \lambda_k. \tag{33}$$

Necessarily, there exist constants $\alpha_{k,j}, \beta_{k,j}$ such that

$$y_{h(k),j-1} = \alpha_{k,j} A_k(y_{k,j}), \quad j \geq 1, \tag{34}$$

$$y_{k,j+1} = \beta_{k,j+1} B_k(y_{h(k),j}), \quad j \geq 0, \tag{35}$$

$$\beta_{k,j} \alpha_{k,j} = \lambda_{k,j} \tag{36}$$

$$\lambda_{h(k),j} = \lambda_{k,j+1} + \lambda_k. \tag{37}$$

In accordance with (7), define

$$P_k(x) = \exp\left(\int^x q_k/p \right) \tag{38}$$

Let $b_k(x)$ denote the shape-invariant factorization gauge; i.e.;

$$A_k(y) = (b_k/\pi_k) \mathcal{W}(\pi_k, y), \tag{39}$$

where

$$\mathcal{W}(f,g) = fg' - f'g \tag{40}$$

Equation (16) implies the following necessary condition,

$$p\,P_k/P_{h(k)} = b_k^2. \tag{41}$$

This is a rather strong constraint, because the left-hand side is a product of quasi-rational functions, while the right-hand side is a rational squared.

3.2. Covariant factorization

Next, we introduce the notion of a covariant isospectral factorization. Let $T_k(y)$ be a shape-invariant family of PES operators, as above. Suppose that $\phi_k(x)$ is an indexed family of isospectral factorization functions. Let

$$T_k = \tilde{B}_k\tilde{A}_k + \tilde{\lambda}_k, \quad \tilde{A}_k(\phi_k) = 0 \tag{42}$$

be the corresponding isospectral factorization. Let

$$\hat{T}_k = \tilde{A}_k\tilde{B}_k + \tilde{\lambda}_k, \quad \tilde{B}_k(\hat{\phi}_k) = 0 \tag{43}$$

be the partner operator and partner eigenfunction, respectively. We say that the factorization with respect to ϕ_k is covariant if

$$A_k(\phi_k) \propto \phi_{h(k)}. \tag{44}$$

The following Lemma furnishes a useful test for covariant factorization. Let us say that a PES operator is formally non-degenerate if for every formal eigenvalue (eigenfunction is quasi-rational) $\lambda_0 \in \mathbb{R}$ there exists at most one linearly independent quasi-rational eigenfunction with that eigenvalue.

Lemma 3.1 *Suppose that $\phi_k(x)$ is continuous with respect to k and formally non-degenerate for generic values of $k \in K$. Furthermore, suppose that*

$$\tilde{\lambda}_{h(k)} = \tilde{\lambda}_k + \lambda_k. \tag{45}$$

Then, the factorization with respect to ϕ_k is covariant.

Proof. By (32) (33) and (45),

$$T_{h(k)}(A_k(\phi_k)) = A_k(T_k(\phi_k)) + \lambda_k A_k(\phi_k) \tag{46}$$

$$= (\lambda_k + \tilde{\lambda}_k)A_k(\phi_k) \tag{47}$$

$$= \tilde{\lambda}_{h(k)}A_k(\phi_k) \tag{48}$$

Since ϕ_k is quasi-rational and since A_k has rational coefficients, $A_k(\phi_k)$ is also quasi-rational. Hence, (44) holds for generic k. Therefore, it holds for all k. QED

4. Laguerre polynomials

Let us introduce the PES operator

$$\mathcal{L}_k(y) := xy'' + (k+1-x)y'. \tag{49}$$

The classical associated Laguerre polynomials, $L_{k,n}(x)$ can be defined as the corresponding eigenpolynomials,

$$\mathcal{L}_k(L_{k,n}) = -nL_{k,n}, \tag{50}$$

normalized by the condition

$$L_{k,n}(x) = \frac{(-1)^n}{n!}x^n + \text{lower degree terms.}$$

The classical Laguerre polynomials are shape-invariant by virtue of the following factorizations

$$\mathcal{L}_k = B_k A_k \tag{51}$$

$$\mathcal{L}_{k+1} = A_k B_k + 1 \qquad \text{where} \tag{52}$$

$$A_k(y) = y' \tag{53}$$

$$B_k(y) = xy' + (k+1-x)y \tag{54}$$

For $k > -1$, the resulting polynomials are orthogonal relative to the weight

$$W_k(x) = x^k\, \mathrm{e}^{-x}, \quad x \in (0,\infty), \tag{55}$$

and can be realized as solutions of a spectral problem [3, 7]. The corresponding L^2 norms are given by

$$\int_0^\infty L_{k,n}^2 W_k\, dx = \Gamma(n+k+1)/n!. \tag{56}$$

The quasi-rational eigenfunctions of $\mathcal{L}_k(y)$ are known [4, Sec. 6.1]:

$$\phi_1(x) = L_{k,m}(x), \qquad \lambda_0 = -m \tag{57}$$

$$\phi_2(x) = x^{-k}L_{-k,m}(x) \qquad \lambda_0 = k-m \tag{58}$$

$$\phi_3(x) = e^x L_{k,m}(-x) \qquad \lambda_0 = k+1+m \tag{59}$$

$$\phi_4(x) = x^{-k}e^x L_{-k,m}(-x), \qquad \lambda_0 = m+1, \tag{60}$$

where $m = 0, 1, 2, \ldots$. The corresponding factorizations were analyzed in [12]. Of these, ϕ_1 with $m = 0$ corresponds to a state-deleting transformation and underlies the shape-invariance of the classical Laguerre polynomials. For $m > 0$, the ϕ_1 eigenfunctions yield singular operators and hence do not yield novel orthogonal polynomials. The ϕ_4 family results in a state-adding transformation. The resulting orthogonal polynomials do not satisfy condition (4); such factorizations were discussed in [13]. The type 2 and type 3 factorizations ϕ_2, ϕ_3 result in novel orthogonal polynomials, although for ϕ_3 it is necessary to assume that $k > m$. These families correspond, respectively, to the type L1, L2 Laguerre polynomials of [25].

Let us consider these two families of factorization on a case-by-case basis. The derivations that follow depend in an elementary fashion on the following well-known identities of the Laguerre polynomials. We will apply them below without further comment.

$$L_{k,0}(x) = 1 \tag{61}$$
$$L_{k,n}(x) = 0, \quad n \leq -1, \tag{62}$$
$$nL_{k,n}(x) + (x - 2n - k + 1)L_{k,n-1}(x) + (n + k - 1)L_{k,n-2}(x) = 0, \tag{63}$$
$$L'_{k,n}(x) = -L_{k+1,n-1}(x), \tag{64}$$
$$L_{k,n}(x) = L_{k+1,n}(x) - L_{k+1,n-1}(x). \tag{65}$$

4.1. The L1 family

Fix an integer $m \geq 0$ and a real $k > -1$. Take $\phi_3(x)$ as the factorization function and

$$\xi_{k,m}(x) = L_{k,m}(-x) \tag{66}$$

as the factorization gauge. Applying (12) (13) (21) (22), the resulting factorization is

$$\mathcal{L}_k = B^{\mathrm{I}}_{k,m} A^{\mathrm{I}}_{k,m} + k + m + 1, \quad \text{where} \tag{67}$$
$$A^{\mathrm{I}}_{k,m}(y) = \xi_{k,m} y' - \xi_{k+1,m} y \tag{68}$$
$$B^{\mathrm{I}}_{k,m}(y) = (xy' + (1 + k)y)/\xi_{k,m} \tag{69}$$

The partner eigenfunction is $\hat{\phi}(z) = z^{-1-k}$. Let us define

$$\mathcal{L}^{\mathrm{I}}_{k,m} = A^{\mathrm{I}}_{k-1,m} B^{\mathrm{I}}_{k-1,m} + k + m \tag{70}$$
$$\mathcal{L}^{\mathrm{I}}_{k,m}(y) = xy'' + (k + 1 - x)y' + my - 2\rho_{k-1,m}(xy' + ky), \quad \text{where} \tag{71}$$
$$\rho_{k,m} = \xi'_{k,m}/\xi_{k,m} = \xi_{k+1,m-1}/\xi_{k,m}. \tag{72}$$

On the basis of the above factorization, we define type I exceptional Laguerre polynomials to be

$$L^{\mathrm{I}}_{k,m,n} = -A^{\mathrm{I}}_{k-1,m}\left(L_{k-1,n-m}\right) \tag{73}$$
$$= \xi_{k,m}\, L_{k-1,n-m} + \xi_{k-1,m}\, L_{k,n-m-1}, \qquad n \geq m \tag{74}$$

By construction, these polynomials satisfy

$$\mathcal{L}^{\mathrm{I}}_{k,m}(L^{\mathrm{I}}_{k,m,n}) = (m - n)L^{\mathrm{I}}_{k,m,n}, \quad n \geq m. \tag{75}$$

By (10) and (28) the sequence $\{L^{\mathrm{I}}_{k,m,n}\}_{n=m}^{\infty}$ constitutes an m-OPS relative to the weight

$$W^{\mathrm{I}}_{k,m}(x) = x^k e^{-x}/\xi^2_{k-1,m}, \quad x \in (0, \infty) \tag{76}$$

Using (30) and (56), we obtain

$$\int_0^{\infty} (L^{\mathrm{I}}_{k,m,n})^2\, W^{\mathrm{I}}_{k,m}\, dx = (k + n)\Gamma(k + n - m)/(n - m)! \tag{77}$$

For $m = 0$ the above definitions reduce to their classical counterparts; to wit,

$$\mathcal{L}^{\mathrm{I}}_{k,0} = \mathcal{L}_k \tag{78}$$
$$L^{\mathrm{I}}_{k,0,n} = L_{k,n}, \tag{79}$$
$$W^{\mathrm{I}}_{k,0}(x) = x^k e^x \tag{80}$$

4.2. The L2 family

Fix an integer $m \geq 0$ and $k > m$, and take $\phi_2(x)$ as the factorization function. Set

$$\eta_{k,m}(x) = L_{-k,m}(x); \tag{81}$$

take $x\,\eta_{k,m}$ as the factorization gauge. The resulting factorization is

$$\mathcal{L}_k = B^{\mathrm{II}}_{k,m} A^{\mathrm{II}}_{k,m} + (k-m), \quad \text{where} \tag{82}$$

$$A^{\mathrm{II}}_{k,m}(y) = x\eta_{k,m} y' + (k-m)\eta_{k+1,m} y \tag{83}$$

$$B^{\mathrm{II}}_{k,m}(y) = (y' - y)/\eta_{k,m} \tag{84}$$

The partner eigenfunction is $\hat{\phi}(z) = e^z$. Based on this factorization, we define

$$\mathcal{L}^{\mathrm{II}}_{k,m} = A^{\mathrm{II}}_{k+1,m} B^{\mathrm{II}}_{k+1,m} + (k+1-m) \tag{85}$$

$$\mathcal{L}^{\mathrm{II}}_{k,m}(y) = xy'' + (k+1-x)y' - my + 2x\sigma_{k+1,m}(y'-y), \quad \text{where} \tag{86}$$

$$\sigma_{k,m} = -\eta'_{k,m}/\eta_{k,m} = \eta_{k-1,m-1}/\eta_{k,m} \tag{87}$$

We now define the type II X_m Laguerre polynomials to be

$$L^{\mathrm{II}}_{k,m,n} = -A^{\mathrm{II}}_{k+1,m}\left(L_{k+1,n-m}\right), \qquad n \geq m \tag{88}$$

$$= x\eta_{k+1,m}\, L_{k+2,n-m-1} + (m-k-1)\eta_{k+2,m}\, L_{k+1,n-m} \tag{89}$$

By construction, these polynomial satisfy

$$\mathcal{L}^{\mathrm{II}}_{k,m}(L^{\mathrm{I}}_{k,m,n}) = (m-n)L^{\mathrm{I}}_{k,m,n}, \quad n \geq m. \tag{90}$$

Thus, the sequence $\{L^{\mathrm{II}}_{k,m,n}\}_{n=m}^{\infty}$ constitutes an m-OPS relative to the weight

$$W^{\mathrm{II}}_{k,m}(x) = x^k e^{-x}/\eta^2_{k+1,m}, \quad x \in (0,\infty) \tag{91}$$

Using (56), we also have

$$\int_0^\infty (L^{\mathrm{II}}_{k,m,n})^2\, W^{\mathrm{II}}_{k,m}\, dx = \frac{(1+k+n-2m)}{(n-m)!}\,\Gamma(2+k+n-m) \tag{92}$$

As above, for $m = 0$ the above definitions reduce to their classical counterparts, albeit the polynomials have a different normalization:

$$L^{\mathrm{II}}_{k,0,n} = -(k+1+n)L_{k,n}. \tag{93}$$

The proof that the sets $\{L^{\mathrm{I}}_{k,m,n}\}_{n=m}^{\infty}$ and $\{L^{\mathrm{II}}_{k,m,n}\}_{n=m}^{\infty}$ span dense subspaces of the Hilbert spaces $L^2([0,\infty), W^{\mathrm{I}}_{k,m}dx)$ and $L^2([0,\infty), W^{\mathrm{II}}_{k,m}dx)$ will be given in a forthcoming publication [18].

5. Shape-invariance of the exceptional polynomials

In this section we prove that above defined X_m polynomials are shape-invariant. The explanation for this remarkable fact is the commutativity/permutability of iterated Darboux transformations, also known as the Darboux-Crum transformation.

Let $T_0(y) = T(y)$ be a given PES operator, and let $\phi_1(x), \ldots, \phi_n(x)$ be quasi-rational eigenfunctions. Let $T_1(y) = \hat{T}(y)$ be the partner PES operator corresponding to ϕ_1. Now $A_1(\phi_2)$ is a quasi-rational eigenfunction for $T_1(y)$. Let $T_2(y) = \hat{T}_1(y)$ be the corresponding partner operator. Continue in like fashion. We arrive at the following chain of factorizations:

$$T_0 = B_1A_1 + \lambda_1 \tag{94}$$

$$T_j = A_jB_j + \lambda_j, \quad j = 1, \dots, n-1, \tag{95}$$

$$= B_{j+1}A_{j+1} + \lambda_{j+1} \tag{96}$$

$$T_n = A_nB_n + \lambda_n, \tag{97}$$

where

$$(A_j \cdots A_2A_1)(\phi_j) = 0, \quad j = 1, 2, \dots, n. \tag{98}$$

In the end, we obtain the following intertwining relations:

$$T_0\mathcal{B} = \mathcal{B}T_n, \qquad \text{where} \quad \mathcal{B} = B_1B_2\cdots B_n \tag{99}$$

$$\mathcal{A}T_0 = T_n\mathcal{A}, \qquad \text{where} \quad \mathcal{A} = A_n\cdots A_2A_1. \tag{100}$$

By construction,

$$\mathcal{A}(\phi_j) = 0, \quad j = 1, 2, \dots, n. \tag{101}$$

Hence,

$$\mathcal{A}(y) = b(x)\mathcal{W}(\phi_1, \dots, \phi_n, y)/\mathcal{W}(\phi_1, \dots, \phi_n), \tag{102}$$

where $b(x)$ is the higher-order rational factorization gauge, and where $\mathcal{W}$ denotes the Wronskian operator. As before, $b(x)$ is uniquely determined (up to scalar multiple) by the requirement that the eigenfunctions of $T_n(y)$ constitute a primitive sequence of polynomials.

The key observation is that up to sign, the above definition of T_n is independent of the order of the factorization functions. Let us exploit this commutativity to prove that the above-defined X_m polynomials are shape-invariant. To do so, requires that we consider a certain 2-step factorization.

Let $T_k(y)$ be a family of shape-invariant PES operators as per (31). Let $\pi_k(x) = y_{k,1}(x)$ denote the corresponding ground-state eigenpolynomials, and let (32) (33) be the corresponding factorizations, where without loss of generality the factorization eigenvalue is set to zero.

Next, let $\phi_k(x)$ be a quasi-rational eigenfunction that corresponds to a covariant, isospectral factorization as per (44). Let $\hat{T}_k(y)$ be the corresponding family of isospectral operators as per (43). We claim that this family is also shape-invariant. Let

$$\hat{T}_k = \hat{B}_k\hat{A}_k, \qquad \hat{A}_k(\tilde{\pi}_k) = 0 \tag{103}$$

be the ground-state factorization of the partner operator, where

$$\tilde{\pi}_k = \tilde{A}_k(\pi_k) \tag{104}$$

is the new ground-state polynomial. Our claim is that

$$\hat{T}_{h(k)} = \hat{A}_k\hat{B}_k + \lambda_k, \tag{105}$$

For convenience, let us set

$$\tilde{T}_k = \hat{A}_k \hat{B}_k \tag{106}$$

The 2nd order intertwining relation is

$$\mathcal{A}_k T_k = \tilde{T}_k \mathcal{A}_k \quad \text{where} \tag{107}$$

$$\mathcal{A}_k(y) = b(x) W(\pi_k, \phi_k, y)/W(\pi_k, \phi_k), \tag{108}$$

and where $b(x)$ is the 2nd order rational factorization gauge whose form is not relevant to our argument. There are two ways to factorize $\mathcal{A}_k$, the 2nd order intertwiner:

$$\mathcal{A}_k = \hat{A}_k \tilde{A}_k \tag{109}$$

$$\mathcal{A}_k = \tilde{A}_{h(k)} A_k \tag{110}$$

The 2nd equation is true because by (44) we have

$$\tilde{A}_{h(k)}(A_k(\phi_k)) \propto \tilde{A}_{h(k)}(\phi_{h(k)}) = 0. \tag{111}$$

Hence,

$$\mathcal{A}_k T_k = \tilde{A}_{h(k)} A_k T_k \tag{112}$$

$$= \tilde{A}_{h(k)}(T_{h(k)} - \lambda_k) A_k \tag{113}$$

$$= (\hat{T}_{h(k)} - \lambda_k)\mathcal{A}_k. \tag{114}$$

Hence, by Equation (107),

$$\tilde{T}_k \mathcal{A}_k = (\hat{T}_{h(k)} - \lambda_k)\mathcal{A}_k. \tag{115}$$

The ring of differential operators with rational coefficients has no zero divisors. Therefore, the desired relation (105) follows.

Next, let us illustrate the above result by explicitly showing the shape-invariant factorization for the type I exceptional Laguerre polynomials defined in the preceding section. The index set consists of real $k > -1$. Let us set

$$\pi_k(x) = 1, \tag{116}$$

$$h(k) = k + 1 \tag{117}$$

$$\lambda_k = 1 \tag{118}$$

$$T_k(y) = \mathcal{L}_k(y) \tag{119}$$

The classical Laguerre polynomials are shape-invariant; relations (32) (33) hold, as per (51)-(54). Let us fix an integer $m \geq 0$ and set

$$\phi_k(x) = e^x \xi_{k,m}, \tag{120}$$

$$\tilde{A}_k(y) = A^{\mathrm{I}}_{k,m}(y) \tag{121}$$

$$\tilde{B}_k(y) = B^{\mathrm{I}}_{k,m}(y) \tag{122}$$

$$\hat{T}_k(y) = \mathcal{L}^{\mathrm{I}}_{k+1,m} \tag{123}$$

$$\tilde{\lambda}_k = k + m \tag{124}$$

These definitions realize a particular instance of the isospectral factorizations shown in (42) (43). By inspection of (57)-(60), the operator $\mathcal{L}_k$ is formally non-degenerate for generic k. Equation (45) is satisfied, and hence by Lemma 3.1 the isospectral factorization with respect to (120) is covariant. Therefore, the operators $\mathcal{L}^{\mathrm{I}}_{k,m}$ are shape-invariant.

Next, we explicitly describe the ground-state factorization for $\mathcal{L}^{\mathrm{I}}_{k,m}$ and verify the shape-invariance property. To determine an explicit form for $\hat{A}_k, \hat{B}_k$ we make use of formula (41). Here,

$$\hat{q}_k(x) = (k+2-x) - 2x\rho_{k+1,m} \tag{125}$$

$$\hat{P}_k(x) = \exp\left(\int^x \hat{q}_k/p\right) = e^{-x}x^{k+2}/\xi^2_{k,m} \tag{126}$$

$$p\,\hat{P}_k/\hat{P}_{k+1} = \xi^2_{k+1,m}/\xi^2_{k,m} \tag{127}$$

In this way, we arrive at the shape-invariant factorization

$$\mathcal{L}^{\mathrm{I}}_{k,m} = \hat{B}^{\mathrm{I}}_{k,m}\hat{A}^{\mathrm{I}}_{k,m}, \quad \hat{A}^{\mathrm{I}}_{k,m}(\xi_{k,m}) = 0, \tag{128}$$

$$\mathcal{L}^{\mathrm{I}}_{k+1,m} = \hat{A}_{k,m}\hat{B}_{k,m} + 1, \quad \hat{B}^{\mathrm{I}}_{k,m}(e^x x^{-1-k}\xi_{k-1,m}) = 0 \tag{129}$$

where

$$\hat{A}^{\mathrm{I}}_{k,m}(y) = (\xi_{k,m}/\xi_{k-1,m})(y' - \rho_{k,m}\,y) \tag{130}$$

$$\hat{B}^{\mathrm{I}}_{k,m}(y) = (\xi_{k-1,m}/\xi_{k,m})(xy' + (1+k)\,y) - xy \tag{131}$$

Thus, the type I polynomials obey the following lowering and raising relations:

$$\hat{A}^{\mathrm{I}}_{k,m}(L^{\mathrm{I}}_{k,m,n}) = -L^{\mathrm{I}}_{k+1,m,n-1}, \quad n \geq m. \tag{132}$$

$$\hat{B}^{\mathrm{I}}_{k,m}(L^{\mathrm{I}}_{k+1,m,n}) = (n+1-m)L^{\mathrm{I}}_{k,m,n+1}, \quad n \geq m. \tag{133}$$

In a similar fashion, we derive the following shape-invariant factorization for the type II polynomials. This time, we let

$$\phi_k(x) = x^{-k}\eta_{k,m}, \tag{134}$$

$$\tilde{\lambda}_k = k - m, \tag{135}$$

$$\hat{T}_k(y) = \mathcal{L}^{\mathrm{II}}_{k-1,m}, \tag{136}$$

$$\tilde{A}_k(y) = A^{\mathrm{II}}_{k,m}, \tag{137}$$

$$\tilde{B}_k(y) = B^{\mathrm{II}}_{k,m}, \tag{138}$$

$$\hat{q}_k(x) = (k-x) + 2x\sigma_{k,m} \tag{139}$$

$$\hat{P}_k(x) = \exp\left(\int^x \hat{q}_k/p\right) = e^{-x}x^{k}/\eta^2_{k,m} \tag{140}$$

$$p\,\hat{P}_k/\hat{P}_{k+1} = \eta^2_{k+1,m}/\eta^2_{k,m} \tag{141}$$

Applying the formulas of section 3, we obtain the following shape-invariant factorization:

$$\mathcal{L}^{\mathrm{II}}_{k,m} = \hat{B}^{\mathrm{II}}_{k,m}\hat{A}^{\mathrm{II}}_{k,m}, \qquad \hat{A}^{\mathrm{II}}_{k,m}(\eta_{k+2,m}) = 0, \tag{142}$$

$$\mathcal{L}^{\mathrm{II}}_{k+1,m} = \hat{A}^{\mathrm{II}}_{k,m}\hat{B}^{\mathrm{II}}_{k,m} + 1, \quad \hat{B}^{\mathrm{II}}_{k,m}(e^x x^{-1-k}\eta_{k+1,m}) = 0 \tag{143}$$

where

$$\hat{A}^{\mathrm{II}}_{k,m}(y) = (\eta_{k+2,m}/\eta_{k+1,m})(y' + \sigma_{k+2,m}\,y) \tag{144}$$

$$\hat{B}^{\mathrm{II}}_{k,m}(y) = (\eta_{k+1,m}/\eta_{k+2,m})(xy' + (1+k-x)\,y) + \tag{145}$$

$$+(\eta_{k,m-1}/\eta_{k+2,m})xy \tag{146}$$

The type II polynomials obey the same lowering and raising relations as in (132).

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