A JACOBI TYPE CHRISTOFFEL–DARBOUX FORMULA FOR MULTIPLE ORTHOGONAL POLYNOMIALS OF MIXED TYPE

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Abstract. An alternative expression for the Christoffel–Darboux formula for multiple orthogonal polynomials of mixed type is derived from the LU factorization of the moment matrix of a given measure and two sets of weights. We use the action of the generalized Jacobi matrix J, also responsible for the recurrence relations, on the linear forms and their duals to obtain the result.

1. Introduction

In this paper we address a natural question that arises from the LU factorization approach to multiple orthogonality [7]. The Gauss decomposition of a Hankel matrix, which plays the role of a moment matrix, leads in the classical case to a natural description of algebraic facts regarding orthogonal polynomials on the real line (OPRL) such as recursion relations and Christoffel–Darboux (CD) formula. In that case we have a chain of orthogonal polynomials $\{P_l(x)\}_{l=0}^{\infty}$ of increasing degree l. In [7] we extended that approach to the multiple orthogonality scenario, and the Gauss decomposition of an appropriate moment matrix led to sequences of families of multiple orthogonal polynomials in the real line (MOPRL), $\big\{Q^{(\mathrm{II},a_1(l))}_{[\vec{\nu}_1(l);\vec{\nu}_2(l-1)]}\big\}_{l=0}^{\infty}$ and $\big\{\bar{Q}^{(\mathrm{I},a_1(l))}_{[\vec{\nu}_2(l);\vec{\nu}_1(l-1)]}\big\}_{l=0}^{\infty}$. These families happens to be biorthogonal, and therefore we will refer to them as biorthogonal sequences of linear forms. The recursion formulae are relations constructed in terms of the linear forms in these sequences. However, the Daems–Kuijlaars Christoffel–Darboux formula given in Proposition 4 –that was re-deduced in [7] by linear algebraic means (Gauss decomposition) and the use of the ABC theorem– was not expressed in terms of linear forms belonging to the mentioned sequences. This situation is rather different to the OPRL case, in that standard scenario the CD formula, call it the ABC type CD formula, is expressed in terms of orthogonal polynomials in the sequence. The aim of this paper is to show that, within that scheme, we can deduce an alternative but equivalent MOPRL Christoffel–Darboux formula constructed in terms of linear forms in the sequences $\big\{Q^{(\mathrm{II},a_1(l))}_{[\vec{\nu}_1(l);\vec{\nu}_2(l-1)]}\big\}_{l=0}^{\infty}$ and $\big\{\bar{Q}^{(\mathrm{I},a_1(l))}_{[\vec{\nu}_2(l);\vec{\nu}_1(l-1)]}\big\}_{l=0}^{\infty}$ as in OPRL situation. Besides we are able to find a OPRL type CD formula, expressed in terms solely of elements in the biorthogonal sequences of MOP of mixed type, there are two prices to pay; firstly, we need, in general, more terms that in the ABC type CD formula for these MOPs and secondly we will need to know the coefficients in the recursion relation; i.e., the Jacobi coefficients. We will refer to these type of CD formulae as Jacobi type CD formula as they are based on the structure of the Jacobi type matrix associated to the biorthogonal sequences which gives their recursion relations.

We must stress that in the OPRL scenario there are many ways to prove the CD formula [27]. In particular, in the one hand we could prove it using the ABC theorem combined with the moment matrix symmetry and in the other hand using the eigen-value properties of the Jacobi matrix. These two approaches –ABC and Jacobi– lead, in this simple situation, two the same result. However, as already mentioned, in the MOP scenario the two approaches leads to different results: the ABC type CD formula (or Daems–Kuijlaars CD formula) and the Jacobi type CD formula.

1.1. Historical background.

Simultaneous rational approximation starts back in 1873 when Hermite proved the transcendence of the Euler number e [21]. Later, K. Mahler delivered at the University of Groningen several lectures [24] where he settled down the foundations of this theory, see also [13] and [22]. Simultaneous rational approximation when expressed in terms of Cauchy transforms leads to multiple orthogonality of polynomials.

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Given an interval $\Delta \subset \mathbb{R}$ of the real line, let $\mathcal{M}(\Delta)$ denote all the finite positive Borel measures with support containing infinitely many points in Δ. Fix $\mu \in \mathcal{M}(\Delta)$, and let us consider a system of weights $\vec{w} = (w_1, \dots, w_p)$ on Δ, with $p \in \mathbb{N}$; i.e. $w_1, \dots, w_p$ being real integrable functions on Δ which does not change sign on Δ. Fix a multi-index $\vec{\nu} = (\nu_1, \dots, \nu_p) \in \mathbb{Z}_+^p$, $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$, and denote $|\vec{\nu}| = \nu_1 + \dots + \nu_p$. Then, there exist polynomials, $A_1, \dots, A_p$ not all identically equal to zero which satisfy the following orthogonality relations

$$\int_\Delta x^j \sum_{a=1}^{p} A_a(x) w_a(x) \mathrm{d}\mu(x) = 0, \qquad \deg A_a \leq \nu_a - 1, \qquad j = 0, \dots, |\vec{\nu}| - 2. \tag{1}$$

Analogously, there exists a polynomial B not identically equal to zero, such that

$$\int_\Delta x^j B(x) w_b(x) \mathrm{d}\mu(x) = 0, \qquad \deg B \leq |\vec{\nu}|, \qquad j = 0, \dots, \nu_b - 1, \quad b = 1, \dots, p. \tag{2}$$

These are the so called multiple orthogonal polynomials of type I and type II, respectively, with respect to the combination $(\mu, \vec{w}, \vec{\nu})$ of the measure μ, the systems of weights $\vec{w}$ and the multi-index $\vec{\nu}$. When $p = 1$ both definitions coincide with standard orthogonal polynomials on the real line. Given a measure $\mu \in \mathcal{M}(\Delta)$ and a system of weights $\vec{w}$ on Δ a multi-index $\vec{\nu}$ is called type I or type II normal if $\deg A_a$ must equal to $\nu_a - 1$, $a = 1, \dots, p$, or $\deg B$ must equal to $|\vec{\nu}| - 1$, respectively. When for a pair $(\mu, \vec{w})$ all the multi-indices are type I or type II normal, then the pair is called type I perfect or type II perfect respectively. Multiple orthogonal of polynomials have been employed in several proofs of irrationality of numbers. For example, in [11] F. Beukers shows that Apery's proof [10] of the irrationality of $\zeta(3)$ can be placed in the context of a combination of type I and type II multiple orthogonality which is called mixed type multiple orthogonality of polynomials. More recently, mixed type approximation has appeared in random matrix and non-intersecting Brownian motion theories, [12], [15], [23]. Sorokin [28] studied a simultaneous rational approximation construction which is closely connected with multiple orthogonal polynomials of mixed type. In [30] a Riemann–Hilbert problem was found that characterizes multiple orthogonality of type I and II, extending in this way the result previously found in [20] for standard orthogonality. In [15] mixed type multiple orthogonality was analyzed from this perspective. For a general study, but not including multiple orthogonality, of Christoffel–Darboux kernels see [27]. In [9] we gave a generalizations of CD formulae to matrix generalized orthogonal polynomials. In [16] MOPRL and some CD kernels are used in the study of average characteristic polynomials and in [17] some properties of models of n one-dimensional, nonintersecting Brownian motions with two prescribed starting points at time $t = 0$ and two prescribed ending points at time $t = 1$ in a critical regime are analyzed with the aid of Hermite MOP. Finally, in [19] a large class of MOPRL are shown to fulfill that its zeros on the real line are simple, lie in the interior of the convex hull of the support of the measure and the zeros of consecutive orthogonal polynomials interlace.

1.2. **Perfect systems and MOPRL of mixed type.** In order to introduce multiple orthogonal polynomials of mixed type we consider two systems of weights $\vec{w}_1 = (w_{1,1}, \dots, w_{1,p_1})$ and $\vec{w}_2 = (w_{2,1}, \dots, w_{2,p_2})$ where $p_1, p_2 \in \mathbb{N}$, and two multi-indices $\vec{\nu}_1 = (\nu_{1,1}, \dots, \nu_{1,p_1}) \in \mathbb{Z}_+^{p_1}$ and $\vec{\nu}_2 = (\nu_{2,1}, \dots, \nu_{2,p_2}) \in \mathbb{Z}_+^{p_2}$ with $|\vec{\nu}_1| = |\vec{\nu}_2| + 1$. There exist polynomials $A_1, \dots, A_{p_1}$, not all identically zero, such that $\deg A_s < \nu_{1,s}$, which satisfy the following relations

$$\int_\Delta \sum_{a=1}^{p_1} A_a(x) w_{1,a}(x) w_{2,b}(x) x^j d\mu(x) = 0, \quad j = 0, \dots, \nu_{2,b} - 1, \quad b = 1, \dots, p_2. \tag{3}$$

In this paper we say that we have p_1 components of type II and p_2 components of type I. They are called mixed multiple-orthogonal polynomials with respect to the combination $(\mu, \vec{w}_1, \vec{w}_2, \vec{\nu}_1, \vec{\nu}_2)$ of the measure μ, the systems of weights $\vec{w}_1$ and $\vec{w}_2$ and the multi-indices $\vec{\nu}_1$ and $\vec{\nu}_2$. It is easy to show that finding the polynomials $A_1, \dots, A_{p_1}$ is equivalent to solving a system of $|\vec{\nu}_2|$ homogeneous linear equations for the $|\vec{\nu}_1|$ unknown coefficients of the polynomials. Since $|\vec{\nu}_1| = |\vec{\nu}_2| + 1$ the system always has a nontrivial solution. The matrix of this system of equations is the so called moment matrix, and the study of its Gauss decomposition will be the cornerstone of this paper. Observe that when $p_1 = 1$ we are in the type II case and if $p_2 = 1$ in type I case. Hence in general we can find a solution of (3) where there is an $a \in \{1, \dots, p_1\}$ such that $\deg A_a < \nu_{1,a} - 1$. When given a combination $(\mu, \vec{w}_1, \vec{w}_2)$ of a measure $\mu \in \mathcal{M}(\Delta)$ and systems of weights $\vec{w}_1$ and $\vec{w}_2$ on Δ if for each pair of multi-indices $(\vec{\nu}_1, \vec{\nu}_2)$ the conditions (3) determine that $\deg A_a = \nu_{1,a} - 1$, $a = 1, \dots, p_1$, then we say that the combination $(\mu, \vec{w}_1, \vec{w}_2)$ is perfect. In this case we can determine a unique system of mixed type orthogonal polynomials $\big(A_1,\dots,A_{p_1}\big)$ satisfying (3) requiring for $a_1\in\{1,\dots p_1\}$ that A_{a_1} monic. Following [15] we say that we have a type II normalization and denote the corresponding system of polynomials by $A_a^{(\mathrm{II},a_1)}$, $j=1,\dots,p_1$. Alternatively, we can proceed as follows, since the system of weights is perfect from (3) we deduce that

$$\int x^{\nu_{1,r_1}}\sum_{a=1}^{p_1}A_a(x)w_{1,a}(x)w_{2,b}(x)\mathrm{d}\mu(x)\neq 0.$$

Then, we can determine a unique system of mixed type of multi-orthogonal polynomials $(A_1^{(\mathrm{I},a_2)},\dots,A_{p_2}^{(\mathrm{I},a_2)})$ imposing that

$$\int x^{\nu_{1,a_2}}\sum_{a=1}^{p_1}A_a^{(\mathrm{I},a_2)}(x)w_{1,a}(x)w_{2,b}(x)\mathrm{d}\mu(x)=1,$$

which is a type I normalization. We will use the notation $A^{(\mathrm{II},a_1)}_{[\vec\nu_1;\vec\nu_2],a}$ and $A^{(\mathrm{I},a_2)}_{[\vec\nu_1;\vec\nu_2],a}$ to denote these multiple orthogonal polynomials with type II and I normalizations, respectively.

A known illustration of perfect combinations $(\mu,\vec w_1,\vec w_2)$ can be constructed with an arbitrary positive finite Borel measure μ and systems of weights formed with exponentials:

$$(\mathrm{e}^{\gamma_1 x},\dots,\mathrm{e}^{\gamma_p x}),\qquad \gamma_i\neq\gamma_j,\qquad i\neq j,\qquad i,j=1,\dots,p, \tag{4}$$

or by binomial functions

$$((1-z)^{\alpha_1},\dots,(1-z)^{\alpha_p}),\qquad \alpha_i-\alpha_j\notin\mathbb{Z},\qquad i\neq j,\qquad i,j=1,\dots,p. \tag{5}$$

or combining both classes, see [25]. Recently, in [18] the authors were able to prove perfectness for a wide class of systems of weights. These systems of functions, now called Nikishin systems, were introduced by E.M. Nikishin [25] and initially named MT-systems (after Markov and Tchebycheff).

1.3. **Gauss decomposition and multiple orthogonality of mixed type. A reminder.** Orthogonal polynomials and the theory of integrable systems have been connected in several ways in the mathematical literature. We are particularly interested in the one based in the Gauss decomposition that was developed in [1]-[5], and applied further in [6]-[8]. These papers set the basis for the method we use in this paper to get an alternative CD formula for MOPRL of mixed type.

In the following we extract from [7] the necessary material for the construction of the mentioned alternative Christoffel–Darboux formula. We introduce the moment matrix and recall how the Gauss decomposition leads to multiple orthogonality. Then, we outline how the recursion relations appears by introducing a Jacobi type semi-infinite matrix and recall the reader the CD formula [14, 15].

1.3.1. *The moment matrix.* We now proceed to define the moment matrix. For that aim we need as starting point two systems of weights $\vec w_\alpha=(w_{\alpha,1},\dots,w_{\alpha,p_\alpha})$, $\alpha=1,2$ and $p_1,p_2\in\{1,2,3,\dots\}$ and a finite Borel measure $\mathrm{d}\mu$ supported all of them on an interval $\Delta\subset\mathbb{R}$. Given two compositions[1] $\vec n_\alpha=(n_{\alpha,1},\dots,n_{\alpha,p_\alpha})$, $\alpha=1,2$, of $|\vec n_\alpha|=n_{\alpha,1}+\dots+n_{\alpha,p_\alpha}$ any given $l\in\mathbb{Z}_+:=\{0,1,2,\dots\}$ determines uniquely, through Euclidean division, the following non-negative integers $k_\alpha(l)\in\mathbb{Z}_+$, $a_\alpha(l)\in\{1,2,\dots,p_\alpha\}$ and $r_\alpha(l)$ such that $r_\alpha(l)\in\{0,1,\dots,n_{\alpha,a_\alpha(l)}-1\}$ and

$$l=\begin{cases}k_\alpha(l)|\vec n_\alpha|+r_\alpha(l), & a_\alpha(l)=1,\\ k_\alpha(l)|\vec n_\alpha|+n_{\alpha,1}+\dots+n_{\alpha,a_\alpha(l)-1}+r_\alpha(l), & a_\alpha(l)\neq 1.\end{cases} \tag{6}$$

[1]Do not confuse with a partition; in Combinatorics, see for example [29], a composition of an integer n is a way of writing n as the sum of a sequence of (strictly) positive integers. Two sequences that differ in the order of their terms define different compositions of their sum, while they are considered to define the same partition of that number. Every integer has finitely many distinct compositions. Given that for the Gauss decomposition description of MOP this order is relevant we have stressed this aspect and preferred the name of composition to that of multi-index, which can be also used.

We define two monomial vectors that may be understood as sequences of monomials according to the composition $\vec{n}_\alpha$, $\alpha=1,2$, introduced previously.

$$\chi_\alpha:=\begin{pmatrix}\chi_{\alpha,[0]}\\ \chi_{\alpha,[1]}\\ \vdots\\ \chi_{\alpha,[k]}\\ \vdots\end{pmatrix} \qquad \text{where} \qquad \chi_{\alpha,[k]}:=\begin{pmatrix}\chi_{\alpha,[k],1}\\ \chi_{\alpha,[k],2}\\ \vdots\\ \chi_{\alpha,[k],a_\alpha}\\ \vdots\\ \chi_{\alpha,[k],p_\alpha}\end{pmatrix} \qquad \text{and} \qquad \chi_{\alpha,[k],a_\alpha}:=\begin{pmatrix}x^{kn_{\alpha,a_\alpha}}\\ x^{kn_{\alpha,a_\alpha}+1}\\ \vdots\\ x^{kn_{\alpha,a_\alpha}+(n_{\alpha,a_\alpha}-1)}\end{pmatrix}.$$

In a similar manner for $\alpha=1,2$ we define the weighted monomial vectors

$$\xi_\alpha:=\begin{pmatrix}\xi_{\alpha,[0]}\\ \xi_{\alpha,[1]}\\ \vdots\\ \xi_{\alpha,[k]}\\ \vdots\end{pmatrix} \qquad \text{where} \qquad \xi_{\alpha,[k]}:=\begin{pmatrix}w_{\alpha,1}\chi_{\alpha,[k],1}\\ w_{\alpha,2}\chi_{\alpha,[k],2}\\ \vdots\\ w_{\alpha,p_\alpha}\chi_{\alpha,[k],p_\alpha}\end{pmatrix}.$$

For example, let us put $p_1=2$ and $p_2=1$ and set the compositions $n_{1,1}=3$, $n_{1,2}=2$ and $n_{2,1}=1$ with weight vectors $\vec{w}_1=(w_{1,1},w_{1,2})$ and $\vec{w}_2=(w_{2,1})$. Then

$$\begin{aligned}\chi_1&=(1,x,x^2,1,x,x^3,x^4,x^5,x^2,x^3,x^6,x^7,x^8,x^4,x^5,\dots)^\top, & \chi_2&=(1,x,x^2,\dots)^\top,\\ \xi_1&=(w_{1,1},w_{1,1}x,w_{1,1}x^2,w_{1,2},w_{1,2}x,w_{1,1}x^3,w_{1,1}x^4,w_{1,1}x^5,w_{1,2}x^2,w_{1,2}x^3,\dots)^\top, & \xi_2&=w_{2,1}(1,x,x^2,\dots)^\top.\end{aligned}$$

We have used two colors, red and blue, for $\alpha=1$, to remark the two ($p_1=2$) forms of growth, in steps of 3 for the monomial powers in red component and of 2 for the blue one. For the corresponding MOP of mixed type (in this case are just of type II as we have choose $p_2=1$) these colors are associated, as we will see, to the two components of type II that this example leads to, the red and blue components. Observe that for the construction of the monomial vectors χ_α and weighted monomial vectors ξ_α, $\alpha=1,2$, only the two compositions $\vec{n}_\alpha$, $\alpha=1,2$, are needed. However, the l-th entries or coefficients, $\chi_\alpha^{(l)}$ and $\xi_\alpha^{(l)}$, of these semi-infinite vectors can be explicitly expressed in terms of the just introduced Euclidean division

$$\chi_\alpha^{(l)}=x^{\nu_{\alpha,a_\alpha(l)}}, \qquad \xi_\alpha^{(l)}=w_{\alpha,a_\alpha(l)}x^{\nu_{\alpha,a_\alpha(l)}}$$

where for any given $l\in\mathbb{Z}_+$ and $a_\alpha:=1,2,\dots,p_\alpha$ we define

$$\nu_{\alpha,a_\alpha}(l):=\begin{cases}k_\alpha(l)|\vec{n}_\alpha|+n_{\alpha,a_\alpha}-1, & a_\alpha<a_\alpha(l),\\ k_\alpha(l)|\vec{n}_\alpha|+r_\alpha(l), & a_\alpha=a_\alpha(l),\\ k_\alpha(l)|\vec{n}_\alpha|-1, & a_\alpha>a_\alpha(l).\end{cases}$$

Notice that $\nu_{\alpha,a_\alpha}(l)$ is the highest degree of all the monomials of type a_α up to the component $\chi_\alpha^{(l)}$ included, of the monomial vector.

We stress that for a given positive integer l the number $a_\alpha(l)\in\{1,\dots,p_\alpha\}$ distinguishes to which of the p_α posible components, or different colors in the previous example, (of type II for $\alpha=1$ and type I for $\alpha=2$) this integer belongs to. Later on for any positive integer l we will need to know which is the closest integer by defect or by excess in a given component $a_\alpha\in\{1,\dots,p_\alpha\}$, $\alpha\in\{1,2\}$. For that aim we introduce the functions

$$\begin{aligned}[\cdot,\cdot]_\alpha^{\lesseqgtr}:&\mathbb{Z}_+\times\{1,\dots,p_\alpha\}\to\mathbb{Z}_+,\\ &(l,\ a)\qquad\qquad\mapsto[l,a]_\alpha^{\lesseqgtr},\end{aligned}$$

where

$$
[l,a]^{<}_{\alpha} := \begin{cases} k_\alpha(l)|\vec n_\alpha| + \sum_{i=1}^{a} n_{\alpha,i} - 1, & a < a_\alpha(l),\\ l, & a = a_\alpha(l),\\ k_\alpha(l)|\vec n_\alpha| - \sum_{i=a+2}^{p_\alpha} n_{\alpha,i} - 1, & a > a_\alpha(l),\end{cases}
$$
$$
[l,a]^{>}_{\alpha} := \begin{cases} (k_\alpha(l)+1)|\vec n_\alpha| + \sum_{i=1}^{a-1} n_{\alpha,i}, & a < a_\alpha(l),\\ l, & a = a_\alpha(l),\\ (k_\alpha(l)+1)|\vec n_\alpha| - \sum_{i=a+1}^{p_\alpha} n_{\alpha,i}, & a > a_\alpha(l).\end{cases} \tag{7}
$$

It can be proven that these are the desired integers; i.e., that $[l,a]^{>}_{\alpha}$ ($[l,a]^{<}_{\alpha}$) is the smallest (largest) integer such that $[l,a]^{>}_{\alpha} \geq l$ ($[l,a]^{<}_{\alpha} \leq l$) and $a_\alpha([l,a]^{>}_{\alpha}) = a$ ($a_\alpha([l,a]^{<}_{\alpha}) = a$).

Finally, given the weighted monomials ξ_α, associated to the compositions $\vec n_\alpha$, $\alpha = 1,2$, we introduce the moment matrix in the following manner

Definition 1. *The moment matrix is given by*

$$
g := \int \xi_1(x)\xi_2(x)^\top \mathrm{d}\mu(x). \tag{8}
$$

1.3.2. *Multiple orthogonality of mixed type: The Gauss decomposition of the moment matrix.*

Definition 2. *For a given perfect combination $(\mu, \vec w_1, \vec w_2)$ we define*

(1) *The Gauss decomposition (also known as LU factorization) of a semi-infinite moment matrix g, determined by $(\mu, \vec w_1, \vec w_2)$, is the problem of finding the solution of*

$$
g = S^{-1}\bar S, \qquad S = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ S_{1,0} & 1 & 0 & \cdots \\ S_{2,0} & S_{2,1} & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad \bar S^{-1} = \begin{pmatrix} \bar S'_{0,0} & \bar S'_{0,1} & \bar S'_{0,2} & \cdots \\ 0 & \bar S'_{1,1} & \bar S'_{1,2} & \cdots \\ 0 & 0 & \bar S'_{2,2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{9}
$$

where $S_{i,j}, \bar S'_{i,j} \in \mathbb{R}$.

(2) *In terms of these matrices we construct the polynomials*

$$
A^{(l)}_a := \sum\nolimits'_{i} S_{l,i} x^{k_1(i)}, \tag{10}
$$

where the sum $\sum'$ is taken for a fixed $a = 1,\dots,p_1$ over those i such that $a = a_1(i)$ and $i \leq l$. We also construct the dual polynomials

$$
\bar A^{(l)}_b := \sum\nolimits'_{j} x^{k_2(j)} \bar S'_{j,l}, \tag{11}
$$

where the sum $\sum'$ is taken for a given b over those j such that $b = a_2(j)$ and $j \leq l$.

(3) *Vectors of linear forms and dual linear forms associated with multiple ortogonal polynomials and their duals are defined by*

$$
Q := \begin{pmatrix} Q^{(0)} \\ Q^{(1)} \\ \vdots \end{pmatrix} = S\xi_1, \qquad \bar Q := \begin{pmatrix} \bar Q^{(0)} \\ \bar Q^{(1)} \\ \vdots \end{pmatrix} = (\bar S^{-1})^\top \xi_2, \tag{12}
$$

Then –see Propositions 3, 4, 5 and 6 in [7]–

Proposition 1. (1) *The linear forms and their duals, introduced in Definition 2, are given by*

$$
Q^{(l)}(x) := \sum_{a=1}^{p_1} A^{(l)}_a(x) w_{1,a}(x), \qquad \bar Q^{(l)}(x) := \sum_{b=1}^{p_2} \bar A^{(l)}_b(x) w_{2,b}(x). \tag{13}
$$

(2) *The orthogonality relations*

$$
\begin{aligned}
&\int Q^{(l)}(x)w_{2,b}(x)x^k\mathrm{d}\mu(x)=0, \quad 0\leq k\leq \nu_{2,b}(l-1)-1, \quad b=1,\dots,p_2,\\
&\int \bar Q^{(l)}(x)w_{1,a}(x)x^k\mathrm{d}\mu(x)=0, \quad 0\leq k\leq \nu_{1,a}(l-1)-1, \quad a=1,\dots,p_1,
\end{aligned}
\tag{14}
$$

are fulfilled.

(3) *We have the following identifications*

$$
A^{(l)}_a=A^{(\mathrm{II},a_1(l))}_{[\vec\nu_1(l);\vec\nu_2(l-1)],a}, \qquad \bar A^{(l)}_b=A^{(\mathrm{I},a_1(l))}_{[\vec\nu_2(l);\vec\nu_1(l-1)],b},
$$

in terms of multiple orthogonal polynomials of mixed type with two normalisations I *and* II*, respectively.*

(4) *The following multiple bi-orthogonality relations among linear forms and their duals*

$$
\int Q^{(\mathrm{II},a_1(l))}_{[\vec\nu_1(l);\vec\nu_2(l-1)]}(x)\bar Q^{(\mathrm{I},a_1(k))}_{[\vec\nu_2(k);\vec\nu_1(k-1)]}(x)\mathrm{d}\mu(x)=\delta_{l,k}, \qquad l,k\in\mathbb{Z}_+,
\tag{15}
$$

hold.

Observe that a major difference between the usual approach to MOPRL of mixed type, in which the orthogonality relations (3) are discussed in its own, and the described Gauss decomposition approach is precisely the biorthogonality conditions given by (15). While for standard OPRL both type orthogonal relations –the perpendicularity of each polynomial P_l to $\{1,x,\dots,x^{l-1}\}$ and the orthogonality of the set of polynomials $\{P_l\}_{l=0}^{\infty}$– are discussed in equal footing this has no parallel before in the MOPRL scenario. Biorthogonality (15) gives such a bridge; i.e., we have two sequences of MORPL –with normalizations of types I and II, respectively– such that its biorthogonality is equivalent to the multiple orthogonality condition of both families.

1.3.3. *Jacobi type matrices and recursion relations.* The moment matrix has a Hankel type symmetry that implies the recursion relations and the Christoffel–Darboux formula. We consider the shift operators Υ_α defined by

$$
(\Upsilon_\alpha)_{l,j}:=\delta_{j,[l+1,a_\alpha(l)]^>}
\tag{16}
$$

Wich satisfy the following relation

$$
\Upsilon_\alpha\chi_\alpha(x)=x\chi_\alpha(x)\Longrightarrow \Upsilon_\alpha\xi_\alpha(x)=x\xi_\alpha(x)
$$

In terms of these shift matrices we can describe the particular Hankel symmetries for the moment (see Proposition 12 in [7]) matrix

Proposition 2. *The moment matrix g satisfies the Hankel type symmetry*

$$
\Upsilon_1 g=g\Upsilon_2^\top.
\tag{17}
$$

From this symmetry we see that the following is consistent

Definition 3. *We define the matrices*

$$
J:=S\Upsilon_1S^{-1}=\bar S\Upsilon_2^\top\bar S^{-1}=J_++J_-, \qquad J_+:=(S\Upsilon_1S^{-1})_+, \qquad J_-:=(\bar S\Upsilon_2^\top\bar S^{-1})_-,
\tag{18}
$$

where the sub-indices $+$ and $-$denote the upper triangular and strictly lower triangular projections.

The matrix J for this MORPL of mixed type is therefore, not a tridiagonal matrix as for the standard OPRL, but more generally a banded matrix with the number of upper and lower diagonal determined by the the number of components and compositions.

The recursion relations follow immediately from the eigenvalue property

$$
JQ(x)=xQ(x) \qquad \bar Q(x)^\top J=x\bar Q(x)^\top,
\tag{19}
$$

which imply for $\big\{Q^{(\mathrm{II},a_1(l))}_{[\vec\nu_1(l);\vec\nu_2(l-1)]}(x)\big\}_{l=0}^{\infty}$ and $\big\{\bar Q^{(\mathrm{I},a_1(k))}_{[\vec\nu_2(k);\vec\nu_1(k-1)]}(x)\big\}_{k=0}^{\infty}$ recursion relations; i. e., each $xQ^{(\mathrm{II},a_1(l))}_{[\vec\nu_1(l);\vec\nu_2(l-1)]}(x)$ is expressed as a finite sum of linear forms in $\big\{Q^{(\mathrm{II},a_1(i))}_{[\vec\nu_1(i);\vec\nu_2(i-1)]}(x)\big\}_{i=0}^{\infty}$ and each $x\bar Q^{(\mathrm{I},a_1(k))}_{[\vec\nu_2(k);\vec\nu_1(k-1)]}(x)$ as a finite combination of dual linear forms in $\big\{\bar Q^{(\mathrm{I},a_1(j))}_{[\vec\nu_2(j);\vec\nu_1(j-1)]}(x)\big\}_{j=0}^{\infty}$.

1.3.4. *The ABC type Christoffel–Darboux formula for MOP of mixed type.*

Definition 4. *The l-th Christoffel–Darboux kernel is defined by*

$$K^{[l]}(x,y):=\sum_{k=0}^{l-1} Q^{(\mathrm{II},a_1(k))}_{[\vec\nu_1(k);\vec\nu_2(k-1)]}(y)\bar Q^{(\mathrm{I},a_1(k))}_{[\vec\nu_2(k);\vec\nu_1(k-1)]}(x). \tag{20}$$

Any semi-infinite vector v can be written in block form as follows

$$v=\left(\begin{array}{c} v^{[l]} \\ \hline v^{[\geq l]}\end{array}\right)$$

$v^{[l]}$ is the finite vector formed with the first l coefficients of v and $v^{[\geq l]}$ the semi-infinite vector formed with the remaining coefficients. This decomposition induces the following block structure for any semi-infinite matrix

$$M=\left(\begin{array}{c|c} M^{[l]} & M^{[l,\geq l]} \\ \hline M^{[\geq l,l]} & M^{[\geq l]}\end{array}\right).$$

In Corollary 2 in [7] we found an ABC (Aitken–Berg–Collar) type theorem —this denomination is the one that appears in [27] for the OPRL case—

Proposition 3. *The Christoffel–Darboux kernel can be expressed in terms of the inverse of the truncated moment matrix as follows*

$$K^{[l]}(x,y)=(\xi_2^{[l]}(x))^\top (g^{[l]})^{-1}\xi_1^{[l]}(y). \tag{21}$$

Finally what we call the ABC type or Kuijlaars–Daems CD formula for MOP of mixed type is (see Proposition 21 in [7])

Proposition 4. *For $l\geq \max(|\vec n_1|,|\vec n_2|)$ the following holds*

$$\begin{aligned}(x-y)K^{[l]}(x,y)=\sum_{b=1}^{p_2}\bar Q^{(\mathrm{II},b)}_{[\vec\nu_2(l-1)+\vec e_{2,b};\vec\nu_1(l-1)]}(x)Q^{(\mathrm{I},b)}_{[\vec\nu_1(l-1);\vec\nu_2(l-1)-\vec e_{2,b}]}(y)\\ -\sum_{a=1}^{p_1}\bar Q^{(\mathrm{I},a)}_{[\vec\nu_2(l-1);\vec\nu_1(l-1)-\vec e_{1,a}]}(x)Q^{(\mathrm{II},a)}_{[\vec\nu_1(l-1)+\vec e_{1,a};\vec\nu_2(l-1)]}(y).\end{aligned} \tag{22}$$

holds.

This ABC type CD formula for MOP of mixed type has been proven by Kuijlaars an Daems using a Riemann–Hilbert problem, see [14, 15]. Later on, in [7] it was proven for the first time by algebraic means –not relying on analytic conditions as in [14, 15]– using the ABC theorem (21) and the symmetry (17). Here $\{\vec e_{i,a}\}_{a=1}^{p_i}\subset\mathbb R^{p_i}$ stands for the vectors in the respective canonical basis, $i=1,2$. We stress the appearance of $\vec\nu_2(l-1)+\vec e_{2,b}$, $\vec\nu_2(l-1)-\vec e_{2,b}$, $\vec\nu_1(l-1)-\vec e_{1,a}]$ and $\vec\nu_1(l-1)+\vec e_{1,a}$ which are multi-indexes that do not belong to the multi–index sequence associated with the sequence of biorthogonal linear forms $\big\{Q^{(\mathrm{II},a_1(l))}_{[\vec\nu_1(l);\vec\nu_2(l-1)]}(x)\big\}_{l=0}^\infty$ and $\big\{\bar Q^{(\mathrm{I},a_1(k))}_{[\vec\nu_2(k);\vec\nu_1(k-1)]}(x)\big\}_{k=0}^\infty$. Our alternative proposal, despite of having a larger number of terms, as we will see below, involves only linear forms in the sequence.

2. Jacobi type Christoffel–Darboux formula for multiple orthogonal polynomials of mixed type

Given any positive integer $l\in\mathbb Z_+$ we consider the arithmetic congruence modulo p_α; i.e.

$$l=\bar l_\alpha \mod p_\alpha, \qquad \bar l_\alpha\in\{0,1,\dots,p_\alpha-1\}\cong\mathbb Z_{p_\alpha}=\mathbb Z/(p_\alpha\mathbb Z), \qquad \alpha=1,2.$$

The result of this paper is the following

Theorem 1. *For $l \geq \max\{|\vec{n}_1|,|\vec{n}_2|\}$ the following Jacobi type Christoffel–Darboux formula holds*

$$(y-x)K^{[l]}(x,y) = \sum_{(i,j)\in\sigma_1[l]} \bar{Q}^{(\mathrm{I},a_1(j))}_{[\vec{\nu}_2(j);\vec{\nu}_1(j-1)]}(x) J_{j,i} Q^{(\mathrm{II},a_1(i))}_{[\vec{\nu}_1(i);\vec{\nu}_2(i-1)]}(y) - \sum_{(i,j)\in\sigma_2[l]} \bar{Q}^{(\mathrm{I},a_1(j))}_{[\vec{\nu}_2(j);\vec{\nu}_1(j-1)]}(x) J_{j,i} Q^{(\mathrm{II},a_1(i))}_{[\vec{\nu}_1(i);\vec{\nu}_2(i-1)]}(y),$$

where

$$\begin{aligned}\sigma_1[l] &:= \big\{l,\dots,[l,\overline{(a_1(l)-1)}_1]_1^{>}\big\}\times\big\{[l-1,\dots,\overline{(a_1(l-1)+1)}_1]_1^{<}\big\},\\ \sigma_2[l] &:= \big\{[l-1,\overline{(a_2(l-1)+1)}_2]_2^{<},\dots,l-1\big\}\times\big\{l,\dots,[l,\overline{(a_2(l)-1)}_2]_2^{>})\big\}.\end{aligned}$$

Proof. Splitting the eigenvalue property (19) into blocks we get

$$\begin{aligned} JQ(y) = yQ(y) &\Longrightarrow J^{[l]}Q(y)^{[l]} + J^{[l,\geq l]}Q(y)^{[\geq l]} = yQ(y)^{[l]}\\ \bar{Q}(x)^\top J = x\bar{Q}(x)^\top &\Longrightarrow [\bar{Q}(x)^\top]^{[l]}J^{[l]} + [\bar{Q}(x)^\top]^{[\geq l]}J^{[\geq l,l]} = x[\bar{Q}(x)^\top]^{[l]}\end{aligned}$$

Multiply the first equation from the left by $[\bar{Q}(x)^\top]^{[l]}$ and the second one from the right by $Q(y)^{[l]}$ substract both results to obtain

$$\begin{aligned}[\bar{Q}(x)^\top]^{[l]}J^{[l,\geq l]}Q(y)^{[\geq l]} - [\bar{Q}(x)^\top]^{[\geq l]}J^{[\geq l,l]}Q(y)^{[l]} &= (y-x)[\bar{Q}(x)^\top]^{[l]}\cdot Q(y)^{[l]}\\ &= (y-x)K^{[l]}(x,y)\end{aligned}$$

A brief study of the shape of J shows that, even though $J^{[l,\geq l]}$ has semi-infinite length rows, most of its elements are 0. Actually it only contains a finite number of nonzero entries that concentrate in the lower left corner of itself. The same reasoning applies to $J^{[\geq l,l]}$. This matrix has semi infinite length columns but again it only contains a finite number of nonzero terms concentrated in the upper right corner of itself. Of course the number of terms involved in this expression will depend on the value of l. To be more precise we proceed as follows.

After a study of the shape of J we can state

Lemma 1. *For $l \geq \max\{|\vec{n}_1|,|\vec{n}_2|\}$ the only nonzero elements of J along a given row or column are*

$$\begin{matrix} & & & & J_{[l-1,\overline{(a_1(l-1)+1)}_1]_1^{<},l} & & & & \\ & & & & * & & & & \\ & & & & \vdots & & & & \\ & & & & * & & & & \\ J_{l,[l-1,\overline{(a_2(l-1)+1)}_2]_2^{<}} & * & \cdots & * & J_{l,l} & * & \cdots & * & J_{l,[l+1,\overline{(a_1(l+1)-1)}_2]_1^{>}} \\ & & & & * & & & & \\ & & & & \vdots & & & & \\ & & & & * & & & & \\ & & & & J_{[l+1,\overline{(a_2(l+1)-1)}_2]_2^{>},l} & & & & \end{matrix}$$

Using this Lemma we get the desired result and the proof is complete. □

Remarkably, this Jacobi type CD formula is expressed uniquely in terms of our sequences of biorthogonal linear forms $\big\{Q^{(\mathrm{II},a_1(l))}_{[\vec{\nu}_1(l);\vec{\nu}_2(l-1)]}(x)\big\}_{l=0}^{\infty}$ and $\big\{\bar{Q}^{(\mathrm{I},a_1(k))}_{[\vec{\nu}_2(k);\vec{\nu}_1(k-1)]}(x)\big\}_{k=0}^{\infty}$, and does not need of alien multi-indexes to it, as $\vec{\nu}_2(l-1)+\vec{e}_{2,b}$, $\vec{\nu}_2(l-1)-\vec{e}_{2,b}$, $\vec{\nu}_1(l-1)-\vec{e}_{1,a}]$ and $\vec{\nu}_1(l-1)+\vec{e}_{1,a}$ that appear in the standard CD formula for MOP of mixed type (22). The price we have to pay to have all the terms in the sequence of biorthogonal polynomials is that we will need more terms that in the formula (22) in where we have (p_1+p_2) summands (where each summand is strange to the biorthogonal sequence of linear forms).

The number of terms N from the biorthogonal sequence that are needed in this Jacobi type CD formula can be expressed in terms of

$$n_\alpha(l) := l - [(l-1),\overline{(a_\alpha(l-1)+1)}_\alpha]_\alpha^{<}, \qquad \alpha = 1,2.$$

as follows

Proposition 5. *We have the following equation*

$$N=\sum_{k=0}^{[l,\overline{(a_1(l)-1)}_1]_1^{>}-l}[n_1(l+k)-k]+\sum_{k=0}^{[l,\overline{(a_2(l)-1)}_2]_2^{>}-l}[n_2(l+k)-k]\,.$$

The worst situation is reached when the compositions are $\vec{n}_1=(1,\dots,1)$ and $\vec{n}_2=(1,\dots,1)$; in this case we have that

$$N=\frac{|\vec{n}_1|(|\vec{n}_1|+1)}{2}+\frac{|\vec{n}_2|(|\vec{n}_2|+1)}{2}.$$

For any other pair of compositions we have less terms. In order to be more clear let us suppose that $p_1=3$ and $p_2=2$ with $\vec{n}_1=(4,3,2)$ and $\vec{n}_2=(3,2)$. The corresponding Jacobi type matrix has the following shape

$$J=\left(\begin{array}{c|c} J^{[12]} & J^{[12,\geq 12]}\\ \hline J^{[\geq 12,12]} & J^{[\geq 12]}\end{array}\right)=\left(\begin{array}{cccccccccccc|ccccccccccccccc}
\ast & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots\\
\ast & \ast & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots\\
0 & \ast & \ast & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots\\
0 & 0 & \ast & \ast & \ast & \ast & \ast & \ast & \ast & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots\\
0 & 0 & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots\\
0 & 0 & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots\\
0 & 0 & 0 & 0 & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots\\
0 & 0 & 0 & 0 & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots\\
0 & 0 & 0 & 0 & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots\\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ast & \ast & \ast & \ast & \ast & \ast & \ast & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & 1 & 0 & 0 & 0 & \dots\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & 0 & 0 & 0 & \dots\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & 1 & \dots\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \dots\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \dots\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \dots\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \dots\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \dots\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \dots\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ast & \ast & \ast & \ast & \dots\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ast & \ast & \ast & \ast & \dots\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right), \tag{23}$$

where $\ast$ denotes a non-necessarily null real number.

In our example ($p_1=3$, $p_2=2$, $\vec{n}_1=(4,3,2)$ and $\vec{n}_2=(3,2)$) for $l=12$ we have

$$(y-x)K^{[12]}(x,y)=\left[\sum_{i=6}^{11}\sum_{j=12}^{16}\bar{Q}(x)^{(i)}J_{i,j}Q(y)^{(j)}\right]-\left[\sum_{i=12}^{13}\sum_{j=9}^{11}\bar{Q}(x)^{(i)}J_{i,j}Q(y)^{(j)}\right].$$

2.1. **Expressing the Jacobi type matrix in terms of factorization factors.** As we have seen we can write J in terms of S or of $\bar{S}$, this means that each term of J has two different expressions, giving relations between S with $\bar{S}$. We are not too concerned about these relations since what we want here is the most simple expression we can get for the elements of J. It is easy to realize that this is achieved if we use the expression involving S in order to calculate the upper part of J and the expression involving $\bar{S}$ to calculate the lower part of it. Hence, for every $J_{l,k}$ we will have expressions in terms of the factorization matrices coefficients and the elements of their inverses –thus, in terms of the MOPRL and associated second kind functions. The only terms from the factorization matrices (or their inverses) that will be involved when calculating any $J_{l,k}$ are just those between the main diagonal and the $l-|\vec{n}_1|$ diagonal (both included) of S and those between the main diagonal and the $l+|\vec{n}_2|$ diagonal (both included) of $\bar{S}$. And not even all of them. As we are about to see there are three different kinds of elements in J. The ones along the main diagonal, the ones along the immediate closest diagonals to the main one, and finally all the remaining diagonals. The recursion relation coefficients $J_{k,l}$ are ultimately related to the MOPRL and its associated second kind functions in the following way

Proposition 6. *The elements of the recursion matrix J can be written in terms of products of the entries of the LU factorization matrices and its inverses as follows*

$$\begin{aligned}
J_{l,l} &= S_{l,[(l-1),a(l)]_1^<} + S^{-1}_{[(l+1),a(l)]_1^>,l} + \sum_{\substack{a=1,\dots,p_1\\ a\neq a_1(l)}} S_{l,[(l-1),a]_1^<} S^{-1}_{[(l+1),a]_1^>,l}\\
&= \bar S_{l,[(l+1),a(l)]_2^>} \bar S^{-1}_{l,l} + \bar S_{l,l} \bar S^{-1}_{[(l-1),a(l)]_2^<,l} + \sum_{\substack{a=1,\dots,p_2\\ a\neq a_2(l)}} \bar S_{l,[(l+1),a]_2^>} \bar S^{-1}_{[(l-1),a]_2^<,l}\\
J_{l,l+1} &= S^{-1}_{[(l+1),a(l)]_1^>,l+1} + \sum_{\substack{a=1,\dots,p_1\\ a\neq a_1(l)}} S_{l,[(l-1),a]_1^<} S^{-1}_{[(l+1),a]_1^>,l+1}\\
J_{l+1,l} &= \bar S_{l+1,[(l+1),a(l)]_2^>} \bar S^{-1}_{l,l} + \sum_{\substack{a=1,\dots,p_2\\ a\neq a_2(l)}} \bar S_{l+1,[(l+1),a]_2^>} \bar S^{-1}_{[(l-1),a]_2^<,l}\\
J_{l,l+k} &= \sum_{a=\overline{(a_1(l+k-1)+1)}_1}^{\overline{(a_1(l)-1)}_1\ p_1} S_{l,[(l-1),a]_1^<} S^{-1}_{[(l+1),a]_1^>,l+k} \qquad 2\leq k\leq [(l+1)\overline{(a_1(l+1)-1}_1)]_1^< - l,\\
J_{l+k,l} &= \sum_{a=\overline{(a_2(l+k-1)+1)}_2}^{\overline{a_2(l)-1)}_2\ p_2} \bar S_{l+k,[(l+1),a]_2^>} \bar S^{-1}_{[(l-1),a]_2^<,l}, \qquad 2\leq k\leq [(l+1),\overline{(a_2(l+1)-1)}_2]_2^< - l.
\end{aligned}$$

Proof. To prove it we just take the definition (16) of Υ_α, $\alpha=1,2$, and the definition of J given in (18) to compute the different coefficients. □

Where, for $r, r' < p_\alpha$, $\alpha\in\{1,2\}$, we have used

$$\sum_{a=r}^{r'\ p_\alpha} X_a = \begin{cases} \displaystyle\sum_{a=r}^{r'} X_a, & r\leq r',\\ \displaystyle\sum_{a=1}^{r'} X_a + \sum_{a=r}^{p_\alpha} X_a, & r > r'.\end{cases}$$

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