

Orthogonal Laurent polynomials in unit circle, extended CMV ordering and 2D Toda type integrable hierarchies

Carlos Álvarez-Fernández^{1,2,†} and Manuel Mañas^{1,‡}

¹Departamento de Física Teórica II, Universidad Complutense,
28040-Madrid, Spain

²Departamento de Métodos Cuantitativos, Universidad Pontificia Comillas,
28015-Madrid, Spain

[†]calvarez@cee.upcomillas.es, [‡]manuel.manas@fis.ucm.es

Abstract

Orthogonal Laurent polynomials in the unit circle and the theory of Toda-like integrable systems are connected using the Gauss–Borel factorization of a Cantero–Moral–Velázquez moment matrix, which is constructed in terms of a complex quasi-definite measure supported in the unit circle. The factorization of the moment matrix leads to orthogonal Laurent polynomials in the unit circle and the corresponding second kind functions. Jacobi operators, 5-term recursion relations and Christoffel–Darboux kernels, projecting to particular spaces of truncated Laurent polynomials, and corresponding Christoffel–Darboux formulae are obtained within this point of view in a completely algebraic way. Cantero–Moral–Velázquez sequence of Laurent monomials is generalized and recursion relations, Christoffel–Darboux kernels, projecting to general spaces of truncated Laurent polynomials and corresponding Christoffel–Darboux formulae are found in this extended context. Continuous deformations of the moment matrix are introduced and it is shown how they induce a time dependant orthogonality problem related to a Toda-type integrable system, which is connected with the well known Toeplitz lattice. Using the classical integrability theory tools the Lax and Zakharov–Shabat equations are obtained. The dynamical system associated with the coefficients of the orthogonal Laurent polynomials is explicitly derived and compared with the classical Toeplitz lattice dynamical system for the Verblunsky coefficients of Szegő polynomials for a positive measure. Discrete flows are introduced and related to Darboux transformations. Finally, the representation of the orthogonal Laurent polynomials (and its second kind functions), using the formalism of Miwa shifts, in terms of τ -functions is presented and bilinear equations are derived.

Contents

1	Introduction	2
2	Orthogonal Laurent polynomials in the circle, LU factorization and the CMV ordering	4
2.1	Biorthogonal Laurent polynomials	5
2.2	Second kind functions	9
2.3	Recursion relations	15
2.4	Projection operators and the Christoffel–Darboux kernel	19
2.5	Associated Laurent polynomials	20
2.6	The Christoffel–Darboux formula	23
3	Extended CMV ordering and orthogonal Laurent polynomials	24
3.1	Extending the CMV ordering	25

3.2	Functions of the second kind	26
3.3	Recursion relations	27
3.4	The Christoffel–Darboux kernel	29
4	Associated 2D Toda type hierarchies	33
4.1	2D Toda flows	34
4.2	Integrable Toda equations	35
4.3	CMV matrices and the Toeplitz lattice	36
4.4	Discrete flows	37
4.5	τ -functions	40
4.6	Bilinear equations	46
	Appendices	47
	A Proofs	47

1 Introduction

This paper puts focus on orthogonal Laurent polynomials in the unit circle (OLPUC), a subject with strong links to that of orthogonal polynomials in the unit circle (OPUC), see [61], [56] and [57]. Despite that is well established that this matter is as source of interesting problems and applications in approximation theory, we are mainly interested in its connections with the general theory of integrable systems.

Let us introduce here some notation that will be used along this article. We will denote the unit circle by $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$, $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ stands for the unit disk and $\Lambda_{[p,q]} := \text{span}\{z^{-p}, z^{-p+1}, \dots, z^q\}$ denotes the linear space of complex Laurent polynomials with their corresponding restrictions on their degrees while $\Lambda_{[\infty]}$ for the infinite set of Laurent polynomials. When $z \in \mathbb{T}$ we will use the parametrization $z = e^{i\theta}$ with $\theta \in [0, 2\pi)$.

A complex Borel measure μ supported in \mathbb{T} is said to be positive definite if it maps measurable sets into non-negative numbers. When the measure μ is absolutely continuous with respect to the Lebesgue measure of the circle $d\theta$, according to the Radon–Nikodym theorem, it can be always expressed using a complex weight function w , so that $d\mu(\theta) = w(\theta)d\theta$. If in addition the measure is positive definite then the weight function should be a non-negative function on \mathbb{T} . For notational simplicity we will use, whenever it is convenient, the complex notation $d\mu(z) = ie^{i\theta}d\mu(\theta)$. If μ is a positive Borel measure supported in \mathbb{T} , then the OPUC or Szegő polynomials are monic polynomials P_n of degree equal or less than n that satisfy the following system of equations, the orthogonality relations,

$$\int_{\mathbb{T}} P_n(z)z^{-k}d\mu(z) = 0, \quad k = 0, 1, \dots, n - 1. \quad (1)$$

It is well known the deep connections between orthogonal polynomials in the real line (OPRL) in $[-1, 1]$ and OPUC (e.g. [31, 17]). Let us observe that for this analysis the use of spectral theory techniques requires the study of the operator of multiplication by z . The study of the matrix associated to this operator leads to recurrence laws. Both, OPRL and OPUC have recurrence laws, but the main difference is that in the real case three term recurrence laws provide a tridiagonal matrix, the so called Jacobi operator, while in the circle case one is lead to a Hessenberg matrix [37], leading to a more involved scenario to deal with than the Jacobi case (as it is not a sparse matrix with a finite number of non vanishing diagonals). More precisely, the study of the recurrence relations for the OPUC requires the definition of the reciprocal or reverse Szegő polynomials $P_l^*(z) := z^l \overline{P_l(z^{-1})}$ and the reflection or Verblunsky¹ coefficients $\alpha_l := P_l(0)$.

¹Schur parameters is another usual name. The definition is not unique and $\alpha_l := -\overline{P_{l+1}(0)}$ is another common definition.

With these elements the recursion relations for the Szegő polynomials can be written as

$$\begin{pmatrix} P_l \\ P_l^* \end{pmatrix} = \begin{pmatrix} z & \alpha_l \\ z\bar{\alpha}_l & 1 \end{pmatrix} \begin{pmatrix} P_{l-1} \\ P_{l-1}^* \end{pmatrix}. \quad (2)$$

There has been a relevant number of studies on the zeroes of the OPUC, see for example [10, 14, 15], or [32, 35, 47, 51], which have interesting applications to signal analysis theory, see [39, 41, 52, 53]. Despite of that the situation is still far from the corresponding state of the art in the OPRL context. A second important issue is the fact that the set of Szegő polynomials is in general not dense in the Hilbert space $L^2(\mathbb{T}, \mu)$. As it follows from Szegő's theorem it holds that for a nontrivial probability measure on $d\mu$ with Verblunsky coefficients $\{\alpha_n\}_{n=0}^\infty$ the Szegő's polynomials are dense in $L^2(\mathbb{T}, \mu)$ if and only if $\prod_{n=0}^\infty (1 - |\alpha_n|^2) = 0$. If $d\mu$ is an absolutely continuous probability measure then the Kolmogorov's density theorem follows: polynomials are dense in $L^2(\mathbb{T}, \mu)$ if and only if the so called Szegő's condition $\int_{\mathbb{T}} \log(w(\theta)) d\theta = -\infty$ is fulfilled [60].

Orthogonal Laurent polynomials in the real line (OLPRL), where introduced in [42, 43] in the context of the strong Stieltjes moment problem, i.e finding a positive measure μ such that its moments

$$m_j = \int_{\mathbb{R}} x^j d\mu(x) \quad j = 0, \pm 1, \pm 2, \dots \quad (3)$$

are known in advance. When the moment problem has a solution, there exist polynomials $\{Q_n\}$ such that

$$\int_{\mathbb{R}} x^{-n+j} Q_n(x) d\mu(x) = 0, \quad j = 0, \dots, n-1, \quad (4)$$

which are called Laurent polynomials or L-polynomials. The theory of Laurent polynomials on the real line was developed in parallel with the theory of orthogonal polynomials, see [22, 29, 40] and [50]. Orthogonal Laurent polynomials theory was carried from the real line to the circle [62] and subsequent works broadened the matter (e.g. [25, 21, 23, 24]), treating subjects like recursion relations, Favard's theorem, quadrature problems, and Christoffel–Darboux formulae.

The analysis of OLPUC, and specially the use of the Cantero–Moral–Velázquez (CMV) [21] representation is very helpful in the study of a number properties of Szegő polynomials. Different reasons support this statement, for example as we mentioned while the OLPUC are always dense in $L^2(\mathbb{T}, \mu)$ this is not true in general for the OPUC, see [18] and [25], and also the bijection between OLPUC in the CMV representation and the ordinary Szegő polynomials allows to replace the complicated recurrence relations with a five-term recurrence relation more alike to the structure of the OPRL. The representation of the operator of multiplication by z is much more natural using CMV matrices than using Hessenberg matrices. This is the main motivation for us in order to take CMV matrices as a essential element in our scheme. Other papers have reviewed and broadened the study of CMV matrices, see for example [58, 44]. Alternative or generic orders in the base used to span $\Lambda_{[\infty]}$ can be found in [24]

The approach to the integrable hierarchies that we use here is based on the Gauss–Borel factorization. The seminal paper of M. Sato [55] and further developments performed by the Kyoto school [26]-[28] settled the Lie-group theoretical description of the integrable hierarchies. It was M. Mulase, in the key paper [49], the one who made the connection between factorization problems, dressing procedures and integrability. In this context, K. Ueno and T. Takasaki [63] performed an analysis of the Toda-type hierarchies and their soliton-like solutions. In a series of papers, M. Adler and P. van Moerbeke [3]-[9], made clear the connection between the Lie-group factorization, applied to Toda-type hierarchies (what they call discrete K-P) and the Gauss–Borel factorization applied to a moment matrix that comes from an orthogonality problems; thus, the corresponding orthogonal polynomials are closely related to specific solutions of the integrable hierarchy. See [45] and [11] for further developments in relation with the factorization problem, multicomponent Toda lattices and generalized orthogonality. In the paper [8] a profound study of the OPUC and the Toda-type associated lattice, called the Toeplitz lattice (TL), was performed. A relevant reduction of the equations of the TL has been found by L. Golinskii [36] in the context of Schur flows when the measure is invariant

under conjugation, another interesting paper on this subject is [48]. The Toeplitz Lattice has been proved equivalent to the Ablowitz–Ladik lattice (ALL), [1, 2], and that work has been generalized to the link between matrix orthogonal polynomials and the non-Abelian ALL in [20]. Both of them have to deal with the Hessenberg operator for the multiplication by z .

Our aim is to explore the connection between Toda-type integrable systems and orthogonality in the circle from a different point of view. As we proof in this paper the CMV representation is a bridge between the factorization techniques used in [12] and the circular case. We will see that many results obtained in [12] about Christoffel–Darboux (CD) formulae, continuous and discrete deformations, and τ -function theory can be extended to the circular case under the suitable choice of moment matrices and shift operators.

Let us recall the reader that measures and linear functionals are closely connected; given a linear functional \mathcal{L} on $\Lambda_{[\infty]}$ we define the corresponding moments of \mathcal{L} as $c_n := \mathcal{L}[z^n]$ for all the possible integer values of $n \in \mathbb{Z}$. The functional \mathcal{L} is said to be Hermitian whenever $c_{-n} = \overline{c_n}$, $\forall n \in \mathbb{Z}$. Moreover, the functional \mathcal{L} is defined as quasi-definite (positive definite) when the principal submatrices of the Toeplitz moment matrix $(\Delta_{i,j})$, $\Delta_{i,j} := c_{i-j}$, associated to the sequence c_n is non-singular (positive definite), i.e. $\forall n \in \mathbb{Z}, \Delta_n := \det(c_{i-j})_{i,j=0}^n \neq 0 (> 0)$. Some aspects on quasi-definite functionals and their perturbations are studied in [13, 19].

It is known [34] that when the linear functional \mathcal{L} is Hermitian and positive definite there exist a finite positive Borel measure with a support lying on \mathbb{T} such that $\mathcal{L}[f] = \int_{\mathbb{T}} f d\mu$, $\forall f \in \Lambda_{[\infty]}$. In addition, an Hermitian positive definite linear functional \mathcal{L} defines a sesquilinear form $\langle \cdot, \cdot \rangle_{\mathcal{L}} : \Lambda_{[\infty]} \times \Lambda_{[\infty]} \mapsto \mathbb{C}$ as $\langle f, g \rangle_{\mathcal{L}} = \mathcal{L}[f\bar{g}]$, $\forall f, g \in \Lambda_{[\infty]}$. Two Laurent polynomials $\{f, g\} \subset \Lambda_{[\infty]}$ are said to be orthogonal with respect to \mathcal{L} if $\langle f, g \rangle_{\mathcal{L}} = 0$. From the properties of \mathcal{L} is easy to see that $\langle \cdot, \cdot \rangle_{\mathcal{L}}$ is a scalar product and if μ is the positive finite Borel measure associated to \mathcal{L} we are lead to the corresponding Hilbert space $L^2(\mathbb{T}, \mu)$, the closure of $\Lambda_{[\infty]}$. As we mentioned before $\{P_l\}_{l=0}^{\infty}$ denotes the set of monic orthogonal polynomials, $\deg P_l \leq l$, for a positive measure μ satisfying (1) and therefore $\{P_l\}_{l=0}^q$ is an orthogonal basis of the space of truncated polynomials $\Lambda_{[0,q]}$.

In this work we allow for a more general setting assuming that \mathcal{L} is just quasi-definite, which is associated to a corresponding quasi-definite complex measure μ , see [33]. As before a sesquilinear form $\langle \cdot, \cdot \rangle_{\mathcal{L}}$ is defined for any such linear functional \mathcal{L} ; thus, we just have the linearity (in the first entry) and skew-linearity (in the second entry) properties. However, we have no symmetry allowing the interchange of the two arguments. We formally broaden the notion of orthogonality and say that f is orthogonal to g if $\langle f, g \rangle_{\mathcal{L}} = 0$, but we must be careful as in this more general situation it could happen that $\langle f, g \rangle_{\mathcal{L}} = 0$ but $\langle g, f \rangle_{\mathcal{L}} \neq 0$.

The layout of this paper goes as follows. In §2 we present the application of the Gauss–Borel factorization of a CMV moment matrix to the construction of OLPUC, associated second kind functions, 5-term recursion relations and CD formulae. In §3 we perform a similar work using a more general sequence that the one used in [21]. This allows us to study snake-shaped (as are denoted in [24]) recursion formulae. Moreover, the CD formulae we derive for these extended cases is the kernel for the orthogonal projection to the general space of truncated Laurent polynomials, $\Lambda_{[p,q]}$ with $p, q \in \mathbb{N}$. This is a large generalization of the CMV situation as the possible spaces of truncated Laurent polynomials is very particular, namely either $\Lambda_{[l,l]}$ or $\Lambda_{[l+1,l]}$ with $l \in \mathbb{N}$. To conclude, in §4 we study the deformations of the moment matrices to obtain the integrable equations associated, the representation of the OLPUC and its associated second kind functions using τ -functions and corresponding bilinear identities.

2 Orthogonal Laurent polynomials in the circle, LU factorization and the CMV ordering

In this section we use the Gauss–Borel, also known as LU or Gaussian, factorization of an infinite dimensional matrix, that we refer to as moment matrix, to derive bi-orthogonal Laurent polynomials in the unit circle (BOLPUC) to the given measure μ —that, as we will see, are closely related to the Szegő polynomials—, associated second kind functions in terms of the Fourier series of the measure, their recursion relations and

corresponding CD formulae very much in the spirit of [12]. The key idea is to use the results of [21] to construct a very specific moment matrix that will lead to the mentioned results.

2.1 Biorthogonal Laurent polynomials

We are ready to, using a CMV moment type matrix, find a set of bi-orthogonal Laurent polynomials, its connection with Szegő polynomials and corresponding determinantal expressions. For this aim, let us consider the basic object fixing the CMV order of the Fourier family $\{z^j\}_{j \in \mathbb{Z}}$. This order allows us to work in realm of the semi-infinite matrices avoiding in this way the less convenient scenario of bi-infinite matrices, very much as in the multiple OPRL situation [12].

Definition 1. *We denote*

$$\chi_1(z) := (1, 0, z, 0, z^2, 0, \dots)^\top, \quad \chi_2(z) := (0, 1, 0, z, 0, z^2, \dots)^\top, \quad \chi_a^*(z) := z^{-1}\chi_a(z^{-1}), \quad a = 1, 2.$$

$$\chi(z) := \chi_1(z) + \chi_2^*(z) = (1, z^{-1}, z, z^{-2}, \dots)^\top, \quad \chi^*(z) := \chi_1^*(z) + \chi_2(z) = (z^{-1}, 1, z^{-2}, z, \dots)^\top.$$

With these sequences at hand and with a given quasi-definite complex Borel measure μ supported on \mathbb{T} we define the CMV moment matrix

Definition 2. *The CMV moment matrix is the following semi-infinite complex-valued matrix²*

$$g := \oint_{\mathbb{T}} \chi(z)\chi(z)^\dagger d\mu(z). \quad (5)$$

The LU factorization, which will play an important role in what follows, is

$$g = S_1^{-1}S_2, \quad (6)$$

where S_1 is a normalized³ lower triangular matrix and S_2 is an upper triangular matrix. Hence we write

$$S_1 = \begin{pmatrix} 1 & 0 & 0 & \dots \\ (S_1)_{10} & 1 & 0 & \dots \\ (S_1)_{20} & (S_1)_{21} & 1 & \dots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix}, \quad S_2 = \begin{pmatrix} (S_2)_{00} & (S_2)_{01} & (S_2)_{02} & \dots \\ 0 & (S_2)_{11} & (S_2)_{12} & \dots \\ 0 & 0 & (S_2)_{22} & \dots \\ \vdots & \vdots & \dots & \ddots \end{pmatrix}.$$

This Gaussian factorization for the moment matrix makes sense if all the principal minors are non-singular, which is precisely the quasi-definiteness condition for the measure μ . More on the algebraic Gauss–Borel (or LU) factorization and its connection with integrable systems can be read in [30].

With the aid of these matrices we consider the sequences

$$\Phi_{1,1} := S_1\chi_1, \quad \Phi_{1,2} := S_1\chi_2^*, \quad \Phi_{2,1} := (S_2^{-1})^\dagger\chi_1, \quad \Phi_{2,2} := (S_2^{-1})^\dagger\chi_2^*,$$

which can be written as semi-infinite vectors

$$\Phi_{1,a}(z) = \begin{pmatrix} \varphi_{1,a}^{(0)}(z) \\ \varphi_{1,a}^{(1)}(z) \\ \vdots \end{pmatrix}, \quad \Phi_{2,a}(z) = \begin{pmatrix} \varphi_{2,a}^{(0)}(z) \\ \varphi_{2,a}^{(1)}(z) \\ \vdots \end{pmatrix},$$

²The reader should notice that if μ is a positive measure then g is a definite positive Hermitian matrix; i.e., $g = g^\dagger$.

³The coefficients in the main diagonal are equal to the unity.

for $a = 1, 2$. The corresponding components $\varphi_{b,1}^{(l)}$ are polynomials of degree l in variable z , while $\varphi_{b,2}^{(l)}$ are polynomials of degree $l + 1$ in the variable z^{-1} which vanish at $z = \infty$. Inspired by the multiple orthogonal case [12] we define the following sequences of Laurent polynomials

$$\Phi_1(z) := S_1 \chi(z) = \Phi_{1,1} + \Phi_{1,2}, \quad \Phi_2(z) := (S_2^{-1})^\dagger \chi(z) = \Phi_{2,1} + \Phi_{2,2},$$

which are semi-infinite vectors that we write in the form

$$\Phi_1(z) = \begin{pmatrix} \varphi_1^{(0)}(z) \\ \varphi_1^{(1)}(z) \\ \vdots \end{pmatrix}, \quad \Phi_2(z) = \begin{pmatrix} \varphi_2^{(0)}(z) \\ \varphi_2^{(1)}(z) \\ \vdots \end{pmatrix},$$

with its coefficients being Laurent polynomials.

As we said in §1 the measure μ has an associated sesquilinear form $\langle \cdot, \cdot \rangle_{\mathcal{L}}$ acting on any pair of Laurent polynomials in \mathbb{T} , $\varphi_1(z)$ and $\varphi_2(z)$, as

$$\langle \varphi_1, \varphi_2 \rangle_{\mathcal{L}} := \oint_{\mathbb{T}} \varphi_1(z) \bar{\varphi}_2(z^{-1}) d\mu(z). \quad (7)$$

From the definition is clear that g (whose principal minors should not vanish) is the matrix associated to $\langle \cdot, \cdot \rangle_{\mathcal{L}}$. The reader can check that the principal minors of g are exactly the Toeplitz minors Δ_n (one matrix is obtained from the other using permutations). As we are going to use quasi-definite measures the factorization condition will hold in the subsequent work.

We recall the reader that given two linear spaces V, V' and a sesquilinear form

$$\begin{aligned} \langle \cdot, \cdot \rangle_{\mathcal{L}} : V \times V' &\rightarrow \mathbb{C} \\ (x, y) &\mapsto \langle x, y \rangle_{\mathcal{L}} \end{aligned}$$

we say that sets $X \subset V$ and $Y \subset V'$ are bi-orthogonal if $\langle x, y \rangle_{\mathcal{L}} = 0$ for all $x \in X$ and $y \in Y$.

Theorem 1. *The sets of Laurent polynomials $\{\varphi_1^{(l)}\}_{l=0}^{\infty}$ and $\{\varphi_2^{(l)}\}_{l=0}^{\infty}$ are bi-orthogonal in the unit circle with respect to the sesquilinear form defined in (7), that is*

$$\langle \varphi_1^{(l)}, \varphi_2^{(k)} \rangle_{\mathcal{L}} = \oint_{\mathbb{T}} \varphi_1^{(l)}(z) \bar{\varphi}_2^{(k)}(z^{-1}) d\mu(z) = \delta_{l,k} \quad l, k = 0, 1, \dots \quad (8)$$

Proof. We compute

$$\begin{aligned} \oint_{\mathbb{T}} \Phi_1(z) \bar{\Phi}_2(z^{-1})^\top d\mu(z) &= \oint_{\mathbb{T}} \Phi_1(z) \Phi_2(z)^\dagger d\mu(z) \\ &= S_1 \left[\oint_{\mathbb{T}} \chi(z) \chi(z)^\dagger d\mu(z) \right] S_2^{-1} \\ &= \mathbb{I}. \end{aligned}$$

□

Orthogonality relations (8) can alternatively be expressed as follows

$$\begin{aligned} \langle \varphi_1^{(2l)}, z^k \rangle_{\mathcal{L}} &= \oint_{\mathbb{T}} \varphi_1^{(2l)}(z) z^{-k} d\mu(z) = 0, \quad k = -l, \dots, l-1, \\ \langle \varphi_1^{(2l+1)}, z^k \rangle_{\mathcal{L}} &= \oint_{\mathbb{T}} \varphi_1^{(2l+1)}(z) z^{-k} d\mu(z) = 0, \quad k = -l, \dots, l, \\ \langle z^k, \varphi_2^{(2l)} \rangle_{\mathcal{L}} &= \oint_{\mathbb{T}} \bar{\varphi}_2^{(2l)}(z^{-1}) z^k d\mu(z) = 0, \quad k = -l, \dots, l-1, \\ \langle z^k, \varphi_2^{(2l+1)} \rangle_{\mathcal{L}} &= \oint_{\mathbb{T}} \bar{\varphi}_2^{(2l+1)}(z^{-1}) z^k d\mu(z) = 0, \quad k = -l, \dots, l. \end{aligned} \quad (9)$$

Proposition 1. *Given a positive definite measure μ there exists $h_l \in \mathbb{R}$, $l = 0, 1, \dots$, such that*

$$\varphi_2^{(l)} = h_l^{-1} \varphi_1^{(l)}$$

Proof. See Appendix A. □

Therefore, are proportional Laurent polynomials and bi-orthogonality (8) implies that $\{\varphi_1^{(l)}\}_{l=0}^\infty$ and $\{\varphi_2^{(l)}\}_{l=0}^\infty$ are sets of orthogonal Laurent polynomials for the positive measure μ , that is

$$\begin{aligned} \langle \varphi_1^{(l)}, \varphi_1^{(k)} \rangle_{\mathcal{L}} &= \oint_{\mathbb{T}} \varphi_1^{(l)}(z) \bar{\varphi}_1^{(k)}(z^{-1}) d\mu(z) = \delta_{l,k} h_l \quad l, k = 0, 1, \dots \\ \langle \varphi_2^{(l)}, \varphi_2^{(k)} \rangle_{\mathcal{L}} &= \oint_{\mathbb{T}} \varphi_2^{(l)}(z) \bar{\varphi}_2^{(k)}(z^{-1}) d\mu(z) = \delta_{l,k} h_l^{-1} \quad l, k = 0, 1, \dots \end{aligned} \quad (10)$$

We are now ready to write the relationship between these CMV Laurent polynomials and the Szegő polynomials P_l introduced previously.

Proposition 2. *If the measure μ is positive definite we have the following identifications between the CMV Laurent polynomials, the Szegő polynomials and their reciprocals*

$$\varphi_1^{(2l)}(z) = z^{-l} P_{2l}(z), \quad \varphi_1^{(2l+1)}(z) = z^{-l-1} P_{2l+1}^*(z). \quad (11)$$

Proof. To check it just observe that

$$\oint_{\mathbb{T}} z^l \varphi_1^{(2l)}(z) z^{-k} d\mu(z) = 0, \quad k = 0, \dots, 2l - 1.$$

Hence, $z^l \varphi_1^{(2l)}(z)$ has the same orthogonality relations that P_{2l} and both are monic polynomials of degree $2l$; uniqueness leads to their identification. In a similar way we proceed for the odd polynomials. Indeed,

$$\oint_{\mathbb{T}} z^{l+1} \varphi_1^{(2l+1)}(z) z^{-k} d\mu(z) = 0, \quad k = 1, \dots, 2l + 1,$$

that is, $z^{l+1} \varphi_1^{(2l+1)}(z)$ has the same orthogonality relations that the polynomial P_{2l+1}^* (that makes them proportional) and both are equal to 1 at $z = 0$, consequently they are the same. □

Using the Verblunsky coefficients⁴ we can write

$$\begin{aligned} \varphi_1^{(2l)}(z) &= \alpha_{2l} z^{-l} + \dots + z^l, \\ \varphi_1^{(2l+1)}(z) &= z^{-l-1} + \dots + \bar{\alpha}_{2l+1} z^l. \end{aligned}$$

For later use, and in addition to the reflection coefficients, it is also useful to define the sequence $\rho_l := \sqrt{1 - |\alpha_l|^2}$, related to $\{h_l\}_{l=0}^\infty$ by

$$\rho_l^2 = \frac{h_l}{h_{l-1}},$$

valid for $l > 0$, with $\rho_0 = 0$.

In the general quasi-definite case, we can perform a very similar construction. We write

$$\begin{aligned} \varphi_1^{(2l)}(z) &= \alpha_{2l}^{(1)} z^{-l} + \dots + z^l, \\ \varphi_1^{(2l+1)}(z) &= z^{-l-1} + \dots + \bar{\alpha}_{2l+1}^{(2)} z^l, \end{aligned} \quad (12)$$

$$\begin{aligned} \varphi_2^{(2l)}(z) &= \bar{h}_{2l}^{-1} \alpha_{2l}^{(2)} z^{-l} + \dots + \bar{h}_{2l}^{-1} z^l, \\ \varphi_2^{(2l+1)}(z) &= \bar{h}_{2l+1}^{-1} z^{-l-1} + \dots + \bar{h}_{2l+1}^{-1} \bar{\alpha}_{2l+1}^{(1)} z^l, \end{aligned} \quad (13)$$

⁴See the Introduction.

and also $\rho_0^2 := 0, \rho_l^2 := \frac{h_l}{h_{l-1}}, l = 1, 2, \dots$. The Hermitian case can be considered a particular reduction where $\alpha_l^{(2)} = \alpha_l^{(1)}$. The reason for the notation stands in the following fact. Given a quasi-definite measure μ we can find two families of monic orthogonal polynomials such that

$$\begin{aligned} \int_{\mathbb{T}} P_l^{(1)}(z) z^{-k} d\mu(z) &= 0 \quad k = 0, 1, \dots, l-1, \\ \int_{\mathbb{T}} z^k \bar{P}_l^{(2)}(z^{-1}) d\mu(z) &= 0 \quad k = 0, 1, \dots, l-1. \end{aligned} \quad (14)$$

If we call $\alpha_l^{(1)} = P_l^{(1)}(0)$, and $\alpha_l^{(2)} = P_l^{(2)}(0)$ then these coefficients are related with the Laurent Polynomial coefficients and we can obtain the quasi-definite version of Proposition 2, that is

$$\begin{aligned} \varphi_1^{(2l)}(z) &= z^{-l} P_{2l}^{(1)}(z), & \varphi_1^{(2l+1)}(z) &= z^{-l-1} P_{2l+1}^{(2)*}(z), \\ \varphi_2^{(2l)}(z) &= \bar{h}_{2l}^{-1} z^{-l} P_{2l}^{(2)}(z), & \varphi_2^{(2l+1)}(z) &= \bar{h}_{2l+1}^{-1} z^{-l-1} P_{2l+1}^{(1)*}(z). \end{aligned} \quad (15)$$

In what follows $g^{[l]} := \sum_{i,j=0}^{l-1} g_{i,j} E_{i,j}$ denotes the $l \times l$ truncated moment matrix and $\chi^{[l]}$ is the truncated vector consisting of the l first components of χ . With this notation we can express the bi-orthogonal Laurent polynomials in different ways:

Proposition 3. *The following expressions hold true*

$$\varphi_1^{(l)}(z) = \chi^{(l)} - (g_{l,0} \ g_{l,1} \ \cdots \ g_{l,l-1}) (g^{[l]})^{-1} \chi^{[l]} \quad (16)$$

$$= (S_2)_l (0 \ 0 \ \dots \ 0 \ 1) (g^{[l+1]})^{-1} \chi^{[l+1]} \quad (17)$$

$$= \frac{1}{\det g^{[l]}} \det \begin{pmatrix} g_{0,0} & g_{0,1} & \cdots & g_{0,l-1} & \chi^{(0)}(z) \\ g_{1,0} & g_{1,1} & \cdots & g_{1,l-1} & \chi^{(1)}(z) \\ \vdots & \vdots & & \vdots & \vdots \\ g_{l,0} & g_{l,1} & \cdots & g_{l,l-1} & \chi^{(l)}(z) \end{pmatrix}, \quad l \geq 1, \quad (18)$$

and

$$\bar{\varphi}_2^{(l)}(\bar{z}) = (S_2)_l^{-1} \left((\chi^{(l)})^\dagger - (\chi^{[l]})^\dagger (g^{[l]})^{-1} \begin{pmatrix} g_{0,l} \\ g_{1,l} \\ \vdots \\ g_{l-1,l} \end{pmatrix} \right) \quad (19)$$

$$= (\chi^{[l+1]})^\dagger (g^{[l+1]})^{-1} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \quad (20)$$

$$= \frac{1}{\det g^{[l+1]}} \det \begin{pmatrix} g_{0,0} & g_{0,1} & \cdots & g_{0,l} \\ g_{1,0} & g_{1,1} & \cdots & g_{1,l} \\ \vdots & \vdots & & \vdots \\ g_{l-1,0} & g_{l-1,1} & \cdots & g_{l-1,l} \\ \hline \chi^{(0)}(z) & \chi^{(1)}(z) & \cdots & \chi^{(l)}(z) \end{pmatrix}, \quad l \geq 1. \quad (21)$$

Proof. See Appendix A. □

Similar expressions hold for $\varphi_{a,1}^{(l)}$ replacing χ by χ_1 , $a = 1, 2$, and for $\varphi_{a,2}^{(l)}$ replacing χ by χ_2^* , $a = 1, 2$.

2.2 Second kind functions

In this subsection we introduce the second kind functions associated with the orthogonal Laurent polynomials discussed before. First we present determinantal expressions, then we connect them with the Fourier series of the measure and also with corresponding Cauchy/Caratheorody transforms.

Definition 3. *The partial second kind sequences are given by*

$$C_{1,1}(z) := (S_1^{-1})^\dagger \chi_1^*(z), \quad C_{1,2}(z) := (S_1^{-1})^\dagger \chi_2(z), \quad C_{2,1}(z) := S_2 \chi_1^*(z), \quad C_{2,2}(z) := S_2 \chi_2(z).$$

and the second kind sequences

$$C_1(z) := (S_1^{-1})^\dagger \chi^*(z), \quad C_2(z) := S_2 \chi^*(z).$$

Observe that

$$C_1 = C_{1,1} + C_{1,2}, \quad C_2 = C_{2,1} + C_{2,2}. \quad (22)$$

We have the expressions as semi-infinite vectors, being its coefficients what we call second kind functions

$$C_{1,a}(z) := \begin{pmatrix} C_{1,a}^{(0)}(z) \\ C_{1,a}^{(1)}(z) \\ \vdots \end{pmatrix}, \quad C_{2,a}(z) := \begin{pmatrix} C_{2,a}^{(0)}(z) \\ C_{2,a}^{(1)}(z) \\ \vdots \end{pmatrix}, \quad C_1(z) := \begin{pmatrix} C_1^{(0)}(z) \\ C_1^{(1)}(z) \\ \vdots \end{pmatrix}, \quad C_2(z) := \begin{pmatrix} C_2^{(0)}(z) \\ C_2^{(1)}(z) \\ \vdots \end{pmatrix}.$$

The expressions just given for the second kind functions are, in principle, formal, as the matrix products lead to series, not necessarily convergent, instead of finite sums. In fact (as we will show in Proposition 5) they are well defined in terms of the bi-orthogonal Laurent polynomials and truncated Fourier series of the measure, with convergence ensured in some annulus centered at the origin of the complex plane. The coefficients of these sequences are called second kind functions.

One can find analogous determinantal type expressions as those for the OLPUC given in Proposition 3, if we define

$$\Gamma_{1,j}^{(l)} := \sum_{k \geq l} g_{jk} \chi^{*(k)}, \quad \Gamma_{2,j}^{(l)} := \sum_{k \geq l} g_{jk}^\dagger \chi^{*(k)}.$$

Proposition 4. *The second kind functions have the following determinantal expressions for $l \geq 1$*

$$\overline{C_1^{(l)}(z)} = \frac{1}{\det g^{[l+1]}} \det \begin{pmatrix} g_{0,0} & g_{0,1} & \cdots & g_{0,l} \\ g_{1,0} & g_{1,1} & \cdots & g_{1,l} \\ \vdots & \vdots & & \vdots \\ g_{l-1,0} & g_{l-1,1} & \cdots & g_{l-1,l} \\ \hline \Gamma_{2,0}^{(l)}(z) & \Gamma_{2,1}^{(l)}(z) & \cdots & \Gamma_{2,l}^{(l)}(z) \end{pmatrix}, \quad (23)$$

and

$$C_2^{(l)}(z) = \frac{1}{\det g^{[l]}} \det \begin{pmatrix} g_{0,0} & g_{0,1} & \cdots & g_{0,l-1} & \Gamma_{1,0}^{(l)}(z) \\ g_{1,0} & g_{1,1} & \cdots & g_{1,l-1} & \Gamma_{1,1}^{(l)}(z) \\ \vdots & \vdots & & \vdots & \vdots \\ g_{l,0} & g_{l,1} & \cdots & g_{l,l-1} & \Gamma_{1,l}^{(l)}(z) \end{pmatrix}. \quad (24)$$

Proof. See Appendix A. □

Now, we introduce the following definition

Definition 4. 1. For the bi-orthogonal Laurent polynomials $\varphi_1^{(l)}$ and $\varphi_2^{(l)}$ we use the notation $\varphi_2^{(l)}(e^{i\theta}) = \sum_{|k| \ll \infty} \varphi_{2,k}^{(l)} e^{ik\theta}$ and $\varphi_1^{(l)}(e^{i\theta}) = \sum_{|k| \ll \infty} \varphi_{1,k}^{(l)} e^{ik\theta}$,⁵

2. Let

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} d\mu(\theta), \quad F_\mu(u) = \sum_{n=-\infty}^{\infty} c_n u^n,$$

be the Fourier coefficients and the Fourier series of the measure μ .

3. For each integer k we introduce the following truncated Fourier series

$$F_{\mu,k}^{(+)}(z) := \sum_{n \geq -k} c_n z^n, \quad F_{\mu,k}^{(-)}(z) := \sum_{n < -k} c_n z^n.$$

Observations

1. It holds that $c_n(\bar{\mu}) = \overline{c_{-n}(\mu)}$, hence $\overline{c_{-n}} = c_n$ for real measures. Consequently,

$$F_{\bar{\mu},k}^{(+)}(z) = \overline{F_{\mu,-k-1}^{(-)}(z^{-1})}, \quad F_{\bar{\mu},k}^{(-)}(z) = \overline{F_{\mu,-k-1}^{(+)}(z^{-1})}, \quad F_{\bar{\mu}}(z) = \overline{F_{\mu}(z^{-1})}.$$

2. The Fourier series always converges in $\mathcal{D}'(\mathbb{T})$, the space of distributions in the circle, so that $\int_0^{2\pi} F_\mu(\theta) f(\theta) d\theta = \int_0^{2\pi} f(\theta) d\mu(\theta)$, $\forall f \in \mathcal{D}(\mathbb{T})$, here $\mathcal{D}(\mathbb{T})$ denotes the linear space of test functions on the circle. For an absolutely continuous measure $d\mu(\theta) = w(\theta) d\theta$ we can write $d\mu(\theta) = F_\mu(\theta) d\theta$.

3. We will also consider the Laurent series $F_\mu(z) = \sum_{n=-\infty}^{\infty} c_n z^n$, for $z \in \mathbb{C}$. Notice that $F_{\mu,k}^{(+)} + F_{\mu,k}^{(-)} = F_\mu$.

4. Let $D(0; r, R) = \{z \in \mathbb{C} : r < |z| < R\}$ denote the annulus around $z = 0$ with interior and exterior radii r and R and $R_\pm := (\limsup_{n \rightarrow \infty} \sqrt[n]{|c_{\pm n}|})^{\mp 1}$. Then, according to the Cauchy–Hadamard theorem, we have

- The series $F_{\mu,k}^{(+)}(z)$ converges uniformly in any compact set K , $K \subset D(0; 0, R_+)$.
- The series $F_{\mu,k}^{(-)}(z)$ converges uniformly in any compact set K , $K \subset D(0; R_-, \infty)$.
- The series $F_\mu(z)$ converges uniformly in any compact set K , $K \subset D(0; R_-, R_+)$.

5. According to F. and M. Riesz theorem if $c_n = 0$, $n < 0$, then μ is absolutely continuous with respect to the Lebesgue measure on \mathbb{T} [54]. In fact if $f(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\mu(\theta)}{1 - ze^{-i\theta}}$, then $w(\theta) = \lim_{r \rightarrow 1} f(re^{i\theta}) \in L^1(\mathbb{T})$ and $d\mu = w(\theta) d\theta$; therefore, in this case we have $f(z) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f(u)}{u-z} du$, a holomorphic function in \mathbb{D} . Moreover, the set of $w(\theta) = f(e^{i\theta}) \in L^2(\mathbb{T})$ with $c_n = 0$, $n < 0$, is isometric to the set H^2 of holomorphic functions in \mathbb{D} with limits when $r \rightarrow 1$ in $L^2(\mathbb{T})$, observe that $w = \sum_{n=0}^{\infty} c_n e^{ni\theta} = F_\mu$.

Proposition 5. The partial second kind functions can be expressed as

$$\begin{aligned} C_{1,1}^{(l)} &= 2\pi \sum_{|k| \ll \infty} \varphi_{2,k}^{(l)} z^{-k-1} \bar{F}_{\mu,-k-1}^{(-)}(z), \quad R_- < |z| < \infty, \quad C_{1,2}^{(l)} = 2\pi \sum_{|k| \ll \infty} \varphi_{2,k}^{(l)} z^{-k-1} \bar{F}_{\mu,-k-1}^{(+)}(z), \quad 0 < |z| < R_+, \\ C_{2,1}^{(l)} &= 2\pi \sum_{|k| \ll \infty} \varphi_{1,k}^{(l)} z^{-k-1} F_{\mu,k}^{(+)}(z^{-1}), \quad R_+^{-1} < |z| < \infty, \quad C_{2,2}^{(l)} = 2\pi \sum_{|k| \ll \infty} \varphi_{1,k}^{(l)} z^{-k-1} F_{\mu,k}^{(-)}(z^{-1}), \quad 0 < |z| < R_-^{-1}, \end{aligned}$$

and the second kind functions as

$$C_1^{(l)} = 2\pi \varphi_2^{(l)}(z^{-1}) z^{-1} \bar{F}_\mu(z), \quad R_- < |z| < R_+, \quad C_2^{(l)} = 2\pi \varphi_1^{(l)}(z^{-1}) z^{-1} F_\mu(z^{-1}), \quad R_+^{-1} < |z| < R_-^{-1}. \quad (25)$$

⁵ Here $\sum_{|k| \ll \infty}$ is used just to indicate that the sum is finite.

Proof. From the formal definition of $C_{a,b}$, $a, b = 1, 2$, and the aid of the Gaussian factorization of the moment matrix g , we have

$$\begin{aligned} C_{1,1} &= (S_2^{-1})^\dagger \oint_{\mathbb{T}} \chi(u) \chi(u)^\dagger \overline{d\mu(u)} \chi_1^*(z), & C_{1,2} &= (S_2^{-1})^\dagger \oint_{\mathbb{T}} \chi(u) \chi(u)^\dagger \overline{d\mu(u)} \chi_2(z) \\ C_{2,1} &= S_1 \oint_{\mathbb{T}} \chi(u) \chi(u)^\dagger d\mu(u) \chi_1^*(z), & C_{2,2} &= S_1 \oint_{\mathbb{T}} \chi(u) \chi(u)^\dagger d\mu(u) \chi_2(z), \end{aligned}$$

We recall that $((S_2^{-1})^\dagger \chi(u))^{(l)} = \varphi_2^{(l)}(u)$ and $(S_1 \chi(u))^{(l)} = \varphi_1^{(l)}(u)$ and expand the matrix products involved, without any interchange of integrals and summation symbols, to get

$$\begin{aligned} C_{1,1}^{(l)} &= \sum_{|k| \ll \infty} \varphi_{2,k}^{(l)} \left(\sum_{n=0}^{\infty} \left[\int_0^{2\pi} e^{i(k-n)\theta} d\bar{\mu}(\theta) \right] z^{-n-1} \right), & C_{1,2}^{(l)} &= \sum_{|k| \ll \infty} \varphi_{2,k}^{(l)} \left(\sum_{n=0}^{\infty} \left[\int_0^{2\pi} e^{i(k+n+1)\theta} d\bar{\mu}(\theta) \right] z^n \right) \\ C_{2,1}^{(l)} &= \sum_{|k| \ll \infty} \varphi_{1,k}^{(l)} \left(\sum_{n=0}^{\infty} \left[\int_0^{2\pi} e^{i(k-n)\theta} d\mu(\theta) \right] z^{-n-1} \right), & C_{2,2}^{(l)} &= \sum_{|k| \ll \infty} \varphi_{1,k}^{(l)} \left(\sum_{n=0}^{\infty} \left[\int_0^{2\pi} e^{i(k+n+1)\theta} d\mu(\theta) \right] z^n \right). \end{aligned}$$

Using the Fourier coefficients c_n we write

$$\begin{aligned} C_{1,1}^{(l)} &= 2\pi \sum_{|k| \ll \infty} \varphi_{2,k}^{(l)} \left(\sum_{n=0}^{\infty} \overline{c_{k-n}} z^{-n-1} \right), & C_{1,2}^{(l)} &= 2\pi \sum_{|k| \ll \infty} \varphi_{2,k}^{(l)} \left(\sum_{n=0}^{\infty} \overline{c_{k+n+1}} z^n \right), \\ C_{2,1}^{(l)} &= 2\pi \sum_{|k| \ll \infty} \varphi_{1,k}^{(l)} \left(\sum_{n=0}^{\infty} c_{n-k} z^{-n-1} \right), & C_{2,2}^{(l)} &= 2\pi \sum_{|k| \ll \infty} \varphi_{1,k}^{(l)} \left(\sum_{n=0}^{\infty} c_{-k-n-1} z^n \right), \end{aligned} \tag{26}$$

from where the desired result follows. \square

Example. Assume that

$$c_n = \frac{1}{2|n|!} + \delta_{n,0} \left(e + \frac{1}{2} \right)$$

so that

$$d\mu(\theta) = w(\theta) d\theta, \quad w = e + \sum_{n=0}^{\infty} \frac{\cos n\theta}{n!} = e + e^{\cos \theta} \cos(\sin \theta),$$

where the weight w is a smooth positive 2π -periodic function. Then

$$\begin{aligned} F_{\mu,k}^{(+)} &= \begin{cases} e + \frac{1}{2} \left(\sum_{n=0}^k \frac{z^{-n}}{n!} + e^z \right), & k \geq 0, \\ \frac{1}{2} \left(- \sum_{n=0}^{|k|-1} \frac{z^n}{n!} + e^z \right) & k < 0, \end{cases} \\ F_{\mu,k}^{(-)} &= \begin{cases} \frac{1}{2} \left(- \sum_{n=0}^k \frac{z^{-n}}{n!} + e^{z^{-1}} \right) & k \geq 0, \\ e + \frac{1}{2} \left(\sum_{n=0}^{|k|-1} \frac{z^n}{n!} + e^{z^{-1}} \right) & k < 0, \end{cases} \end{aligned}$$

with annulus of convergence $D = \{z \in \mathbb{C} : 0 < |z| < 1\}$. Notice that in this example

$$F_{\mu}(z) = e + \frac{e^z + e^{z^{-1}}}{2}$$

and therefore

$$C_1^{(l)}(z) = 2\pi z^{-1} \varphi_2^{(l)}(z^{-1}) F_{\mu}(z), \quad C_2^{(l)}(z) = 2\pi z^{-1} \varphi_1^{(l)}(z^{-1}) F_{\mu}(z).$$

Proposition 6. *The formal series for $\Gamma_{a,j}^l(z)$ (where $a = 1, 2$) defined in Proposition 4 can be expressed in terms of the Fourier series of μ and consequently are convergent in corresponding annulus in the complex plane. More precisely*

$$\begin{aligned}\Gamma_{1,j}^{(l)}(z) &= 2\pi z^{-J(j)-1} \left(F_{J(j)-l,\mu}^{(+)}(z^{-1}) + F_{J(j)+l,\mu}^{(-)}(z^{-1}) \right), & R_+^{-1} < |z| < R_-^{-1}, \\ \Gamma_{2,j}^{(l)}(z) &= 2\pi z^{-J(j)-1} \left(\bar{F}_{l-J(j)-1,\mu}^{(-)}(z) + \bar{F}_{-l-J(j)-1,\mu}^{(+)}(z) \right), & R_- < |z| < R_+.\end{aligned}$$

where⁶ $J(j) = [(-1)^{a(j)-1} \frac{j}{2}]$, being $[p]$ the integer part of p .

Proof. See Appendix A. □

Notice that for $l = 0$ we have

$$\begin{aligned}\Gamma_{1,j}^{(0)}(z) &= 2\pi z^{-J(j)-1} F_\mu(z^{-1}), & R_+^{-1} < |z| < R_-^{-1}, \\ \Gamma_{2,j}^{(0)}(z) &= 2\pi z^{-J(j)-1} \bar{F}_\mu(z), & R_- < |z| < R_+.\end{aligned}$$

Now we will justify the name we have given to these functions and show a Cauchy integral representation of them. We will prove it in two different scenarios depending on the measure. First, with a more measure theory taste, assuming that μ is positive and using the Lebesgue dominated convergence theorem. Second, with a more complex analysis taste, considering absolutely continuous complex measures $d\mu = w(\theta)d\theta$ when w is a continuous complex function.

Theorem 2. *Assume a positive measure $d\mu(\theta)$ or a complex measure $d\mu(\theta) = w(\theta)d\theta$ with w a continuous function. Then, the second kind functions can be written as the following Cauchy integrals*

$$\begin{aligned}C_{1,1}^{(l)} &= z^{-1} \oint_{\mathbb{T}} \frac{u\varphi_2^{(l)}(u) \overline{d\mu(u)}}{u - z^{-1}}, & C_{2,1}^{(l)} &= z^{-1} \oint_{\mathbb{T}} \frac{u\varphi_1^{(l)}(u)}{u - z^{-1}} d\mu(u), & |z| > 1, \\ C_{1,2}^{(l)} &= -z^{-1} \oint_{\mathbb{T}} \frac{u\varphi_2^{(l)}(u) \overline{d\mu(u)}}{u - z^{-1}}, & C_{2,2}^{(l)} &= -z^{-1} \oint_{\mathbb{T}} \frac{u\varphi_1^{(l)}(u)}{u - z^{-1}} d\mu(u), & |z| < 1.\end{aligned}$$

Proof. From the definition of $C_{a,b}$ and the aid of the Gaussian factorization of the moment matrix g , $(S_1^{-1})^\dagger = (S_2^{-1})^\dagger g^\dagger$ or $S_2 = S_1 g$ we get

$$\begin{aligned}C_{1,1} &= (S_2^{-1})^\dagger \oint_{\mathbb{T}} \chi(u) \chi(u)^\dagger \overline{d\mu(u)} \chi_1^*(z) = \sum_{n=0}^{\infty} \left(\oint_{\mathbb{T}} \Phi_2(u) u^{-n} z^{-n-1} \overline{d\mu(u)} \right), \\ C_{1,2} &= (S_2^{-1})^\dagger \oint_{\mathbb{T}} \chi(u) \chi(u)^\dagger \overline{d\mu(u)} \chi_2(z) = \sum_{n=0}^{\infty} \left(\oint_{\mathbb{T}} \Phi_2(u) u^{n+1} z^n \overline{d\mu(u)} \right), \\ C_{2,1} &= S_1 \oint_{\mathbb{T}} \chi(u) \chi(u)^\dagger d\mu(u) \chi_1^*(z) = \sum_{n=0}^{\infty} \left(\oint_{\mathbb{T}} \Phi_1(u) u^{-n} z^{-n-1} d\mu(u) \right), \\ C_{2,2} &= S_1 \oint_{\mathbb{T}} \chi(u) \chi(u)^\dagger d\mu(u) \chi_2(z) = \sum_{n=0}^{\infty} \left(\oint_{\mathbb{T}} \Phi_1(u) u^{n+1} z^n d\mu(u) \right).\end{aligned}$$

For the series in these expressions we have:

1. The series $\sum_{n=0}^{\infty} u^{-n} z^{-n-1}$ converges uniformly in the u variable in any compact set $K \subset \{u \in \mathbb{C} : |u| > |z|^{-1}\}$ to $(z - u^{-1})^{-1}$ and if $|z| > 1$ then we can take K such that $\mathbb{T} \subset K$.
2. The series $\sum_{n=0}^{\infty} u^{n+1} z^n$ converges uniformly in the u variable in any compact set $K \subset \{u \in \mathbb{C} : |u| < |z|^{-1}\}$ to $-(z - u^{-1})^{-1}$ and if $|z| < 1$ then we can take K such that $\mathbb{T} \subset K$.

⁶The reader can check that $z^{J(j)} = \chi^{(j)}(z)$.

Let us assume a positive measure. The corresponding m -th partial sums are

$$\sum_{n=0}^m u^{-n} z^{-n-1} = z^{-1} \frac{1 - (uz)^{-m-1}}{1 - (uz)^{-1}}, \quad \sum_{n=0}^m u^{n+1} z^n = u \frac{1 - (uz)^{m+1}}{1 - (uz)}.$$

If we write $u = e^{i\theta}$ and $z = |z| e^{i \arg z}$ we have

$$\left| \frac{1 - (uz)^{-m-1}}{1 - (uz)^{-1}} \right|^2 = \frac{1 - 2|z|^{-(m+1)} \cos((m+1)(\theta + \arg z)) + |z|^{-2(m+1)}}{1 - 2|z|^{-1} \cos(\theta + \arg z) + |z|^{-2}}.$$

For $|z|^{-1} < 1$ we have the following inequalities

$$\begin{aligned} 0 < 1 - 2|z|^{-(m+1)} \cos((m+1)(\theta + \arg z)) + |z|^{-2(m+1)} &\leq (1 + |z|^{-(m+1)})^2 < 4, \\ 0 < 1 - 2|z|^{-1} \cos(\theta + \arg z) + |z|^{-2} &\geq (1 - |z|^{-1})^2, \end{aligned}$$

so that, for $u \in \mathbb{T}$, we infer

$$\left| z^{-1} \frac{1 - (uz)^{-m-1}}{1 - (uz)^{-1}} \right| < \frac{2|z|^{-1}}{1 - |z|^{-1}}, \quad |z| > 1.$$

Similarly, we conclude that

$$\left| u \frac{1 - (uz)^{m+1}}{1 - uz} \right| < \frac{2}{1 - |z|}, \quad |z| < 1.$$

Thus, for $u \in \mathbb{T}$ and $a = 1, 2$, we have the control bounds

$$\begin{aligned} \left| \sum_{n=0}^m \varphi_a^{(l)}(u) u^{-n} z^{-n-1} \right| &< \left| \frac{2}{1 - |z|} \varphi_a^{(l)}(u) \right|, & |z| > 1, \\ \left| \sum_{n=0}^m \varphi_a^{(l)}(u) u^{n+1} z^n \right| &< \left| \frac{2}{1 - |z|} \varphi_a^{(l)}(u) \right|, & |z| < 1. \end{aligned}$$

Consequently, as the Laurent polynomials φ_a are measurable functions in \mathbb{T} , the Lebesgue dominated convergence theorem leads to the sated result.

Finally, if we assume that $d\mu = w(\theta)d\theta$, with w a continuous complex function; we can always write $w(\theta)d\theta = F(u) \frac{du}{iu}$, $u = e^{i\theta}$, with F a continuous function on \mathbb{T} . Then, recalling the uniform convergence of the geometrical series involved and the fact that the Laurent polynomials φ_a are continuous functions on \mathbb{T} , we can interchange integral and series symbols arriving to the expressions

1. $\int_{\mathbb{T}} \left(\sum_{n=0}^{\infty} \Phi_a(u) u^{-n} z^{-n-1} \right) d\mu(u) = \sum_{n=0}^{\infty} \left(\int_{\mathbb{T}} \Phi_a(u) u^{-n} z^{-n-1} d\mu(u) \right)$ for $|z| > 1$.
2. $\int_{\mathbb{T}} \left(\sum_{n=0}^{\infty} \Phi_a(u) u^{n+1} z^n \right) d\mu(u) = \sum_{n=0}^{\infty} \left(\int_{\mathbb{T}} \Phi_a(u) u^{n+1} z^n d\mu(u) \right)$ for $|z| < 1$.

□

The result motivates the name given to these functions [46]. These expressions can be also written as Geronimus transforms, for that aim we just need to recall that for $u \in \mathbb{T}$

$$\frac{1}{z - u^{-1}} = \frac{1}{2z} \left(1 + \frac{u + z^{-1}}{u - z^{-1}} \right),$$

and therefore for $l \geq 1$

$$\begin{aligned} C_{1,1}^{(l)} &= \frac{1}{2z} \oint_{\mathbb{T}} \frac{u+z^{-1}}{u-z^{-1}} \varphi_2^{(l)}(u) \overline{d\mu(u)}, & C_{2,1}^{(l)} &= \frac{1}{2z} \oint_{\mathbb{T}} \frac{u+z^{-1}}{u-z^{-1}} \varphi_1^{(l)}(u) d\mu(u), & |z| > 1, \\ C_{1,2}^{(l)} &= -\frac{1}{2z} \oint_{\mathbb{T}} \frac{u+z^{-1}}{u-z^{-1}} \varphi_2^{(l)}(u) d\mu(u), & C_{2,2}^{(l)} &= -\frac{1}{2z} \oint_{\mathbb{T}} \frac{u+z^{-1}}{u-z^{-1}} \varphi_1^{(l)}(u) d\mu(u), & |z| < 1. \end{aligned}$$

For $l = 0$, we just obtain (up to constants)

$$\frac{1}{2z} (|\mu| + \mathcal{C}(z^{-1})),$$

where $\mathcal{C}(z)$ is the Carathéodory transform of the measure:

$$\mathcal{C}(z) := \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(\theta).$$

Example. Let us assume that for $|z| > 1$ we have

$$C_{2,1}^{(l)} = z^{-1} \oint_{\mathbb{T}} \frac{u\varphi_1^{(l)}(u)}{u-z^{-1}} F(u) \frac{du}{iu}, \quad (27)$$

and that the only singularity of F inside \mathbb{D} can lay exclusively at $z = 0$. On the one hand, as F has a Laurent expansion of the form $F(z) = \sum_{k \in \mathbb{Z}} F_k z^k$ and there no singularities different from $z = 0$ we conclude that the coefficients of the Laurent series are just the Fourier coefficients of the measure⁷

$$F_k = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{F(u)}{u^{k+1}} du = \frac{1}{2\pi} \int_0^{2\pi} F(\exp i\theta) e^{-ik\theta} d\theta = c_k.$$

On the other hand, as the Laurent polynomial φ_1 has its singularities at $z = 0$, and $z^{-1} \in \mathbb{D}$, then the set of singularities of the integrand that lay in \mathbb{D} in (27) is $\{0, z^{-1}\}$. Hence, the residue theorem gives

$$C_{2,1}^{(l)} = 2\pi z^{-1} \left(\text{Res}_{u=z^{-1}} \left(\frac{\varphi_1^{(l)}(u)F(u)}{u-z^{-1}} \right) + \text{Res}_{u=0} \left(\frac{\varphi_1^{(l)}(u)F(u)}{u-z^{-1}} \right) \right).$$

We now evaluate the residues, the first one may be computed noticing that $\varphi_1^{(l)}(u)F(u)$ is analytic at $u = z^{-1}$

$$2\pi z^{-1} \text{Res}_{u=z^{-1}} \left(\frac{\varphi_1^{(l)}(u)F(u)}{u-z^{-1}} \right) = \varphi_1^{(l)}(z^{-1})F(z^{-1}) = C_1^{(l)},$$

where we have used Proposition 5. For the residue at $u = 0$ we use the Laurent series around this point

$$\varphi_1^{(l)}(u) = \sum_k \varphi_{1,k}^{(l)} u^k, \quad F(u) = \sum_j c_j u^j, \quad \frac{1}{u-z^{-1}} = -\sum_{n=0}^{\infty} z^{n+1} u^n,$$

so that

$$2\pi z^{-1} \text{Res}_{u=0} \left(\frac{\varphi_1^{(l)}(u)F(u)}{u-z^{-1}} \right) = -\sum_{\substack{k+j+n=-1 \\ n \geq 0}} \varphi_{1,k}^{(l)} c_j z^{n+1} = -\sum_k \varphi_{1,k}^{(l)} \sum_{n=0}^{\infty} c_{-k-n-1} z^{n+1} = -C_{2,2}^{(l)}$$

and we finally get

$$C_{2,1} = C_2 - C_{2,2},$$

in agreement with (22).

The application of the residue theorem to the formulae in Theorem 2 leads to expressions for the second kind functions in terms of residues:

⁷This property does not hold if F has different singularities from $z = 0$ inside \mathbb{D} , as we can not choose \mathbb{T} as the integration circuit to apply the Cauchy formula for the coefficients of the Laurent series.

Proposition 7. *Let us assume $d\mu(\theta) = F(u)\frac{du}{iu}$ then:*

1. *When F is an analytic function in $\mathbb{C} \setminus \mathbb{D}$ but for a set of isolated singularities, and if denote by $\{z_{j,-}\}_{j=0}^{p_-}$ the set of distinct points obtained from the union of $z_{0,-} = 0$ and the set of singularities of F at $\mathbb{C} \setminus \mathbb{D}$, then, for $z \notin \mathbb{T}$,*

$$C_{1,1}^{(l)}(z) = 2\pi z^{-1} \left[\varphi_2^{(l)}(z^{-1}) \bar{F}(z) \theta(|z| - 1) - \text{Res}_{u=0} (\varphi_2^{(l)}(u) \bar{F}(u^{-1})) z + \sum_{j=1}^{p_-} \left(\frac{\varphi_2^{(l)}(\bar{z}_{j,-}^{-1})}{\bar{z}_{j,-}^{-1} - z^{-1}} \text{Res}_{u=\bar{z}_{j,-}^{-1}} \bar{F}(u^{-1}) \right) \right],$$

$$C_{1,2}^{(l)}(z) = 2\pi z^{-1} \left[\varphi_2^{(l)}(z^{-1}) \bar{F}(z) \theta(1 - |z|) + \text{Res}_{u=0} (\varphi_2^{(l)}(u) \bar{F}(u^{-1})) z - \sum_{j=1}^{p_-} \left(\frac{\varphi_2^{(l)}(\bar{z}_{j,-}^{-1})}{\bar{z}_{j,-}^{-1} - z^{-1}} \text{Res}_{u=\bar{z}_{j,-}^{-1}} \bar{F}(u^{-1}) \right) \right]$$

2. *If F is an analytic function in \mathbb{D} but for a set of isolated singularities, and denote by $\{z_{j,+}\}_{j=0}^{p_+}$ the set of different points obtained from the union of $z_0 = 0$ and the set of singularities of F at \mathbb{D} . Then, for $z \notin \mathbb{T}$,*

$$C_{2,1}^{(l)}(z) = 2\pi z^{-1} \left[\varphi_1^{(l)}(z^{-1}) F(z^{-1}) \theta(|z| - 1) - \text{Res}_{u=0} (\varphi_1^{(l)}(u) F(u)) z + \sum_{j=1}^{p_+} \left(\frac{\varphi_1^{(l)}(z_{j,+})}{z_{j,+} - z^{-1}} \text{Res}_{u=z_j} F(u) \right) \right],$$

$$C_{2,2}^{(l)}(z) = 2\pi z^{-1} \left[\varphi_1^{(l)}(z^{-1}) F(z^{-1}) \theta(1 - |z|) + \text{Res}_{u=0} (\varphi_1^{(l)}(u) F(u)) z - \sum_{j=1}^{p_+} \left(\frac{\varphi_1^{(l)}(z_{j,+})}{z_{j,+} - z^{-1}} \text{Res}_{u=z_j} F(u) \right) \right].$$

To conclude this section we give some summation rules that are derived using the geometrical series

Proposition 8. *The OLPUC and its corresponding partial second kind functions satisfy*

$$\sum_{l=0}^{\infty} \overline{C_{a,1}^{(l)}}(z) \varphi_{a,1}^{(l)}(z') = \frac{1}{z - z'}, \quad |z'| > |z|,$$

$$\sum_{l=0}^{\infty} \overline{C_{a,2}^{(l)}}(z) \varphi_{a,2}^{(l)}(z') = -\frac{1}{z - z'}, \quad |z'| < |z|,$$

$$\sum_{l=0}^{\infty} \overline{C_{a,1}^{(l)}}(z) \varphi_{a,2}^{(l)}(z') = \sum_{l=0}^{\infty} \overline{C_{a,2}^{(l)}}(z) \varphi_{a,1}^{(l)}(z') = 0.$$

for $a = 1, 2$.

Proof. See Appendix A. □

2.3 Recursion relations

We are about to derive, using the Gaussian factorization, the CMV recursion relations and obtain in this way the well known CMV five diagonal Jacobi type matrix for the recursion of the Szegő polynomials. Let us begin with the following

Definition 5. *Given the canonical basis for semi-infinite matrices $E_{i,j}$, $i, j \in \mathbb{Z}_+$, we define the projections*

$$\Pi_1 := \sum_{j=0}^{\infty} E_{2j,2j}, \quad \Pi_2 := \sum_{j=0}^{\infty} E_{2j+1,2j+1},$$

and the matrices

$$\begin{aligned}\Lambda_1 &:= \sum_{j=0}^{\infty} E_{2j,2+2j}, & \Lambda_2 &:= \sum_{j=0}^{\infty} E_{1+2j,3+2j}, \\ \Lambda &:= \sum_{j=0}^{\infty} E_{j,j+1}, & \Upsilon &:= \Lambda_1 + \Lambda_2^\top + E_{1,1}\Lambda^\top.\end{aligned}$$

The matrix Υ ,

$$\Upsilon = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

is a central object in this paper, as its *dressing* –its orbit by conjugations– gives the pentadiagonal CMV Jacobi type matrix.

It is immediate to check that

Proposition 9. *The following relations hold*

$$\begin{aligned}\Lambda_1\chi(z) &= z\Pi_1\chi(z), & \Lambda_2\chi(z) &= z^{-1}\Pi_2\chi(z),\end{aligned}\tag{28}$$

$$\Lambda_1^\top\chi(z) = (z^{-1}\Pi_1 - E_{0,0}\Lambda)\chi(z), \quad \Lambda_2^\top\chi(z) = (z\Pi_2 - E_{1,1}\Lambda^\top)\chi(z),$$

$$\begin{aligned}\Upsilon\chi(z) &= z\chi(z), \\ \Upsilon^\top\chi(z) &= z^{-1}\chi(z).\end{aligned}\tag{29}$$

With the aid of these conditions we characterize the moment matrix g as verifying a symmetry constraint, which we call *string* equation, from where the recursion as well as the CD formulae will be derived. The symmetry is detailed in the following

Proposition 10. *The CMV moment matrix fulfils the following condition*

$$\Upsilon g = g\Upsilon.\tag{30}$$

Proof. To prove the previous result we proceed as follows

$$\begin{aligned}\Upsilon g &= \oint_{\mathbb{T}} (\Lambda_1 + \Lambda_2^\top + E_{1,1}\Lambda^\top)\chi(z)\chi(z)^\dagger d\mu(z) = \oint_{\mathbb{T}} z\chi(z)\chi(z)^\dagger d\mu(z) \\ &= \oint_{\mathbb{T}} \chi(z)(z^{-1}\chi(z))^\dagger d\mu(z) \\ &= \oint_{\mathbb{T}} \chi(z)((\Lambda_1^\top + \Lambda_2 + E_{0,0}\Lambda)\chi(z))^\dagger d\mu(z) \\ &= g(\Lambda_1 + \Lambda_2^\top + \Lambda^\top E_{0,0}) \\ &= g(\Lambda_1 + \Lambda_2^\top + E_{1,1}\Lambda^\top) \\ &= g\Upsilon.\end{aligned}$$

□

We now proceed to dress the Υ matrix in two ways

Definition 6. We use the notation

$$J_1 := S_1 \Upsilon S_1^{-1}, \quad J_2 := S_2 \Upsilon S_2^{-1}.$$

The following proposition is trivially derived from the above definition and (30)

Proposition 11.

$$J_1 = J_2.$$

Consequently, we introduce the CMV Jacobi type matrix

Definition 7. We define $J := J_1 = J_2$.

The matrix J has a five diagonal structure; as easily follows when one observes that J_1 has zero coefficients over the third upper-diagonal and that J_2 has all its coefficients equal to zero under the third lower-diagonal. More specifically, the structure is

$$J = \begin{pmatrix} * & * & | & 1 & 0 & | & 0 & 0 & | & 0 & 0 & | & 0 & 0 & \cdots \\ + & * & | & * & 0 & | & 0 & 0 & | & 0 & 0 & | & 0 & 0 & \cdots \\ \hline 0 & * & | & * & * & | & 1 & 0 & | & 0 & 0 & | & 0 & 0 & \cdots \\ 0 & + & | & * & * & | & * & 0 & | & 0 & 0 & | & 0 & 0 & \cdots \\ \hline 0 & 0 & | & 0 & * & | & * & * & | & 1 & 0 & | & 0 & 0 & \cdots \\ 0 & 0 & | & 0 & + & | & * & * & | & * & 0 & | & 0 & 0 & \cdots \\ \hline 0 & 0 & | & 0 & 0 & | & 0 & * & | & * & * & | & 1 & 0 & \cdots \\ 0 & 0 & | & 0 & 0 & | & 0 & + & | & * & * & | & * & 0 & \cdots \\ \hline 0 & 0 & | & 0 & 0 & | & 0 & 0 & | & 0 & * & | & * & * & \cdots \\ 0 & 0 & | & 0 & 0 & | & 0 & 0 & | & 0 & + & | & * & * & \cdots \\ \vdots & \vdots & | & \vdots & \vdots & | & \vdots & \vdots & | & \vdots & \vdots & | & \vdots & \ddots & \end{pmatrix}$$

where, $*$ is a possibly non-vanishing term and $+$ is a positive term. In fact, using the LU factorization problem we are able to completely characterize J in terms of the Verblunsky coefficients.

Proposition 12. 1. The non-vanishing coefficients of J are

$$\begin{aligned} J_{2k,2k-1} &= -\rho_{2k}^2 \alpha_{2k+1}^{(1)}, & J_{2k,2k} &= -\bar{\alpha}_{2k}^{(2)} \alpha_{2k+1}^{(1)}, & J_{2k,2k+1} &= -\alpha_{2k+2}^{(1)}, & J_{2k,2k+2} &= 1, \\ J_{2k+1,2k-1} &= \rho_{2k+1}^2 \rho_{2k}^2, & J_{2k+1,2k} &= \rho_{2k+1}^2 \bar{\alpha}_{2k}^{(2)}, & J_{2k+1,2k+1} &= -\bar{\alpha}_{2k+1}^{(2)} \alpha_{2k+2}^{(1)}, & J_{2k+1,2k+2} &= \bar{\alpha}_{2k+1}^{(2)}. \end{aligned}$$

2. We have the recursion relations

$$z\varphi_1^{(2k)} = \varphi_1^{(2k+2)} - \alpha_{2k+2}^{(1)} \varphi_1^{(2k+1)} - \bar{\alpha}_{2k}^{(2)} \alpha_{2k+1}^{(1)} \varphi_1^{(2k)} - \rho_{2k}^2 \alpha_{2k+1}^{(1)} \varphi_1^{(2k-1)}, \quad (31)$$

$$z\varphi_1^{(2k+1)} = \bar{\alpha}_{2k+1}^{(2)} \varphi_1^{(2k+2)} - \bar{\alpha}_{2k+1}^{(2)} \alpha_{2k+2}^{(1)} \varphi_1^{(2k+1)} + \rho_{2k+1}^2 \bar{\alpha}_{2k}^{(2)} \varphi_1^{(2k)} + \rho_{2k+1}^2 \rho_{2k}^2 \varphi_1^{(2k-1)}, \quad (32)$$

Proof. 1. See Appendix A.

2. We have

$$\begin{aligned} z\varphi_1^{(2k)} &= J_{2k,2k+2} \varphi_1^{(2k+2)} + J_{2k,2k+1} \varphi_1^{(2k+1)} + J_{2k,2k} \varphi_1^{(2k)} + J_{2k,2k-1} \varphi_1^{(2k-1)}, \\ z\varphi_1^{(2k+1)} &= J_{2k+1,2k+2} \varphi_1^{(2k+2)} + J_{2k+1,2k+1} \varphi_1^{(2k+1)} + J_{2k+1,2k} \varphi_1^{(2k)} + J_{2k+1,2k-1} \varphi_1^{(2k-1)}. \end{aligned}$$

□

As $J_1\Phi_1 = S_1\Upsilon S_1^{-1}S_1\chi(z) = z\Phi_1$, the sequences Φ_1, Φ_2 have a five term recurrence formula. However, although there are five non-vanishing diagonals the recurrence relations do not have more than four non-zero terms, explicitly

$$\begin{aligned} z\varphi_1^{(2k)} &= J_{2k,2k+2}\varphi_1^{(2k+2)} + J_{2k,2k+1}\varphi_1^{(2k+1)} + J_{2k,2k}\varphi_1^{(2k)} + J_{2k,2k-1}\varphi_1^{(2k-1)} \\ &= \varphi_1^{(2k+2)} - \alpha_{2k+2}^{(1)}\varphi_1^{(2k+1)} - \bar{\alpha}_{2k}^{(2)}\alpha_{2k+1}^{(1)}\varphi_1^{(2k)} - \rho_{2k}^2\alpha_{2k+1}^{(1)}\varphi_1^{(2k-1)}, \\ z\varphi_1^{(2k+1)} &= J_{2k+1,2k+2}\varphi_1^{(2k+2)} + J_{2k+1,2k+1}\varphi_1^{(2k+1)} + J_{2k+1,2k}\varphi_1^{(2k)} + J_{2k+1,2k-1}\varphi_1^{(2k-1)} \\ &= \bar{\alpha}_{2k+1}^{(2)}\varphi_1^{(2k+2)} - \bar{\alpha}_{2k+1}^{(2)}\alpha_{2k+2}^{(1)}\varphi_1^{(2k+1)} + \rho_{2k+1}^2\bar{\alpha}_{2k}^{(2)}\varphi_1^{(2k)} + \rho_{2k+1}^2\rho_{2k}^2\varphi_1^{(2k-1)}, \end{aligned} \quad (33)$$

to them we can add the truncated relations for $k = 0, 1$

$$\begin{aligned} z\varphi_1^{(0)} &= \varphi_1^{(2)} - \alpha_2^{(1)}\varphi_1^{(1)} - \alpha_1^{(1)}\varphi_1^{(0)}, \\ z\varphi_1^{(1)} &= \bar{\alpha}_1^{(2)}\varphi_1^{(2)} - \bar{\alpha}_1^{(2)}\alpha_2^{(1)}\varphi_1^{(1)} + \rho_1^2\varphi_1^{(0)}. \end{aligned}$$

It is also possible to build recursion relations multiplying by z^{-1} , thus we have

Proposition 13. *The OLPUC have the following recursion relations*

$$z^{-1}\varphi_1^{(2k)} = \alpha_{2k}^{(1)}\varphi_1^{(2k+1)} - \alpha_{2k}^{(1)}\bar{\alpha}_{2k+1}^{(2)}\varphi_1^{(2k)} + \rho_{2k}^2\alpha_{2k-1}^{(1)}\varphi_1^{(2k-1)} + \rho_{2k-1}^2\rho_{2k}^2\varphi_1^{(2k-2)}, \quad (34)$$

$$z^{-1}\varphi_1^{(2k+1)} = \varphi_1^{(2k+3)} - \bar{\alpha}_{2k+3}^{(2)}\varphi_1^{(2k+2)} - \alpha_{2k+1}^{(1)}\bar{\alpha}_{2k+2}^{(2)}\varphi_1^{(2k+1)} - \rho_{2k+1}^2\bar{\alpha}_{2k+2}^{(2)}\varphi_1^{(2k)}, \quad (35)$$

where we have to add the truncated relation

$$z^{-1}\varphi_1^{(0)} = \varphi_1^{(1)} - \bar{\alpha}_1^{(2)}\varphi_1^{(0)}.$$

Proof. Using that $S_1\Upsilon^\top S_1^{-1}\Phi_1 = S_1\Upsilon^\top S_1^{-1}S_1\chi(z) = z^{-1}\Phi_1$ we need to calculate the coefficients of $S_1\Upsilon^\top S_1^{-1} = S_2\Upsilon^\top S_2^{-1}$ as we did with J_1 , to obtain the desired result. \square

With the previous result we can get

Proposition 14. *The coefficients ρ_k^2 verify the following relations*

$$\rho_k^2 = 1 - \alpha_k^{(1)}\bar{\alpha}_k^{(2)}, \quad k \geq 0.$$

Proof. See Appendix A. \square

The following results were found previously in [21] using an alternative derivation. Recalling the explicit form of the CMV Jacobi type matrix J provided in Proposition 12 we get

Proposition 15. *When the measure μ is positive the recursion relations for the OLPUC can be expressed in terms of the Verblunsky coefficients as follows*

$$\begin{aligned} z\varphi_1^{(2k)} &= \varphi_1^{(2k+2)} - \alpha_{2k+2}\varphi_1^{(2k+1)} - \bar{\alpha}_{2k}\alpha_{2k+1}\varphi_1^{(2k)} - \rho_{2k}^2\alpha_{2k+1}\varphi_1^{(2k-1)}, \\ z\varphi_1^{(2k+1)} &= \bar{\alpha}_{2k+1}\varphi_1^{(2k+2)} - \bar{\alpha}_{2k+1}\alpha_{2k+2}\varphi_1^{(2k+1)} + \rho_{2k+1}^2\bar{\alpha}_{2k}\varphi_1^{(2k)} + \rho_{2k+1}^2\rho_{2k}^2\varphi_1^{(2k-1)}, \end{aligned} \quad (36)$$

$$\begin{aligned} z^{-1}\varphi_1^{(2k)} &= \alpha_{2k}\varphi_1^{(2k+1)} - \alpha_{2k}\bar{\alpha}_{2k+1}\varphi_1^{(2k)} + \rho_{2k}^2\alpha_{2k-1}\varphi_1^{(2k-1)} + \rho_{2k-1}^2\rho_{2k}^2\varphi_1^{(2k-2)}, \\ z^{-1}\varphi_1^{(2k+1)} &= \varphi_1^{(2k+3)} - \bar{\alpha}_{2k+3}\varphi_1^{(2k+2)} - \alpha_{2k+1}\bar{\alpha}_{2k+2}\varphi_1^{(2k+1)} - \rho_{2k+1}^2\bar{\alpha}_{2k+2}\varphi_1^{(2k)}, \end{aligned} \quad (37)$$

where $k \geq 0$.

Using the Szegő polynomials and their reciprocals recursion relations (31) and (32) can be expressed like

$$zP_{2k} = z^{-1}(P_{2k+2} - \alpha_{2k+2}(k)P_{2k+1}^*) + -\bar{\alpha}_{2k}\alpha_{2k+1}P_{2k} - (1 - |\alpha_{2k}|^2)\alpha_{2k+1}P_{2k-1}^*, \quad (38)$$

$$zP_{2k+1}^* = \bar{\alpha}_{2k+1}P_{2k+2} - \bar{\alpha}_{2k+1}\alpha_{2k+2}P_{2k+1}^* + z((1 - |\alpha_{2k+1}|^2)\bar{\alpha}_{2k}P_{2k} + (1 - |\alpha_{2k+1}|^2)(1 - |\alpha_{2k}|^2)P_{2k-1}^*), \quad (39)$$

relations that can be obtained also with the classical Szegő recurrence formulae and their reciprocals in \mathbb{T} .

2.4 Projection operators and the Christoffel–Darboux kernel

Here we discuss the CD kernel; i.e., the integral kernel of the *quasi-orthogonal projection*, according to the sesquilinear form $\langle \cdot, \cdot \rangle_{\mathcal{L}}$ defined by the measure μ , to the space of OLPUC.

Definition 8. *We use the following notation*

$$\Lambda^{[l]} := \mathbb{C}\{\chi^{(0)}, \dots, \chi^{(l-1)}\} = \begin{cases} \Lambda_{[k, k-1]}, & l = 2k, \\ \Lambda_{[k, k]}, & l = 2k + 1. \end{cases} \quad (40)$$

Notice that

$$\Lambda^{[l]} = \mathbb{C}\{\varphi_1^{(0)}, \dots, \varphi_1^{(l-1)}\} = \mathbb{C}\{\varphi_2^{(0)}, \dots, \varphi_2^{(l-1)}\}.$$

Associated with these spaces of truncated Laurent polynomials we consider the following related spaces, *quasi-orthogonal complements*,

$$(\Lambda^{[l]})^{\perp_2} := \left\{ \sum_{l \leq k < \infty} c_k \varphi_1^{(k)}, c_k \in \mathbb{C} \right\}, \quad (\Lambda^{[l]})^{\perp_1} := \left\{ \sum_{l \leq k < \infty} c_k \varphi_2^{(k)}, c_k \in \mathbb{C} \right\}.$$

Formally, we can express the following *bi-quasi-orthogonality* relations

$$\langle \Lambda^{[l]}, (\Lambda^{[l]})^{\perp_1} \rangle_{\mathcal{L}} = 0, \quad \langle (\Lambda^{[l]})^{\perp_2}, \Lambda^{[l]} \rangle_{\mathcal{L}} = 0,$$

and the corresponding splittings

$$\Lambda_{[\infty]} = \Lambda^{[l]} \oplus (\Lambda^{[l]})^{\perp_1} = \Lambda^{[l]} \oplus (\Lambda^{[l]})^{\perp_2},$$

induce the associated *quasi-orthogonal projections*

$$\pi_1^{(l)} : \Lambda_{[\infty]} \rightarrow \Lambda^{[l]}, \quad \pi_2^{(l)} : \Lambda_{[\infty]} \rightarrow \Lambda^{[l]}.$$

The reader should notice that we cannot properly talk of an orthogonal complement and an orthogonal projection if the measure is not positive and consequently we do not have an scalar product. If μ is a positive measure then $(\Lambda^{[l]})^{\perp_1} = (\Lambda^{[l]})^{\perp_2} = (\Lambda^{[l]})^{\perp}$ and both projections are truly orthogonal and coincide.

Definition 9. *The CD kernel is defined by*⁸

$$K^{[l]}(z, z') := \sum_{k=0}^{l-1} \varphi_1^{(k)}(z') \overline{\varphi_2^{(k)}(\bar{z})}. \quad (41)$$

As the CD kernel is expressed in terms of Laurent polynomials the definition makes sense as long as $z, z' \neq 0$. This is the kernel of the integral representation of the projections $\pi_1^{(l)}, \pi_2^{(l)}$:

Proposition 16. *The integral representation*

$$\begin{aligned} (\pi_1^{(l)} f)(z') &= \oint_{\mathbb{T}} K^{[l]}(z, z') f(z) d\mu(z), & \forall f \in \Lambda_{[\infty]}, \\ \overline{(\pi_2^{(l)} f)(z)} &= \oint_{\mathbb{T}} K^{[l]}(z, z') \bar{f}(\bar{z}') d\mu(z'), & \forall f \in \Lambda_{[\infty]}, \end{aligned}$$

holds.

⁸In case that we have a positive measure μ then we can define the orthonormal Laurent polynomials $\tilde{\varphi}^{(l)} = (h_l)^{-\frac{1}{2}} \varphi_1^{(l)} = (h_l)^{\frac{1}{2}} \varphi_2^{(l)}$ so that $K^{[l]}(z, z') = \sum_{k=0}^{l-1} h_k^{-1} \varphi_1^{(k)}(z') \overline{\varphi_1^{(k)}(\bar{z})} = \sum_{k=0}^{l-1} h_k \varphi_2^{(k)}(z') \overline{\varphi_2^{(k)}(\bar{z})} = \sum_{k=0}^{l-1} \tilde{\varphi}^{(k)}(z') \overline{\tilde{\varphi}^{(k)}(\bar{z})}$.

Proof. It follows from the bi-orthogonality condition (8). □

This CD kernel has the reproducing property

Proposition 17. *The kernel $K^{[l]}(z, z')$ fulfills*

$$K^{[l]}(z, z') = \oint_{\mathbb{T}} K^{[l]}(z, u) K^{[l]}(u, z') d\mu(u).$$

Proof. See Appendix A. □

2.5 Associated Laurent polynomials

In order to find CD formulae for the CD kernel just discussed we need the following definitions introducing what we call associated Laurent polynomials

Definition 10. *In the l even case the associated Laurent polynomials are*

$$\begin{aligned} \varphi_{1,+1}^{(l)} &:= \chi^{(l)} - (g_{l,0} \ g_{l,1} \ \cdots \ g_{l,l-1}) (g^{[l]})^{-1} \chi^{[l]}, & \varphi_{1,-2}^{(l-1)} &:= e_{l-1}^\top (g^{[l]})^{-1} \chi^{[l]}, \\ \varphi_{2,+2}^{(l)} &:= \chi^{(l+1)} - (\bar{g}_{0,l+1} \ \bar{g}_{1,l+1} \ \cdots \ \bar{g}_{l-1,l+1}) ((g^{[l]})^{-1})^\dagger \chi^{[l]}, & \varphi_{2,-1}^{(l-1)} &:= e_{l-2}^\top ((g^{[l]})^{-1})^\dagger \chi^{[l]}, \end{aligned}$$

while for the l odd case they are

$$\begin{aligned} \varphi_{1,+1}^{(l)} &:= \chi^{(l+1)} - (g_{l+1,0} \ g_{l+1,1} \ \cdots \ g_{l+1,l-1}) (g^{[l]})^{-1} \chi^{[l]}, & \varphi_{1,-2}^{(l-1)} &:= e_{l-2}^\top (g^{[l]})^{-1} \chi^{[l]}, \\ \varphi_{2,+2}^{(l)} &:= \chi^{(l)} - (\bar{g}_{0,l} \ \bar{g}_{1,l} \ \cdots \ \bar{g}_{l-1,l}) ((g^{[l]})^{-1})^\dagger \chi^{[l]}, & \varphi_{2,-1}^{(l-1)} &:= e_{l-1}^\top ((g^{[l]})^{-1})^\dagger \chi^{[l]}. \end{aligned}$$

The associated polynomials can be expressed in terms of the Laurent polynomials in different alternative manners

Theorem 3. *If μ is a positive measure the associated Laurent polynomials in the l even case are*

$$\begin{aligned} \varphi_{1,+1}^{(l)}(z) &= \varphi_1^{(l)}(z), & \varphi_{1,-2}^{(l-1)}(z) &= h_{l-1}^{-1} \varphi_1^{(l-1)}(z), \\ \varphi_{2,+2}^{(l)}(z) &= z^{-1} h_l \bar{\varphi}_2^{(l)}(z^{-1}), & \varphi_{2,-1}^{(l-1)}(z) &= z^{-1} \bar{\varphi}_2^{(l-1)}(z^{-1}), \end{aligned} \quad (42)$$

$$= z^{-1} (\bar{\alpha}_l h_l \varphi_2^{(l)}(z) + h_{l-1} \rho_l^2 \varphi_2^{(l-1)}(z)), \quad = z^{-1} (\rho_l^2 \varphi_2^{(l)}(z) - \alpha_l \varphi_2^{(l-1)}(z)), \quad (43)$$

$$= h_{l+1} \varphi_2^{(l+1)}(z) - h_l \bar{\alpha}_{l+1} \varphi_2^{(l)}(z), \quad = \alpha_{l-1} \varphi_2^{(l-1)}(z) + \varphi_2^{(l-2)}(z), \quad (44)$$

and in the l odd case

$$\begin{aligned} \varphi_{2,+2}^{(l)}(z) &= h_l^{-1} \varphi_2^{(l)}(z), & \varphi_{2,-1}^{(l-1)}(z) &= \varphi_2^{(l-1)}(z), \\ \varphi_{1,+1}^{(l)}(z) &= \bar{\varphi}_1^{(l)}(z^{-1}), & \varphi_{1,-2}^{(l-1)}(z) &= h_{l-1}^{-1} \bar{\varphi}_1^{(l-1)}(z^{-1}), \end{aligned} \quad (45)$$

$$= z (\alpha_l \varphi_1^{(l)}(z) + \rho_l^2 \varphi_1^{(l-1)}(z)), \quad = z (h_l^{-1} \rho_l^2 \varphi_1^{(l)}(z) - \bar{\alpha}_l h_{l-1}^{-1} \varphi_1^{(l-1)}(z)) \quad (46)$$

$$= \varphi_1^{(l+1)}(z) - \alpha_{l+1} \varphi_1^{(l)}(z), \quad = h_{l-1}^{-1} \bar{\alpha}_{l-1} \varphi_1^{(l-1)}(z) + h_{l-2}^{-1} \varphi_1^{(l-2)}(z). \quad (47)$$

Proof. 1. To prove (42) and (45) we proceed as follows. On the one hand, when l is even (48) implies

$$\oint_{\mathbb{T}} z \varphi_{2,+2}^{(l)}(z) z^{-j} d\mu(z) = 0, \quad j = -\frac{l}{2} + 1, \dots, \frac{l}{2},$$

and on the other hand, due to the Hermitian property of the scalar product, it follows for $\varphi_2^{(l)}$ that

$$\oint_{\mathbb{T}} \bar{\varphi}_2^{(l)}(z^{-1}) z^{-j} d\mu(z) = 0, \quad j = -\frac{l}{2} + 1, \dots, \frac{l}{2}.$$

Hence, $z\varphi_{2,+2}^{(l)} \in \Lambda_{[l/2,l/2]}$ and solves the same linear system of equations that $\bar{\varphi}_2^{(l)}(z^{-1}) \in \Lambda_{[l/2,l/2]}$ does. Consequently both Laurent polynomials are proportional. The equality is obtained from the coefficients in the power $z^{-\frac{l}{2}}$. In a similar way, from (49) we see that

$$\oint_{\mathbb{T}} z^j z^{-1} \bar{\varphi}_{2,-1}^{(l-1)}(z^{-1}) d\mu(z) = 0, \quad j = -\frac{l}{2} + 1, \dots, \frac{l}{2} - 1, \quad \oint_{\mathbb{T}} z^{\frac{l}{2}} z^{-1} \bar{\varphi}_{2,-1}^{(l-1)}(z^{-1}) d\mu(z) = 1,$$

this means that $z\varphi_{2,-1}^{(l-1)}(z) \in \Lambda_{[l/2-1,l/2]}$ has the same orthogonality relations and normalization condition that $\bar{\varphi}_2^{(l-1)}(z^{-1}) \in \Lambda_{[l/2-1,l/2]}$, so they coincide. Analogously, in the odd case we obtain (45).

2. For (43) and (46) we argue in the following manner. Using orthogonality relations for $\bar{\varphi}_1^{(l)}(z^{-1})$ and $\bar{\varphi}_2^{(l-1)}(z^{-1})$ we conclude that

$$\begin{aligned} \bar{\varphi}_1^{(l)}(z^{-1}) &\in \text{span}\{\varphi_1^{(l)}, \varphi_1^{(l-1)}\}, \\ \bar{\varphi}_2^{(l-1)}(z^{-1}) &\in \text{span}\{\varphi_2^{(l)}, \varphi_2^{(l-1)}\}, \end{aligned}$$

and identifying coefficients

$$\begin{aligned} z\varphi_{2,+2}^{(l)}(z) &= h_l \bar{\varphi}_2^{(l)}(z^{-1}) = \bar{\varphi}_1^{(l)}(z^{-1}) = \bar{\alpha}_l \varphi_1^{(l)}(z) + \rho_l^2 \varphi_1^{(l-1)}(z) = \bar{\alpha}_l h_l \varphi_2^{(l)}(z) + \rho_l^2 h_{l-1} \varphi_2^{(l-1)}(z), \\ z\varphi_{2,-1}^{(l-1)}(z) &= \bar{\varphi}_2^{(l-1)}(z^{-1}) = \rho_l^2 \varphi_2^{(l)}(z) - \alpha_l \varphi_2^{(l-1)}(z), \end{aligned}$$

that concludes the proof of (43), (46) follows similarly.

3. Finally, we proceed now to prove (44) and (47). For the even case we compute the following integral

$$\begin{aligned} \oint_{\mathbb{T}} \chi^{[l]}(z) \bar{\varphi}_{2,+2}^{(l)}(\bar{z}) d\mu(z) &= \oint_{\mathbb{T}} \chi^{[l]}(z) (\chi^{(l+1)}(z))^{\dagger} d\mu(z) - \\ &\quad - \oint_{\mathbb{T}} \chi^{[l]}(z) \chi^{[l]}(z)^{\dagger} d\mu(z) (g^{[l]})^{-1} \begin{pmatrix} g_{0,l+1} \\ g_{1,l+1} \\ \dots \\ g_{l-1,l+1} \end{pmatrix} \\ &= (g_{0,l+1} \quad g_{1,l+1} \quad \dots \quad g_{l-1,l+1})^{\top} - (g_{0,l+1} \quad g_{1,l-1} \quad \dots \quad g_{l-1,l+1})^{\top} \\ &= (0 \quad 0 \quad \dots \quad 0)^{\top}, \end{aligned}$$

which written componentwise reads

$$\oint_{\mathbb{T}} z^j \bar{\varphi}_{2,+2}^{(l)}(\bar{z}) d\mu(z) = 0, \quad j = -\frac{l}{2}, \dots, \frac{l}{2} - 1. \quad (48)$$

It also follows from the definition that $\bar{\varphi}_{2,+2}^{(l)}(z^{-1}) \in \Lambda_{[\frac{l}{2}-1, \frac{l}{2}+1]}$, and $(\bar{\varphi}_{2,+2}^{(l)} - z^{\frac{l}{2}+1}) \in \Lambda_{[\frac{l}{2}-1, \frac{l}{2}]}$. For the other associated Laurent polynomials, the orthogonality relations are

$$\oint_{\mathbb{T}} z^j \bar{\varphi}_{2,-1}^{(l-1)}(\bar{z}) d\mu(z) = 0, \quad j = -\frac{l}{2}, \dots, \frac{l}{2} - 2, \quad \oint_{\mathbb{T}} z^{\frac{l}{2}-1} \bar{\varphi}_{2,-1}^{(l-1)}(\bar{z}) d\mu(z) = 1. \quad (49)$$

To get this result we proceed as before

$$\oint_{\mathbb{T}} \chi^{[l]}(z) \bar{\varphi}_{2,-1}^{(l-1)}(\bar{z}) d\mu(z) = \left(\oint_{\mathbb{T}} \chi^{[l]}(z) \chi^{[l]}(z)^{\dagger} d\mu(z) \right) (g^{[l]})^{-1} e_{l-2} = e_{l-2},$$

that is the matrix version of the orthogonality relations.

These orthogonality relations lead to

$$\varphi_{2,+2}^{(l)} = a_l \varphi_2^{(l+1)} + b_l \varphi_2^{(l)}, \quad \varphi_{2,-1}^{(l-1)} = c_{l-1} \varphi_2^{(l-1)} + d_{l-1} \varphi_2^{(l-2)}.$$

Let us prove this statement. As $\varphi_{2,+2}^{(l)} \in \Lambda_{[\frac{l}{2}+1, \frac{l}{2}-1]}$ then $\varphi_{2,+2}^{(l)} \in \text{span}\{\varphi_2^{(0)}, \varphi_2^{(1)}, \dots, \varphi_2^{(l+1)}\}$, but due to the orthogonality relations all the coefficients vanish except for the ones corresponding to $\varphi_2^{(l+1)}$ and $\varphi_2^{(l)}$. Comparing the coefficients of $z^{-\frac{l}{2}-1}$ and $z^{\frac{l}{2}}$ we get the system of equations

$$1 = (S_2)_{l+1, l+1}^{-1} a_l + 0, \quad 0 = a_l (S_2)_{l+1, l+1}^{-1} \bar{\alpha}_{l+1} + (S_2)_{l, l}^{-1} b_l,$$

from where we conclude $a_l = (S_2)_{l+1, l+1}$ and $b_l = -\bar{\alpha}_{l+1} (S_2)_{l, l}$. Now, we notice that $\varphi_{2,-1}^{(l-1)} \in \text{span}\{\varphi_2^{(0)}, \varphi_2^{(1)}, \dots, \varphi_2^{(l-1)}\}$ and also that the orthogonality relations imply that

$$\varphi_{2,-1}^{(l-1)} \perp \text{span}\{\varphi_2^{(0)}, \varphi_2^{(1)}, \dots, \varphi_2^{(l-3)}\}.$$

Therefore, $\varphi_{2,-1}^{(l-1)} \in \text{span}\{\varphi_2^{(l-2)}, \varphi_2^{(l-1)}\}$ and we only need to find the expression of the associated Laurent polynomials as a linear combination of these two Laurent polynomials. For that aim, we take the complex conjugate, multiply by $\varphi_1^{(l-1)}$ and $\varphi_1^{(l-2)}$, and integrate to obtain

$$\begin{aligned} \bar{c}_{l-1} &= \oint_{\mathbb{T}} \varphi_1^{(l-1)}(z) \bar{\varphi}_{2,-1}^{(l-1)}(\bar{z}) d\mu(z) = \oint_{\mathbb{T}} z^{-\frac{l}{2}} \bar{\varphi}_{2,-1}^{(l-1)}(\bar{z}) d\mu(z) + \bar{\alpha}_{l-1} \oint_{\mathbb{T}} z^{\frac{l}{2}-1} \bar{\varphi}_{2,-1}^{(l-1)}(\bar{z}) d\mu(z) = 0 + \bar{\alpha}_{l-1}, \\ \bar{d}_{l-1} &= \oint_{\mathbb{T}} \varphi_1^{(l-2)}(z) \bar{\varphi}_{2,-1}^{(l-1)}(\bar{z}) d\mu(z) = \oint_{\mathbb{T}} z^{\frac{l}{2}-1} \bar{\varphi}_{2,-1}^{(l-1)}(\bar{z}) d\mu(z) = 1, \end{aligned}$$

so we conclude $c_{l-1} = \alpha_{l-1}$ and $d_{l-1} = 1$. For the odd case one proceeds in an analogous form.

A different proof for the same formula can be found in Appendix A. \square

Finally we give determinantal expressions for these polynomials

Proposition 18. *The associated Laurent polynomials have the following determinantal expressions*

$$\varphi_{1,+a}^{(l)}(z) = \frac{1}{\det g^{[l]}} \det \left(\begin{array}{cccc|c} g_{0,0} & g_{0,1} & \cdots & g_{0,l-1} & \chi^{(0)}(z) \\ g_{1,0} & g_{1,1} & \cdots & g_{1,l-1} & \chi^{(1)}(z) \\ \vdots & \vdots & & \vdots & \vdots \\ g_{l-1,0} & g_{l-1,1} & \cdots & g_{l-1,l-1} & \chi^{(l-1)}(z) \\ \hline g_{l+a,0} & g_{l+a,1} & \cdots & g_{l+a,l-1} & \chi^{(l+a)}(z) \end{array} \right), \quad l \geq 1. \quad (50)$$

$$\varphi_{1,-a}^{(l)}(z) = \frac{(-1)^{l+l-a}}{\det g^{[l+1]}} \det \left(\begin{array}{cccc|ccc} g_{0,0} & g_{0,1} & \cdots & g_{0,l-a-1} & g_{0,l-a+1} & \cdots & g_{0,l} & \chi^{(0)}(z) \\ g_{1,0} & g_{1,1} & \cdots & g_{1,l-a-1} & g_{1,l-a+1} & \cdots & g_{1,l} & \chi^{(1)}(z) \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ g_{l,0} & g_{l,1} & \cdots & g_{l,l-a-1} & g_{l,l-a+1} & \cdots & g_{l+l,a,l} & \chi^{(l)}(z) \end{array} \right), \quad l \geq 1. \quad (51)$$

and

$$\bar{\varphi}_{2,+a}^{(l)}(\bar{z}) = \frac{1}{\det g^{[l]}} \det \left(\begin{array}{cccc|c} g_{0,0} & g_{0,1} & \cdots & g_{0,l-1} & g_{0,l+a} \\ g_{1,0} & g_{1,1} & \cdots & g_{1,l-1} & g_{1,l+a} \\ \vdots & \vdots & & \vdots & \vdots \\ g_{l-1,0} & g_{l-1,1} & \cdots & g_{l-1,l-1} & g_{l-1,l+a} \\ \hline (\chi^{(0)}(z))^\dagger & (\chi^{(1)}(z))^\dagger & \cdots & (\chi^{(l-1)}(z))^\dagger & (\chi^{(l+a)}(z))^\dagger \end{array} \right), \quad l \geq 1. \quad (52)$$

$$\bar{\varphi}_{2,-a}^{(l)}(\bar{z}) = \frac{(-1)^{l+l-a}}{\det g^{[l+1]}} \det \begin{pmatrix} g_{0,0} & g_{0,1} & \cdots & g_{0,l} \\ g_{1,0} & g_{1,1} & \cdots & g_{1,l} \\ \vdots & \vdots & & \vdots \\ g_{l-a-1,0} & g_{l-a-1,1} & \cdots & g_{l-a-1,l} \\ g_{l-a+1,0} & g_{l-a+1,1} & \cdots & g_{l-a+1,l} \\ \vdots & \vdots & & \vdots \\ g_{l,0} & g_{l,1} & \cdots & g_{l,l} \\ \hline (\chi^{(0)}(z))^\dagger & (\chi^{(1)}(z))^\dagger & \cdots & (\chi^{(l)}(z))^\dagger \end{pmatrix}, \quad l \geq 1. \quad (53)$$

Proof. See Appendix A. □

2.6 The Christoffel–Darboux formula

To obtain CD formula in this context we need a number of preliminary lemmas. First we consider a version of the Aitken–Berg–Collar theorem [59]

Lemma 1. *The following ABC type formula*

$$K^{[l]}(z, z') = \chi^{[l]}(z)^\dagger (g^{[l]})^{-1} \chi^{[l]}(z')$$

is fulfilled.

Proof. See the Appendix A. □

The CD formula can be obtained using the previous expressions for the CD kernel.

Lemma 2. *For the CD kernel one has*

$$\begin{aligned} (z' - \bar{z}^{-1})K^{[l]}(z, z') &= \chi^{[l]}(z)^\dagger (g^{[l]})^{-1} z' \chi^{[l]}(z') - \bar{z}^{-1} \chi^{[l]}(z)^\dagger (g^{[l]})^{-1} \chi^{[l]}(z') \\ &= (\chi^{[l]}(z)^\dagger (g^{[l]})^{-1} g^{[l, \geq l]} - \chi^{[\geq l]}(z)^\dagger) \Upsilon^{[\geq l, l]} (g^{[l]})^{-1} \chi^{[l]}(z') - \\ &\quad - \chi^{[l]}(z)^\dagger (g^{[l]})^{-1} \Upsilon^{[l, \geq l]} (g^{[l, \geq l]} (g^{[l]})^{-1} \chi^{[l]}(z') - \chi^{[\geq l]}(z')). \end{aligned}$$

Proof. See Appendix A. □

The reader can easily check

Lemma 3. *If l is an even number*

$$\Upsilon^{[l, \geq l]} = E_{l-2, l-l} = e_{l-2} e_0^\top, \quad \Upsilon^{[\geq l, l]} = E_{l+1-l, l-1} = e_1 e_{l-1}^\top,$$

while for the l odd case he have

$$\Upsilon^{[l, \geq l]} = E_{l-1, l+1-l} = e_{l-1} e_1^\top, \quad \Upsilon^{[\geq l, l]} = E_{l-l, l-2} = e_0 e_{l-2}^\top.$$

Theorem 4. *The following CD formula holds*

$$K^{[l]}(z, z') = \frac{\bar{\varphi}_{2,+2}^{(l)}(\bar{z}) \bar{z} \varphi_{1,-2}^{(l-1)}(z') - \varphi_{1,+1}^{(l)}(z') \bar{z} \bar{\varphi}_{2,-1}^{(l-1)}(\bar{z})}{(1 - z' \bar{z})}, \quad z' \bar{z} \neq 1. \quad (54)$$

Proof. The proof of (54) relies in Lemmas 2 and 3. Let us first study the l even case; with Lemma 2 and 3 we obtain a more explicit expression for the CD kernel given by

$$\begin{aligned} (\bar{z}^{-1} - z')K^{[l]}(z, z') &= (\chi^{(l+1)}(z)^\dagger - \chi^{[l]}(z)^\dagger (g^{[l]})^{-1} g^{[l, \geq l]} e_1) e_{l-1}^\top (g^{[l]})^{-1} \chi^{[l]}(z') - \\ &\quad - \chi^{[l]}(z)^\dagger (g^{[l]})^{-1} e_{l-2} (\chi^{(l)}(z') - e_0^\top g^{[\geq l, l]} (g^{[l]})^{-1} \chi^{[l]}(z')), \end{aligned}$$

then using Definition 10 for the associated Laurent polynomials and Proposition 3 we conclude our claim. that leads to (54).

For the l odd case, reasoning again with Lemmas 2 and 3 we obtain the expression

$$\begin{aligned} (\bar{z}^{-1} - z')K^{[l]}(z, z') &= (\chi^{(l)}(z)^\dagger - \chi^{[l]}(z)^\dagger (g^{[l]})^{-1} g^{[l, \geq l]} e_0) e_{l-2}^\top (g^{[l]})^{-1} \chi^{[l]}(z') - \\ &\quad - \chi^{[l]}(z)^\dagger (g^{[l]})^{-1} e_{l-1} (\chi^{(l+1)}(z') - e_1^\top g^{[\geq l, l]} (g^{[l]})^{-1} \chi^{[l]}(z')). \end{aligned}$$

and recalling with Definition 10 we immediately get the claimed result. \square

Recalling the different expressions for the associated Laurent polynomials in Theorem 3, one easily notice that

Corollary 1. *For a positive measure μ the CD kernel can be written in the following alternative forms. In the l even case it can be written as*

$$K^{[l]}(z, z') = \frac{\varphi_1^{(l)}(\bar{z}^{-1}) \varphi_2^{(l-1)}(z') - \varphi_1^{(l)}(z') \varphi_2^{(l-1)}(\bar{z}^{-1})}{(1 - z' \bar{z})} \quad (55)$$

$$= \frac{(\bar{\alpha}_l \varphi_1^{(l)}(z) + \rho_l^2 \varphi_1^{(l-1)}(z)) \varphi_2^{(l-1)}(z') - \varphi_1^{(l)}(z') (\rho_l^2 \varphi_2^{(l)}(z) - \alpha_l \varphi_2^{(l-1)}(z))}{(1 - z' \bar{z})}, \quad (56)$$

$$= \frac{z(\varphi_1^{(l+1)}(z) - \bar{\alpha}_{l+1} \varphi_1^{(l)}(z)) \varphi_2^{(l-1)}(z') - \varphi_1^{(l)}(z') z(\alpha_{l-1} \varphi_2^{(l-1)}(z) + \varphi_2^{(l-2)}(z))}{(1 - z' \bar{z})} \quad (57)$$

while in the l odd case it can be written as

$$K^{[l]}(z, z') = \frac{\bar{z} \bar{\varphi}_1^{(l)}(\bar{z}) \bar{\varphi}_2^{(l-1)}(z'^{-1}) - \bar{\varphi}_1^{(l)}(z'^{-1}) \bar{z} \bar{\varphi}_2^{(l-1)}(\bar{z})}{(1 - z' \bar{z})} \quad (58)$$

$$= \frac{\bar{z} z' \bar{\varphi}_1^{(l)}(\bar{z}) (\rho_l^2 \varphi_2^{(l)}(z') - \bar{\alpha}_l \varphi_2^{(l-1)}(z')) - (\alpha_l \varphi_1^{(l)}(z') + \rho_l^2 \varphi_1^{(l-1)}(z')) \bar{\varphi}_2^{(l-1)}(\bar{z})}{(1 - z' \bar{z})} \quad (59)$$

$$= \frac{z \varphi_1^{(l)}(z) (\bar{\alpha}_{l-1} \varphi_2^{(l-1)}(z') + \varphi_2^{(l-2)}(z')) - (\varphi_1^{(l+1)}(z') - \alpha_{l+1} \varphi_1^{(l)}(z')) z \varphi_2^{(l-1)}(z)}{(1 - z' \bar{z})}. \quad (60)$$

Formulae (56) and (59) were found⁹ in [25], however the other expressions, to the best of our knowledge, are new.

3 Extended CMV ordering and orthogonal Laurent polynomials

This section is devoted to an extension of the CMV ordering that allows to extend CMV results to more general situations. The main result is an extension of the CD formula allowing for projecting kernels to general spaces of Laurent polynomials, avoiding the CMV restriction on degrees.

⁹However, the authors use the orthonormal sequence instead of the dual monic orthogonal sequences

3.1 Extending the CMV ordering

Let us consider a vector $\vec{n} \in \mathbb{Z}_+^2$, $\vec{n} = (n_+, n_-)$, and the associated alternated sequence with n_+ positive increasing powers of z followed by n_- negative decreasing powers of z , that is

$$\chi_{\vec{n}}(z) := (1, z, \dots, z^{n_+-1}, z^{-1}, z^{-2}, \dots, z^{-n_-}, z^{n_+}, z^{n_++1}, \dots)^\top.$$

The CMV case presented above corresponds to the particular choice $n_+ = n_- = 1$. Given $l \geq 0$ if $\chi_{\vec{n}}^{(l)}$ is a non-negative power of z we say that $a(l) = 1$ and if $\chi_{\vec{n}}^{(l)}$ is a negative power of z we define $a(l) = 2$. In addition, for any $l \geq 0$ we will denote $\nu_+(l)$ ($\nu_-(l)$) as the number of elements in the set $\{\chi_{\vec{n}}^{(l')}, 0 \leq l' \leq l\}$ with $a(l') = 1$ ($a(l') = 2$). That is

$$\nu_+(l) := \#\{\chi_{\vec{n}}^{(l')}, a(l') = 1, 0 \leq l' \leq l\}, \quad \nu_-(l) := \#\{\chi_{\vec{n}}^{(l')}, a(l') = 2, 0 \leq l' \leq l\}. \quad (61)$$

We will use the notation

$$\vec{\nu} = (\nu_+, \nu_-), \quad |\vec{\nu}| := \nu_+ + \nu_-, \quad |\vec{n}| := n_+ + n_-.$$

Additional expressions for (61) can be found using the Euclidean division, ([12], [16]), as we now explain. For any given $l \geq 0$ there exists unique non-negative integers $q(l)$ and $r(l)$ such that

$$\begin{aligned} l &= q(l)|\vec{n}| + r(l), & 0 \leq r(l) < n_+, & \text{if } a(l) = 1, \\ l &= q(l)|\vec{n}| + n_+ + r(l), & 0 \leq r(l) < n_-, & \text{if } a(l) = 2, \end{aligned}$$

so that

$$\nu_+(l) = \begin{cases} q(l)n_+ + r(l) + 1, & a(l) = 1, \\ (q(l) + 1)n_+, & a(l) = 2, \end{cases} \quad \nu_-(l) = \begin{cases} q(l)n_-, & a(l) = 1, \\ q(l)n_- + r(l) + 1, & a(l) = 2, \end{cases}$$

from where $|\vec{\nu}(l)| = l + 1$. If $\{e_k\}_{k=0}^\infty$ is the formal canonical basis of \mathbb{R}^∞ we consider $\{e_a(k)\}_{k=0}^\infty$, with $a = 1, 2$, defined as

$$e_1(\nu_+(l) - 1) := e_l, \quad e_2(\nu_-(l) - 1) := e_l,$$

these are new labels for $\{e_k\}_{k=0}^\infty$ adapted to \vec{n} . Given a non-negative integer l there exist a unique non-negative integer k and a number $a \in \{1, 2\}$ such that $e_a(k) = e_l$. This basis allows for a natural decomposition of $\chi_{\vec{n}}$ using positive and negative powers. In particular

$$\chi_{\vec{n},a}(z) := \sum_{k=0}^{\infty} e_a(k) z^k, \quad a = 1, 2, \quad \chi_{\vec{n}} = \chi_{\vec{n},1} + \chi_{\vec{n},2}^*.$$

With those sequences we define the extended CMV moment matrix

Definition 11. *The extended moment matrix is the following semi-infinite matrix*

$$g_{\vec{n}} := \oint_{\mathbb{T}} \chi_{\vec{n}}(z) \chi_{\vec{n}}(z)^\dagger d\mu(z). \quad (62)$$

Notice that this moment matrix is a definite positive Hermitian matrix if μ is positive. The Gaussian factorization for the semi-infinite matrix $g_{\vec{n}}$ is

$$g_{\vec{n}} = (S_{\vec{n},1})^{-1} S_{\vec{n},2},$$

where $S_{\vec{n},1}$ is a normalized lower triangular matrix and $S_{\vec{n},2}$ is an upper triangular matrix. The associated sequences of Laurent polynomials are

$$\Phi_{\vec{n},1}(z) := S_{\vec{n},1}\chi_{\vec{n}}(z), \quad \Phi_{\vec{n},2}(z) := (S_{\vec{n},2}^{-1})^\dagger \chi_{\vec{n}}(z),$$

where

$$\Phi_{\vec{n},1}(z) = \begin{pmatrix} \varphi_{\vec{n},1}^{(0)}(z) \\ \varphi_{\vec{n},1}^{(1)}(z) \\ \vdots \end{pmatrix}, \quad \Phi_{\vec{n},2}(z) = \begin{pmatrix} \varphi_{\vec{n},2}^{(0)}(z) \\ \varphi_{\vec{n},2}^{(1)}(z) \\ \vdots \end{pmatrix}.$$

As in the CMV case, the sets of Laurent polynomials $\{\varphi_{\vec{n},1}^{(l)}\}_{l=0}^\infty$ and $\{\varphi_{\vec{n},2}^{(l)}\}_{l=0}^\infty$ are bi-orthogonal with respect to the sesquilinear form $\langle \cdot, \cdot \rangle_{\mathcal{L}}$, that is

$$\langle \varphi_{\vec{n},1}^{(l)}, \varphi_{\vec{n},2}^{(k)} \rangle_{\mathcal{L}} = \oint_{\mathbb{T}} \varphi_{\vec{n},1}^{(l)}(z) \bar{\varphi}_{\vec{n},2}^{(k)}(z^{-1}) d\mu(z) = \delta_{l,k}, \quad l, k = 0, 1, \dots \quad (63)$$

In addition if the measure μ is positive we have that $\varphi_{\vec{n},2}^{(l)} = h_l^{-1} \varphi_{\vec{n},1}^{(l)}$ and both families are proportional Laurent polynomials. In addition

$$\langle \varphi_{\vec{n},1}^{(l)}, \varphi_{\vec{n},1}^{(k)} \rangle_{\mathcal{L}} = \oint_{\mathbb{T}} \varphi_{\vec{n},1}^{(l)}(z) \bar{\varphi}_{\vec{n},1}^{(k)}(z^{-1}) d\mu(z) = \delta_{l,k} h_l, \quad l, k = 0, 1, \dots \quad (64)$$

so in this case $\{\varphi_{\vec{n},1}^{(l)}\}_{l=0}^\infty$ is a set of orthogonal Laurent polynomials with $\varphi_{\vec{n},1}^{(l)}(z) \in \Lambda_{[\nu_-(l), \nu_+(l)-1]}$. The orthogonality relations (64) read as follows

$$\langle \varphi_{\vec{n},1}^{(l)}, z^k \rangle_{\mathcal{L}} = \oint_{\mathbb{T}} \varphi_{\vec{n},1}^{(l)}(z) z^{-k} d\mu(z) = 0, \quad k = -\nu_-(l-1), \dots, \nu_+(l-1) - 1. \quad (65)$$

In terms of the Szegő polynomials we have

Proposition 19. *For a positive measure μ we have the following identifications between the extended CMV Laurent polynomials, the Szegő polynomials and their reciprocals*

$$\begin{aligned} z^{\nu_-(l)} \varphi_{\vec{n},1}^{(l)}(z) &= P_l(z), & a(l) &= 1, \\ z^{\nu_-(l)} \varphi_{\vec{n},1}^{(l)}(z) &= P_l^*(z), & a(l) &= 2. \end{aligned} \quad (66)$$

Proof. See Appendix A. □

The reader should notice that in the CMV case, $n_+ = n_- = 1$, for $l = 2k$ we have $a(l) = 1$, $\nu_-(l) = k$ and $\nu_+(l) = k + 1$, and when $l = 2k + 1$ then $a(l) = 2$, $\nu_-(l) = \nu_+(l) = k + 1$.

3.2 Functions of the second kind

Here we just give a brief description account of this extended case

Definition 12. *The partial second kind sequences with the extended ordering are given by*

$$C_{\vec{n},1,1}(z) := (S_{\vec{n},1}^{-1})^\dagger \chi_{\vec{n},1}^*(z), \quad C_{\vec{n},1,2}(z) := (S_{\vec{n},1}^{-1})^\dagger \chi_{\vec{n},2}(z), \quad C_{\vec{n},2,1}(z) := S_{\vec{n},2} \chi_{\vec{n},1}^*(z), \quad C_{\vec{n},2,2}(z) := S_{\vec{n},2} \chi_{\vec{n},2}(z).$$

and the second kind sequences

$$C_{\vec{n},1}(z) := (S_{\vec{n},1}^{-1})^\dagger \chi_{\vec{n}}^*(z), \quad C_{\vec{n},2}(z) := S_{\vec{n},2} \chi_{\vec{n}}^*(z).$$

Generalized determinantal formulae can be obtained

Proposition 20. *The extended second kind functions have the following determinantal expressions for $l \geq 1$.*

$$\overline{C_{\bar{n},1}^{(l)}}(z) = \frac{1}{\det g_{\bar{n}}^{[l+1]}} \det \begin{pmatrix} g_{\bar{n},0,0} & g_{\bar{n},0,1} & \cdots & g_{\bar{n},0,l} \\ g_{\bar{n},1,0} & g_{\bar{n},1,1} & \cdots & g_{\bar{n},1,l} \\ \vdots & \vdots & & \vdots \\ g_{\bar{n},l-1,0} & g_{\bar{n},l-1,1} & \cdots & g_{\bar{n},l-1,l} \\ \hline \bar{\Gamma}_{\bar{n},2,0}^{(l)} & \bar{\Gamma}_{\bar{n},2,1}^{(l)} & \cdots & \bar{\Gamma}_{\bar{n},2,l}^{(l)} \end{pmatrix}, \quad (67)$$

and

$$C_{\bar{n},2}^{(l)}(z) = \frac{1}{\det g_{\bar{n}}^{[l]}} \det \begin{pmatrix} g_{\bar{n},0,0} & g_{\bar{n},0,1} & \cdots & g_{\bar{n},0,l-1} & \left| \begin{array}{c} \Gamma_{\bar{n},1,0}^{(l)} \\ \Gamma_{\bar{n},1,1}^{(l)} \\ \vdots \\ \Gamma_{\bar{n},1,l}^{(l)} \end{array} \right. \\ g_{\bar{n},1,0} & g_{\bar{n},1,1} & \cdots & g_{\bar{n},1,l-1} & \\ \vdots & \vdots & & \vdots & \\ g_{\bar{n},l,0} & g_{\bar{n},l,1} & \cdots & g_{\bar{n},l,l-1} & \end{pmatrix}, \quad (68)$$

where $\Gamma_{\bar{n},1,j}^{(l)} := \sum_{k \geq l} g_{\bar{n},j,k} \chi_{\bar{n}}^{*(k)}$ and $\Gamma_{\bar{n},2,j}^{(l)} := \sum_{k \geq l} g_{\bar{n},j,k}^\dagger \chi_{\bar{n}}^{*(k)}$.

The same can be said about the relationship between the second kind functions, the Fourier series of the measures, and the integral representation, that can be found in

Proposition 21. *The partial second kind functions can be expressed as*

$$\begin{aligned} C_{\bar{n},1,1}^{(l)} &= 2\pi \sum_{|k| \ll \infty} \varphi_{\bar{n},2,k}^{(l)} z^{-k-1} \bar{F}_{\mu,-k-1}^{(-)}(z), & R_- < |z| < \infty & \quad C_{\bar{n},1,2}^{(l)} &= 2\pi \sum_{|k| \ll \infty} \varphi_{\bar{n},2,k}^{(l)} z^{-k-1} \bar{F}_{\mu,-k-1}^{(+)}(z), & 0 < |z| < R_+ \\ C_{\bar{n},2,1}^{(l)} &= 2\pi \sum_{|k| \ll \infty} \varphi_{\bar{n},1,k}^{(l)} z^{-k-1} F_{\mu,k}^{(+)}(z^{-1}), & R_+^{-1} < |z| < \infty & \quad C_{\bar{n},2,2}^{(l)} &= 2\pi \sum_{|k| \ll \infty} \varphi_{\bar{n},1,k}^{(l)} z^{-k-1} F_{\mu,k}^{(-)}(z^{-1}), & 0 < |z| < R_-^{-1} \end{aligned}$$

and the second kind functions as

$$C_{\bar{n},1}^{(l)} = 2\pi \varphi_{\bar{n},2}^{(l)}(z^{-1}) z^{-1} \bar{F}_{\mu}(z), \quad R_- < |z| < R_+ \quad C_{\bar{n},2}^{(l)} = 2\pi \varphi_{\bar{n},1}^{(l)}(z^{-1}) z^{-1} F_{\mu}(z^{-1}) \quad R_+^{-1} < |z| < R_-^{-1}. \quad (69)$$

and in

Proposition 22. *Assume a positive measure μ or a complex measure $w(\theta)d\theta$ with w a continuous function. Then, the second kind sequences can be written as the following Cauchy integrals*

$$\begin{aligned} C_{\bar{n},1,1}^{(l)} &= z^{-1} \oint_{\mathbb{T}} \frac{u \varphi_{\bar{n},2}^{(l)}(u)}{u - z^{-1}} d\mu(u), & C_{\bar{n},2,1}^{(l)} &= z^{-1} \oint_{\mathbb{T}} \frac{u \varphi_{\bar{n},1}^{(l)}(u)}{u - z^{-1}} d\mu(u), & |z| > 1, \\ C_{\bar{n},1,2}^{(l)} &= -z^{-1} \oint_{\mathbb{T}} \frac{u \varphi_{\bar{n},2}^{(l)}(u)}{u - z^{-1}} d\mu(u), & C_{\bar{n},2,2}^{(l)} &= -z^{-1} \oint_{\mathbb{T}} \frac{u \varphi_{\bar{n},1}^{(l)}(u)}{u - z^{-1}} d\mu(u), & |z| < 1. \end{aligned}$$

3.3 Recursion relations

As we already commented above the recursion relations among extended Laurent polynomials are more involved than in the CMV case.

Definition 13. Given $\vec{n} \in \mathbb{Z}_+^2$ we define the projections

$$\Pi_{\vec{n},a} := \sum_{k=0}^{\infty} e_a(k) e_a(k)^\top, \quad a = 1, 2,$$

and the shift matrices

$$\begin{aligned} \Lambda_{\vec{n},a} &:= \sum_{k=0}^{\infty} e_a(k) e_a(k+1)^\top, \\ \Upsilon_{\vec{n}} &:= \Lambda_{\vec{n},1} + \Lambda_{\vec{n},2}^\top + E_{n_+,n_+} (\Lambda^\top)^{n_+}. \end{aligned}$$

In the context of section 3 recursion relations can be obtained using the same technique. Our objective is to have an expression for the multiplication by z and by z^{-1} using the shift operators.

Proposition 23. 1. The moment matrix $g_{\vec{n}}$ has the following symmetry

$$\Upsilon_{\vec{n}} g_{\vec{n}} = g_{\vec{n}} \Upsilon_{\vec{n}}. \quad (70)$$

2. The operator of multiplication by z has a $|\vec{n}| + 3$ diagonal band structure in the basis given by $\Phi_{\vec{n},1}$ or $\Phi_{\vec{n},2}$

Proof. See Appendix A □

Now we introduce the following associate integers. For $a = 1, 2$ we shall call l_{+a} and l_{-a} to the smallest (largest) integer l' that verifies $l' \geq l$ ($l' \leq l$) and $a_1(l') = a$. For instance, in the previous case with $n_+ = n_- = 1$, if l is an even number $l_{+1} = l_{-1} = l$, $l_{+2} = l + 1$, and $l_{-2} = l - 1$, in the case that l is an odd number then $l_{+2} = l_{-2} = l$, $l_{+1} = l + 1$, and $l_{-1} = l - 1$. This structure leads to the following recursion laws for $k \geq 1$ and $0 \leq l \leq |\vec{n}| - 1$ (the $k = 0$ case corresponds to the truncated recurrence relations).

$$z \varphi_{\vec{n},1}^{(|\vec{n}|k+l)} = J_{\vec{n},0}^l(k) \varphi_{\vec{n},1}^{((|\vec{n}|k+l+1)_{+1})} + J_{\vec{n},1}^l(k) \varphi_{\vec{n},1}^{((|\vec{n}|k+l+1)_{+1}-1)} + \dots + J_{\vec{n},m_{\vec{n}}(l)-1}^l(k) \varphi_{\vec{n},1}^{((|\vec{n}|k+l-1)_{-2})},$$

where $m_{\vec{n}}(l)$ is the number of non vanishing terms in the recurrence formulae. The coefficients $J_{\vec{n},j}^l(k)$ are again labeled with the index j that accounts for the $m_{\vec{n}}(l)$ non vanishing coefficients for each l ; as there are only $|\vec{n}|$ “different” recursion relations (the equivalent to the recurrences for the odd and even polynomials) then $l = 0, 1, \dots, |\vec{n}| - 1$. The connection with the elements of the Jacobi operator is the following $J_{\vec{n},j}^l(k) = J_{\vec{n},|\vec{n}|k+l, (|\vec{n}|k+l)_{+1}-j}$. The reader can check that $m_{\vec{n}}(l) = (|\vec{n}|k+l+1)_{+1} - (|\vec{n}|k+l-1)_{-2} + 1$. Due to the fact that $(|\vec{n}|k+l+1)_{+1} \leq |\vec{n}|k+l+n_-+1$ and that $(|\vec{n}|k+l-1)_{-2} \geq |\vec{n}|k+l-1-n_+$ then $m_{\vec{n}}(l) \leq |\vec{n}| + 3$ that agrees with the structure of $|\vec{n}| + 3$ diagonals. Furthermore it is possible to calculate $m_{\vec{n}}(l)$ more explicitly and show that actually it does not depend on k . We can see that

$$\begin{aligned} (|\vec{n}|k+l+1)_{+1} &= \begin{cases} |\vec{n}|k+l+1, & 0 \leq l \leq n_+ - 2, \\ |\vec{n}|(k+1), & n_+ - 1 \leq l \leq |\vec{n}| - 1, \end{cases} \\ (|\vec{n}|k+l-1)_{-2} &= \begin{cases} |\vec{n}|k-1, & 0 \leq l \leq n_+, \\ |\vec{n}|k+l-1 & n_+ + 1 \leq l \leq |\vec{n}| - 1, \end{cases} \end{aligned}$$

and consequently

$$m_{\vec{n}}(l) = \begin{cases} l+3 & l = 0, \dots, n_+ - 2, \\ |\vec{n}| + 2 & l = n_+ - 1, n_+, \\ |\vec{n}| - l + 2 & l = n_+ + 1, \dots, |\vec{n}| - 1, \end{cases}$$

so in fact $m_{\vec{n}}(l) \leq |\vec{n}| + 2$.

The expressions for the coefficients $J_{\vec{n},p}^l(k)$ with $l = 0, \dots, n_+$ are

$$J_{\vec{n},p}^l(k) = \begin{cases} (S_{\vec{n},1}^{-1})_{(|\vec{n}|k+l+1)_{+1}, (|\vec{n}|k+l+1)_{+1}-p} & p = 0, \dots, (|\vec{n}|k+l+1)_{+1} - (|\vec{n}|k+l) - 1 \\ (S_{\vec{n},2}^{-1})_{|\vec{n}|k+l, (|\vec{n}|k+l)_{+2}}^{-1} (S_{\vec{n},2}^{-1})_{(|\vec{n}|k+l)_{-2}, (|\vec{n}|k+l+1)_{+1}-p} & p = (|\vec{n}|k+l+1)_{+1} - (|\vec{n}|k+l), \dots, m_{\vec{n}}(l) - 1, \end{cases}$$

and for $l = n_+, \dots, |\vec{n}| - 1$, the expressions are

$$J_{\vec{n},p}^l(k) = \begin{cases} (S_{\vec{n},1}^{-1})_{|\vec{n}|k+l, (|\vec{n}|k+l)_{-1}} (S_{\vec{n},1}^{-1})_{(|\vec{n}|k+l)_{+1}, (|\vec{n}|k+l)_{+1}-p} & p = 0, \dots, (|\vec{n}|k+l)_{+1} - (|\vec{n}|k+l) \\ (h_{\vec{n}})_{|\vec{n}|k+l} (S_{\vec{n},2}^{-1})_{(|\vec{n}|k+l-1)_{-2}, (|\vec{n}|k+l)_{+1}-p} & p = (|\vec{n}|k+l+1)_{+1} - (|\vec{n}|k+l+1) \dots, m_{\vec{n}}(l) - 1. \end{cases}$$

The particular case of $n_- = n_+ = 1$ gives the 5 diagonal CMV matrix with only four non-vanishing coefficients. As a consequence, the standard CMV case has the shortest possible recurrence formula.

3.4 The Christoffel–Darboux kernel

We discuss the CD kernel for this extended case

Definition 14. For each non-negative integer l we define the set of truncated Laurent polynomial subspace as the following span

$$\Lambda_{\vec{n}}^{[l]} := \mathbb{C}\{\chi_{\vec{n}}^{(0)}, \dots, \chi_{\vec{n}}^{(l-1)}\} = \Lambda_{[\nu_-(l-1), \nu_+(l-1)-1]}.$$

As before, we define *quasi-orthogonal* subspaces

$$(\Lambda_{\vec{n}}^{[l]})^{\perp_1} := \left\{ \sum_{l \leq k < \infty} c_k \varphi_{\vec{n},2}^{(k)}, c_k \in \mathbb{C} \right\}, \quad (\Lambda_{\vec{n}}^{[l]})^{\perp_2} := \left\{ \sum_{l \leq k < \infty} c_k \varphi_{\vec{n},1}^{(k)}, c_k \in \mathbb{C} \right\},$$

so that the following *bi-quasi-orthogonality* relations are satisfied

$$\langle \Lambda_{\vec{n}}^{[l]}, (\Lambda_{\vec{n}}^{[l]})^{\perp_1} \rangle_{\mathcal{L}} = 0, \quad \langle (\Lambda_{\vec{n}}^{[l]})^{\perp_2}, \Lambda_{\vec{n}}^{[l]} \rangle_{\mathcal{L}} = 0,$$

and the corresponding splittings

$$\Lambda_{[\infty]} = \Lambda_{\vec{n}}^{[l]} \oplus (\Lambda_{\vec{n}}^{[l]})^{\perp_1} = \Lambda_{\vec{n}}^{[l]} \oplus (\Lambda_{\vec{n}}^{[l]})^{\perp_2},$$

that induce the associated projections

$$\pi_{\vec{n},1}^{(l)} : \Lambda_{[\infty]} \rightarrow \Lambda_{\vec{n}}^{[l]}, \quad \pi_{\vec{n},2}^{(l)} : \Lambda_{[\infty]} \rightarrow \Lambda_{\vec{n}}^{[l]},$$

hold. In the positive definite case this extended version allows for the projection in more general spaces of truncated Laurent polynomials. Recall that for the CMV case the space of truncated polynomials given in (40) includes only a very particular class of these truncations, excluding the majority of cases. The introduction of the extended ordering allows to include in the discussion all the possible situations of truncation. In fact the space $\Lambda_{[p,q]}$ can be achieved in a number of ways, and always by the choice $n_+ = q + 1$, $n_- = p$.

Definition 15. The CD kernel is

$$K_{\vec{n}}^{[l]}(z, z') := \sum_{k=0}^{l-1} \varphi_{\vec{n},1}^{(k)}(z') \bar{\varphi}_{\vec{n},2}^{(k)}(\bar{z}), \quad (71)$$

and, whenever the measure μ is positive definite, we have the equivalent expressions

$$K_{\vec{n}}^{[l]}(z, z') = \sum_{k=0}^{l-1} h_k^{-1} \varphi_{\vec{n},1}^{(k)}(z') \bar{\varphi}_{\vec{n},1}^{(k)}(\bar{z}) = \sum_{k=0}^{l-1} h_k \varphi_{\vec{n},2}^{(k)}(z') \bar{\varphi}_{\vec{n},2}^{(k)}(\bar{z}) = \sum_{k=0}^{l-1} \tilde{\varphi}_{\vec{n}}^{(k)}(z') \overline{\tilde{\varphi}_{\vec{n}}^{(k)}(z)}.$$

Proceeding as in the CMV ordering we conclude the following results

Proposition 24. 1. This is the integral kernel of the integral representation of the projections $\pi_{\bar{n},1}^{(l)}$, $\pi_{\bar{n},2}^{(l)}$

$$\begin{aligned} (\pi_{\bar{n},1}^{(l)}f)(z') &= \oint_{\mathbb{T}} K_{\bar{n}}^{[l]}(z, z')f(z)d\mu(z), & \forall f \in \Lambda_{[\infty]}, \\ \overline{(\pi_{\bar{n},2}^{(l)}f)(z)} &= \oint_{\mathbb{T}} K_{\bar{n}}^{[l]}(z, z')\bar{f}(\bar{z}')d\mu(z'), & \forall f \in \Lambda_{[\infty]}. \end{aligned}$$

2. This CD kernel $K_{\bar{n}}^{[l]}(z, z')$ has the reproducing property

$$K_{\bar{n}}^{[l]}(z, z') = \oint_{\mathbb{T}} K_{\bar{n}}^{[l]}(z, u)K_{\bar{n}}^{[l]}(u, z')d\mu(u).$$

3. The following version of the ABC theorem holds

$$K_{\bar{n}}^{[l]}(z, z') = \chi_{\bar{n}}^{[l]}(z)\dagger(g_{\bar{n}}^{[l]})^{-1}\chi_{\bar{n}}^{[l]}(z').$$

4. The CD kernel has the following expression

$$\begin{aligned} (z' - \bar{z}^{-1})K_{\bar{n}}^{[l]}(z, z') &= \chi_{\bar{n}}^{[l]}(z)\dagger(g_{\bar{n}}^{[l]})^{-1}z'\chi_{\bar{n}}^{[l]}(z') - \bar{z}^{-1}\chi_{\bar{n}}^{[l]}(z)\dagger(g_{\bar{n}}^{[l]})^{-1}\chi_{\bar{n}}^{[l]}(z') \\ &= (\chi_{\bar{n}}^{[l]}(z)\dagger(g_{\bar{n}}^{[l]})^{-1}g_{\bar{n}}^{[l, \geq l]} - \chi_{\bar{n}}^{[\geq l]}(z)\dagger\Upsilon_{\bar{n}}^{[\geq l, l]}(g_{\bar{n}}^{[l]})^{-1}\chi_{\bar{n}}^{[l]}(z') - \\ &\quad - \chi_{\bar{n}}^{[l]}(z)\dagger(g_{\bar{n}}^{[l]})^{-1}\Upsilon_{\bar{n}}^{[l, \geq l]}(g_{\bar{n}}^{[\geq l, l]}(g_{\bar{n}}^{[l]})^{-1}\chi_{\bar{n}}^{[l]}(z') - \chi_{\bar{n}}^{[\geq l]}(z')). \end{aligned}$$

The integers $l_{\pm a}$ can be used to calculate the Υ blocks in this case. The reader can check

Proposition 25. The formula for $\Upsilon_{\bar{n}}$ is the following

$$\Upsilon_{\bar{n}}^{[l, \geq l]} = e_{(l-1)_{-1}}e_{l_{+1}-l}^{\top}, \quad \Upsilon_{\bar{n}}^{[\geq l, l]} = e_{l_{+2}-l}e_{(l-1)_{-2}}^{\top}.$$

Thus, the expression of the CD kernel is

$$\begin{aligned} (\bar{z}^{-1} - z')K_{\bar{n}}^{[l]}(z, z') &= (\chi_{\bar{n}}^{[\geq l]}(z)\dagger - \chi_{\bar{n}}^{[l]}(z)\dagger(g_{\bar{n}}^{[l]})^{-1}g_{\bar{n}}^{[l, \geq l]})e_{l_{+2}-l}e_{(l-1)_{-2}}^{\top}(g_{\bar{n}}^{[l]})^{-1}\chi_{\bar{n}}^{[l]}(z') - \\ &\quad - \chi_{\bar{n}}^{[l]}(z)\dagger(g_{\bar{n}}^{[l]})^{-1}e_{(l-1)_{-1}}e_{l_{+1}-l}^{\top}(\chi_{\bar{n}}^{[\geq l]}(z') - g_{\bar{n}}^{[\geq l, l]}(g_{\bar{n}}^{[l]})^{-1}\chi_{\bar{n}}^{[l]}(z')), \end{aligned} \tag{72}$$

that suggests the definition of the following associated polynomials

Definition 16. The associated Laurent polynomials are defined by

$$\begin{aligned} \varphi_{\bar{n},1,+a}^{(l)} &:= \chi_{\bar{n}}^{(l+a)} - (g_{\bar{n},l+a,0} \quad g_{\bar{n},l+a,1} \quad \cdots \quad g_{\bar{n},l+a,l-1})(g_{\bar{n}}^{[l]})^{-1}\chi_{\bar{n}}^{[l]}, & \varphi_{\bar{n},1,-a}^{(l)} &:= e_{l-a}^{\top}(g_{\bar{n}}^{[l+1]})^{-1}\chi_{\bar{n}}^{[l+1]}, \\ \varphi_{\bar{n},2,+a}^{(l)} &:= \chi_{\bar{n}}^{(l+a)} - (\bar{g}_{\bar{n},0,l+a} \quad \bar{g}_{\bar{n},1,l+a} \quad \cdots \quad \bar{g}_{\bar{n},l-1,l+a})((g_{\bar{n}}^{[l]})^{-1})\dagger\chi_{\bar{n}}^{[l]}, & \varphi_{\bar{n},2,-a}^{(l)} &:= e_{l-a}^{\top}((g_{\bar{n}}^{[l+1]})^{-1})\dagger\chi_{\bar{n}}^{[l+1]}, \end{aligned}$$

where $a = 1, 2$.

It is easy to see that $\varphi_{\bar{n},1,+a(l)}^{(l)} = \varphi_{\bar{n},1}^{(l)}, \varphi_{\bar{n},2,+a(l)}^{(l)} = (\bar{S}_2)_{ll}\varphi_{\bar{n},2}^{(l)}, \varphi_{\bar{n},1,-a(l)}^{(l)} = (S_2)_{ll}^{-1}\varphi_{\bar{n},1}^{(l)}$ and $\varphi_{\bar{n},2,-a(l)}^{(l)} = \varphi_{\bar{n},2}^{(l)}$.

Theorem 5. For the associated Laurent polynomials $\varphi_{\bar{n},+a}^{(l)}, \varphi_{\bar{n},-a}^{(l)}$ we have two alternative expressions

1. The reciprocal type form (valid for positive definite cases)

$$\begin{aligned}\varphi_{\bar{n},1,+2}^{(l)}(z) &= \varphi_{\bar{n},2,+2}^{(l)}(z) = z^{\nu_+(l)-\nu_-(l)-2}\bar{\varphi}_{\bar{n},1}^{(l)}(z^{-1}), \\ \varphi_{\bar{n},1,-2}^{(l)}(z) &= \varphi_{\bar{n},2,-2}^{(l)}(z) = z^{\nu_+(l)-\nu_-(l)-1}\bar{\varphi}_{\bar{n},2}^{(l)}(z^{-1}),\end{aligned}\tag{73}$$

when $a(l) = 1$ and

$$\begin{aligned}\varphi_{\bar{n},1,+1}^{(l)}(z) &= \varphi_{\bar{n},2,+1}^{(l)}(z) = z^{\nu_+(l)-\nu_-(l)}\bar{\varphi}_{\bar{n},1}^{(l)}(z^{-1}), \\ \varphi_{\bar{n},1,-1}^{(l)}(z) &= \varphi_{\bar{n},2,-1}^{(l)}(z) = z^{\nu_+(l)-\nu_-(l)-1}\bar{\varphi}_{\bar{n},2}^{(l)}(z^{-1}),\end{aligned}\tag{74}$$

for $a(l) = 2$.

2. The linear combination form (valid for quasi-definite cases)

$$\varphi_{\bar{n},1,+a}^{(l)} = (S_1^{-1})_{l+a,l+a}\varphi_{\bar{n},1}^{(l+a)} + (S_1^{-1})_{l+a,l+a-1}\varphi_{\bar{n},1}^{(l+a-1)} + \cdots + (S_1^{-1})_{l+a,l}\varphi_{\bar{n},1}^{(l)},\tag{75}$$

$$\varphi_{\bar{n},2,+a}^{(l)} = (\bar{S}_2)_{l+a,l+a}\varphi_{\bar{n},2}^{(l+a)} + (\bar{S}_2)_{l+a-1,l+a}\varphi_{\bar{n},2}^{(l+a-1)} + \cdots + (\bar{S}_2)_{l,l+a}\varphi_{\bar{n},2}^{(l)},\tag{76}$$

$$\varphi_{\bar{n},1,-a}^{(l)} = (S_2^{-1})_{l-a,l-a}\varphi_{\bar{n},1}^{(l-a)} + (S_2^{-1})_{l-a,l-a+1}\varphi_{\bar{n},1}^{(l-a+1)} + \cdots + (S_2^{-1})_{l-a,l}\varphi_{\bar{n},1}^{(l)},\tag{77}$$

$$\varphi_{\bar{n},2,-a}^{(l)} = (\bar{S}_1)_{l-a,l-a}\varphi_{\bar{n},2}^{(l-a)} + (\bar{S}_1)_{l-a+1,l-a}\varphi_{\bar{n},2}^{(l-a+1)} + \cdots + (\bar{S}_1)_{l,l-a}\varphi_{\bar{n},2}^{(l)}.\tag{78}$$

Proof. 1. Let us suppose that $a(l) = 1$. In that case we have $\varphi_{\bar{n},1,+1}^{(l)} = \varphi_{\bar{n},1}^{(l)}$ and $\varphi_{\bar{n},1,+2}^{(l)} \in \Lambda_{[\nu_-(l)+1,\nu_+(l)-2]}$. Consequently, $z^{\nu_-(l)-\nu_+(l)+2}\varphi_{\bar{n},1,+2}^{(l)} \in \Lambda_{[\nu_+(l)-1,\nu_-(l)]}$ and $a(l) = 1$, $\nu_+(l) = \nu_+(l-1) + 1$, $\nu_-(l) = \nu_-(l-1)$. For the dual polynomials $\varphi_{\bar{n},2,-1}^{(l)} = \varphi_{\bar{n},2}^{(l)}$ and $\varphi_{\bar{n},2,-2}^{(l)} \in \Lambda_{[\nu_-(l),\nu_+(l)-1]}$, hence $z^{\nu_-(l)-\nu_+(l)+1}\varphi_{\bar{n},2,-2}^{(l)} \in \Lambda_{[\nu_+(l)-1,\nu_-(l)]}$. Using (79) we conclude that the following orthogonality relations hold true

$$\begin{aligned}\oint_{\mathbb{T}} z^{\nu_-(l)-\nu_+(l)+2}\varphi_{\bar{n},1,+2}^{(l)}(z)z^{-k}d\mu(z) &= 0, & k = -\nu_+(l-1) + 1, \dots, \nu_-(l-1), \\ \oint_{\mathbb{T}} z^{\nu_-(l)-\nu_+(l)+1}\varphi_{\bar{n},2,-2}^{(l)}(z)z^{-k}d\mu(z) &= 0, & k = -\nu_+(l-1) + 1, \dots, \nu_-(l-1), \\ \oint_{\mathbb{T}} z^{\nu_-(l)-\nu_+(l)+1}\varphi_{\bar{n},2,-2}^{(l)}(z)z^{\nu_+(l-1)}d\mu(z) &= 1,\end{aligned}$$

and we get the result.

Now let us suppose that $a(l) = 2$. In this case we have $\varphi_{\bar{n},1,+2}^{(l)} = \varphi_{\bar{n},1}^{(l)}$ and $\varphi_{\bar{n},1,+1}^{(l)} \in \Lambda_{[\nu_-(l)-1,\nu_+(l)]}$. Consequently, $z^{\nu_-(l)-\nu_+(l)}\varphi_{\bar{n},1,+2}^{(l)} \in \Lambda_{[\nu_+(l)-1,\nu_-(l)]}$. Now, as $a(l) = 2$, we have $\nu_+(l) = \nu_+(l-1)$ and $\nu_-(l) = \nu_-(l-1) + 1$. For the dual polynomials $\varphi_{\bar{n},2,-2}^{(l)} = \varphi_{\bar{n},2}^{(l)}$ and $\varphi_{\bar{n},2,-1}^{(l)} \in \Lambda_{[\nu_-(l),\nu_+(l)-1]}$, so $z^{\nu_-(l)-\nu_+(l)+1}\varphi_{\bar{n},2,-1}^{(l)} \in \Lambda_{[\nu_+(l)-1,\nu_-(l)]}$. Now using again (79) we get

$$\begin{aligned}\oint_{\mathbb{T}} z^{\nu_-(l)-\nu_+(l)}\varphi_{\bar{n},1,+1}^{(l)}(z)z^{-k}d\mu(z) &= 0, & k = -\nu_+(l-1) + 1, \dots, \nu_-(l-1), \\ \oint_{\mathbb{T}} z^{\nu_-(l)-\nu_+(l)}\varphi_{\bar{n},2,-1}^{(l)}(z)z^{-k}d\mu(z) &= 0, & k = -\nu_+(l-1) + 1, \dots, \nu_-(l-1), \\ \oint_{\mathbb{T}} z^{\nu_-(l)-\nu_+(l)+1}\varphi_{\bar{n},2,-1}^{(l)}(z)z^{-\nu_-(l-1)-1}d\mu(z) &= 1.\end{aligned}$$

2. For $\varphi_{\bar{n},1,+a}^{(l)}$ direct computation gives

$$\begin{aligned}\oint_{\mathbb{T}} \varphi_{\bar{n},1,+a}^{(l)}(z)\chi_{\bar{n}}^{[l]}(z)^\dagger d\mu(z) &= \oint_{\mathbb{T}} (\chi_{\bar{n}}^{(l+a)}(z) - (g_{\bar{n},l+a,0} \quad g_{\bar{n},l+a,1} \quad \cdots \quad g_{\bar{n},l+a,l-1})(g_{\bar{n}}^{[l]})^{-1}\chi_{\bar{n}}^{[l]}(z))\chi_{\bar{n}}^{[l]}(z)^\dagger d\mu(z) \\ &= (g_{\bar{n},l+a,0} \quad g_{\bar{n},l+a,1} \quad \cdots \quad g_{\bar{n},l+a,l-1}) - (g_{\bar{n},l+a,0} \quad g_{\bar{n},l+a,1} \quad \cdots \quad g_{\bar{n},l+a,l-1}) \\ &= 0,\end{aligned}$$

and for $\varphi_{\bar{n},2,-a}^{(l)}$ we have

$$\oint_{\mathbb{T}} \chi_{\bar{n}}^{[l+1]}(z) \bar{\varphi}_{\bar{n},2,-a}(\bar{z}) d\mu(z) = \oint_{\mathbb{T}} \chi_{\bar{n}}^{[l+1]}(z) \chi_{\bar{n}}^{[l+1]}(z)^\dagger (g_{\bar{n}}^{[l+1]})^{-1} e_{l-a} d\mu(z) = e_{l-a},$$

so that we get orthogonality relations for the associated polynomials

$$\begin{aligned} \oint_{\mathbb{T}} \varphi_{\bar{n},1,+a}^{(l)}(z) z^{-k} d\mu(z) &= 0, \quad k = -\nu_-(l-1), \dots, \nu_+(l-1) - 1, \\ \oint_{\mathbb{T}} \chi_{\bar{n}}^{(k)}(z) \bar{\varphi}_{\bar{n},2,-a}(\bar{z}) d\mu(z) &= \delta_{k,l-a}, \quad k = 0, 1, \dots, l. \end{aligned} \tag{79}$$

Therefore,

$$\varphi_{\bar{n},1,+a}^{(l)} \in \text{span}\{\varphi_{\bar{n},1}^{(l)}, \varphi_{\bar{n},1}^{(l+1)}, \dots, \varphi_{\bar{n},1}^{(l+a)}\},$$

i.e., $\varphi_{\bar{n},1,+a}^{(l)} = \sum_{j=l}^{l+a} A_j^{(l)} \varphi_{\bar{n},1}^{(j)}$ for a set of coefficients $\{A_j^{(l)}\}$. Comparing the powers of z that appear in the subsequence $\{\chi^{(j)}\}_{l \leq j \leq l+a}$ on both sides of the equation, the following linear system of equations is obtained

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ (S_1)_{l+a,l+a-1} & 1 & 0 & 0 & 0 & 0 \\ (S_1)_{l+a,l+a-2} & (S_1)_{l+a-1,l+a-2} & 1 & 0 & 0 & 0 \\ (S_1)_{l+a,l+a-3} & (S_1)_{l+a-1,l+a-3} & \cdots & 1 & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ (S_1)_{l+a,l} & (S_1)_{l+a-1,l} & \cdots & (S_1)_{l+2,l} & (S_1)_{l+1,l} & 1 \end{pmatrix} \begin{pmatrix} A_{l+a}^{(l)} \\ A_{l+a-1}^{(l)} \\ A_{l+a-2}^{(l)} \\ A_{l+a-3}^{(l)} \\ \vdots \\ A_l^{(l)} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

calling \mathcal{M} the coefficient matrix, the solution can be written as

$$\begin{pmatrix} A_{l+a}^{(l)} \\ A_{l+a-1}^{(l)} \\ \vdots \\ A_l^{(l)} \end{pmatrix} = \begin{pmatrix} (\mathcal{M}^{-1})_{0,0} \\ (\mathcal{M}^{-1})_{1,0} \\ \vdots \\ (\mathcal{M}^{-1})_{l+a-l,0} \end{pmatrix}$$

From the structure of \mathcal{M} we conclude that

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix} \mathcal{M}^\top \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix} := \mathcal{M}' = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ (S_1)_{l+1,l} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ (S_1)_{l+a-1,l} & (S_1)_{l+a-1,l+1} & \cdots & 1 & 0 \\ (S_1)_{l+a,l} & (S_1)_{l+a,l+1} & \cdots & (S_1)_{l+a,l+a-1} & 1 \end{pmatrix}.$$

From the triangular structure of S_1 we deduce that $(\mathcal{M}'^{-1})_{i,j} = (S_1^{-1})_{i+l,j+l}$ for $i, j = 0, 1, \dots, l+a-l$ and consequently $(\mathcal{M}^{-1})_{j,0} = (\mathcal{M}'^{-1})_{l+a-l,l+a-l-j} = (S_1^{-1})_{l+a,l+a-j}$ for $i, j = 0, 1, \dots, l+a-l$, which proves (75). The expression for (78) is obtained using a similar technique. Using again (79) we conclude that $\varphi_{\bar{n},2,-a}^{(l)} \in \text{span}\{\varphi_{\bar{n},2}^{(l-a)}, \varphi_{\bar{n},2}^{(l-a+1)}, \dots, \varphi_{\bar{n},2}^{(l)}\}$; i.e., $\varphi_{\bar{n},2,-a}^{(l)} = \sum_{j=l-a}^l B_j^{(l)} \varphi_{\bar{n},2}^{(j)}$. Bi-orthogonality and normalization properties imply

$$\bar{B}_j^{(l)} = \int_{\mathbb{T}} \varphi_{\bar{n},1}^{(j)}(z) \bar{\varphi}_{\bar{n},2,-a}^{(l)}(z^{-1}) d\mu(z) = (S_1)_{j,l-a}, \quad j = l-a, \dots, l,$$

that proves (78). The other two equations are obtained using the same idea. \square

The polynomials that appear in the CD formula are now clearly identified as

$$(\chi_{\bar{n}}^{[\geq l]}(z)^\dagger - \chi_{\bar{n}}^{[l]}(z)^\dagger (g_{\bar{n}}^{[l]})^{-1} g_{\bar{n}}^{[l, \geq l]}) e_{l+2-l} = \bar{\varphi}_{\bar{n}, 2, +2}^{(l)}(\bar{z}), \quad e_{(l-1)-2}^\top (g_{\bar{n}}^{[l]})^{-1} \chi_{\bar{n}}^{[l]}(z') = \varphi_{\bar{n}, 1, -2}^{(l-1)}(z'), \quad (80)$$

$$e_{l+1-l}^\top (\chi_{\bar{n}}^{[\geq l]}(z') - g_{\bar{n}}^{[\geq l, l]} (g_{\bar{n}}^{[l]})^{-1} \chi_{\bar{n}}^{[l]}(z')) = \varphi_{\bar{n}, 1, +1}^{(l)}(z'), \quad \chi_{\bar{n}}^{[l]}(z)^\dagger (g_{\bar{n}}^{[l]})^{-1} e_{(l-1)-1} = \bar{\varphi}_{\bar{n}, 2, -1}^{(l-1)}(\bar{z}), \quad (81)$$

and consequently the final result is the following

Theorem 6. *The CD formula for the extended ordering is the following*

$$K_{\bar{n}}^{[l]}(z, z') = \frac{\bar{z} \bar{\varphi}_{\bar{n}, 2, +2}^{(l)}(\bar{z}) \varphi_{\bar{n}, 1, -2}^{(l-1)}(z') - \varphi_{\bar{n}, 1, +1}^{(l)}(z') \bar{z} \bar{\varphi}_{\bar{n}, 2, -1}^{(l-1)}(\bar{z})}{(1 - z' \bar{z})}. \quad (82)$$

We get the followings corollaries when we have a positive Borel measure μ

Corollary 2. *Given a positive measure μ , the CD kernel can be expressed using*

$$K_{\bar{n}}^{[l]}(z, z') = \frac{\bar{z}^{\nu_+(l)-\nu_-(l)-1} \varphi_{\bar{n}, 1}^{(l)}(\bar{z}^{-1}) z'^{\nu_+(l)-\nu_-(l)-2} \bar{\varphi}_{\bar{n}, 2}^{(l-1)}(z'^{-1}) - \varphi_{\bar{n}, 1}^{(l)}(z') \bar{z} \bar{\varphi}_{\bar{n}, 2}^{(l-1)}(\bar{z})}{(1 - z' \bar{z})},$$

in the case $a(l) = a(l-1) = 1$,

$$K_{\bar{n}}^{[l]}(z, z') = \frac{\bar{z}^{\nu_+(l)-\nu_-(l)-1} \varphi_{\bar{n}, 1}^{(l)}(\bar{z}^{-1}) \varphi_{\bar{n}, 2}^{(l-1)}(z') - \varphi_{\bar{n}, 1}^{(l)}(z') \bar{z}^{\nu_+(l)-\nu_-(l)-1} \varphi_{\bar{n}, 2}^{(l-1)}(\bar{z}^{-1})}{(1 - z' \bar{z})},$$

in the case $a(l) = 1, a(l-1) = 2$,

$$K_{\bar{n}}^{[l]}(z, z') = \frac{\bar{z} \bar{\varphi}_{\bar{n}, 1}^{(l)}(\bar{z}) z'^{\nu_+(l)-\nu_-(l)} \bar{\varphi}_{\bar{n}, 2}^{(l-1)}(z'^{-1}) - z'^{\nu_+(l)-\nu_-(l)} \bar{\varphi}_{\bar{n}, 1}^{(l)}(z'^{-1}) \bar{z} \bar{\varphi}_{\bar{n}, 2}^{(l-1)}(\bar{z})}{(1 - z' \bar{z})},$$

in the case $a(l) = 2, a(l-1) = 1$,

$$K_{\bar{n}}^{[l]}(z, z') = \frac{\bar{z} \bar{\varphi}_{\bar{n}, 1}^{(l)}(\bar{z}) \varphi_{\bar{n}, 2}^{(l-1)}(z') - z'^{\nu_+(l)-\nu_-(l)} \bar{\varphi}_{\bar{n}, 1}^{(l)}(z'^{-1}) \bar{z}^{\nu_+(l)-\nu_-(l)+1} \varphi_{\bar{n}, 2}^{(l-1)}(\bar{z}^{-1})}{(1 - z' \bar{z})},$$

in the case $a(l) = a(l-1) = 2$.

and

Corollary 3. *The CD formula for a positive Borel measure μ can be expressed in terms on the Szegő polynomials as*

$$K_{\bar{n}}^{[l]}(z, z') = h_{l-1}^{-1} \bar{z}^{a(l)+\nu_+(l)-2} z'^{a(l)-1-\nu_-(l)} \frac{P_l(\bar{z}^{-1}) P_{l-1}^*(z') - P_l(z') P_{l-1}^*(\bar{z}^{-1})}{1 - z' \bar{z}}. \quad (83)$$

4 Associated 2D Toda type hierarchies

Here we analyze the link between the previous constructions on OLPUC and integrable systems of Toda type. Our driving idea is the presence of the Borel-Gauss factorization problem in the theoretical construction of both, OLPUC and Toda.

4.1 2D Toda flows

We will consider a set of complex deformation parameters¹⁰ $t = \{t_{1j}, t_{2j}\}_{j \in \mathbb{N}}$ and with it two semi-infinite matrices that we will define now.¹¹

Definition 17. 1. The deformation matrices are defined as follows

$$W_{1,0}(t) := \exp\left(\sum_{j=1}^{\infty} t_{1j} \Upsilon^j\right), \quad W_{2,0}(t) := \exp\left(\sum_{j=1}^{\infty} t_{2j} (\Upsilon^\top)^j\right), \quad (84)$$

2. For each t we will consider the matrix $g(t)$

$$g(t) := W_{1,0}(t)g(W_{2,0}(t))^{-1},$$

3. and the corresponding time dependant Gauss–Borel factorization¹²

$$g(t) := W_{1,0}(t)g(W_{2,0}(t))^{-1}, \quad g(t) = (S_1(t))^{-1}S_2(t).$$

As we show now the deformed moment matrix is a moment matrix of a deformed measure.

Proposition 26. The “deformed” moment matrix can be understood as a moment matrix for a “deformed” (that is, a “time” dependant) measure given by

$$d\mu(t, z) := \exp\left(\sum_{j=1}^{\infty} t_{1j} z^j - t_{2j} z^{-j}\right) d\mu(z). \quad (85)$$

Proof. First expanding the exponentials in (84) we obtain

$$W_{1,0}(t) = \sum_{k=0}^{\infty} \sigma_1^k(t) \Upsilon^k, \quad (W_{2,0}(t))^{-1} = \sum_{l=0}^{\infty} \sigma_2^l(t) (\Upsilon^\top)^l,$$

then using the definition of g and $g(t)$ we get the desired result

$$\begin{aligned} W_{1,0}(t)g(W_{2,0}(t))^{-1} &= \sum_{k,l=0}^{\infty} \sigma_1^k(t) \Upsilon^k g(\Upsilon^\top)^l \sigma_2^l(t) = \oint_{\mathbb{T}} \sum_{k=0}^{\infty} \sigma_1^k(t) z^k \chi(z) \chi(z)^\dagger \sum_{l=0}^{\infty} \sigma_2^l(t) z^{-l} d\mu(z) \\ &= \oint_{\mathbb{T}} \chi(z) \chi(z)^\dagger \exp\left(\sum_{j=0}^{\infty} (t_{1j} z^j - t_{2j} z^{-j})\right) d\mu(z). \end{aligned}$$

□

From this result we conclude at least for absolutely continuous measures

$$F_{\mu(t)} = \exp\left(\sum_{j=1}^{\infty} t_{1j} z^j - t_{2j} z^{-j}\right) F_{\mu}(z),$$

¹⁰In the framework of the theory on integrable systems these parameters are understood as an infinite set of times, being the independent variables in an associated nonlinear hierarchy of partial differential-difference equations

¹¹We shall drop the subindex \vec{n} from g and Υ as the definitions are valid for any value of \vec{n} . It will be supposed that a particular \vec{n} is chosen and the whole §4 will be built using that \vec{n} .

¹²For the sake of notation simplicity in some situations we drop the time dependence of S_1, S_2 and they will not denote the factors within the Gauss–Borel factorization of the initial condition but for the “deformed” one. Consequently, in this section S_1, S_2 will always depend on “time” parameters.

from where we deduce that the radii defining the annulus of convergence is time independent; i.e., $R_{\pm}(t) = R_{\pm}$

Given a positive definite initial measure μ in order to ensure that the evolved measure $\mu(t)$ is also definite positive for all times is enough to request to the exponential to be real; i.e., setting $t_{2j} = -\bar{t}_{1j}$, so that

$$\exp\left(\sum_{j=0}^{\infty}(t_{1j}z^j + \bar{t}_{1j}z^{-j})\right) = \exp\left(\sum_{j=0}^{\infty}2\operatorname{Re}(t_{1j}z^j)\right).$$

4.2 Integrable Toda equations

Definition 18. *Associated with the deformed Gauss–Borel factorization we consider*

1. *Wave semi-infinite matrices*

$$W_1(t) := S_1(t)W_{1,0}(t), \quad W_2(t) := S_2(t)W_{2,0}(t). \quad (86)$$

2. *Partial wave and partial adjoint (denoted the adjoint by $*$) wave semi-infinite vector functions¹³,*

$$\begin{aligned} \Psi_{1,1}(z, t) &:= W_1(t)\chi_1(z), & \Psi_{2,1}^*(z, t) &:= (W_2(t)^{-1})^\dagger\chi_1(z), \\ \Psi_{1,2}(z, t) &:= W_1(t)\chi_2^*(z), & \Psi_{2,2}^*(z, t) &:= (W_2(t)^{-1})^\dagger\chi_2^*(z), \\ \Psi_{1,1}^*(z, t) &:= (W_1(t)^{-1})^\dagger\chi_1^*(z), & \Psi_{2,1}(z, t) &:= W_2(t)\chi_1^*(z), \\ \Psi_{1,2}^*(z, t) &:= (W_1(t)^{-1})^\dagger\chi_2^*(z), & \Psi_{2,2}(z, t) &:= W_2(t)\chi_2(z), \end{aligned} \quad (87)$$

and wave and adjoint wave functions

$$\begin{aligned} \Psi_1(z, t) &:= W_1(t)\chi(z) = (\Psi_{1,1} + \Psi_{1,2})(z, t), & \Psi_2^*(z, t) &:= (W_2(t)^{-1})^\dagger\chi(z) = (\Psi_{2,1}^* + \Psi_{2,2}^*)(z, t), \\ \Psi_1^*(z, t) &:= (W_1(t)^{-1})^\dagger\chi^*(z) = (\Psi_{1,1}^* + \Psi_{1,2}^*)(z, t), & \Psi_2(z, t) &:= W_2(t)\chi^*(z) = (\Psi_{2,1} + \Psi_{2,2})(z, t). \end{aligned} \quad (88)$$

3. *Lax semi-infinite matrices*

$$L_1(t) := S_1(t)\Upsilon S_1(t)^{-1}, \quad L_2(t) := S_2(t)\Upsilon^\top S_2(t)^{-1}. \quad (89)$$

4. *Zakharov–Shabat semi-infinite matrices*

$$B_{1,j} := (L_1^j)_+, \quad B_{2,j} := (L_2^j)_-, \quad (90)$$

where the subindex $+$ indicates the projection in the upper triangular matrices while the subindex $-$ the projection in the strictly lower triangular matrices.

Theorem 7. *For $j, j' = 1, 2, \dots$, the following differential relations hold*

1. *Auxiliary linear systems for the wave matrices*

$$\frac{\partial W_1}{\partial t_{1j}} = B_{1,j}W_1, \quad \frac{\partial W_1}{\partial t_{2j}} = B_{2,j}W_1, \quad \frac{\partial W_2}{\partial t_{1j}} = B_{1,j}W_2, \quad \frac{\partial W_2}{\partial t_{2j}} = B_{2,j}W_2. \quad (91)$$

2. *Linear systems for the wave and adjoint wave semi-infinite functions*

$$\begin{aligned} \frac{\partial \Psi_1}{\partial t_{1j}} &= B_{1,j}\Psi_1, & \frac{\partial \Psi_1}{\partial t_{2j}} &= B_{2,j}\Psi_1, & \frac{\partial \Psi_2^*}{\partial t_{1j}} &= -B_{1,j}^\dagger\Psi_2^*, & \frac{\partial \Psi_2^*}{\partial t_{2j}} &= -B_{2,j}^\dagger\Psi_2^*, \\ \frac{\partial \Psi_1^*}{\partial t_{1j}} &= -B_{1,j}^\dagger\Psi_1^*, & \frac{\partial \Psi_1^*}{\partial t_{2j}} &= -B_{2,j}^\dagger\Psi_1^*, & \frac{\partial \Psi_2}{\partial t_{1j}} &= B_{1,j}\Psi_2, & \frac{\partial \Psi_2}{\partial t_{2j}} &= B_{2,j}\Psi_2. \end{aligned} \quad (92)$$

¹³Also called Baker, or Baker–Akhiezer, functions

3. Lax equations

$$\frac{\partial L_1}{\partial t_{1j}} = [B_{1,j}, L_1], \quad \frac{\partial L_1}{\partial t_{2j}} = [B_{2,j}, L_1], \quad \frac{\partial L_2}{\partial t_{1j}} = [B_{1,j}, L_2], \quad \frac{\partial L_2}{\partial t_{2j}} = [B_{2,j}, L_2]. \quad (93)$$

4. Zakharov–Shabat equations

$$\frac{\partial B_{1,j}}{\partial t_{1j'}} - \frac{\partial B_{1,j'}}{\partial t_{1j}} + [B_{1,j}, B_{1,j'}] = 0, \quad (94)$$

$$\frac{\partial B_{2,j}}{\partial t_{2j'}} - \frac{\partial \bar{B}_{2,j'}}{\partial t_{2j}} + [B_{2,j}, B_{2,j'}] = 0, \quad (95)$$

$$\frac{\partial B_{1,j}}{\partial t_{2j'}} - \frac{\partial B_{2,j'}}{\partial t_{1j}} + [B_{1,j}, B_{2,j'}] = 0. \quad (96)$$

Proof. The proof can be made using the same idea used in [12], so we do not repeat them here again. \square

From the definition it is clear that the wave functions are associated to the OLPUC for the evolved measure

Proposition 27. *The wave functions are linked to the OLPUC and the Fourier series of the measure through*

$$\Psi_1^{(n)}(z, t) = \varphi_1^{(n)}(z, t) e^{\sum_{j=1}^{\infty} t_{1j} z^j}, \quad (97)$$

$$(\Psi_2^*)^{(n)}(z, t) = \varphi_2^{(n)}(z, t) e^{-\sum_{j=1}^{\infty} \bar{t}_{2j} z^j},$$

$$\begin{aligned} (\Psi_1^*)^{(n)}(z, t) &= 2\pi \varphi_2^{(n)}(z^{-1}, t) z^{-1} \bar{F}_{\mu(t)}(z) e^{-\sum_{j=1}^{\infty} \bar{t}_{1j} z^j} \\ &= 2\pi \varphi_2^{(n)}(z^{-1}, t) z^{-1} \bar{F}_{\mu}(z) e^{-\sum_{j=1}^{\infty} \bar{t}_{2j} z^{-j}}, \end{aligned} \quad (98)$$

$$\begin{aligned} \Psi_2^{(n)}(z, t) &= 2\pi \varphi_2^{(n)}(z^{-1}, t) z^{-1} F_{\mu(t)}(z^{-1}) e^{\sum_{j=1}^{\infty} t_{2j} z^j} \\ &= 2\pi \varphi_2^{(n)}(z^{-1}, t) z^{-1} F_{\mu}(z^{-1}) e^{\sum_{j=1}^{\infty} t_{1j} z^{-j}}. \end{aligned}$$

Moreover, the wave functions are eigen-functions of the Lax and adjoint Lax matrices

$$\begin{aligned} L_1 \Psi_1 &= z \Psi_1, & L_2^\dagger \Psi_2^* &= z \Psi_2^*, \\ L_1^\dagger \Psi_1^* &= z \Psi_1^*, & L_2 \Psi_2 &= z \Psi_2. \end{aligned}$$

4.3 CMV matrices and the Toeplitz lattice

For the CMV ordering of the Laurent basis, Lax equations (93) can be written as a nonlinear dynamical system that is a version in the CMV context of the Toeplitz lattice discussed by Mark Adler and Pierre van Moerbeke [8]

Proposition 28. *For the case $\vec{n} = (1, 1)$ Lax equations (93) have as a consequence the following nonlinear dynamical system for the Verblunsky coefficients*

$$\begin{aligned} \frac{\partial \alpha_k^{(1)}}{\partial t_{11}} &= \alpha_{k+1}^{(1)} (1 - \alpha_k^{(1)} \bar{\alpha}_k^{(2)}), & \frac{\partial \bar{\alpha}_k^{(2)}}{\partial t_{11}} &= -\bar{\alpha}_{k-1}^{(2)} (1 - \alpha_k^{(1)} \bar{\alpha}_k^{(2)}), \\ \frac{\partial \alpha_k^{(1)}}{\partial t_{21}} &= \alpha_{k-1}^{(1)} (1 - \alpha_k^{(1)} \bar{\alpha}_k^{(2)}), & \frac{\partial \bar{\alpha}_k^{(2)}}{\partial t_{21}} &= -\bar{\alpha}_{k+1}^{(2)} (1 - \alpha_k^{(1)} \bar{\alpha}_k^{(2)}), \end{aligned} \quad (99)$$

with $k = 1, 2, \dots$

Proof. See Appendix A. \square

If the initial measure μ is positive definite then the sequences $\{\alpha_k^{(1)}\}$ and $\{\bar{\alpha}_k^{(2)}\}$ are identical in $t = 0$. Furthermore, if we set $t_{21} = -\bar{t}_{11}$, the evolved measure is always real and there is only one family of time-dependent functions. That is the reduction studied by L. B. Golinskii [36] in the context of Schur flows.

We have obtained a CMV version of the Toeplitz lattice, but for any \vec{n} the integrable hierarchy obtained is always equivalent to the one discussed in [8]. The reason for this fact relies in the observation that for any positive Borel measure μ and for any \vec{n} , there is a bijection between the set of OLPU $\{\varphi_{\vec{n},1}^{(l)}\}$ and the set of OPUC $\{P_l\}$. All the coefficients of any P_l are determined in terms of the set of reflection coefficients $\{\alpha_l\}$, so the time evolution for the Szegő polynomials under Toda-type flows is determined by the evolution of the reflection coefficients. As the measure evolution does not depend on \vec{n} , it is natural to always obtain the very same evolution for the family $\{\alpha_l\}$ under the Toda flows. We conjecture that a similar result holds for the quasi-definite case.

4.4 Discrete flows

We now consider discrete flows associated to the moment matrix. Given two integers s_1, s_2 and $s := (s_1, s_2)$ is then possible to make a new deformation of the moment matrix that depends on s .

Definition 19. We introduce for each s and the deformed moment matrix $g(s)$ and its deformed Gauss–Borel factorization

$$g(s) := D_{1,0}(s)g(D_{2,0}(s))^{-1}, \quad g(s) = S_1^{-1}(s)S_2(s),$$

where $D_{1,0}(s), D_{2,0}(s)$ are discrete deformation operators to be determined later on.

We consider the operator T_1 responsible of the shift $s_1 \mapsto s_1 + 1$ and T_2 corresponding to the shift $s_2 \mapsto s_2 + 1$. Let us suppose that matrices q_1, q_2 exist and satisfy

$$\begin{aligned} T_1(D_{1,0}) &= q_1 D_{1,0}, & T_1(D_{2,0}) &= D_{2,0}, \\ T_2(D_{1,0}) &= D_{1,0}, & T_2(D_{2,0}) &= q_2 D_{2,0}, \end{aligned}$$

then we define

$$\delta_1 := S_1(s)q_1 S_1(s)^{-1}, \quad \delta_2 := S_2(s)q_2 S_2(s)^{-1}.$$

If δ_1, δ_2 can be LU factorized, then there exist semi-infinite matrices $\delta_{1,+}, \delta_{1,-}, \delta_{2,+}, \delta_{2,-}$ such that

$$\delta_1 = \delta_{1,-}^{-1} \delta_{1,+}, \quad \delta_2^{-1} = \delta_{2,-}^{-1} \delta_{2,+}.$$

In this case we introduce

$$\omega_1 := \delta_{1,+}, \quad \omega_2 := \delta_{2,-}.$$

Proposition 29. The operators T_1, T_2 and the matrices $S_1(s), S_2(s)$ satisfy the following equations

$$\begin{aligned} T_1(S_1(s))(S_1(s))^{-1} &= \delta_{1,-} & T_1(S_2(s))(S_2(s))^{-1} &= \delta_{1,+} \\ T_2(S_1(s))(S_1(s))^{-1} &= \delta_{2,-} & T_2(S_2(s))(S_2(s))^{-1} &= \delta_{2,+} \end{aligned}$$

Proof. First using T_1 and T_2 on the factorization $D_{1,0}gD_{2,0}^{-1} = S_1^{-1}S_2$ we obtain

$$\begin{aligned} T_1(D_{1,0})gT_1(D_{2,0}^{-1}) &= T_1(S_1^{-1})T_1(S_2) \quad \Rightarrow \quad (T_1(S_1)S_1^{-1})^{-1}T_1(S_2)S_2^{-1} = \delta_1, \\ T_2(D_{1,0})gT_2(D_{2,0}^{-1}) &= T_2(S_1^{-1})T_2(S_2) \quad \Rightarrow \quad (T_2(S_1)S_1^{-1})^{-1}T_2(S_2)S_2^{-1} = \delta_2^{-1}, \end{aligned}$$

then using the factorization for δ_1 and δ_2^{-1} and its uniqueness we can identify the upper and lower triangular parts and prove the claimed result. \square

It is also possible to define wave matrices W_1 and W_2 in this discrete context,

$$W_1 := S_1 D_{1,0} \qquad W_2 := S_2 D_{2,0}.$$

To ensure the consistency between both continuous and discrete flows we only need to replace $D_{1,0} \mapsto D_{1,0} W_{1,0}$ and $D_{2,0} \mapsto D_{2,0} W_{2,0}$. The next results are valid in case continuous evolution is also present or not, for the proof one only needs a slight modification of the one in [12].

Theorem 8. 1. *The next linear system for W_1 and W_2 is satisfied*

$$T_a(W_{a'}) = \omega_a W_{a'} \qquad a, a' = 1, 2. \qquad (100)$$

2. *The discrete versions of the Lax equations are the following*

$$T_a(L_{a'}) = \omega_a L_{a'} \omega_a^{-1} \qquad a, a' = 1, 2. \qquad (101)$$

3. *The compatibility equations for the discrete flows in the linear system (100) are*

$$T_1(\omega_2) \omega_1 = T_2(\omega_1) \omega_2 \qquad (102)$$

if there are also continuous deformation parameters the mixed compatibility equations are

$$\begin{aligned} T_a(B_{1,j}) &= \frac{\partial \omega_a}{\partial t_{1j}} \omega_a^{-1} + \omega_a B_{1,j} \omega_a^{-1} \quad a = 1, 2 \quad j = 1, 2, \dots \\ T_a(B_{2,j}) &= \frac{\partial \omega_a}{\partial t_{2j}} \omega_a^{-1} + \omega_a B_{2,j} \omega_a^{-1} \quad a = 1, 2 \quad j = 1, 2, \dots \end{aligned} \qquad (103)$$

Now we give examples of some discrete flows operators. Let be $\{\lambda_1(j)\}_{j \in \mathbb{Z}}$ and $\{\lambda_2(j)\}_{j \in \mathbb{Z}}$ two complex sequences with $\lambda_1(j), \lambda_2(j) \in \mathbb{D}$, then

$$D_{1,0}^{(1)} := \begin{cases} \prod_{j=0}^{n_1} (\Upsilon - \lambda_1(j) \mathbb{I}) & n_1 > 0 \\ \mathbb{I} & n_1 = 0 \\ \prod_{j=0}^{|n_1|} (\Upsilon - \lambda_1(-j) \mathbb{I})^{-1} & n_1 < 0 \end{cases} \quad (D_{2,0}^{(1)})^{-1} := \begin{cases} \prod_{j=0}^{n_2} (\Upsilon^\top - \lambda_2(j) \mathbb{I}) & n_2 > 0 \\ \mathbb{I} & n_2 = 0 \\ \prod_{j=0}^{|n_2|} (\Upsilon^\top - \lambda_2(-j) \mathbb{I})^{-1} & n_2 < 0 \end{cases}$$

the evolution of the measure is then

$$d\mu(z, s) = \mathcal{D}_1^{(1)}(z, s_1) (\mathcal{D}_2^{(1)})^{-1}(z, s_2) d\mu(z)$$

where

$$\mathcal{D}_1^{(1)}(z, s) = \begin{cases} \prod_{j=0}^{n_1} (z - \lambda_1(j)) & s_1 > 0 \\ 1 & s_1 = 0 \\ \prod_{j=0}^{|s_1|} (z - \lambda_1(-j))^{-1} & s_1 < 0 \end{cases} \quad (\mathcal{D}_2^{(1)})^{-1}(z, s) = \begin{cases} \prod_{j=0}^{n_2} (z^{-1} - \lambda_2(j)) & s_2 > 0 \\ 1 & s_2 = 0 \\ \prod_{j=0}^{|s_2|} (z^{-1} - \lambda_2(-j))^{-1} & s_2 < 0 \end{cases}$$

in that case

$$\begin{aligned} q_1^{(1)} &= \Upsilon - \lambda_1(s_1 + 1) \mathbb{I} & q_2^{(1)} &= \Upsilon^\top - \lambda_2(s_2 + 1) \mathbb{I} \\ \delta_1^{(1)} &= L_1 - \lambda_1(s_1 + 1) \mathbb{I} & \delta_2^{(1)} &= L_2 - \lambda_2(s_2 + 1) \mathbb{I} \end{aligned}$$

The evolution of the wave functions is associated to the evolved Laurent polynomials

$$\begin{aligned} \Psi_1(z, s) &= W_1(s) \chi(z) = S_1(s) D_{1,0}(s) \chi(z) = \Phi_1(z, s) \mathcal{D}_1^{(1)}(z, s), \\ \Psi_2^*(z, s) &= (W_2(s)^{-1})^\dagger \chi(z) = (S_2(s)^{-1})^\dagger (D_{2,0}(s)^{-1})^\dagger \chi(z) = \Phi_2(z, s) (\mathcal{D}_2^{(1)})^{-1}(z, s), \end{aligned}$$

where $\Phi_1(z, t)$ and $\Phi_2(z, t)$ are the Laurent polynomials associated to the evolved measure.

Lemma 4. *We have the following structure for the matrices ω_1, ω_2*

$$\begin{aligned}\omega_1 &= \omega_{1,0} + \omega_{1,1}\Lambda + \cdots + \omega_{1,n_-+1}\Lambda^{n_-+1} \\ \omega_2 &= \omega_{2,0} + \omega_{2,1}\Lambda^\top + \cdots + \omega_{1,n_++1}(\Lambda^\top)^{n_++1} \\ \omega_1^\dagger &= \rho_{1,0} + \rho_{1,1}\Lambda^\top + \cdots + \rho_{1,n_-+1}(\Lambda^\top)^{n_-+1} \\ \omega_2^\dagger &= \rho_{2,0} + \rho_{2,1}\Lambda + \cdots + \rho_{1,n_++1}\Lambda^{n_++1}\end{aligned}$$

for some semi-infinite matrices

$$\begin{aligned}\omega_{1,j} &= \text{diag}(\omega_{1,j}(0), \omega_{1,j}(1), \dots) & j &= 0, \dots, n_- + 1 \\ \omega_{2,j} &= \text{diag}(\omega_{2,j}(0), \omega_{2,j}(1), \dots) & j &= 0, \dots, n_+ + 1 \\ \rho_{1,j} &= \text{diag}(\rho_{1,j}(0), \rho_{1,j}(1), \dots) & j &= 0, \dots, n_- + 1 \\ \rho_{2,j} &= \text{diag}(\rho_{2,j}(0), \rho_{2,j}(1), \dots) & j &= 0, \dots, n_+ + 1\end{aligned}$$

Proof. Immediate from (104). □

Defining

$$\gamma_1(z, s) := z - \lambda_1(s_1 + 1) \qquad \gamma_2(z, s) := z^{-1} - \lambda_2(s_2 + 1)$$

the previous Lemma allows us to compute the action of the operators T_1 and T_2 on the OLPU $\varphi_1^{(l)}(z, s), \varphi_2^{(l)}(z, s)$

Proposition 30. *The following equations hold*

$$\begin{aligned}(T_1\varphi_1^{(l)})\gamma_1 &= \omega_{1,0}(l)\varphi_1^{(l)} + \omega_{1,1}(l)\varphi_1^{(l+1)} + \cdots + \omega_{1,n_-+1}(l)\varphi_1^{(l+n_-+1)} \\ (T_2\varphi_1^{(l)}) &= \omega_{2,0}(l)\varphi_1^{(l)} + \omega_{2,1}(l)\varphi_1^{(l-1)} + \cdots + \omega_{1,n_++1}(l)\varphi_1^{(l-n_+-1)} \\ \varphi_2^{(l)} &= \rho_{1,0}(l)(T_1\varphi_2^{(l)}) + \rho_{1,1}(l)(T_1\varphi_2^{(l-1)}) + \cdots + \rho_{1,n_-+1}(l)(T_1\varphi_2^{(l-n_- -1)}) \\ \varphi_2^{(l)} &= \left(\rho_{2,0}(l)(T_2\varphi_2^{(l)}) + \rho_{2,1}(l)(T_2\varphi_2^{(l+1)}) + \cdots + \rho_{1,n_++1}(l)(T_2\varphi_2^{(l+n_++1)}) \right) \gamma_2\end{aligned}$$

Proof. The first two equations come from (100) and Lemma 4. For the last two equations we use that

$$\omega_a^\dagger T_a((W_{a'}^{-1})^\dagger) = (W_{a'}^{-1})^\dagger \qquad a, a' = 1, 2.$$

□

Another possible option that preserves the reality of the measure is using pairs of conjugate transforms as follows

$$\begin{aligned}D_{1,0}^{(2)} &:= \begin{cases} \prod_{j=0}^{s_1} (\Upsilon - \lambda_1(j)\mathbb{I})(\Upsilon^\top - \bar{\lambda}_1(j)\mathbb{I}) & s_1 > 0 \\ \mathbb{I} & s_1 = 0 \\ \prod_{j=0}^{|s_1|} (\Upsilon - \lambda_1(-j)\mathbb{I})^{-1}(\Upsilon^\top - \bar{\lambda}_1(-j)\mathbb{I})^{-1} & s_1 < 0 \end{cases} \\ (D_{2,0}^{(2)})^{-1} &:= \begin{cases} \prod_{j=0}^{s_2} (\Upsilon^\top - \lambda_2(j)\mathbb{I})(\Upsilon - \bar{\lambda}_2(j)\mathbb{I}) & s_2 > 0 \\ \mathbb{I} & s_2 = 0 \\ \prod_{j=0}^{|s_2|} (\Upsilon^\top - \lambda_2(-j)\mathbb{I})^{-1}(\Upsilon - \bar{\lambda}_2(-j)\mathbb{I})^{-1} & s_2 < 0 \end{cases}\end{aligned}$$

the evolution of the measure is then

$$d\mu(z, s) = \mathcal{D}_1^{(2)}(z, s_1)(\mathcal{D}_2^{(2)})^{-1}(z, s_2)d\mu(z)$$

where

$$\mathcal{D}_1^{(2)}(z, s) = \begin{cases} \prod_{j=0}^{s_1} |z - \lambda_1(j)|^2 & s_1 > 0 \\ 1 & s_1 = 0 \\ \prod_{j=0}^{|s_1|} |z - \lambda_1(-j)|^{-2} & s_1 < 0 \end{cases} \quad (\mathcal{D}_2^{(2)})^{-1}(z, s) = \begin{cases} \prod_{j=0}^{s_2} |z^{-1} - \lambda_2(j)|^2 & s_2 > 0 \\ 1 & s_2 = 0 \\ \prod_{j=0}^{|s_2|} |z^{-1} - \lambda_2(-j)|^{-2} & s_2 < 0 \end{cases} \quad (104)$$

in that case

$$\begin{aligned} q_1^{(2)} &= (\Upsilon - \lambda_1(s_1 + 1)\mathbb{I})(\Upsilon^\top - \bar{\lambda}_1(s_1 + 1)\mathbb{I}) & q_2^{(2)} &= (\Upsilon^\top - \lambda_2(s_2 + 1)\mathbb{I})(\Upsilon - \bar{\lambda}_2(s_2 + 1)\mathbb{I}) \\ \delta_1^{(2)} &= (L_1 - \lambda_1(s_1 + 1)\mathbb{I})(L_1^{-1} - \bar{\lambda}_1(s_1 + 1)\mathbb{I}) & \delta_2^{(2)} &= (L_2 - \lambda_2(s_2 + 1)\mathbb{I})(L_2^{-1} - \bar{\lambda}_2(s_2 + 1)\mathbb{I}) \end{aligned} \quad (105)$$

Observe that these discrete flows leads to extended Geronimus transformations [46]. When the sequences $\{\lambda_1(j)\}, \{\lambda_2(j)\}$ are constant and thus q_1, q_2 are invariant under the action of T_1 and T_2 we can make an interpretation in terms of Darboux transformations in the context of [7]. In that case what we obtain is

$$\begin{aligned} \delta_1 &= \delta_{1,-}^{-1} \delta_{1,+} & T_1 \delta_1 &= T_1(W_1) q_1 T_1(W_1^{-1}) = \omega_1 \delta_1 \omega_1^{-1} = \delta_{1,+} \delta_{1,-}^{-1} \delta_{1,+} \delta_{1,-}^{-1} = \delta_{1,+} \delta_{1,-}^{-1} \\ \delta_2^{-1} &= \delta_{2,-}^{-1} \delta_{2,+} & T_2 \delta_2^{-1} &= T_2(W_2) q_2^{-1} T_2(W_2^{-1}) = \omega_2 \delta_2^{-1} \omega_2^{-1} = \delta_{2,-} \delta_{2,-}^{-1} \delta_{2,+} \delta_{2,-}^{-1} = \delta_{2,+} \delta_{2,-}^{-1} \end{aligned}$$

that is a change in the LU factorization into UL .

4.5 τ -functions

As is well known τ -functions is an essential ingredient of the theory of integrable systems. Not only for the use of Hirota of these functions in the construction of soliton solutions [38] but also for its relevant geometrical insight [26]-[28], also the bilinear equations discussed in the mentioned papers are fundamental in the construction of solutions. The determinantal expressions for the OLPUC, and the associated Laurent polynomials and the corresponding second kind functions lead to a τ -function representation of these objects. For that aim one consider the action of coherent shifts in the time variables, the so called Miwa shifts

Definition 20. 1. *The Miwa shifts are the following time shifts*

$$t \mapsto t \pm [w]_1 := t_{1j} \mapsto t_{1j} \pm \frac{w^j}{j}, \quad t_{2j} \mapsto t_{2j},$$

2. *And the Miwa dual shifts*

$$t \mapsto t \pm [w]_2 := t_{1j} \mapsto t_{1j}, \quad t_{2j} \mapsto t_{2j} \pm \frac{w^j}{j}.$$

A very important property of this Miwa shifts is the form in what they act on the deformed measure

Proposition 31. *The evolved measure $\mu(z, t)$ has the following behaviour*

$$\begin{aligned} \mu(z, t \pm [w^{-1}]_1) &= \left(1 - \frac{z}{w}\right)^{\mp 1} \mu(z, t), & |z| < |w|, \\ \mu(z, t \pm [w]_2) &= \left(1 - \frac{w}{z}\right)^{\pm 1} \mu(z, t) & |z| > |w|. \end{aligned}$$

Proof. Using the series expansion of the logarithm, the evolution factors change under Miwa time shifts like

$$\begin{aligned} \exp\left(\sum_{j=1}^{\infty} (t_{1j} z^j - t_{2j} z^{-j})\right) &\mapsto \exp\left(\sum_{j=1}^{\infty} \left((t_{1j} \pm \frac{1}{j w^j}) z^j - t_{2j} z^{-j}\right)\right) = \left(1 - \frac{z}{w}\right)^{\mp 1} \exp\left(\sum_{j=1}^{\infty} (t_{1j} z^j - t_{2j} z^{-j})\right), \quad |z| < |w| \\ \exp\left(\sum_{j=1}^{\infty} (t_{1j} z^j - t_{2j} z^{-j})\right) &\mapsto \exp\left(\sum_{j=1}^{\infty} \left(t_{1j} z^j - (t_{2j} \pm \frac{w^j}{j}) z^{-j}\right)\right) = \left(1 - \frac{w}{z}\right)^{\pm 1} \exp\left(\sum_{j=1}^{\infty} (t_{1j} z^j - t_{2j} z^{-j})\right), \quad |z| > |w|. \end{aligned}$$

□

We introduce the main and associated τ -functions as determinants

Definition 21. *The τ -function is*

$$\tau^{(0)}(t) := 1, \quad \tau^{(l)}(t) := \det g^{[l]}(t), \quad l = 1, 2, \dots,$$

while the associated τ -functions are

$$\tau_{1,-a}^{(l)}(t) := (-1)^{l+l-a} \det \left(\begin{array}{cccc|ccc} g_{0,0} & g_{0,1} & \cdots & g_{0,l-a-1} & g_{0,l-a+1} & \cdots & g_{0,l} \\ g_{1,0} & g_{1,1} & \cdots & g_{1,l-a-1} & g_{1,l-a+1} & \cdots & g_{1,l} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ g_{l-1,0} & g_{l-1,1} & \cdots & g_{l-1,l-a-1} & g_{l-1,l-a+1} & \cdots & g_{l-1,l} \end{array} \right) \quad l = |\vec{n}|, |\vec{n}| + 1, \dots,$$

$$\tau_{2,-a}^{(l)}(t) := (-1)^{l+l-a} \det \left(\begin{array}{cccc} g_{0,0} & g_{0,1} & \cdots & g_{0,l} \\ g_{1,0} & g_{1,1} & \cdots & g_{1,l} \\ \vdots & \vdots & & \vdots \\ \frac{g_{l-a-1,0}}{g_{l-a+1,0}} & \frac{g_{l-a-1,1}}{g_{l-a+1,1}} & \cdots & \frac{g_{l-a-1,l}}{g_{l-a+1,l}} \\ \vdots & \vdots & & \vdots \\ g_{l,0} & g_{l,1} & \cdots & g_{l,l} \end{array} \right), \quad l = |\vec{n}|, |\vec{n}| + 1, \dots,$$

$$\tau_{2,+a}^{(l)}(t) := \det \left(\begin{array}{cccc|ccc} g_{0,0} & g_{0,1} & \cdots & g_{0,l-2} & g_{0,(l-1)+a} \\ g_{1,0} & g_{1,1} & \cdots & g_{1,l-2} & g_{1,(l-1)+a} \\ \vdots & \vdots & & \vdots & \vdots \\ g_{l-1,0} & g_{l-1,1} & \cdots & g_{l-1,l-2} & g_{l-1,(l-1)+a} \end{array} \right), \quad l = |\vec{n}|, |\vec{n}| + 1, \dots,$$

$$\tau_{1,+a}^{(l)}(t) := \det \left(\begin{array}{cccc} g_{0,0} & g_{0,1} & \cdots & g_{0,l-1} \\ g_{1,0} & g_{1,1} & \cdots & g_{1,l-1} \\ \vdots & \vdots & & \vdots \\ \frac{g_{l-2,0}}{g_{(l-1)+a,0}} & \frac{g_{l-2,1}}{g_{(l-1)+a,1}} & \cdots & \frac{g_{l-2,l-1}}{g_{(l-1)+a,l-1}} \end{array} \right), \quad l = |\vec{n}|, |\vec{n}| + 1, \dots$$

To find expressions for the OLPU in terms of these τ -functions we need the following

Lemma 5. *Let be $r_j(t)$ the j -th row of the matrix $g(t)$, then for $j \in \mathbb{Z}_+ \setminus \{0, n_+\}$*

$$r_j(t - [z^{-1}]_1) = \begin{cases} r_j(t) - z^{-1}r_{(j+1)+1}(t) & a(j) = 1, \\ r_j(t) - z^{-1}r_{(j-1)-2}(t) & a(j) = 2, \end{cases}$$

$$r_j(t + [z]_2) = \begin{cases} r_j(t) - zr_{(j-1)-1}(t) & a(j) = 1, \\ r_j(t) - zr_{(j+1)+2}(t) & a(j) = 2, \end{cases}$$

is satisfied. For $j = 0$ or $j = n_+$ one has

$$\begin{aligned} r_0(t - [z^{-1}]_1) &= r_0(t) - z^{-1}r_{1+1}(t) \\ r_0(t + [z]_2) &= r_0(t) - zr_{0+2}(t) \\ r_{n_+}(t - [z^{-1}]_1) &= r_{n_+}(t) - z^{-1}r_0(t) \\ r_{n_+}(t + [z]_2) &= r_{n_+}(t) - zr_{(n_++1)+2}(t) \end{aligned}$$

Proof. Immediate from Proposition 31. □

Let us recall the skew multi-linear character of determinants and the consequent formulation in terms of wedge products of covectors. Observe that

Lemma 6. *Given a set of covectors $\{r_1, \dots, r_n\}$ it can be shown that*

$$\bigwedge_{j=1}^n (zr_j - r_{j+1}) = \sum_{j=1}^{n+1} (-1)^{n+1-j} z^{j-1} r_1 \wedge r_2 \wedge \dots \wedge \hat{r}_j \wedge \dots \wedge r_{n+1}, \quad (106)$$

where the notation \hat{r}_j means that we have erased the covector r_j in the wedge product $r_1 \wedge \dots \wedge r_{n+1}$.

Proof. It can be done directly by induction. □

These two lemmas are the key property to characterize deformed OLPUC using τ -functions

Theorem 9. *Given $l \geq |\vec{n}|$, one has the following τ -function representation of the OLPUC*

$$\begin{aligned} \varphi_1^{(l)}(z, t) &= \varphi_{1,+1}^{(l)}(z, t) = (S_2)_U \varphi_{1,-1}^{(l)}(z, t) = z^{\nu_+(l)-1} \frac{\tau^{(l)}(t - [z^{-1}]_1)}{\tau^{(l)}(t)}, \quad a(l) = 1, \\ \varphi_1^{(l)}(z, t) &= \varphi_{1,+2}^{(l)}(z, t) = (S_2)_U \varphi_{1,-2}^{(l)}(z, t) = z^{-\nu_-(l)} \frac{\tau^{(l)}(t + [z]_2)}{\tau^{(l)}(t)}, \quad a(l) = 2, \end{aligned} \quad (107)$$

for the other “+” associated polynomials we have

$$\begin{aligned} \varphi_{1,+2}^{(l)}(z, t) &= z^{-\nu_-(l+2)} \frac{\tau^{(l)}(t + [z]_2)}{\tau^{(l)}(t)}, \quad a(l) = 1, \\ \varphi_{1,+1}^{(l)}(z, t) &= z^{\nu_+(l+1)-1} \frac{\tau^{(l)}(t - [z^{-1}]_1)}{\tau^{(l)}(t)}, \quad a(l) = 2, \end{aligned} \quad (108)$$

and finally, the remaining “-” polynomials can be written as follows

$$\begin{aligned} \varphi_{1,-2}^{(l)}(z, t) &= z^{\nu_+(l)-1} \frac{\tau_{1,-2}^{(l)}(t - [z^{-1}]_1)}{\tau^{(l+1)}(t)}, \quad a(l) = 1, \\ \varphi_{1,-1}^{(l)}(z, t) &= z^{-\nu_-(l)} \frac{\tau_{1,-1}^{(l)}(t + [z]_2)}{\tau^{(l+1)}(t)}, \quad a(l) = 2. \end{aligned} \quad (109)$$

Proof. Let us prove (107). If $a(l) = 1$ we can use Lemma 6 with $r_1 = r_{(l-1)_{-2}}$ and $r_n = r_{(l-1)_{-1}}$ to expand

$$\begin{aligned} z^{\nu_+(l)-1} \tau^{(l)}(t - [z^{-1}]_1) &= z^{\nu_+(l)-1} (M_U^{(l+1)} + (-1)^{l+(l-1)-1} z^{-1} M_{(l-1)_{-1}l}^{(l+1)} + \dots + (-1)^{l+(l-1)-2} z^{-l} M_{(l-1)_{-2}l}^{(l+1)}) \\ &= \sum_{j=0}^l (-1)^{l+j} M_{jl}^{(l+1)} \chi^{(j)}(z) \\ &= \det(g^{[l]}(t)) \varphi_1^{(l)}(z, t) \\ &= \tau^{(l)}(t) \varphi_1^{(l)}(z, t). \end{aligned}$$

If $a(l) = 2$ the same procedure works with $r_1 = r_{(l-1)_{-1}}$ and $r_n = r_{(l-1)_{-2}}$. Now the expansion is

$$\begin{aligned} z^{-\nu_-(l)} \tau^{(l)}(t + [z]_2) &= z^{-\nu_-(l)} (M_U^{(l+1)} + (-1)^{l+(l-1)-2} z M_{(l-1)_{-2}l}^{(l+1)} + \dots + (-1)^{l+(l-1)-1} z^l M_{(l-1)_{-1}l}^{(l+1)}) \\ &= \sum_{j=0}^l (-1)^{l+j} M_{jl}^{(l+1)} \chi^{(j)}(z) \\ &= \det(g^{[l]}(t)) \varphi_1^{(l)}(z, t) \\ &= \tau^{(l)}(t) \varphi_1^{(l)}(z, t). \end{aligned}$$

The proof of expressions in (108) can be performed using the same technique, expanding the right hand side. If $a(l) = 1$ we have to consider Lema 6 with the rows $r_1 = r_{(l-1)_{-1}}$ and $r_n = r_{(l-1)_{-2}}$. In the case $a(l) = 2$ the rows $r_1 = r_{(l-1)_{-2}}$ and $r_n = r_{(l-1)_{-1}}$ should be considered.

To conclude we prove (109), repeating previous arguments with adequate τ -functions, that is, in the case $a(l) = 1$

$$\begin{aligned} z^{\nu_+(l)-1} \tau_{1,-2}^{(l)}(t - [z^{-1}]_1) &= z^{\nu_+(l)-1} \left((-1)^{l+l-2} M_{ll-2}^{(l+1)} + (-1)^{l-2+(l-1)-1} z^{-1} M_{(l-1)_{-1}l-2}^{(l+1)} \right. \\ &\quad \left. + \dots + (-1)^{l-2+(l-1)-2} z^{-l} M_{(l-1)_{-2}l-2}^{(l+1)} \right) \\ &= \sum_{j=0}^l (-1)^{l-2+j} M_{jl-2}^{(l+1)} \chi^{(j)}(z) \\ &= \tau^{(l+1)}(t) \varphi_{1,-2}^{(l)}(z, t). \end{aligned}$$

and in the case $a(l) = 2$ then

$$\begin{aligned} z^{-\nu_-(l)} \tau_{1,-1}^{(l)}(t + [z]_2) &= z^{-\nu_-(l)} \left((-1)^{l+l-1} M_{ll-1}^{(l+1)} + (-1)^{l-1+(l-1)-2} z M_{(l-1)_{-2}l-1}^{(l+1)} \right. \\ &\quad \left. + \dots + (-1)^{l-1+(l-1)-1} z^l M_{(l-1)_{-1}l-1}^{(l+1)} \right) = \\ &= \sum_{j=0}^l (-1)^{l-1+j} M_{jl-1}^{(l+1)} \chi^{(j)}(z) \\ &= \tau^{(l+1)}(t) \varphi_{1,-1}^{(l)}(z, t). \end{aligned}$$

□

To obtain the τ -function representation of the dual Laurent polynomials $\varphi_2^{(l)}$ and their associated ones we would proceed using again Lemma 5 and Lemma 6 interchanging the role of rows and columns, that would lead to the final expression

Theorem 10. *For any $l \geq |\vec{n}|$ the dual Laurent polynomials φ_2 have the following expressions in terms of τ -functions*

$$\begin{aligned} \overline{\varphi_2^{(l)}(z, t)} &= (S_2)_{ll}^{-1} \overline{\varphi_{2,+1}^{(l)}(z, t)} = \overline{\varphi_{2,-1}^{(l)}(z, t)} = \bar{z}^{\nu_+(l)-1} \frac{\tau^{(l)}(t + [\bar{z}^{-1}]_2)}{\tau^{(l+1)}(t)}, \quad a(l) = 1, \\ \overline{\varphi_2^{(l)}(z, t)} &= (S_2)_{ll}^{-1} \overline{\varphi_{2,+2}^{(l)}(z, t)} = \overline{\varphi_{2,-2}^{(l)}(z, t)} = \bar{z}^{-\nu_-(l)} \frac{\tau^{(l)}(t - [\bar{z}]_1)}{\tau^{(l+1)}(t)}, \quad a(l) = 2, \end{aligned}$$

the “+” labeled associated polynomials can be written as

$$\begin{aligned} \overline{\varphi_{2,+2}^{(l)}(z, t)} &= \bar{z}^{-\nu_-(l+2)} \frac{\tau^{(l)}(t - [\bar{z}]_1)}{\tau^{(l)}(t)}, \quad a(l) = 1, \\ \overline{\varphi_{2,+1}^{(l)}(z, t)} &= \bar{z}^{\nu_+(l+1)-1} \frac{\tau^{(l)}(t + [\bar{z}^{-1}]_2)}{\tau^{(l)}(t)}, \quad a(l) = 2, \end{aligned}$$

to conclude, the “-” labeled polynomials have the following representation

$$\begin{aligned} \overline{\varphi_{2,-2}^{(l)}(z, t)} &= \bar{z}^{\nu_+(l)-1} \frac{\tau_{2,-2}^{(l)}(t + [\bar{z}^{-1}]_2)}{\tau^{(l+1)}(t)}, \quad a(l) = 1, \\ \overline{\varphi_{2,-1}^{(l)}(z, t)} &= \bar{z}^{-\nu_-(l)} \frac{\tau_{2,-1}^{(l)}(t - [\bar{z}]_1)}{\tau^{(l+1)}(t)}, \quad a(l) = 2. \end{aligned}$$

We will end this section with results regarding the τ -function representation of the second kind functions in the way we did in [12].

Lemma 7. *The following identity*

$$\bigwedge_{j=1}^n \left(\sum_{i=0}^{\infty} r_{j+i} z^{-i} \right) = r_1 \wedge \cdots \wedge r_{n-1} \wedge \left(\sum_{i=0}^{\infty} r_{n+i} z^{-i} \right) \quad (110)$$

holds.

Proof. Use induction in n . □

Theorem 11. *Let μ be a positive measure supported in \mathbb{T} , then the following statements hold true*

1. *The second kind functions have the following representation involving τ -functions*

$$\begin{aligned} \overline{C_{1,1}^{(l)}(z, t)} &= \bar{z}^{-\nu_+(l+1)} \frac{\tau_{1,+1}^{(l+1)}(t + [\bar{z}^{-1}]_1)}{\tau^{(l+1)}(t)}, & R_- < |z|, \\ \overline{C_{1,2}^{(l)}(z, t)} &= \bar{z}^{\nu_-(l+2)-1} \frac{\tau_{1,+2}^{(l+1)}(t - [\bar{z}]_2)}{\tau^{(l+1)}(t)}, & |z| < R_+, \\ \overline{C_1^{(l)}(z, t)} &= \frac{\bar{z}^{-\nu_+(l+1)} \tau_{1,+1}^{(l+1)}(t + [\bar{z}^{-1}]_1) + \bar{z}^{\nu_-(l+2)-1} \tau_{1,+2}^{(l+1)}(t - [\bar{z}]_2)}{\tau^{(l+1)}(t)}, & R_- < |z| < R_+. \end{aligned} \quad (111)$$

$$\begin{aligned} C_{2,1}^{(l)}(z, t) &= z^{-\nu_+(l+1)} \frac{\tau_{2,+1}^{(l+1)}(t - [z^{-1}]_2)}{\tau^{(l)}(t)}, & R_+^{-1} < |z|, \\ C_{2,2}^{(l)}(z, t) &= z^{\nu_-(l+2)-1} \frac{\tau_{2,+2}^{(l+1)}(t + [z]_1)}{\tau^{(l)}(t)}, & |z| < R_-^{-1} \\ C_2^{(l)}(z, t) &= \frac{z^{-\nu_+(l+1)} \tau_{2,+1}^{(l+1)}(t - [z^{-1}]_2) + z^{\nu_-(l+2)-1} \tau_{2,+2}^{(l+1)}(t + [z]_1)}{\tau^{(l)}(t)}, & R_+^{-1} < |z| < R_-^{-1}. \end{aligned} \quad (112)$$

2. *For $R_- < |z| < R_+$ the Fourier series of the measure can be expressed in terms of τ -functions in the following way*

$$\begin{aligned} F_{\mu(t)}(z) &= \frac{\tau_{2,+1}^{(l+1)}(t - [z]_2) + z^{-\nu_+(l+1)-\nu_-(l+2)+1} \tau_{2,+2}^{(l+1)}(t + [z^{-1}]_1)}{2\pi \tau^{(l)}(t - [z^{-1}]_1)} \\ &= \frac{\tau_{1,+1}^{(l+1)}(t + [z^{-1}]_1) + z^{\nu_+(l+1)+\nu_-(l+2)-1} \tau_{1,+2}^{(l+1)}(t - [z]_2)}{2\pi \tau^{(l)}(t + [z]_2)}, & a(l) = 1, \\ F_{\mu(t)}(z) &= \frac{z^{\nu_+(l+1)+\nu_-(l+2)-1} \tau_{2,+1}^{(l+1)}(t - [z]_2) + \tau_{2,+2}^{(l+1)}(t + [z^{-1}]_1)}{2\pi \tau^{(l)}(t + [z]_2)} \\ &= \frac{z^{1-\nu_+(l+1)-\nu_-(l+2)} \tau_{1,+1}^{(l+1)}(t + [z^{-1}]_1) + \tau_{1,+2}^{(l+1)}(t - [z]_2)}{2\pi \tau^{(l)}(t - [z^{-1}]_1)}, & a(l) = 2, \end{aligned} \quad (113)$$

Proof. We will prove (112) only, and the proof of (111) that goes analogously is left to the reader. The expression from Proposition 4 can be arranged using the truncated columns of the moment matrix, that is

$c_j^{[l]} := \int_{\mathbb{T}} \chi^{[l]}(\chi^{(l)})^\dagger d\mu(t)$. Using this notation

$$\begin{aligned} g^{[l]}(t)C_{2,1}^{(l)}(z, t) &= \tau^{(l)}(t)C_{2,1}^{(l)}(z, t) = c_0^{[l]} \wedge c_1^{[l]} \wedge \cdots \wedge c_{l-1}^{[l]} \wedge \sum_{j=l}^{\infty} c_j^{[l]} (\chi_1^*)^{(j)} \\ &= c_0^{[l]} \wedge c_1^{[l]} \wedge \cdots \wedge c_{l-1}^{[l]} \wedge \sum_{j=l+1}^{\infty} z^{-\nu_+(j)} c_j^{[l]} \delta_{a(j),1}. \end{aligned}$$

We define $\mathbb{Z}_{+,i} := \{j \in \mathbb{Z}_+, a(j) = i\}$, $i = 1, 2$ and notice that $\mathbb{Z}_+ = \mathbb{Z}_{+,1} \cup \mathbb{Z}_{+,2}$. The restrictions $\nu_+|_{\mathbb{Z}_{+,1}}, \nu_-|_{\mathbb{Z}_{+,2}}$ of the mappings $\nu_+, \nu_- : \mathbb{Z}_+ \mapsto \mathbb{N}$ are bijections; hence, they have a well defined inverse, $(\nu_+)^{-1}$ and $(\nu_-)^{-1}$. Therefore,

$$\begin{aligned} \tau^{(l)}(t)C_{2,1}^{(l)}(z, t) &= c_0^{[l]} \wedge c_1^{[l]} \wedge \cdots \wedge c_{l-1}^{[l]} \wedge z^{-\nu_+(l+1)} \sum_{j=0}^{\infty} z^{-j} c_{(\nu_+)^{-1}(\nu_+(l+1)+j)}^{[l]} \\ &= z^{-\nu_+(l+1)} c_0^{[l]} \wedge c_1^{[l]} \wedge \cdots \wedge c_{l-1}^{[l]} \wedge \sum_{j=0}^{\infty} z^{-j} c_{(\nu_+)^{-1}(\nu_+(l+1)+j)}^{[l]} \\ &= z^{-\nu_+(l+1)} \tau_{2,+1}^{(l+1)}(t - [z^{-1}]_2), \end{aligned}$$

where the last step requires the use of Lemma 7 with $c_0^{[l]}, c_1^{[l]}, \dots, c_{l-1}^{[l]}, c_{l+1}^{[l]}$. Proceeding in a very similar way, we have

$$\begin{aligned} \det(g^{[l]}(t))C_{2,2}^{(l)}(z, t) &= \tau^{(l)}(t)C_{2,2}^{(l)}(z, t) = c_0^{[l]} \wedge c_1^{[l]} \wedge \cdots \wedge c_{l-1}^{[l]} \wedge \sum_{j=l}^{\infty} c_j^{[l]} \chi_2^{(j)} \\ &= c_0^{[l]} \wedge c_1^{[l]} \wedge \cdots \wedge c_{l-1}^{[l]} \wedge \sum_{j=l+2}^{\infty} z^{\nu(j)-1} c_j^{[l]} \delta_{a(j),2} \\ &= c_0^{[l]} \wedge c_1^{[l]} \wedge \cdots \wedge c_{l-1}^{[l]} \wedge z^{\nu(l+2)-1} \sum_{j=0}^{\infty} z^j c_{(\nu_-)^{-1}(\nu_-(l+2)+j)}^{[l]} \\ &= z^{\nu_-(l+2)-1} c_0^{[l]} \wedge c_1^{[l]} \wedge \cdots \wedge c_{l-1}^{[l]} \wedge \sum_{j=0}^{\infty} z^j c_{(\nu_-)^{-1}(\nu_-(l+2)+j)}^{[l]} \\ &= z^{\nu_-(l+2)} \tau_{2,+2}^{(l+1)}(t + [z]_1). \end{aligned}$$

with $c_0^{[l]}, c_1^{[1]}, \dots, c_{l+2}^{[l+2]}$ as adequate entries for Lemma 7. Finally, (113) are obtained combining equations (111) and (112) with Proposition 5, Theorem 9 and Theorem 10. \square

A comment on the convergence of the expressions in Theorem 11 and it's proof is needed at this point. The main tool used in the proof is the series expansion of $(1 - z)^{-1}$, that is convergent only for $|z| < 1$ but can be analytically extended outside \mathbb{T} . For instance, the τ -function expression for $C_{2,1}^{(l)}$ is only strictly valid outside \mathbb{T} . Nevertheless, it can be analytically extended inside the circle up to R_- that is where the series for $C_{2,1}$ is convergent. As this extension is unique, we can talk about the analytically extended ‘‘Miwa-shifted’’ τ -function. The same can be said about the other equalities involving τ -functions; they are formally correct and convergent outside or inside \mathbb{T} , but there is an analytical continuation for the shifted τ -functions that converges where the Cauchy transforms do. The differences between expressions for the Cauchy transforms (111) and (112) and their equivalent in the real line (e.g. [9],[12]) is due to the existence of a positive and a negative part in the Laurent expansion around $z = 0$. The series expansion of $(1 - z)^{-1}$ generates the positive part and the expansion of $(1 - z^{-1})^{-1}$ gives a negative power series that generates the singular part of the Laurent expansion.

4.6 Bilinear equations

For the derivation of a bilinear identity we proceed similarly as we did in [12] proving several lemmas. For the first one, let W_1, W_2 be the wave matrices associated with the moment matrix g ; so that, $W_1 g = W_2$. Then, we have

Lemma 8. *The wave matrices associated with different times satisfy*

$$W_1(t)W_1(t')^{-1} = W_2(t)W_2(t')^{-1}, \quad (114)$$

Proof. We consider simultaneously the following equations

$$\begin{aligned} W_1(t)g &= W_2(t), \\ W_1(t')g &= W_2(t'), \end{aligned}$$

and we get

$$W_1(t)^{-1}W_2(t) = W_1(t')^{-1}W_2(t') = g,$$

and the result becomes evident. \square

Lemma 9. 1. *For the vectors χ, χ^* the following formulae hold*

$$\text{Res}_{z=0} (\chi(\chi^*)^\top) = \text{Res}_{z=0} (\chi^* \chi^\top) = \mathbb{I},$$

2. [12] *For any couple of semi-infinite matrices U and V we have*

$$UV = \text{Res}_{z=0} \left((U\chi)(V^\top \chi^*)^\top \right) \quad (115)$$

$$= \text{Res}_{z=0} \left((U\chi^*)(V^\top \chi)^\top \right) \quad (116)$$

We have the following

Theorem 12. *For any t, t'*

1. *Wave functions satisfy*

$$\text{Res}_{z=0} \left(\Psi_1(z, t)(\Psi_1^*(\bar{z}, t'))^\dagger \right) = \text{Res}_{z=0} \left(\Psi_2(z, t)(\Psi_2^*(\bar{z}, t'))^\dagger \right)$$

2. *OLPUC fulfill*

$$\begin{aligned} \text{Res}_{z=0} \left(\varphi_1^{(k)}(z, t) \bar{\varphi}_2^{(l)}(z^{-1}, t') z^{-1} F_\mu(z) e^{\sum_{j=1}^{\infty} (t_{1j} z^j - t'_{2j} z^{-j})} \right) \\ = \text{Res}_{z=\infty} \left(\varphi_1^{(k)}(z, t) \bar{\varphi}_2^{(l)}(z^{-1}, t') z^{-1} F_\mu(z) e^{\sum_{j=1}^{\infty} (t_{1j} z^j - t'_{2j} z^{-j})} \right), \end{aligned} \quad (117)$$

Proof. 1. First we notice that (115) and (116) can be written as

$$\begin{aligned} UV &= \text{Res}_{z=0} \left((U\chi(z))(V^\dagger \chi^*(\bar{z}))^\dagger \right) \\ &= \text{Res}_{z=0} \left((U\chi^*(z))(V^\dagger \chi(\bar{z}))^\dagger \right) \end{aligned}$$

If we set in (115) $U = W_1(t)$ and $V = W_1(t')^{-1}$ and in (116) we put $U = W_2(t)$ and $V = W_2(t')^{-1}$ attending to (114), recalling that $\Psi_1 = W_1 \chi$, $\Psi_2 = W_2 \chi^*$ and observing that $\Psi_1^* = (W_1^{-1})^\dagger \chi^*$ and $\Psi_2^* = (W_2^{-1})^\dagger \chi$ we get the stated bilinear equation for the wave functions.

2. We can substitute the expressions (97) and (98) to prove the second part of the result.

We can reformulate this result using the residue theorem \square

Corollary 4. *If $R_- = 0$ and $R_+ = \infty$ and we take γ_0 and γ_∞ small zero-index cycles around $z = 0$ and $z = \infty$, respectively; then*

$$\oint_{\gamma_0} \Psi_1^{(n)}(z, t) (\bar{\Psi}_1^*)^{(m)}(z, t') dz = \oint_{\gamma_0} \Psi_2^{(n)}(z, t) (\bar{\Psi}_2^*)^{(m)}(z, t') dz, \quad (118)$$

$$\begin{aligned} \oint_{\gamma_0} \varphi_1^{(k)}(z, t) \bar{\varphi}_2^{(l)}(z^{-1}, t') e^{\sum_{j=1}^{\infty} (t_{1j} z^j - t'_{2j} z^{-j})} z^{-1} F_\mu(z) dz \\ = \oint_{\gamma_\infty} \varphi_1^{(k)}(z, t) \bar{\varphi}_2^{(l)}(z^{-1}, t') e^{\sum_{j=1}^{\infty} (t_{1j} z^j - t'_{2j} z^{-j})} z^{-1} F_\mu(z) dz, \end{aligned}$$

Alternatively the bilinear equation can be expressed using τ -functions

$$\begin{aligned} \oint_{\gamma_0} \tau_{1,+1}^{(l)}(t - [z^{-1}]_1) \left(\tau_{1,+1}^{(l+1)}(t + [z^{-1}]_1) + z^{\nu_+(l+1) + \nu_-(l+2) - 1} \tau_{1,+2}^{(l+1)}(t - [z]_2) \right) e^{\sum_{j=1}^{\infty} (t_{1j} - t'_{1j}) z^j} \frac{dz}{z} \\ = \oint_{\gamma_0} \tau_{2,+1}^{(l)}(t + [z^{-1}]_2) \left(\tau_{2,+1}^{(l+1)}(t - [z^{-1}]_2) + z^{\nu_+(l+1) + \nu_-(l+2) - 1} \tau_{2,+2}^{(l+1)}(t + [z]_1) \right) e^{\sum_{j=1}^{\infty} (t_{2j} - t'_{2j}) z^j} \frac{dz}{z}, \end{aligned}$$

if $a(l)=1$ and

$$\begin{aligned} \oint_{\gamma_0} \tau_{1,+2}^{(l)}(t + [z]_2) \left(z^{-\nu_+(l+1) - \nu_-(l+2) + 1} \tau_{1,+1}^{(l+1)}(t + [z^{-1}]_1) + \tau_{1,+2}^{(l+1)}(t - [z]_2) \right) e^{\sum_{j=1}^{\infty} (t_{1j} - t'_{1j}) z^j} \frac{dz}{z} \\ = \oint_{\gamma_0} \tau_{2,+2}^{(l)}(t - [z]_1) \left(z^{-\nu_+(l+1) - \nu_-(l+2) + 1} \tau_{2,+1}^{(l+1)}(t - [z^{-1}]_2) + \tau_{2,+2}^{(l+1)}(t + [z]_1) \right) e^{\sum_{j=1}^{\infty} (t_{2j} - t'_{2j}) z^j} \frac{dz}{z}, \end{aligned}$$

if $a(l)=2$.

Acknowledgements

MM thanks economical support from the Spanish Ministerio de Ciencia e Innovación, research project FIS2008-00200.

Appendices

A Proofs

Proof of Proposition 1 In this case g is positive definite and Hermitian, if we write $S_2 = h \hat{S}_2$, where $h = \text{diag}(h_0, h_1, \dots)$ is a diagonal matrix and $\hat{S}_2 = \begin{pmatrix} 1 & (\hat{S}_2)_{01} & (\hat{S}_2)_{02} & \dots \\ 0 & 1 & (\hat{S}_2)_{12} & \dots \\ 0 & 0 & 1 & \dots \\ \vdots & \vdots & & \ddots \end{pmatrix}$, the uniqueness of the factorization implies that $\hat{S}_2 = (S_1^{-1})^\dagger$ and $h_l \in \mathbb{R}$ and we get the stated result.

Proof of Proposition 3 Expressions (16), (17), (19) and (20) are obtained expressing the factorization problem as a system of equations. From (17) we deduce

$$(S_1)_{lk} = (S_2)_{ll}((g^{[l+1]})^{-1})_{l,k} = \frac{(S_2)_{ll}}{\det g^{[l+1]}} (-1)^{l+k} M_{k,l}^{(l+1)},$$

so that

$$\varphi_1^{(l)}(z) = \sum_{k=0}^l (S_1)_{lk} \chi^{(k)} = \frac{1}{\det g^{[l]}} \sum_{k=0}^l (-1)^{l+k} M_{k,l}^{(l+1)} \chi^{(k)},$$

as stated in (18). To prove (21) we consider

$$(S_2^{-1})_{kl} = ((g^{[l+1]})^{-1})_{k,l} = \frac{1}{\det g^{[l+1]}} (-1)^{l+k} M_{l,k}^{(l+1)},$$

so that

$$\bar{\varphi}_2^{(l)}(\bar{z}) = \sum_{k=0}^l (S_2^{-1})_{kl} (\chi^{(k)})^\dagger = \frac{1}{\det g^{[l+1]}} \sum_{k=0}^l (-1)^{l+k} M_{l,k}^{(l+1)} (\chi^{(k)})^\dagger,$$

that leads to (21).

Proof of Proposition 4 Using the definition for C_1 , we have that

$$C_1^{(l)}(z) = \sum_{k \geq l} (S_1^{-1})_{lk}^\dagger \chi^{*(k)}(z) = \sum_{j=0}^l (S_2^{-1})_{lj}^\dagger \sum_{k \geq l} g_{jk}^\dagger \chi^{*(k)}(z) = \sum_{j=0}^l (S_2^{-1})_{lj}^\dagger \Gamma_{2,j}^{(l)}(z),$$

so

$$\overline{C_1^{(l)}(z)} = \sum_{j=0}^l (S_2^{-1})_{jl} \bar{\Gamma}_2^{(j)}(\bar{z}).$$

For the other set of functions we have

$$C_2^{(l)}(z) = \sum_{k \geq l} (S_2)_{l,k} \chi^{*(k)}(z) = \sum_{j=0}^l (S_1)_{lj} \sum_{k \geq l} g_{jk} \chi^{*(k)}(z) = \sum_{j=0}^l (S_1)_{lj} \Gamma_{1,j}^{(l)}(z).$$

Comparing the expressions with those in Proposition 3 we see that they are formally identical, so we conclude the stated result.

Proof of Proposition 6 Using the Fourier coefficients of the measure and the definition for the $\Gamma_{a,j}^{(l)}$ we obtain

$$\begin{aligned} \Gamma_{1,j}^{(0)}(z) &= \sum_{k \geq 0} g_{j,k} \chi^{*(k)}(z) = \sum_{k \geq 0} g_{j,k} \chi_1^{*(k)}(z) + \sum_{k \geq 0} g_{j,k} \chi_2^{*(k)}(z) \\ &= \sum_{k \geq 0} \int_0^{2\pi} e^{i(J(j)-k)\theta} d\mu(\theta) z^{-k-1} + \sum_{k \geq 0} \int_0^{2\pi} e^{i(J(j)+k+1)\theta} d\mu(\theta) z^k \\ &= 2\pi \sum_{k \geq 0} (c_{k-J(j)} z^{-k-1} + c_{-k-J(j)-1} z^k) \\ &= 2\pi z^{-J(j)-1} \left(\sum_{k \geq -J(j)} c_k (z^{-1})^k + \sum_{k < -J(j)} c_k (z^{-1})^k \right) \\ &= 2\pi z^{-J(j)-1} F_\mu(z^{-1}). \end{aligned}$$

$$\begin{aligned}
\Gamma_{2,j}^{(0)}(z) &= \sum_{k \geq 0} g_{j,k}^\dagger \chi^{*(k)}(z) = \sum_{k \geq 0} g_{j,k}^\dagger \chi_1^{*(k)}(z) + \sum_{k \geq 0} g_{j,k}^\dagger \chi_2^{*(k)}(z) \\
&= \sum_{k \geq 0} \int_0^{2\pi} e^{i(J(j)-k)\theta} d\bar{\mu}(\theta) z^{-k-1} + \sum_{k \geq 0} \int_0^{2\pi} e^{i(J(j)+k+1)\theta} d\bar{\mu}(\theta) z^k \\
&= 2\pi \sum_{k \geq 0} (\overline{c_{J(j)-k}} z^{-k-1} + \overline{c_{k+J(j)+1}} z^k) \\
&= 2\pi z^{-J(j)-1} \left(\sum_{k < J(j)+1} \overline{c_k} z^k + \sum_{k \geq J(j)+1} \overline{c_k} z^k \right) \\
&= 2\pi z^{-J(j)-1} \bar{F}_\mu(z).
\end{aligned}$$

From here we obtain the rest of the expressions

$$\begin{aligned}
\Gamma_{1,j}^{(l)}(z) &= 2\pi z^{-J(j)-1} \left(\sum_{k \geq -J(j)+l} c_k (z^{-1})^k + \sum_{k < -J(j)-l} c_k (z^{-1})^k \right) = 2\pi z^{-J(j)-1} \left(F_{J(j)-l,\mu}^{(+)}(z^{-1}) + F_{J(j)+l,\mu}^{(-)}(z^{-1}) \right), \\
\Gamma_{2,j}^{(l)}(z) &= 2\pi z^{-J(j)-1} \left(\sum_{k < J(j)+1-l} \overline{c_k} z^k + \sum_{k \geq J(j)+1+l} \overline{c_k} z^k \right) = 2\pi z^{-J(j)-1} \left(\bar{F}_{l-J(j)-1,\mu}^{(-)}(z) + \bar{F}_{-l-J(j)-1,\mu}^{(+)}(z) \right).
\end{aligned}$$

Proof of Proposition 8 From the definitions we have

$$\begin{aligned}
(C_{1,1})^\dagger(z) \varphi_{1,1}(z') &= \chi_1^*(z) S_1^{-1} S_1 \chi_1(z') = (\chi_1^*)^\dagger(z) \chi_1(z') = \sum_{n=0}^{\infty} z^{-n-1} (z')^n, \\
(C_{2,1})^\dagger(z) \varphi_{2,1}(z') &= \chi_1^*(z) S_2^\dagger (S_2^\dagger)^{-1} \chi_1(z') = (\chi_1^*)^\dagger(z) \chi_1(z') = \sum_{n=0}^{\infty} z^{-n-1} (z')^n, \\
(C_{1,2})^\dagger(z) \varphi_{1,2}(z') &= \chi_2(z) S_1^{-1} S_1 \chi_2^*(z') = (\chi_2^*)^\dagger(z) \chi_2(z') = \sum_{n=0}^{\infty} z^n (z')^{-n-1}, \\
(C_{2,2})^\dagger(z) \varphi_{2,2}(z') &= \chi_2(z) S_2^\dagger (S_2^\dagger)^{-1} \chi_2^*(z') = (\chi_2^*)^\dagger(z) \chi_2(z') = \sum_{n=0}^{\infty} z^n (z')^{-n-1},
\end{aligned}$$

which, after the study the region of convergence of the series involved, leads to the sated result. Then other identities derived from $(\chi_1^*)^\dagger(z) \chi_2(z') = (\chi_2^*)^\dagger(z) \chi_1(z') = 0$.

Proof of Proposition 12 It can be deduced as follows

$$\begin{aligned}
J_{2k,2k+2} &= (S_1 E_{2k,2k+2} S_1^{-1})_{2k,2k+2} = (S_1)_{2k,2k} (S_1^{-1})_{2k+2,2k+2} = 1, \\
J_{2k,2k+1} &= (S_1 E_{2k,2k+2} S_1^{-1})_{2k,2k+1} = (S_1)_{2k,2k} (S_1^{-1})_{2k+2,2k+1} = -(S_1)_{2k+2,2k+1} = -\alpha_{2k+2}^{(1)}, \\
J_{2k,2k} &= (S_2 E_{2k+1,2k-1} S_2^{-1})_{2k,2k} = (S_2)_{2k,2k+1} (S_2^{-1})_{2k-1,2k} = \\
&= -(S_2)_{2k,2k} (S_2^{-1})_{2k,2k+1} (S_2)_{2k+1,2k+1} (S_2^{-1})_{2k-1,2k} = -\bar{\alpha}_{2k}^{(2)} \alpha_{2k+1}^{(1)}, \\
J_{2k,2k-1} &= (S_2 E_{2k+1,2k-1} S_2^{-1})_{2k,2k-1} = (S_2)_{2k,2k+1} (S_2^{-1})_{2k-1,2k-1} = \\
&= -(S_2)_{2k,2k} (S_2^{-1})_{2k,2k+1} (S_2)_{2k+1,2k+1} (S_2^{-1})_{2k-1,2k-1} = -\rho_{2k}^2 \alpha_{2k+1}^{(1)}.
\end{aligned}$$

$$\begin{aligned}
J_{2k+1,2k-1} &= (S_2 E_{2k+1,2k-1} S_2^{-1})_{2k+1,2k-1} = (S_2)_{2k+1,2k+1} (S_2^{-1})_{2k,2k} (S_2)_{2k,2k} (S_2^{-1})_{2k-1,2k-1} \\
&= \rho_{2k+1}^2 \rho_{2k}^2, \\
J_{2k+1,2k} &= (S_2 E_{2k+1,2k-1} S_2^{-1})_{2k+1,2k} = (S_2)_{2k+1,2k+1} (S_2^{-1})_{2k-1,2k} \\
&= (S_2)_{2k+1,2k+1} (S_2^{-1})_{2k,2k} (S_2)_{2k,2k} (S_2^{-1})_{2k-1,2k} \\
&= \rho_{2k+1}^2 \bar{\alpha}_{2k}^{(2)}, \\
J_{2k+1,2k+1} &= (S_1 E_{2k,2k+2} S_1^{-1})_{2k+1,2k+1} = (S_1)_{2k+1,2k} (S_1^{-1})_{2k+2,2k+1} = -(S_1)_{2k+1,2k} (S_1)_{2k+2,2k+1} \\
&= -\bar{\alpha}_{2k+1}^{(2)} \alpha_{2k+2}^{(1)}, \\
J_{2k+1,2k+2} &= (S_1 E_{2k,2k+2} S_1^{-1})_{2k+1,2k+2} = (S_1)_{2k+1,2k} (S_1^{-1})_{2k+2,2k+2} = \bar{\alpha}_{2k+1}^{(2)}.
\end{aligned}$$

Proof of Proposition 14 For $k = 0$ the result comes from the definition of ρ_0^2 . For $k = 1, 2$ we have to use the truncated recursion relations,

$$\begin{aligned}
z^{-1} \varphi_1^{(0)} &= \varphi_1^{(1)} - \bar{\alpha}_1^{(2)} \varphi_1^{(0)}, & z \varphi_1^{(0)} &= \varphi_1^{(2)} - \alpha_2^{(1)} \varphi_1^{(1)} - \alpha_1^{(1)} \varphi_1^{(0)}, \\
z^{-1} \varphi_1^{(1)} &= \varphi_1^{(3)} - \bar{\alpha}_3^{(2)} \varphi_1^{(2)} - \alpha_1^{(1)} \bar{\alpha}_2^{(2)} \varphi_1^{(1)} - \rho_1^2 \bar{\alpha}_2^{(2)} \varphi_1^{(0)}, & z \varphi_1^{(1)} &= \bar{\alpha}_1^{(2)} \varphi_1^{(2)} - \alpha_2^{(1)} \bar{\alpha}_1^{(2)} \varphi_1^{(1)} + \rho_1^2 \varphi_1^{(0)},
\end{aligned}$$

multiplying by z and integrating we obtain $h_0 = h_1 - \bar{\alpha}_1^{(2)} \int \varphi_1^{(0)} z d\mu$, and $\int \varphi_1^{(0)} z d\mu = -\alpha_1^{(1)} h_0$, from where we have $h_0 = h_1 + \bar{\alpha}_1^{(2)} \alpha_1^{(1)} h_0$. Now multiplying by z^{-1} and integrating we obtain $0 = \bar{\alpha}_1^{(2)} h_2 - \alpha_2^{(1)} \bar{\alpha}_1^{(2)} \int \varphi_1^{(1)} z^{-1} d\mu + \rho_1^2 \int \varphi_1^{(0)} z^{-1} d\mu$, $\int \varphi_1^{(1)} z^{-1} d\mu = -\rho_1^2 \bar{\alpha}_2^{(2)} h_0$ and $\int \varphi_1^{(0)} z^{-1} d\mu = -\bar{\alpha}_1^{(2)} h_0$ leading to $0 = h_2 + \alpha_2^{(1)} \bar{\alpha}_2^{(2)} h_1 - h_1$.

The other $k \geq 2$ can be proved by induction. For odd case we multiply (34) by z^{k+1} to obtain

$$0 = \alpha_{2k}^{(1)} h_{2k+1} - \alpha_{2k}^{(1)} \bar{\alpha}_{2k+1}^{(2)} \int_{\mathbb{T}} \varphi_1^{(2k)} z^{k+1} d\mu + \rho_{2k}^2 \alpha_{2k-1}^{(1)} \int_{\mathbb{T}} \varphi_1^{(2k-1)} z^{k+1} d\mu + \rho_{2k-1}^2 \rho_{2k}^2 \int_{\mathbb{T}} \varphi_1^{(2k-2)} z^{k+1} d\mu$$

then multiplying by z^k the recurrence expressions (31) and (32) for $z \varphi_1^{(2k)}$, $z \varphi_1^{(2k-1)}$, $z \varphi_1^{(2k-2)}$ and integrating we substitute, to get

$$0 = h_{2k+1} + \alpha_{2k+1}^{(1)} \bar{\alpha}_{2k+1}^{(2)} h_{2k} - \alpha_{2k-1}^{(1)} \bar{\alpha}_{2k-1}^{(2)} h_{2k} - \rho_{2k-1}^2 h_{2k},$$

from where using the induction principle the result is proven. Should we want to obtain the rest of the equations (those with even k), then it is necessary to multiply by z^{-k-1} the odd recurrence relations for $z \varphi_1^{(2k+1)}$ (32) and use the same procedure.

Proof from Proposition 17 From the definition we have

$$\begin{aligned}
\int_{\mathbb{T}} K^{[l]}(u, z') K^{[l]}(z, u) d\mu(u) &:= \int_{\mathbb{T}} \sum_{j=0}^{l-1} \varphi_1^{(j)}(z') \bar{\varphi}_2^{(j)}(\bar{u}) \sum_{k=0}^{l-1} \varphi_1^{(k)}(u) \bar{\varphi}_2^{(k)}(\bar{z}) d\mu(u) \\
&= \sum_{j=0}^{l-1} \varphi_1^{(j)}(z') \sum_{k=0}^{l-1} \bar{\varphi}_2^{(k)}(\bar{z}) \int_{\mathbb{T}} \varphi_1^{(k)}(u) \bar{\varphi}_2^{(j)}(\bar{u}) d\mu(u) \\
&= \sum_{j=0}^{l-1} \varphi_1^{(j)}(z') \sum_{k=0}^{l-1} \bar{\varphi}_2^{(k)}(\bar{z}) \delta_{j,k} = K^{[l]}(z, z').
\end{aligned}$$

Alternate proof for Theorem 3 Due to Gerardo Ariznabarreta.

From the block factorization problem $g^{[l]} = (S_1^{[l]})^{-1}S_2^{[l]}$. As S_2 is upper-triangular, we can write $S_1^{[\geq l, l]}g^{[l]} + S_1^{[\geq l, \geq l]}g^{[\geq l, l]} = S_2^{[\geq l, l]} = 0$ from where

$$S_1^{[\geq l, l]} = -S_1^{[\geq l, \geq l]}g^{[\geq l, l]}(g^{[l]})^{-1},$$

then using the definition for $\varphi_1^{(l)}$ and using the previous formula, we get

$$\begin{aligned}\varphi_1^{(l)} &= \chi^{(l)} + \sum_{j=0}^{l-1} (S_1^{[\geq l, l]})_{0,j} \chi^{(j)} = \chi^{(l)} - \sum_{r,j=0}^{l-1} \sum_{k \geq 0} (S_1^{[\geq l, \geq l]})_{0,k} (g^{[\geq l, l]})_{k,r} (g^{[l]})_{r,j}^{-1} \chi^{(j)} \\ &= \chi^{(l)} - \sum_{r,j=0}^{l-1} (g^{[\geq l, l]})_{0,r} (g^{[l]})_{r,j}^{-1} \chi^{(j)} = \chi^{(l)} - (g_{l,0} \ g_{l,1} \ \cdots \ g_{l,l-1}) (g^{[l]})^{-1} \chi^{[l]}.\end{aligned}$$

In addition, we can express the formula for $\varphi^{(l+1)}$ in the following way as

$$\begin{aligned}\varphi^{(l+1)} &= \chi^{(l+1)} + (S_1^{[\geq l+1, l+1]})_{0l} \chi^{(l)} + \sum_{j=0}^{l-1} (S_1^{[\geq l+1, l+1]})_{0j} \chi^{(j)} \\ &= \chi^{(l+1)} + (S_1)_{l+1, l} \chi^{(l)} + \sum_{j=0}^{l-1} (S_1^{[\geq l+1, l+1]})_{0,j} \chi^{(j)} \\ &= \chi^{(l+1)} + (S_1)_{l+1, l} \chi^{(l)} + \sum_{j=0}^{l-1} (S_1^{[\geq l+1, l+1]})_{1,j} \chi^{(j)} \\ &= \chi^{(l+1)} + (S_1)_{l+1, l} \chi^{(l)} - \sum_{r,j=0}^{l-1} \sum_{k \geq 0} (S_1^{[\geq l, \geq l]})_{1,k} (g^{[\geq l, l]})_{k,r} (g^{[l]})_{r,j}^{-1} \chi^{(j)} \\ &= (S_1)_{l+1, l} (\chi^{(l)} - (g_{l,0} \ g_{l,1} \ \cdots \ g_{l,l-1}) (g^{[l]})^{-1} \chi^{[l]}) + \\ &\quad + (\chi^{(l+1)} - (g_{l+1,0} \ g_{l+1,1} \ \cdots \ g_{l+1, l-1}) (g^{[l]})^{-1} \chi^{[l]}),\end{aligned}$$

from where we obtain (if l is odd) $\varphi_{1,+1}^{(l)}(z) = \varphi_1^{(l+1)}(z) - \alpha_{l+1} \varphi_1^{(l)}(z)$.

Proof of Lemma 1 If we denote by $\Pi^{[l]} = \sum_{i=0}^{l-1} E_{i,i}$ (the projection over the first l components) we find

$$K^{[l]}(z, z') = (\Pi^{[l]} \Phi_2(z))^\dagger \Pi^{[l]} \Phi_1(z') = \Phi_2(z)^\dagger \Pi^{[l]} \Phi_1(z') = \chi(z)^\dagger S_2^{-1} \Pi^{[l]} S_1 \chi(z').$$

From the block factorization $g^{[l]} = (S_1^{[l]})^{-1}S_2^{[l]}$ and its inverse $(g^{[l]})^{-1} = (S_2^{[l]})^{-1}S_1^{[l]}$ we can get an expression for $K^{[l]}$ using only finite size matrices, that is

$$\begin{aligned}K^{[l]}(z, z') &= \chi(z)^\dagger \Pi^{[l]} S_2^{-1} \Pi^{[l]} S_1 \Pi^{[l]} \chi(z') \\ &= \chi^{[l]}(z)^\dagger (S_2^{[l]})^{-1} S_1^{[l]} \chi^{[l]}(z') \\ &= \chi^{[l]}(z)^\dagger (g^{[l]})^{-1} \chi^{[l]}(z').\end{aligned}$$

Proof of Lemma 2 The symmetry of g in (30) can be expressed using the block structure

$$\Upsilon = \left(\begin{array}{c|c} \Upsilon^{[l]} & \Upsilon^{[l, \geq l]} \\ \hline \Upsilon^{[\geq l, l]} & \Upsilon^{[\geq l]} \end{array} \right), \quad g = \left(\begin{array}{c|c} g^{[l]} & g^{[l, \geq l]} \\ \hline g^{[\geq l, l]} & g^{[\geq l]} \end{array} \right),$$

With this block structure we get

$$\Upsilon^{[l]}g^{[l]} + \Upsilon^{[l,\geq l]}g^{[\geq l,l]} = g^{[l]}\Upsilon^{[l]} + g^{[l,\geq l]}\Upsilon^{[\geq l,l]},$$

or equivalently, recalling the Gaussian factorization, we arrive to

$$(g^{[l]})^{-1}\Upsilon^{[l]} - \Upsilon^{[l]}(g^{[l]})^{-1} = (g^{[l]})^{-1}(g^{[l,\geq l]}\Upsilon^{[\geq l,l]} - \Upsilon^{[l,\geq l]}g^{[\geq l,l]})(g^{[l]})^{-1}.$$

We have also the equations

$$\Upsilon^{[l]}\chi^{[l]}(z) + \Upsilon^{[l,\geq l]}\chi^{[\geq l,l]}(z) = z\chi^{[l]}(z), \quad \chi^{[l]}(z)^\dagger \Upsilon^{[l]} + \chi^{[\geq l,l]}(z)^\dagger \Upsilon^{[\geq l,l]} = \bar{z}^{-1}\chi^{[l]}(z)^\dagger,$$

that leads to

$$\begin{aligned} (z' - \bar{z}^{-1})K^{[l]}(z, z') &= \chi^{[l]}(z)^\dagger (g^{[l]})^{-1} z' \chi^{[l]}(z') - \bar{z}^{-1} \chi^{[l]}(z)^\dagger (g^{[l]})^{-1} \chi^{[l]}(z') \\ &= \chi^{[l]}(z)^\dagger (g^{[l]})^{-1} (\Upsilon^{[l]}\chi^{[l]}(z') + \Upsilon^{[l,\geq l]}\chi^{[\geq l,l]}(z')) - \\ &\quad - (\chi^{[l]}(z)^\dagger \Upsilon^{[l]} + \chi^{[\geq l,l]}(z)^\dagger \Upsilon^{[\geq l,l]})(g^{[l]})^{-1} \chi^{[l]}(z') \\ &= \chi^{[l]}(z)^\dagger ((g^{[l]})^{-1} \Upsilon^{[l]} - \Upsilon^{[l]}(g^{[l]})^{-1}) \chi^{[l]}(z') + \\ &\quad + \chi^{[l]}(z)^\dagger (g^{[l]})^{-1} \Upsilon^{[l,\geq l]} \chi^{[\geq l,l]}(z') - \chi^{[\geq l,l]}(z)^\dagger \Upsilon^{[\geq l,l]}(g^{[l]})^{-1} \chi^{[l]}(z') \\ &= \chi^{[l]}(z)^\dagger ((g^{[l]})^{-1} (g^{[l,\geq l]}\Upsilon^{[\geq l,l]} - \Upsilon^{[l,\geq l]}g^{[\geq l,l]})(g^{[l]})^{-1}) \chi^{[l]}(z') + \\ &\quad + \chi^{[l]}(z)^\dagger (g^{[l]})^{-1} \Upsilon^{[l,\geq l]} \chi^{[\geq l,l]}(z') - \chi^{[\geq l,l]}(z)^\dagger \Upsilon^{[\geq l,l]}(g^{[l]})^{-1} \chi^{[l]}(z') \\ &= (\chi^{[l]}(z)^\dagger (g^{[l]})^{-1} g^{[l,\geq l]} - \chi^{[\geq l,l]}(z)^\dagger) \Upsilon^{[\geq l,l]}(g^{[l]})^{-1} \chi^{[l]}(z') - \\ &\quad - \chi^{[l]}(z)^\dagger (g^{[l]})^{-1} \Upsilon^{[l,\geq l]}(g^{[l]})^{-1} \chi^{[l]}(z') - \chi^{[\geq l,l]}(z')^\dagger. \end{aligned}$$

Proof of Proposition 19 If $a(l) = 1$ then $z^{\nu_-(l)}\varphi_1^{(l)}(z)$ is a monic polynomial of degree $\nu_-(l) + \nu_+(l) - 1$, while when $a(l) = 2$ then $z^{\nu_+(l)-1}\bar{\varphi}_1^{(l)}(z^{-1})$ is a monic polynomial of degree $\nu_-(l) + \nu_+(l) - 1$. The orthogonality relations for $z^{\nu_-(l)}\varphi_1^{(l)}(z)$ and $z^{\nu_+(l)-1}\bar{\varphi}_1^{(l)}(z^{-1})$ can be obtained from (65)

$$\begin{aligned} \oint_{\mathbb{T}} z^{\nu_-(l)}\varphi_{\bar{n},1}^{(l)}(z)z^{-k}d\mu(z) &= 0, \quad k = 0, \dots, |\bar{\nu}(l)| - 1, \quad a(l) = 1, \\ \oint_{\mathbb{T}} z^{\nu_+(l)-1}\bar{\varphi}_{\bar{n},1}^{(l)}(z^{-1})z^{-k}d\mu(z) &= 0, \quad k = 0, \dots, |\bar{\nu}(l)| - 1, \quad a(l) = 2, \end{aligned} \tag{119}$$

that means that

$$\begin{aligned} z^{\nu_-(l)}\varphi_{\bar{n},1}^{(l)}(z) &= P_{|\bar{\nu}(l)|-1}(z), & a(l) &= 1, \\ z^{\nu_+(l)-1}\bar{\varphi}_{\bar{n},1}^{(l)}(z^{-1}) &= P_{|\bar{\nu}(l)|-1}(z), & a(l) &= 2, \end{aligned}$$

recalling that $|\bar{\nu}(l)| - 1 = l$ we get the desired result.

Proof of Proposition 23

1. The shift operators defined fulfill

$$\begin{aligned} \Lambda_{\bar{n},1}\chi_{\bar{n}}(z) &= z\Pi_{\bar{n},1}\chi_{\bar{n}}(z), & \Lambda_{\bar{n},2}\chi_{\bar{n}}(z) &= z^{-1}\Pi_{\bar{n},2}\chi_{\bar{n}}(z), \\ \Lambda_{\bar{n},1}^\top\chi_{\bar{n}}(z) &= (z^{-1}\Pi_{\bar{n},1} - E_{0,0}\Lambda^{n^+})\chi_{\bar{n}}(z), & \Lambda_{\bar{n},2}^\top\chi_{\bar{n}}(z) &= (z\Pi_{\bar{n},2} - E_{n^+,n^+}(\Lambda^\top)^{n^+})\chi_{\bar{n}}(z), \end{aligned}$$

that means that

$$(\Lambda_{\bar{n},1} + \Lambda_{\bar{n},2}^\top + E_{n^+,n^+}(\Lambda^\top)^{n^+})\chi_{\bar{n}}(z) = z\chi_{\bar{n}}(z), \quad (\Lambda_{\bar{n},1}^\top + \Lambda_{\bar{n},2} + E_{0,0}\Lambda^{n^+})\chi_{\bar{n}}(z) = z^{-1}\chi_{\bar{n}}(z),$$

from there it follows that

$$\begin{aligned}
\Upsilon_{\bar{n}} g_{\bar{n}} &= \oint_{\mathbb{T}} z \chi_{\bar{n}}(z) \chi_{\bar{n}}(z)^\dagger d\mu(z) \\
&= \oint_{\mathbb{T}} \chi_{\bar{n}}(z) (z^{-1} \chi_{\bar{n}}(z))^\dagger d\mu(z) \\
&= g_{\bar{n}} (\Lambda_{\bar{n},1}^\top + \Lambda_{\bar{n},2} + E_{0,0} \Lambda^{n_+})^\dagger \\
&= g_{\bar{n}} (\Lambda_{\bar{n},1} + \Lambda_{\bar{n},2}^\top + E_{n_+,n_+} (\Lambda^\top)^{n_+}) \\
&= g_{\bar{n}} \Upsilon_{\bar{n}}.
\end{aligned}$$

2. With the definitions

$$J_{\bar{n},1} := S_{\bar{n},1} \Upsilon_{\bar{n}} S_{\bar{n},1}^{-1}, \quad J_{\bar{n},2} := S_{\bar{n},2} \Upsilon_{\bar{n}} S_{\bar{n},2}^{-1},$$

the use of Proposition 23 leads easily to

$$J_{\bar{n}} := J_{\bar{n},1} = J_{\bar{n},2}. \quad (120)$$

The matrix $J_{\bar{n},1}$ has $n_- + 1$ diagonals over the main diagonal and the matrix $J_{\bar{n},2}$ has $n_+ + 1$ diagonals under the main diagonal (in both computations we have excluded the main diagonal itself), so both $J_{\bar{n},1}, J_{\bar{n},2}$ have $n_+ + n_- + 3$ diagonal band.

Proof of Proposition 28 We have calculated previously $J = L_1$ and using the same method L_2 can be calculated, both are five-diagonal matrices given by

$$L_1 = J = \begin{pmatrix} -\alpha_1^{(1)} & -\alpha_2^{(1)} & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ \rho_1^2 & -\alpha_2^{(1)} \bar{\alpha}_1^{(2)} & \bar{\alpha}_1^{(2)} & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & -\rho_2^2 \alpha_3^{(1)} & -\alpha_3^{(1)} \bar{\alpha}_2^{(2)} & -\alpha_4^{(1)} & 1 & 0 & 0 & 0 & \dots \\ 0 & \rho_2^2 \rho_3^2 & \rho_3^2 \bar{\alpha}_2^{(2)} & -\alpha_4^{(1)} \bar{\alpha}_3^{(2)} & \bar{\alpha}_3^{(2)} & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & -\rho_4^2 \alpha_5^{(1)} & -\alpha_5^{(1)} \bar{\alpha}_4^{(2)} & -\alpha_6^{(1)} & 1 & 0 & \dots \\ 0 & 0 & 0 & \rho_4^2 \rho_5^2 & \rho_5^2 \bar{\alpha}_4^{(2)} & -\alpha_6^{(1)} \bar{\alpha}_5^{(2)} & \bar{\alpha}_5^{(2)} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$L_2 = \begin{pmatrix} -\bar{\alpha}_1^{(2)} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ -\rho_1^2 \bar{\alpha}_2^{(2)} & -\alpha_1^{(1)} \bar{\alpha}_2^{(2)} & -\bar{\alpha}_3^{(2)} & 1 & 0 & 0 & 0 & 0 & \dots \\ \rho_1^2 \rho_2^2 & \rho_2^2 \alpha_1^{(1)} & -\alpha_2^{(1)} \bar{\alpha}_3^{(2)} & \alpha_2^{(1)} & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & -\rho_3^2 \bar{\alpha}_4^{(2)} & -\alpha_3^{(1)} \bar{\alpha}_4^{(2)} & -\bar{\alpha}_5^{(2)} & 1 & 0 & 0 & \dots \\ 0 & 0 & \rho_3^2 \rho_4^2 & \rho_4^2 \alpha_3^{(1)} & -\alpha_4^{(1)} \bar{\alpha}_5^{(2)} & \alpha_4^{(1)} & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & -\rho_5^2 \bar{\alpha}_6^{(2)} & -\alpha_5^{(1)} \bar{\alpha}_6^{(2)} & -\bar{\alpha}_7^{(2)} & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

as we are looking only for time flows associated to t_{11} and t_{21} then $B_{1,1} = (L_1)_+$ and $B_{2,1} = (L_2)_-$. Using (93) in the matrix elements $(L_1)_{k,k+1}$ for $k \geq 0$ we obtain

$$\begin{aligned}\frac{\partial \alpha_{2k+2}^{(1)}}{\partial t_{11}} &= -\frac{\partial (L_1)_{2k,2k+1}}{\partial t_{11}} = -[B_{1,1}, L_1]_{2k,2k+1} = \alpha_{2k+3}^{(1)}(1 - \alpha_{2k+2}^{(1)} \bar{\alpha}_{2k+2}^{(2)}) \\ \frac{\partial \alpha_{2k+2}^{(1)}}{\partial t_{21}} &= -\frac{\partial (L_1)_{2k,2k+1}}{\partial t_{21}} = -[B_{2,1}, L_1]_{2k,2k+1} = \alpha_{2k+1}^{(1)}(1 - \alpha_{2k+2}^{(1)} \bar{\alpha}_{2k+2}^{(2)}) \\ \frac{\partial \bar{\alpha}_{2k+1}^{(2)}}{\partial t_{11}} &= \frac{\partial (L_1)_{2k+1,2k+2}}{\partial t_{11}} = [B_{1,1}, L_1]_{2k+1,2k+2} = -\bar{\alpha}_{2k}^{(2)}(1 - \alpha_{2k+1}^{(1)} \bar{\alpha}_{2k+1}^{(2)}) \\ \frac{\partial \bar{\alpha}_{2k+1}^{(2)}}{\partial t_{21}} &= \frac{\partial (L_1)_{2k+1,2k+2}}{\partial t_{21}} = [B_{2,1}, L_1]_{2k+1,2k+2} = -\bar{\alpha}_{2k+2}^{(2)}(1 - \alpha_{2k+1}^{(1)} \bar{\alpha}_{2k+1}^{(2)})\end{aligned}$$

and now looking at $(L_1)_{2k,2k}$ and $(L_2)_{2k+1,2k+1}$ for $k \geq 0$ we obtain the rest of the equations

$$\begin{aligned}\frac{\partial (\bar{\alpha}_{2k}^{(2)} \alpha_{2k+1}^{(1)})}{\partial t_{11}} &= -\frac{\partial (L_1)_{2k,2k}}{\partial t_{11}} = -[B_{1,1}, L_1]_{2k,2k} \Rightarrow \frac{\partial \alpha_{2k+1}^{(1)}}{\partial t_{11}} = \alpha_{2k+2}^{(1)}(1 - \alpha_{2k+1}^{(1)} \bar{\alpha}_{2k+1}^{(2)}) \\ \frac{\partial (\bar{\alpha}_{2k}^{(2)} \alpha_{2k+1}^{(1)})}{\partial t_{21}} &= -\frac{\partial (L_1)_{2k,2k}}{\partial t_{21}} = -[B_{2,1}, L_1]_{2k,2k} \Rightarrow \frac{\partial \alpha_{2k+1}^{(1)}}{\partial t_{21}} = \alpha_{2k}^{(1)}(1 - \alpha_{2k+1}^{(1)} \bar{\alpha}_{2k+1}^{(2)}) \\ \frac{\partial (\alpha_{2k+1}^{(1)} \bar{\alpha}_{2k+2}^{(2)})}{\partial t_{11}} &= -\frac{\partial (L_2)_{2k+1,2k+1}}{\partial t_{11}} = -[B_{1,1}, L_2]_{2k+1,2k+1} \Rightarrow \frac{\partial \bar{\alpha}_{2k+2}^{(2)}}{\partial t_{11}} = -\bar{\alpha}_{2k+1}^{(2)}(1 - \alpha_{2k+2}^{(1)} \bar{\alpha}_{2k+2}^{(2)}) \\ \frac{\partial (\alpha_{2k+1}^{(1)} \bar{\alpha}_{2k+2}^{(2)})}{\partial t_{21}} &= -\frac{\partial (L_2)_{2k+1,2k+1}}{\partial t_{21}} = -[B_{2,1}, L_2]_{2k+1,2k+1} \Rightarrow \frac{\partial \bar{\alpha}_{2k+2}^{(2)}}{\partial t_{21}} = -\bar{\alpha}_{2k+3}^{(2)}(1 - \alpha_{2k+2}^{(1)} \bar{\alpha}_{2k+2}^{(2)})\end{aligned}$$

considering all the equations we obtain (99).

Proof of Proposition 18 First let us look to (50). Using Definition 16 it can be expressed as

$$\begin{aligned}\varphi_{1,+a}^{(l)}(z) &= \chi^{(l+a)}(z) - \sum_{i,j=0}^{l-1} g_{l+a,i} (g^{[l]})_{ij}^{-1} \chi^{(j)}(z) \\ &= \frac{1}{\det g^{[l]}} \left(\chi^{(l+a)}(z) \det g^{[l]} - \sum_{i,j=0}^{l-1} g_{l+a,i} (-1)^{i+j} M_{ji}^{(l)} \chi^{(j)}(z) \right) \\ &= \frac{1}{\det g^{[l]}} \left(\chi^{(l+a)}(z) \det g^{[l]} + \sum_{j=0}^{l-1} (-1)^{j+l} \sum_{i=0}^{l-1} (-1)^{l-1+j} g_{l+a,i} M_{ji}^{(l)} (-1)^{i+j} \chi^{(j)}(z) \right),\end{aligned}$$

that is the expansion of (50). Using the same idea with (53)

$$\varphi_{2,-a}^{(l)}(z) = \sum_{j=0}^l (g^{[l+1]})_{l-a,j}^{-1} \chi^{(j)}(z) = \frac{1}{\det g^{[l+1]}} \sum_{j=0}^l (-1)^{l+l-a} (-1)^{l+j} M_{jl-a}^{(l+1)} \chi^{(j)}(z),$$

we arrive at the expansion of (53) taking the complex conjugate. Both (51) and (52) can be proved using the same ideas.

References

- [1] M. J. Ablowitz and J. F. Ladik, *Nonlinear differential-difference equations*. J. Math. Phys. **16** (1975), 598-603. 1

- [2] M. J. Ablowitz, J. F. Ladik, *Nonlinear differential-difference equations and Fourier analysis*. J. Math. Phys. **17** (1976), 1011-1018. 1
- [3] M. Adler, P. van Moerbeke, *Group factorization, moment matrices and Toda lattices*, Int. Math. Res. Notices **12** (1997) 556-572. 1
- [4] M. Adler, P. van Moerbeke, *Generalized orthogonal polynomials, discrete KP and Riemann–Hilbert problems*, Commun. Math. Phys. **207** (1999) 589-620.
- [5] M. Adler, P. van Moerbeke, *The spectrum of coupled random matrices*, Ann. Math. **149** (1999) 921-976.
- [6] M. Adler, P. van Moerbeke, *Vertex operator solutions to the discrete KP hierarchy*, Commun. Math. Phys. **203** (1999) 185-210.
- [7] M. Adler, P. van Moerbeke, *Darboux transforms on band matrices, weights and associated polynomials*, Int. Math. Res. Not. **18** (2001) 935-984. 4.4
- [8] M. Adler, P. van Moerbeke, *Integrals over classical groups, random permutations, Toda and Toeplitz lattices* Comm. Pure Appl. Math. **54** (2001) 153-205. 1, 4.3, 4.3
- [9] M. Adler, P. van Moerbeke, P. Vanhaecke, *Moment matrices and multi-component KP, with applications to random matrix theory*, Commun. Math. Phys. **286** (2009) 1-38. 1, 4.5
- [10] M. Alfaro, *Una expresión de los polinomios ortogonales sobre la circunferencia unidad*, Actas III J.M.H.L. (Sevilla, 1974) 2 (1982) 1-8. 1
- [11] C. Álvarez-Fernández, U. Fidalgo and M. Mañas, *The multicomponent 2D Toda hierarchy: generalized matrix orthogonal polynomials, multiple orthogonal polynomials and Riemann–Hilbert problems*, Inverse Problems **26** (2010) 055009 (17 pp.) 1
- [12] C. Álvarez-Fernández, U. Fidalgo, M. Mañas, *Multiple orthogonal polynomials of mixed type: Gauss-Borel factorization and the multi-component 2D Toda hierarchy*, Adv. Math. **227** (2011) 1451-1525. 1, 2, 2.1, 2.1, 3.1, 4.2, 4.4, 4.5, 4.5, 4.6, 2
- [13] R. Álvarez-Nodarse, J. Arvesú, F. Marcellán, *Modifications of quasi-definite linear functionals via addition of delta and derivatives of delta Dirac functions*, Indagationes Mathematicae, **15** (2004), 1-20. 1
- [14] M. Ambroladze, *On exceptional sets of asymptotics relations for general orthogonal polynomials*, J. Approx. Theory **82** (1995) 257-273. 1
- [15] D. Barrios, G. López, *Ratio asymptotics for orthogonal polynomials on arcs of the unit circle*, Constr. Approx. **15** (1999) 1-31. 1
- [16] M. J. Bergvelt, A. P. E. ten Kroode, *Partitions, vertex operators constructions and multi-component KP equations*, Pacific J. Math. **171** (1995) 23-88. 3.1
- [17] E. Berriochoa, A. Cachafeiro, J. Garcia-Amor *Connection between orthogonal polynomials on the unit circle and bounded interval*, Journal of Computational and Applied Mathematics **177** (2005) 205-223. 1
- [18] A. Bultheel, P. González-Vera, E. Hendriksen, O. Njåstad, *Orthogonal rational functions*, Cambridge Monographs on Applied and Computational Mathematics, vol.5, Cambridge University Press, Cambridge, **1999**. 1
- [19] A. Cachafeiro, F. Marcellán, C. Pérez, *Lebesgue perturbation of a quasi-definite Hermitian functional. The positive definite case*, Linear Algebra and Appl. **369** (2003), 235-250. 1

- [20] M. Cafasso, *Matrix Biorthogonal Polynomials on the unit circle and the non-Abelian Ablowitz-Ladik hierarchy*, J. Phys. A: Math. Theor. **42** (2009), 365211. 1
- [21] M.J. Cantero, L. Moral, L. Velázquez *Five-diagonal matrices and zeros of orthogonal polynomials on the unit circle* Linear Algebra and Appl. **362** (2003), 29-56. 1, 2, 2.3
- [22] L. Cochran, S.C. Cooper, *Orthogonal Laurent polynomials on the real line*, in: S.C. Cooper, W.J. Thron (Eds.), Continued Fractions and Orthogonal Functions, Lecture Notes in Pure and Applied Mathematics Series, **154**, Marcel Dekker, NewYork, 1994, p. 47100. 1
- [23] R. Cruz-Barroso, L. Daruis, Pablo González-Vera, O. Njåstadb, *Sequences of orthogonal Laurent polynomials, bi-orthogonality and quadrature formulas on the unit circle*, Journal of Computational and Applied Mathematics **200** (2007) 424-440. 1
- [24] R. Cruz-Barroso, S. Delvaux *Orthogonal Laurent polynomials on the unit circle and snake-shaped matrix factorizations* Journal of Approximation Theory **161** (2009), 65-87. 1
- [25] R. Cruz-Barroso, P. González-Vera, *A Christoffel–Darboux formula and a Favard’s theorem for Laurent orthogonal polynomials on the unit circle* (2005) J. Comput. Appl. Math. **179**, 157-173. 1, 2.6
- [26] E. Date, M. Jimbo, M. Kashiwara, T. Miwa, *Operator approach to the Kadomtsev-Petviashvili equation. Transformation groups for soliton equations. III* J. Phys. Soc. Jpn. **50**, (1981) 3806-3812. 1, 4.5
- [27] E. Date, M. Jimbo, M. Kashiwara, T. Miwa, *Transformation groups for soliton equations. Euclidean Lie algebras and reduction of the KP hierarchy*, Publ. Res. Inst. Math. Sci. **18** (1982) 1077-1110.
- [28] E. Date, M. Jimbo, M. Kashiwara, T. Miwa, *Transformation groups for soliton equations in Nonlinear Integrable Systems-Classical Theory and Quantum Theory* M. Jimbo and T. Miwa (eds.) World Scientific, Singapore, 1983, pp. 39-120. 1, 4.5
- [29] C. Díaz-Mendoza, P. González-Vera, M. Jiménez-Paiz, *Strong Stieltjes distributions and orthogonal Laurent polynomials with applications to quadratures and Padé approximation*, Math. Comput. **74** (2005) 1843-1870. 1
- [30] R. Felipe, F. Ongay, *Algebraic aspects of the discrete KP hierarchy*, Linear Algebra App. **338** (2001) 1-17. 2.1
- [31] G. Freud, *Orthogonal Polynomials*, Akadémiai Kiadó, Budapest and Pergamon Press, Oxford, 1971, 1985. 1
- [32] P. García, F. Marcellán, *On zeros of regular orthogonal polynomials on the unit circle*, Ann. Polon. Math. **58** (1993) 287-298. 1
- [33] W. Gautschi, *Orthogonal Polynomials: computation and approximation*, Oxford University Press, New York (2004). 1
- [34] Ya. L. Geronimus, *Polynomials orthogonal on a circle and their applications, Series and Approximations*, Amer. Math. Soc. Transl., serie 1, vol. 3, Amer. Math. Soc., Providence, RI (1962), 1-78. 1
- [35] E. Godoy, F. Marcellán, *Orthogonal polynomials on the unit circle: distribution of zeros*, J. Comput. Appl. Math. **37** (1991) 195-208. 1
- [36] L. Golinskii, *Schur flows and orthogonal polynomials on the unit circle*, (2006) Sb. Math. **197** 1145. 1, 4.3
- [37] G. H. Golub, C. F. Van Loan, *Matrix Computations*, third ed., The John Hopkins University Press, Baltimore, CA, 1996. 1

- [38] R. Hirota, *Exact solutions of the KdV equation for multiple collisions of solitons*, Phys. Rev. Lett. **27** (1971) 1192-94. 4.5
- [39] W. B. Jones, O. Njåstad, *Applications of Szegő polynomials to digital signal processing*, Rocky Mountain J. Math. **21** (1991) 387-436. 1
- [40] W. B. Jones, O. Njåstad, W.J. Thron, *Two-point Pade expansions for a family of analytic functions*, J. Comput. Appl. Math. **9** (1983) 105.123. 1
- [41] W.B. Jones, O. Njåstad, W. J. Thron, H. Waadeland, *Szegő polynomials applied to frequency analysis*, Comput. Appl. Math. **46** (1993) 217-228. 1
- [42] W. B. Jones, W.J. Thron, *Orthogonal Laurent polynomials and Gaussian quadrature*, in: K.E. Gustafson, W.P. Reinhardt (Eds.), *Quantum Mechanics in Mathematics Chemistry and Physics*, Plenum, NewYork, 1981, pp. 449-455. 1
- [43] W. B. Jones, W.J. Thron, H. Waadeland, *A strong stieltjes moment problem*, Trans. Amer. Math. Soc. **261** (1980) 503-528. 1
- [44] R. Killip and I. Nenciu, *CMV: The unitary analogue of Jacobi matrices*, Commun. Pure Appl. Math. **60** (2006) 1148-1188. 1
- [45] M. Mañas, L. Martínez Alonso and C. Álvarez-Fernández, *The multicomponent 2D Toda hierarchy: discrete flows and string equations*, Inv. Prob. **25** (2009) 065007 (31 pp). 1
- [46] F. Marcellan, J. Hernandez *Geronimus spectral transforms and measures on the complex plane*, J. Comp. App. Math. **219** (2008) 441-456. 2.2, 4.4
- [47] H. N. Mhaskar, E.B. Saff, *On the distribution of zeros of polynomials orthogonal on the unit circle*, J. Approx. Theory. **63** (1990) 30-38. 1
- [48] A. Mukaihira, Y. Nakamura, *Schur flow for orthogonal polynomials on the unit circle and its integrable discretization*, Journal of Computational and Applied Mathematics **139** (2002) 75-94. 1
- [49] M. Mulase, *Complete integrability of the Kadomtsev–Petviashvili equation*, Adv. Math. **54** (1984) 57-66. 1
- [50] O. Njåstad, W. J. Thron, *The theory of sequences of orthogonal L-polynomials*, in: H. Waadeland, H. Wallin (Eds.), *Pade Approximants and Continued Fractions*, Det Kongelige Norske Videnskabers Selskabs Skrifter, 1983, pp. 54.91. 1
- [51] P. Nevai, V. Totik, *Orthogonal polynomials and their zeros*, Acta Scientiarum Mathematicarum Szeged **53** (1-2) (1989) 99-104. 1
- [52] K. Pan, *Asymptotics for Szegő polynomials associated with Wiener–Levinson filters*, J. Comput. Appl. Math. **46** (1993) 387-394. 1
- [53] K. Pan, E. B. Saff, *Asymptotics for zeros of Szegő polynomials associated with trigonometric polynomials signals*, J. Approx. Theory **71** (1992) 239-251. 1
- [54] F. and M. Riesz, *Über die Randwerte einer analytischen Funktion*, Quatrième Congrès des Mathématiciens Scandinaves, Stockholm, (1916), pp. 27-44. 5
- W. Rudin, *Real and Complex Analysis* (International Series in Pure and Applied Mathematics) McGraw Hill (1986).

- [55] M. Sato, *Soliton equations as dynamical systems on infinite dimensional Grassmann manifolds*, Res. Inst. Math. Sci. Kokyuroku **439**, (1981) 30-46 . 1
- [56] B. Simon, *Orthogonal Polynomials on the Unit Circle, Part 1: Classical Theory*, AMS Colloquium Series, American Mathematical Society, Providence, RI, (2005). 1
- [57] B. Simon, *Orthogonal Polynomials on the Unit Circle, Part 2: Spectral Theory*, AMS Colloquium Series, American Mathematical Society, Providence, RI, (2005). 1
- [58] B. Simon, *CMV matrices: Five years after*, Journal of Computational and Applied Mathematics **208** (2007) 120-154. 1
- [59] B. Simon, *The Christoffel–Darboux Kernel*, Proceedings of Symposia in Pure Mathematics **79**:“Perspectives in Partial Differential Equations, Harmonic Analysis and Applications: A Volume in Honor of Vladimir G. Maz’ya’s 70th Birthday”, (2008) 295-336. [arXiv:0806.1528](https://arxiv.org/abs/0806.1528) 2.6
- [60] B. Simon *Szegő’s Theorem and its descendants*, Princeton University Press, Princeton, New Jersey (2011). 1
- [61] G. Szegő, *Orthogonal Polynomials*, American Mathematical Society Colloquium Publications, Vol. XXIII. American Mathematical Society, Providence, Rhode Island, (1975). 1
- [62] W. J. Thron, *L-polynomials orthogonal on the unit circle*, in: A. Cuyt (Ed.), *Nonlinear Methods and Rational Approximation*, Reidel Publishing Company, Dordrecht, 1988, pp. 271-278. 1
- [63] K. Ueno and K. Takasaki, *Toda lattice hierarchy*, in *Group Representations and Systems of Differential Equations*, Adv. Stud. Pure Math. **4** (1984) 1-95. 1