Spaceability and operator ideals

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Definition (Aron, Gurariy, Seoane, 2005)

Given a topological vector space $X$, a subset $A \subset X$ is said to be **lineable** if $A \cup \{0\}$ contains an infinite-dimensional linear subspace.

The subset $A$ will be called **spaceable** if $A \cup \{0\}$ contains an infinite-dimensional closed linear subspace.
Lineability and spaceability

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Recent results for operator ideals

Theorem (Puglisi, Seoane, 2008)
If $E$ and $F$ are Banach spaces where $E$ has the two series property, then $L(E, F^*) \setminus \Pi_1(E, F^*)$ is lineable.

Theorem (Botelho, Diniz, Pellegrino, 2009)
If $E$ is a superreflexive Banach space containing a complemented infinite-dimensional subspace with unconditional basis, or $F$ is a Banach space having an infinite unconditional basic sequence, then $K(E, F) \setminus \Pi_p(E, F)$ is lineable for every $p \geq 1$.

Theorem (Kitson, Timoney, 2011)
If $E$ is a superreflexive Banach space, then $K(E, F) \setminus \bigcup_{p \geq 1} \Pi_p(E, F)$ is spaceable.
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Our aim

We will consider operator ideals in the sense of Pietsch $I_1$ and $I_2$ and Banach spaces $E$ and $F$ such that $I_1(E,F) \setminus I_2(E,F)$ is non-empty.

Is $I_1(E,F) \setminus I_2(E,F)$ spaceable?

If $I_2$ is not closed in $I_1$, then $I_1(E,F) \setminus I_2(E,F)$ is spaceable.
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- We will consider operator ideals in the sense of Pietsch \( I_1 \) and \( I_2 \) and Banach spaces \( E \) and \( F \) such that \( I_1(E, F) \setminus I_2(E, F) \) is non-empty.
- Is \( I_1(E, F) \setminus I_2(E, F) \) spaceable?
- If \( I_1 \) and \( I_2 \) are Banach operator ideals such that \( I_2 \subset I_1 \) continuously and \( I_2 \) is not closed in \( I_1 \), then \( I_1(E, F) \setminus I_2(E, F) \) is spaceable.
Operator ideals

Definition

Let $B$ denote the class of all Banach spaces and let $L$ denote the class of all bounded linear operators between Banach spaces.

An operator ideal $I$ is a "mapping" $I : B \times B \rightarrow 2^L$ satisfying the following conditions:

1. For each pair of Banach spaces $E$ and $F$, $I(E, F)$ (or $I(E)$ if $E = F$) is a subspace of the space $L(E, F)$ (or $L(E)$ if $E = F$) of bounded linear operators from $E$ to $F$ containing all finite-rank operators.

2. If in a scheme of bounded linear operators $E_0 \xrightarrow{S_1} E \xrightarrow{T} F \xrightarrow{S_2} F_0$ we have $T \in I(E, F)$, then $S_2 \circ T \circ S_1 \in I(E_0, F_0)$.
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- If in a scheme of bounded linear operators $E_0 \xrightarrow{S_1} E \xrightarrow{T} F \xrightarrow{S_2} F_0$ we have $T \in I(E, F)$, then $S_2 \circ T \circ S_1 \in I(E_0, F_0)$.
Ideal norms

An ideal norm defined on an ideal $I$ is a rule $\| \cdot \|_I$ that assigns to every operator $T \in I$ a non-negative number $\| T \|_I$ satisfying the following conditions:

$$\| x^* \otimes y \|_I = \| x^* \|_{E^*} \| y \|_F \text{ for } x^* \in E^*, y \in F$$

$$\| S + T \|_I \leq \| S \|_I + \| T \|_I \text{ for } S, T \in I (E, F).$$

$$\| S^2 \circ T \circ S^1 \|_I \leq \| S^2 \| \| T \|_I \| S^1 \| \text{ for } S^2 \in L(F, F_0), T \in I (E, F) \text{ and } S^1 \in L(E_0, E).$$

An ideal norm is a norm.

The usual operator norm of $L(E, F)$ is an ideal norm.

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\| S_2 \circ T \circ S_1 \|_I \leq \| S_2 \| \| T \|_I \| S_1 \| \quad \text{for} \quad S_2 \in L(F, F_0), \quad T \in I(E, F) \quad \text{and} \quad S_1 \in L(E_0, E)
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\[ \| S + T \|_I \leq \| S \|_I + \| T \|_I \text{ for } S, T \in I. \]

\[ \| S_2 \circ T \circ S_1 \|_I \leq \| S_2 \| \| T \|_I \| S_1 \| \text{ for } S_2 \in L(F,F_0), T \in I(E,F) \text{ and } S_1 \in L(E_0,E). \]
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Genericity and small sets in analysis

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2. $\| S + T \|_I \leq \| S \|_I + \| T \|_I$ for $S, T \in I(E, F)$.
3. $\| S_2 \circ T \circ S_1 \|_I \leq \| S_2 \| \| T \|_I \| S_1 \|$ for $S_2 \in L(F, F_0)$, $T \in I(E, F)$ and $S_1 \in L(E_0, E)$.
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- $\| S + T \|_I \leq \| S \|_I + \| T \|_I$ for $S, T \in I(E, F)$.
- $\| S_2 \circ T \circ S_1 \|_I \leq \| S_2 \| \| T \|_I \| S_1 \|$ for $S_2 \in L(F, F_0)$, $T \in I(E, F)$ and $S_1 \in L(E_0, E)$.

- An ideal norm is a norm.
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- $\| T \| \leq \| T \|_I$ for $T \in I$. 
The Definition

A Banach space $E$ is said to be $\sigma$-reproducible if there exists a sequence $(E_n)_{n \in \mathbb{N}}$ of complemented subspaces, where $P_n: E \to E_n$ is a bounded projection, satisfying the following conditions:

1. Each $E_n$ is isomorphic to $E$.
2. $P_i \circ P_j = 0$ if $i \neq j$.
3. The projections $\tilde{P}_k = \sum_{n=1}^{k} P_n: E \to \bigoplus_{n=1}^{k} E_n$ are uniformly bounded for all $k \in \mathbb{N}$.

This is an isomorphic property. If $E$ and $F$ are $\sigma$-reproducible Banach spaces, then $E \oplus F$ and $E^*$ are also $\sigma$-reproducible.
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The projections $\tilde{P}_k = \sum_{n=1}^{k} P_n : E \to \bigoplus_{n=1}^{\infty} E_n$ are uniformly bounded for all $k \in \mathbb{N}$. This is an isomorphic property.

If $E$ and $F$ are $\sigma$-reproducible Banach spaces, then $E \oplus F$ and $E^*$ are also $\sigma$-reproducible.
A Banach space $E$ is said to be $\sigma$-reproducible if there exists a sequence $(E_n)_{n \in \mathbb{N}}$ of complemented subspaces, where $P_n : E \rightarrow E_n$ is a bounded projection, satisfying the following conditions:

- Each $E_n$ is isomorphic to $E$.
- $P_i \circ P_j = 0$ if $i \neq j$.
- The projections $\widehat{P}_k = \sum_{n=1}^{k} P_n : E \rightarrow \bigoplus_{n=1}^{k} E_n$ are uniformly bounded for all $k \in \mathbb{N}$.
A Banach space $E$ is said to be $\sigma$-reproducible if there exists a sequence $(E_n)_{n \in \mathbb{N}}$ of complemented subspaces, where $P_n : E \to E_n$ is a bounded projection, satisfying the following conditions:

- Each $E_n$ is isomorphic to $E$.
- $P_i \circ P_j = 0$ if $i \neq j$.
- The projections $\widetilde{P}_k = \sum_{n=1}^{k} P_n : E \to \bigoplus_{n=1}^{k} E_n$ are uniformly bounded for all $k \in \mathbb{N}$.

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A Banach space $E$ is said to be \textit{\(\sigma\)-reproducible} if there exists a sequence $(E_n)_{n \in \mathbb{N}}$ of complemented subspaces, where $P_n : E \rightarrow E_n$ is a bounded projection, satisfying the following conditions:

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This is an isomorphic property.

If $E$ and $F$ are \(\sigma\)-reproducible Banach spaces, then $E \oplus F$ and $E^*$ are also \(\sigma\)-reproducible.
Rearrangement invariant spaces

Definition

Given a measure space \((\Omega, \lambda)\), where \(\Omega = [0, 1)\) or \(\Omega = \mathbb{N}\) and \(\lambda\) is the Lebesgue measure, or \(\Omega = \mathbb{N}\) and \(\lambda\) is the counting measure, the distribution function \(\lambda x\) associated to a scalar measurable function \(x\) on \(\Omega\) is defined by

\[
\lambda x(s) = \lambda \{ t \in \Omega : |x(t)| > s \}.
\]

And the decreasing rearrangement function \(x^*\) of \(x\) is defined by

\[
x^*(t) = \inf \{ s \in [0, \infty) : \lambda x(s) \leq t \}.
\]

Definition

A Banach space \((E, \| \cdot \|_E)\) of measurable functions defined on \(\Omega\) is said to be a rearrangement invariant space if the following conditions are satisfied:

1. If \(y \in E\) and \(|x| \leq |y|\) \(\lambda\)-a.e. on \(\Omega\), then \(x \in E\) and \(\|x\|_E \leq \|y\|_E\).
2. If \(y \in E\) and \(\lambda x = \lambda y\), then \(x \in E\) and \(\|x\|_E = \|y\|_E\).
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Given a measure space \((\Omega, \lambda)\), where \(\Omega = [0, 1], [0, \infty)\) and \(\lambda\) is the Lebesgue measure, or \(\Omega = \mathbb{N}\) and \(\lambda\) is the counting measure, the distribution function \(\lambda_x\) associated to a scalar measurable function \(x\) on \(\Omega\) is defined by

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- If $y \in E$ and $\lambda_x = \lambda_y$, then $x \in E$ and $\|x\|_E = \|y\|_E$. 
Proposition

Every rearrangement invariant space $E$ is $\sigma$-reproducible.
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Proof

First, let $E$ be a rearrangement invariant space on $[0, 1]$. For every $a \in [0, 1)$ and $r \in (0, 1-a]$ we consider the complemented subspace $E_{a, r} = \{ x \in E : \text{supp } x \subseteq [a, a+r] \}$ and the bounded projection $P_{a, r} : E \to E_{a, r}$ given by $P_{a, r}(x) = x \chi_{[a, a+r]}$ for $x \in E$. 
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and the bounded projection $P_{a,r} : E \rightarrow E_{a,r}$ given by $P_{a,r}(x) = x \chi_{[a,a+r]}$ for $x \in E$. 
Proof (cont.)

For a measurable function $x$ we define the linear operators

$$T_{a,r}(x)(t) = x \left( \frac{t - a}{r} \right) \chi_{(a,a+r]}(t)$$

$$S_{a,r}(x)(t) = x((1 - t)a + t(a + r))$$

which are bounded from $L^\infty$ to $L^\infty$, from $L^1$ to $L^1$ and, then, from $E$ to $E$ (Calderón-Mitjagin interpolation theorem).
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$$S_{a,r}(x)(t) = x((1-t)a + t(a+r))$$

which are bounded from $L^\infty$ to $L^\infty$, from $L^1$ to $L^1$ and, then, from $E$ to $E$ (Calderón-Mitjagin interpolation theorem). $(S_{a,r} \circ T_{a,r})(x) = x$ for every $x \in E$, $(T_{a,r} \circ S_{a,r})(x) = x$ for every $x \in E_{a,r}$ and $T_{a,r} : E \to E_{a,r}$ and $S_{a,r} : E_{a,r} \to E$ are isomorphisms. For every $n \in \mathbb{N}$ we consider $a_n = 1 - \frac{1}{2^{n-1}}$ and $r_n = \frac{1}{2^n}$. Let $E_n = E_{a_n,r_n}$ and $P_n = P_{a_n,r_n}$. Since $\|\widetilde{P_k}\| = 1$ for all $k \in \mathbb{N}$, $E$ is $\sigma$-reproducible.
Now, let $E$ be a rearrangement invariant space on $[0, \infty)$. Let $\{A_n : n \in \mathbb{N}\}$ be a disjoint sequence of subsets of $[0, \infty)$ where $A_n = \bigcup_{k=1}^{\infty} (a_n,k, a_n,k+1)$ for an increasing sequence $(a_n,k)_{k \in \mathbb{N}} \subset \mathbb{N}$, and the complemented subspaces $E_n = \{x \in E: \text{supp } x \subseteq A_n\}$.

Given a measurable function $x$, we define $T_n(x)(t) = \sum_{k=1}^{\infty} x(t+k-1-a_n,k) \chi((a_n,k,a_n,k+1](t))$ and $S_n(x)(t) = \sum_{k=1}^{\infty} x(t+k-1-k) \chi([k-1,k](t))$.

$T_n(x)^* = x^*$, $S_n(x)^* \leq x^*$ and $(S_n \circ T_n)(x) = x$. Then, $T_n : E \to E_n$ is an isometry and $S_n : E_n \to E$ is an isomorphism. Follow the $[0,1]$ case.
Proof (cont.)

Now, let $E$ be a rearrangement invariant space on $[0, \infty)$. Let $\{A_n : n \in \mathbb{N}\}$ a disjoint sequence of subsets of $[0, \infty)$ where $A_n = \bigcup_{k=1}^{\infty} (a_{n,k}, a_{n,k} + 1]$ for an increasing sequence $(a_{n,k})_{k \in \mathbb{N}} \subset \mathbb{N}$, and the complemented subspaces $E_n = \{x \in E : \text{supp } x \subseteq A_n\}$.
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$$T_n(x)(t) = \sum_{k=1}^{\infty} x(t + k - 1 - a_{n,k}) \chi(a_{n,k};a_{n,k}+1](t)$$

$$S_n(x)(t) = \sum_{k=1}^{\infty} x(t + a_{n,k} - k - 1) \chi(k-1,k](t).$$
Proof (cont.)

Now, let $E$ be a rearrangement invariant space on $[0, \infty)$. Let $\{A_n : n \in \mathbb{N}\}$ a disjoint sequence of subsets of $[0, \infty)$ where $A_n = \bigcup_{k=1}^{\infty} (a_{n,k}, a_{n,k} + 1]$ for an increasing sequence $(a_{n,k})_{k \in \mathbb{N}} \subset \mathbb{N}$, and the complemented subspaces $E_n = \{x \in E : \text{supp } x \subseteq A_n\}$. Given a measurable function $x$, we define

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Finally, we consider a symmetric sequence space. Let \( \{A_k : k \in \mathbb{N}\} \) be a disjoint partition of \( \mathbb{N} \) where the subset \( A_k \) is the rank of an injective map \( \varphi_k : \mathbb{N} \to \mathbb{N} \) for every \( k \in \mathbb{N} \). For \( x = (x_n)_{n \in \mathbb{N}} \) we define the linear operators \( T_k(x) = (a_n)_{n \in \mathbb{N}} \) with

\[
a_n = \begin{cases} 
    x_m & \text{if } \varphi_k(m) = n \\
    0 & \text{if } n \notin A_k
\end{cases}
\]

\( S_k(x) = (x_{\varphi_k(n)})_{n \in \mathbb{N}}. \)
Finally, we consider a symmetric sequence space. Let \( \{A_k : k \in \mathbb{N}\} \) be a disjoint partition of \( \mathbb{N} \) where the subset \( A_k \) is the rank of an injective map \( \varphi_k : \mathbb{N} \to \mathbb{N} \) for every \( k \in \mathbb{N} \). For \( x = (x_n)_{n \in \mathbb{N}} \) we define the linear operators \( T_k(x) = (a_n)_{n \in \mathbb{N}} \) with

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\( S_k(x) = (x_{\varphi_k(n)})_{n \in \mathbb{N}} \). If \( E_k = \{x \in E : \text{supp } x \subseteq A_k\} \), then \( T_k : E \to E_k \) is an isometry and \( S_k : E_k \to E \) is an isomorphism. And reasoning again as in the \([0, 1]\) case we obtain the result.
The space $C[0, 1]$
Proposition

The space $C[0, 1]$ is $\sigma$-reproducible.
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Proof

Given $0 < a < b < 1$, $C[a, b]$ is isomorphic to its closed subspace $C_0[a, b] = \{f \in C[a, b] : f(a) = f(b) = 0\}$. A bounded projection $P[a, b] : C[0, 1] \to \hat{C}[a, b]$ with $\|P[a, b]\| \leq 2$ is defined by $P[a, b](f)(x) = (f(x) - f(a) - f(b) - f(a)(x-a))^\chi[a, b](x)$. For an increasing sequence $(a_n)_{n \in \mathbb{N}} \subset (0, 1)$, let $E_n = \hat{C}[a_n, a_{n+1}]$. 
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$$P_{[a,b]}(f)(x) = \left( f(x) - f(a) - \frac{f(b) - f(a)}{b - a} (x - a) \right) \chi_{[a,b]}(x).$$
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$$P_{[a,b]}(f)(x) = \left( f(x) - f(a) - \frac{f(b) - f(a)}{b-a} (x-a) \right) \chi_{[a,b]}(x).$$

For an increasing sequence $(a_n)_{n \in \mathbb{N}} \subset (0,1)$, let $E_n = \hat{C}[a_n, a_{n+1}]$. 
Theorem

Let $I_1$ and $I_2$ be operator ideals such that $I_1(E, F) \setminus I_2(E, F)$ is non-empty for a couple of Banach spaces $E$ and $F$.

If $E$ or $F$ is $\sigma$-reproducible and $I_1(E, F)$ is complete for an ideal norm, then $I_1(E, F) \setminus I_2(E, F)$ is spaceable.

Proof

Let $T \in I_1(E, F) \setminus I_2(E, F)$.

If $E$ is a $\sigma$-reproducible Banach space with isomorphisms $\phi_n : E_n \rightarrow E$ and bounded projections $P_n : E \rightarrow E_n$, for every $n \in \mathbb{N}$ we consider the operator $T_n = T \circ \phi_n \circ P_n$ which belongs to $I_1(E, F) \setminus I_2(E, F)$. If $T_n \in I_2(E, F)$, then $T_n|_{E_n} = T \circ \phi_n \in I_2(E_n, F)$.

The sequence $(T_n)_{n \in \mathbb{N}}$ is formed by linearly independent operators. To show this, if $\sum_{k=1}^n a_k T_k = 0$, restricting to $E_j$ we obtain $a_j = 0$ with $1 \leq j \leq k$. Thus, $I_1(E, F) \setminus I_2(E, F)$ is lineable.
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Proof

Let $T \in I_1(E, F) \setminus I_2(E, F)$. If $E$ is a $\sigma$-reproducible Banach space with isomorphisms $\phi_n : E_n \to E$ and bounded projections $P_n : E \to E_n$, for every $n \in \mathbb{N}$ we consider the operator $T_n = T \circ \phi_n \circ P_n$ which belongs to $I_1(E, F) \setminus I_2(E, F)$. If $T_n \in I_2(E, F)$, then $T_n |_{E_n} = T \circ \phi_n \in I_2(E_n, F)$. The sequence $(T_n)_{n \in \mathbb{N}}$ is formed by linearly independent operators. To show this, if $\sum k_n a_n T_n = 0$, restricting to $E_j$ we obtain $a_j = 0$ with $1 \leq j \leq k$. Thus, $I_1(E, F) \setminus I_2(E, F)$ is lineable.
Theorem

Let $I_1$ and $I_2$ be operator ideals such that $I_1(E, F) \setminus I_2(E, F)$ is non-empty for a couple of Banach spaces $E$ and $F$. If $E$ or $F$ is $\sigma$-reproducible and $I_1(E, F)$ is complete for an ideal norm, then $I_1(E, F) \setminus I_2(E, F)$ is spaceable.
The Theorem

Let $I_1$ and $I_2$ be operator ideals such that $I_1(E, F) \setminus I_2(E, F)$ is non-empty for a couple of Banach spaces $E$ and $F$. If $E$ or $F$ is $\sigma$-reproducible and $I_1(E, F)$ is complete for an ideal norm, then $I_1(E, F) \setminus I_2(E, F)$ is spaceable.

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Let $T \in I_1(E, F) \setminus I_2(E, F)$.
Theorem

Let $I_1$ and $I_2$ be operator ideals such that $I_1(E, F) \setminus I_2(E, F)$ is non-empty for a couple of Banach spaces $E$ and $F$. If $E$ or $F$ is $\sigma$-reproducible and $I_1(E, F)$ is complete for an ideal norm, then $I_1(E, F) \setminus I_2(E, F)$ is spaceable.

Proof

Let $T \in I_1(E, F) \setminus I_2(E, F)$. If $E$ is a $\sigma$-reproducible Banach space with isomorphisms $\phi_n : E_n \to E$ and bounded projections $P_n : E \to E_n$, for every $n \in \mathbb{N}$ we consider the operator $T_n = T \circ \phi_n \circ P_n$ which belongs to $I_1(E, F) \setminus I_2(E, F)$. If $T_n \in I_2(E, F)$, then $T_n|_{E_n} = T \circ \phi_n \in I_2(E_n, F)$. 
Theorem

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Proof

Let $T \in I_1(E, F) \setminus I_2(E, F)$. If $E$ is a $\sigma$-reproducible Banach space with isomorphisms $\phi_n : E_n \rightarrow E$ and bounded projections $P_n : E \rightarrow E_n$, for every $n \in \mathbb{N}$ we consider the operator $T_n = T \circ \phi_n \circ P_n$ which belongs to $I_1(E, F) \setminus I_2(E, F)$. If $T_n \in I_2(E, F)$, then $T_n|_{E_n} = T \circ \phi_n \in I_2(E_n, F)$. The sequence $(T_n)_{n \in \mathbb{N}}$ is formed by linearly independent operators. To show this, if $\sum_{n=1}^{k} a_n T_n = 0$, restricting to $E_j$ we obtain $a_j = 0$ with $1 \leq j \leq k$. Thus, $I_1(E, F) \setminus I_2(E, F)$ is lineable.
Furthermore, \((T_n)_{n \in \mathbb{N}}\) is a basic sequence in \(l_1(E, F)\). Indeed, for any integers \(k < m\) and any choice of scalars \((\lambda_n)_{n \in \mathbb{N}}\) we have

\[
\left\| \sum_{n=1}^{k} \lambda_n T_n \right\|_{l_1} = \left\| \sum_{n=1}^{m} \lambda_n T_n \circ \widetilde{P}_k \right\|_{l_1} \leq \left\| \sum_{n=1}^{m} \lambda_n T_n \right\|_{l_1} \left\| \widetilde{P}_k \right\|.
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\left\| \sum_{n=1}^{k} \lambda_n T_n \right\|_{l_1} = \left\| \sum_{n=1}^{m} \lambda_n T_n \circ \tilde{P}_k \right\|_{l_1} \leq \left\| \sum_{n=1}^{m} \lambda_n T_n \right\|_{l_1} \left\| \tilde{P}_k \right\|.
\]

Let \(S \in [T_n : n \in \mathbb{N}] \subset l_1(E, F)\) with \(S = \sum_{n=1}^{\infty} \lambda_n T_n \neq 0\). Then there exists \(n_0 \in \mathbb{N}\) such that \(\lambda_{n_0} \neq 0\). We have that \(S|_{E_{n_0}} = \lambda_{n_0} T \circ \phi_{n_0} \notin l_2(E_{n_0}, F)\). Thus, \(S \notin l_2(E, F)\) and \([T_n : n \in \mathbb{N}] \subset l_1(E, F) \setminus l_2(E, F)\).
If $F$ is $\sigma$-reproducible with isomorphisms $(\phi_n)_{n \in \mathbb{N}}$, for each $n \in \mathbb{N}$ we consider the operator $T_n = \phi_n^{-1} \circ T$ which belongs to $l_1(E, F) \setminus l_2(E, F)$. The sequence $(T_n)_{n \in \mathbb{N}}$ is formed by linearly independent operators. Thus, we obtain that $l_1(E, F) \setminus l_2(E, F)$ is lineable. Indeed, for any integers $k < m$ and any choice of scalars $(\lambda_n)_{n \in \mathbb{N}}$ we have
\[
\|k \sum_{n=1}^{m} \lambda_n T_n\|_{l_1} \leq \|\tilde{P}_k \circ \sum_{n=1}^{m} \lambda_n T_n\|_{l_1}.
\]
Let $S \in [T_n: n \in \mathbb{N}] \subset l_1(E, F)$, $S = \sum_{n=1}^{\infty} \lambda_n T_n \neq 0$. There exists $n_0 \in \mathbb{N}$ such that $\lambda_{n_0} \neq 0$. If $S \in l_2(E, F)$, then $P_{n_0} \circ S \in l_2(E, F)$, but this is not true because $P_{n_0} \circ S = \lambda_{n_0} T_{n_0}$. Then $[T_n: n \in \mathbb{N}] \subset l_1(E, F) \setminus l_2(E, F)$.
If $F$ is $\sigma$-reproducible with isomorphisms $(\phi_n)_{n \in \mathbb{N}}$, for each $n \in \mathbb{N}$ we consider the operator $T_n = \phi_n^{-1} \circ T$ which belongs to $I_1(E, F) \setminus I_2(E, F)$. The sequence $(T_n)_{n \in \mathbb{N}}$ is formed by linearly independent operators. Thus, we obtain that $I_1(E, F) \setminus I_2(E, F)$ is lineable.
If $F$ is $\sigma$-reproducible with isomorphisms $(\phi_n)_{n \in \mathbb{N}}$, for each $n \in \mathbb{N}$ we consider the operator $T_n = \phi_n^{-1} \circ T$ which belongs to $I_1(E, F) \setminus I_2(E, F)$. The sequence $(T_n)_{n \in \mathbb{N}}$ is formed by linearly independent operators. Thus, we obtain that $I_1(E, F) \setminus I_2(E, F)$ is lineable. And $(T_n)_{n \in \mathbb{N}}$ is a basic sequence. Indeed, for any integers $k < m$ and any choice of scalars $(\lambda_n)_{n \in \mathbb{N}}$ we have

$$\left\| \sum_{n=1}^{k} \lambda_n T_n \right\|_{I_1} = \left\| \widetilde{P}_k \circ \sum_{n=1}^{m} \lambda_n T_n \right\|_{I_1} \leq \left\| \widetilde{P}_k \right\| \left\| \sum_{n=1}^{m} \lambda_n T_n \right\|_{I_1}.$$
Proof (cont.)

If $F$ is $\sigma$-reproducible with isomorphisms $(\phi_n)_{n \in \mathbb{N}}$, for each $n \in \mathbb{N}$ we consider the operator $T_n = \phi_n^{-1} \circ T$ which belongs to $I_1(E, F) \setminus I_2(E, F)$. The sequence $(T_n)_{n \in \mathbb{N}}$ is formed by linearly independent operators. Thus, we obtain that $I_1(E, F) \setminus I_2(E, F)$ is lineable. And $(T_n)_{n \in \mathbb{N}}$ is a basic sequence. Indeed, for any integers $k < m$ and any choice of scalars $(\lambda_n)_{n \in \mathbb{N}}$ we have

$$\left\| \sum_{n=1}^{k} \lambda_n T_n \right\|_{l_1} = \left\| \widetilde{\mathcal{P}}_k \circ \sum_{n=1}^{m} \lambda_n T_n \right\|_{l_1} \leq \left\| \widetilde{\mathcal{P}}_k \right\| \left\| \sum_{n=1}^{m} \lambda_n T_n \right\|_{l_1} .$$

Let $S \in \overline{\{T_n : n \in \mathbb{N}\}} \subset I_1(E, F)$, $S = \sum_{n=1}^{\infty} \lambda_n T_n \neq 0$. There exists $n_0 \in \mathbb{N}$ such that $\lambda_{n_0} \neq 0$. If $S \in I_2(E, F)$, then $P_{n_0} \circ S \notin I_2(E, F)$, but this is not true because $P_{n_0} \circ S = \lambda_{n_0} T_{n_0}$. Then $\overline{\{T_n : n \in \mathbb{N}\}} \subset I_1(E, F) \setminus I_2(E, F)$.
Theorem

If $E$ or $F$ is a $\sigma$-reproducible Banach space, $I$ is an operator ideal such that $I(E,F)$ is complete for an ideal norm, and $(I_n)_{n \in \mathbb{N}}$ is a sequence of operator ideals such that $I(E,F) \setminus \bigcup_{n=1}^{\infty} I_n(E,F)$ is non-empty for every $n \in \mathbb{N}$, then the set $I(E,F) \setminus \bigcup_{n=1}^{\infty} I_n(E,F)$ is spaceable.

Proof

Let $S_n \in I(E,F) \setminus I_n(E,F)$ for every $n \in \mathbb{N}$. If $E$ is a $\sigma$-reproducible Banach space with isomorphisms $(\varphi_n)_{n \in \mathbb{N}}$ and bounded projections $(P_n)_{n \in \mathbb{N}}$, let us consider the operators $S_n \circ \varphi_n \circ P_n \in I(E,F) \setminus I_n(E,F)$ for every $n \in \mathbb{N}$.

Then $T = \sum_{n=1}^{\infty} S_n \circ \varphi_n \circ P_n$ is bounded.
Theorem
If $E$ or $F$ is a $\sigma$-reproducible Banach space, $I$ is an operator ideal such that $I(E, F)$ is complete for an ideal norm, and $(I_n)_{n \in \mathbb{N}}$ is a sequence of operator ideals such that $I(E, F) \setminus I_n(E, F)$ is non-empty for every $n \in \mathbb{N}$, then the set $I(E, F) \setminus \bigcup_{n=1}^{\infty} I_n(E, F)$ is spaceable.
Consequences

Theorem

If \( E \) or \( F \) is a \( \sigma \)-reproducible Banach space, \( I \) is an operator ideal such that \( I(E, F) \) is complete for an ideal norm, and \( (I_n)_{n\in\mathbb{N}} \) is a sequence of operator ideals such that \( I(E, F) \setminus I_n(E, F) \) is non-empty for every \( n \in \mathbb{N} \), then the set \( I(E, F) \setminus \bigcup_{n=1}^{\infty} I_n(E, F) \) is spaceable.

Proof

Let \( S_n \in I(E, F) \setminus I_n(E, F) \) for every \( n \in \mathbb{N} \).
**Theorem**

If $E$ or $F$ is a $\sigma$-reproducible Banach space, $I$ is an operator ideal such that $I(E, F)$ is complete for an ideal norm, and $(I_n)_{n \in \mathbb{N}}$ is a sequence of operator ideals such that $I(E, F) \setminus I_n(E, F)$ is non-empty for every $n \in \mathbb{N}$, then the set $I(E, F) \setminus \bigcup_{n=1}^{\infty} I_n(E, F)$ is spaceable.

**Proof**

Let $S_n \in I(E, F) \setminus I_n(E, F)$ for every $n \in \mathbb{N}$. If $E$ is a $\sigma$-reproducible Banach space with isomorphisms $(\phi_n)_{n \in \mathbb{N}}$ and bounded projections $(P_n)_{n \in \mathbb{N}}$, let us consider the operators $S_n \circ \phi_n \circ P_n \in I(E, F) \setminus I_n(E, F)$ for every $n \in \mathbb{N}$. 
Consequences

**Theorem**

If $E$ or $F$ is a $\sigma$-reproducible Banach space, $I$ is an operator ideal such that $I(E,F)$ is complete for an ideal norm, and $(I_n)_{n\in\mathbb{N}}$ is a sequence of operator ideals such that $I(E,F) \setminus I_n(E,F)$ is non-empty for every $n \in \mathbb{N}$, then the set $I(E,F) \setminus \bigcup_{n=1}^{\infty} I_n(E,F)$ is spaceable.

**Proof**

Let $S_n \in I(E,F) \setminus I_n(E,F)$ for every $n \in \mathbb{N}$. If $E$ is a $\sigma$-reproducible Banach space with isomorphisms $(\phi_n)_{n\in\mathbb{N}}$ and bounded projections $(P_n)_{n\in\mathbb{N}}$, let us consider the operators $S_n \circ \phi_n \circ P_n \in I(E,F) \setminus I_n(E,F)$ for every $n \in \mathbb{N}$. Then

$$T = \sum_{n=1}^{\infty} \frac{S_n \circ \phi_n \circ P_n}{2^n \|S_n \circ \phi_n \circ P_n\|_I} \in I(E,F) \setminus I_n(E,F).$$
Consequences

Proof (cont.)

Now, reasoning as in the proof of the main theorem we can construct a sequence \((T_k)_{k \in \mathbb{N}}\) such that \([T_k : k \in \mathbb{N}] \subset I(E, F) \setminus I_n(E, F)\) for every \(n \in \mathbb{N}\).
Now, reasoning as in the proof of the main theorem we can construct a sequence \((T_k)_{k \in \mathbb{N}}\) such that \(\overline{\{T_k : k \in \mathbb{N}\}} \subset I(E, F) \setminus I_n(E, F)\) for every \(n \in \mathbb{N}\). If \(F\) is a \(\sigma\)-reproducible Banach space with isomorphisms \((\phi_n)_{n \in \mathbb{N}}\), let us consider the operators \(\phi_n^{-1} \circ S_n \in I(E, F) \setminus I_n(E, F)\) for every \(n \in \mathbb{N}\).
Now, reasoning as in the proof of the main theorem we can construct a sequence \((T_k)_{k \in \mathbb{N}}\) such that \([T_k : k \in \mathbb{N}] \subset I(E, F) \setminus I_n(E, F)\) for every \(n \in \mathbb{N}\). If \(F\) is a \(\sigma\)-reproducible Banach space with isomorphisms \((\phi_n)_{n \in \mathbb{N}}\), let us consider the operators \(\phi_n^{-1} \circ S_n \in I(E, F) \setminus I_n(E, F)\) for every \(n \in \mathbb{N}\). Then

\[
T = \sum_{n=1}^{\infty} \frac{\phi_n^{-1} \circ S_n}{2^n \|\phi_n^{-1} \circ S_n\|_I}
\]

belongs to \(I(E, F) \setminus \bigcup_{n=1}^{\infty} I_n(E, F)\).
Corollary

Let $E$ and $F$ be Banach spaces, and \{\(I_p : p \in [a, b]\)\} be a family of operator ideals such that $I_p(E, F) \not\subset I_q(E, F)$ if $p < q$ with continuous inclusion.
Corollary

Let $E$ and $F$ be Banach spaces, and $\{I_p : p \in [a, b]\}$ be a family of operator ideals such that $I_p(E, F) \not\subseteq I_q(E, F)$ if $p < q$ with continuous inclusion. If $E$ or $F$ is a $\sigma$-reproducible Banach space and $I_b(E, F)$ is complete for an ideal norm, then the set $I_b(E, F) \setminus \bigcup_{p<b} I_p(E, F)$ is spaceable.
Applications: strictly singular operators

Definition

A linear operator $T$ between two Banach spaces $E$ and $F$ is called strictly singular (SS) if it fails to be an isomorphism on any infinite-dimensional subspace of $E$.

$K \subset SS$.

If $1 \leq p, q < \infty$ with $p \neq q$ or $p = q \neq 2$, then the set $SS(\mathcal{L}_p, \mathcal{L}_q) \setminus K(\mathcal{L}_p, \mathcal{L}_q)$ is spaceable.

If $1 \leq p < q < \infty$, then the set $SS(\ell_p, \ell_q) \setminus K(\ell_p, \ell_q)$ is spaceable.

The set $SS(E, c_0) \setminus K(E, c_0)$ is spaceable for every symmetric sequence space $E \neq c_0$. 
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Applications: strictly singular operators

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\[ 1 \leq p < q < \infty \text{ with } p \neq q \text{ or } p = q \neq 2, \text{ then the set } SS(\ell^p, \ell^q) \setminus K(\ell^p, \ell^q) \text{ is spaceable.} \]

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Applications: strictly singular operators

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Applications: strictly singular operators

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A linear operator $T$ between two Banach spaces $E$ and $F$ is called **strictly singular** (SS) if it fails to be an isomorphism on any infinite-dimensional subspace of $E$.

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Definition

A Banach space $E$ has the **Kato property** when $SS(E) = K(E)$. 
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If $E$ is a $\sigma$-reproducible Banach space, then the set $SS(E) \setminus K(E)$ is spaceable if and only if $E$ does not have the Kato property.
Applications: strictly singular operators

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A Banach space $E$ has the **Kato property** when $SS(E) = K(E)$.

**Corollary**
If $E$ is a $\sigma$-reproducible Banach space, then the set $SS(E) \setminus K(E)$ is spaceable if and only if $E$ does not have the Kato property.

- For Lorentz function spaces $L^{p,q}[0,1]$ with $1 < p < \infty, 1 \leq q \leq \infty$, the set $SS(L^{p,q}) \setminus K(L^{p,q})$ is spaceable if and only if $q \neq 2$. 

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- For 2-convex (or 2-concave) Orlicz spaces $L^\varphi[0,1]$ it holds that $SS(L^\varphi) \setminus K(L^\varphi)$ is spaceable if and only if the associated set $E^\varphi_\infty \neq \{t^2\}$. 
Applications: finitely strictly singular operators

Definition

A linear operator \( T \) between two Banach spaces \( E \) and \( F \) is called finitely strictly singular (FSS) if there do not exist a number \( c > 0 \) and a sequence of subspaces \( E_n \) of \( E \) with \( \dim(E_n) = n \) such that \( \|T(x)\| \geq c \|x\| \) for all \( x \in \bigcup_{n=1}^\infty E_n \).

\( K \subset FSS \subset SS \).

If \( 1 < p < q < \infty \), the sets \( SS(\ell^p, \ell^q) \setminus FSS(\ell^p, \ell^q) \) and \( FSS(\ell^p, \ell^q) \setminus K(\ell^p, \ell^q) \) are spaceable.

For the disc algebra \( A(D) \), the set \( FSS(A(D)) \setminus K(A(D)) \) is spaceable but the set \( SS(A(D)) \setminus FSS(A(D)) \) is not spaceable.
Definition

A linear operator $T$ between two Banach spaces $E$ and $F$ is called **finitely strictly singular** (FSS) if there do not exist a number $c > 0$ and a sequence of subspaces $E_n$ of $E$ with $\dim(E_n) = n$ such that $\|T(x)\| \geq c\|x\|$ for all $x \in \bigcup_{n=1}^{\infty} E_n$. 
A linear operator $T$ between two Banach spaces $E$ and $F$ is called \textbf{finitely strictly singular} (FSS) if there do not exist a number $c > 0$ and a sequence of subspaces $E_n$ of $E$ with $\dim(E_n) = n$ such that $\|T(x)\| \geq c\|x\|$ for all $x \in \bigcup_{n=1}^{\infty} E_n$.

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Applications: finitely strictly singular operators

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- For the disc algebra $A(D)$, the set $FSS(A(D)) \setminus K(A(D))$ is spaceable but the set $SS(A(D)) \setminus FSS(A(D))$ is not spaceable.
Applications: disjointly strictly singular operators

Definition

A linear operator $T$ from a Banach lattice $E$ to a Banach space $F$ is said to be disjointly strictly singular (DSS) if there is no disjoint sequence of non-null vectors in $E$ such that the restriction of $T$ to the closed subspace spanned by them is an isomorphism.

This closed subspace of operators is stable with respect to the composition on the left with bounded linear operators. $SS \subset DSS$.

For $L^p[0,1]$-spaces, $DSS(L^p) \setminus SS(L^p)$ is spaceable if $1 < p \neq 2$ (the projection of $L^p$ over the closed subspace spanned by Rademacher functions is DSS but not SS).

$DSS(L^q, L^p) \setminus SS(L^q, L^p)$ is spaceable if $1 \leq p < q < \infty$ (the inclusion $L^q \hookrightarrow L^p$ is DSS but not SS).
A linear operator $T$ from a Banach lattice $E$ to a Banach space $F$ is said to be **disjointly strictly singular** (DSS) if there is no disjoint sequence of non-null vectors in $E$ such that the restriction of $T$ to the closed subspace spanned by them is an isomorphism.
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Applications: $(q, p)$-summing operators

Definition

If $1 \leq p \leq q < \infty$, an operator $T \in L(E, F)$ is called $(q, p)$-summing (or $p$-summing if $p = q$) if there is a constant $C$ so that, for every choice of an integer $n$ and vectors $(x_i)_{i=1}^n = 1$ in $E$, we have

$$\left( \sum_{i=1}^n \| T(x_i) \|^q \right)^{1/q} \leq C \sup \| x^* \| \leq 1 \left( \sum_{i=1}^n | x^*(x_i) |^p \right)^{1/p}.$$  

The smallest possible constant $C$ defines a complete ideal norm on this operator ideal, denoted by $\Pi_{q, p}$. For $1 \leq p \leq r \leq q$, it holds $\Pi_{q, q} \subset \Pi_{q, r} \subset \Pi_{q, 1}$. If $H$ is a Hilbert space, then the set $\Pi_{q, 1}(H) \setminus \bigcup_{1 < p \leq q} \Pi_{q, p}(H)$ is spaceable if and only if $1 < q < 2$. 

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Applications: \((q, p)\)-summing operators

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