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Transformations of quadrilateral lattices

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Motivated by the classical studies on transformations of conjugate nets, we develop the general geometric theory of transformations of their discrete analogs: the multidimensional quadrilateral lattices, i.e., lattices $\mathbf{x}:\mathbb{Z}^N\rightarrow\mathbb{R}^M$, $N\leqslant M$, whose elementary quadrilaterals are planar. Our investigation is based on the discrete analog of the theory of the rectilinear congruences, which we also present in detail. We study, in particular, the discrete analogs of the Laplace, Combescure, Lévy, radial, and fundamental transformations and their interrelations. The composition of these transformations and their permutability is also investigated from a geometric point of view. The deep connections between ''transformations'' and ''discretizations'' is also investigated for quadrilateral lattices. We finally interpret these results within the $\bar{\partial}$ formalism.
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I. INTRODUCTION

An interesting topic developed by distinguished geometers of the turn of the last century is the theory of submanifolds equipped with conjugate systems of coordinates (conjugate nets),[1–3] i.e., mappings $\mathbf{x}:\mathbb{R}^N\rightarrow\mathbb{R}^M$, $N\leqslant M$, satisfying the Laplace equations

$$\frac{\partial^2\mathbf{x}}{\partial u_i\,\partial u_j}=\frac{1}{H_i}\frac{\partial H_i}{\partial u_j}\frac{\partial x}{\partial u_i}+\frac{1}{H_j}\frac{\partial H_j}{\partial u_i}\frac{\partial x}{\partial u_j},\quad i,j=1,...,N,\quad i\neq j, \tag{1.1}$$

whose compatibility for $N>2$ gives the Darboux equations

$$\frac{\partial^2 H_k}{\partial u_i\,\partial u_j}=\frac{1}{H_i}\frac{\partial H_i}{\partial u_j}\frac{\partial H_k}{\partial u_i}+\frac{1}{H_j}\frac{\partial H_j}{\partial u_i}\frac{\partial H_k}{\partial u_j},\quad i\neq j\neq k\neq j. \tag{1.2}$$

Imposing suitable geometric constraints on the conjugate nets, one obtains significant reductions like the orthogonal systems of coordinates.[4,2] It was recently shown by Zakharov and Manakov[5] that the Darboux equations can be solved using the $\bar{\partial}$ method and that a suitable constraint on the

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associated $\bar{\partial}$ datum allows one to solve its orthogonality reduction.[6,7] These examples show once more the deep connections between geometry and integrability, which was observed in the past in other cases.[8,9]

During the last years some of these results have been generalized to a discrete level.[10–13] Based on a result by Sauer, which introduced the proper discrete analog of a conjugate net on a surface,[14] Doliwa and Santini introduced the notion of ''Multidimensional Quadrilateral Lattice'' (MQL), i.e., a lattice $\mathbf{x}:\mathbb{Z}^N \to \mathbb{R}^M$, $N \leqslant M$, with all its elementary quadrilaterals planar, which is the discrete analog of a multidimensional conjugate net.[15] Furthermore, they showed that the planarity constraint (which is a linear constraint) provides a way to construct the lattice uniquely, once a suitable set of initial data is given. Therefore this lattice, generated by a set of linear constraints, is ''geometrically integrable.'' They also found that the discrete nonlinear equations characterizing the MQL had been already introduced, using the $\bar{\partial}$ formalism, by Bogdanov and Konopelchenko[16] as a natural integrable discrete analog of the Darboux equations.

Also, the orthogonality constraint has been successfully discretized. This discretization consists in imposing that the elementary quadrilaterals of the MQL are inscribed in circles. This notion was first proposed in Refs. 17, 18 for $N=2$, $M=3$, as a discrete analog of surfaces parametrized by curvature lines (see also Ref. 12); later, by Bobenko for $N=M=3$[19] and, finally, for arbitrary $N \leqslant M$ by Cieśliński, Doliwa, and Santini.[20] These lattices are now called ''Multidimensional Circular Lattices'' (MCL) or discrete orthogonal lattices. In Ref. 20 it was also shown that the geometric integrability scheme for MQLs is consistent with the circularity reduction, thus proving the integrability of the MCL in pure geometric terms. Soon after that, Doliwa, Manakov, and Santini have proven in Ref. 21 the (analytic) integrability of the MCL generalizing to a discrete level the method of solution, proposed in Ref. 7, for the Lamé system and for other reductions of the Darboux equations. More recently, Konopelchenko and Schief have obtained a convenient set of equations characterizing the circular lattices in $\mathbb{E}^3$.[22]

An extensive literature exists on the classes of transformations of the conjugate nets, which provide an effective way to construct new (and more complicated) conjugate nets from given (simple) ones. The basic classes of transformations of conjugate nets, listed for instance in Ref. 3, include the so-called Laplace, Combescure, Lévy, radial, and fundamental transformations. The transformations preserving additional geometric constraints were also extensively investigated; in particular, the reduction of the fundamental transformation compatible with the orthogonality constraint is called the Ribaucour transformation.[23] We finally remark that the classical transformations of conjugate nets provide an interesting geometric interpretation to the basic operations associated with the multicomponent KP hierarchy.[24]

Guided by Sauer's definition of a two-dimensional (2-D) discrete conjugate net[14] and by the studies of Darboux on the Laplace transformations of two-dimensional conjugate nets,[1,3] Doliwa has found in Ref. 25 the discrete analog of the Laplace transform of a 2-D quadrilateral lattice, which provides the geometric interpretation of the Hirota equation[26] (discrete 2-D Toda system). Motivated by the general theory of transformations of conjugate nets, in this paper we make a detailed study of the geometric and analytic properties of the classes of transformations of MQLs. These transformations turn out to be particular cases of a general algebraic formulation recently proposed by us in Ref. 27.

In order to construct the geometric theory of transformations of MQLs, one has first to develop the discrete analog of the theory of rectilinear congruences, which we present in Sec. II. In Sec. II we also define two basic relations between quadrilateral lattices and congruences: *focal lattices of a congruence* and *lattices conjugate to a congruence*. In the subsequent sections (Secs. III–VII), we construct and study (the discrete analog of) the Laplace, Combescure, Lévy, adjoint Lévy, radial, and fundamental transformations of MQLs, emphasizing the geometric significance of all the ingredients of these transformations and explaining the geometric steps involved in the construction of a new MQL from a given one. These transformations are the natural analogs of the corresponding transformations of the conjugate nets, and their definitions can be obtained from the corresponding definitions, replacing the expressions ''focal net'' and ''net conjugate to a congruence'' by ''focal lattice'' and ''lattice conjugate to a congruence,'' respectively. In Sec. VII, in addition, we also give the geometric meaning of the composition of fundamental transformations. The interpretation of the Lévy, adjoint Lévy, and Laplace transformations as geometrically distinguished limits of the fundamental transformation is also used to describe analytically these limits (Sec. VIII). In Sec. IX we show how all these transformations are particular cases of the general vectorial transformation obtained in Ref. 27. A very successful, but empirical, rule used in the literature[28] to build integrable discrete analogs of integrable differential equations consists in finding the finite transformations of the differential systems and in interpreting them as integrable discretizations; the validity of this rule is confirmed as a consequence of our theory. Section X is dedicated to the formulation of the geometric results of the paper within the $\bar{\partial}$ formalism.

We remark that the Combescure and fundamental transformations of quadrilateral lattices have been recently defined independently by Konopelchenko and Schief in Ref. 22 (see Secs. IV and VII of the present paper); in that work they also found the discrete analog of the Ribaucour transformation.

In the rest of this introductory section, we recall the necessary results on MQLs. For details, see Refs. 15 and 27.

Consider a MQL; i.e., a mapping $\mathbf{x}\colon \mathbb{Z}^N \to \mathbb{R}^M$, $N \leqslant M$, with all elementary quadrilaterals planar.[15] The planarity condition can be formulated in terms of the Laplace equations,

$$\Delta_i \Delta_j \mathbf{x} = (T_i A_{ij}) \Delta_i \mathbf{x} + (T_j A_{ji}) \Delta_j \mathbf{x}, \quad i \neq j, \quad i,j = 1,...,N, \tag{1.3}$$

where T_i is the translation operator in the i direction and $\Delta_i = T_i - 1$ is the corresponding difference operator, which are compatible only for the special class of data $A_{ij}\colon \mathbb{Z}^N \to \mathbb{R}$ satisfying the MQL equation,

$$\Delta_k A_{ij} = (T_j A_{jk}) A_{ij} + (T_k A_{kj}) A_{ik} - (T_k A_{ij}) A_{ik}, \quad i \neq j \neq k \neq i. \tag{1.4}$$

It is often convenient to reformulate equations (1.3) as first-order systems.[15] To do so, we introduce the suitably scaled tangent vectors $\mathbf{X}_i$, $i = 1,...,N$,

$$\Delta_i \mathbf{x} = (T_i H_i) \mathbf{X}_i, \tag{1.5}$$

in such a way that the jth variation of $\mathbf{X}_i$ is proportional to $\mathbf{X}_j$ only:

$$\Delta_j \mathbf{X}_i = (T_j Q_{ij}) \mathbf{X}_j, \quad i \neq j. \tag{1.6}$$

The compatibility condition for the system (1.6) gives the following new form of the MQL equations,

$$\Delta_k Q_{ij} = (T_k Q_{ik}) Q_{kj}, \quad i \neq j \neq k \neq i. \tag{1.7}$$

The scaling factors H_i, called the Lamé coefficients, solve the linear equations,

$$\Delta_i H_{ij} = (T_i H_i) Q_{ij}, \quad i \neq j, \tag{1.8}$$

whose compatibility gives equations (1.7) again; moreover,

$$A_{ij} = \frac{\Delta_j H_i}{H_i}, \quad i \neq j.$$

The Laplace equations (1.3) and the MQL equations (1.4) read as

$$\Delta_i \Delta_j \mathbf{x} = T_i((\Delta_j H_i) H_i^{-1}) \Delta_i \mathbf{x} + T_j((\Delta_i H_j) H_j^{-1}) \Delta_j \mathbf{x}, \quad i \neq j, \tag{1.9}$$

$$\Delta_i \Delta_j H_k = T_i((\Delta_j H_i) H_i^{-1}) \Delta_i H_k + T_j((\Delta_i H_j) H_j^{-1}) \Delta_j H_k, \quad i \neq j \neq k \neq i, \tag{1.10}$$

in terms of the Lamé coefficients.

In a recent paper[27] we proved the following basic results.

Theorem 1.1: *Let Q_{ij}, $i,j=1,...,N$, $i\neq j$, be a solution of the MQL equations (1.7) and $\mathbf{Y}_i$ and $\mathbf{Y}_i^*$, $i=1,...,N$, be solutions of the associated linear systems (1.6) and (1.8), taking values in a linear space $\mathbb{W}$ and in its adjoint $\mathbb{W}^*$, respectively. Let $\mathbf{\Omega}[\mathbf{Y},\mathbf{Y}^*]\in L(\mathbb{W})$ be a linear operator in $\mathbf{W}$ defined by the compatible equations*

$$\Delta_i\mathbf{\Omega}[\mathbf{Y},\mathbf{Y}^*]=\mathbf{Y}_i\otimes(T_i\mathbf{Y}_i^*),\quad i=1,...,N. \tag{1.11}$$

If the potential $\mathbf{\Omega}$ is invertible, $\mathbf{\Omega}[\mathbf{Y},\mathbf{Y}^*]\in GL(\mathbb{W})$, then the functions

$$\hat{Q}_{ij}=Q_{ij}-\langle\mathbf{Y}_j^*|\mathbf{\Omega}^{-1}|\mathbf{Y}_i\rangle,\quad i,j=1,...,N,\quad i\neq j, \tag{1.12}$$

are new solutions of the equation (1.7), and

$$\hat{\mathbf{Y}}_i=\mathbf{\Omega}^{-1}\mathbf{Y}_i,\quad i=1,...,N, \tag{1.13}$$

$$\hat{\mathbf{Y}}_i^*=\mathbf{Y}_i^*\mathbf{\Omega}^{-1},\quad i=1,...,N, \tag{1.14}$$

are corresponding new solutions of the equations (1.6), (1.8). In addition,

$$\mathbf{\Omega}[\hat{\mathbf{Y}},\hat{\mathbf{Y}}^*]=C-\mathbf{\Omega}[\mathbf{Y},\mathbf{Y}^*]^{-1}, \tag{1.15}$$

where C is a constant operator.

Proposition 1.1: Consider a constant vector $w\in\mathbb{W}$ and the projection operator P on an M-dimensional subspace $\mathbb{V}$ of $\mathbb{W}$, then the vector function $\mathbf{x}:\mathbb{Z}^N\rightarrow\mathbb{V}\equiv\mathbb{R}^M$, defined by

$$\mathbf{x}=P(\mathbf{\Omega}[\mathbf{Y},\mathbf{Y}^*]\mathbf{w}), \tag{1.16}$$

defines an N-dimensional quadrilateral lattice whose Lamé coefficients and scaled tangent vectors are of the form

$$H_i=\langle\mathbf{Y}_i^*|\mathbf{w}\rangle, \tag{1.17}$$

$$\mathbf{X}_i=P(\mathbf{Y}_i). \tag{1.18}$$

As we shall see in the following sections, the vectorial transformations obtained in Theorem 1.1 contain all the transformations studied in this paper as particular and/or limiting cases.

II. RECTILINEAR CONGRUENCES AND QUADRILATERAL LATTICES

It is well known that rectilinear congruences play a fundamental role in the theory of transformations of multiconjugate systems.[3] In this section we discretize the theory of congruences whose importance in the theory of transformations of MQLs will be evident in the following sections.

Study of families of lines was motivated by the theory of optics, and mathematicians like Monge, Malus, and Hamilton initiated the general theory of rays. However, it was Plücker, who first considered straight lines in $\mathbb{R}^3$ as points of some space; he also found a convenient way to parametrize that space. In the second half of the 19th century, this subject was very popular and was studied, after Plücker, by many distinguished geometers; to mention Klein, Lie, Bianchi, and Darboux only.[29,23,1,3,30]

It turns out (see Chap. XII of Ref. 31 for more details) that, for a generic two-parameter family of lines in $\mathbb{R}^3$ (called *rectilinear congruence*), there exist, roughly speaking, two surfaces (called *focal surfaces of the congruence*) characterized by the property that every line of the family is tangent to both surfaces. This fact does not hold for bigger dimensions of the ambient space and, by definition, *a two-parameter family of straight lines in $\mathbb{R}^M$ is called (rectilinear)*

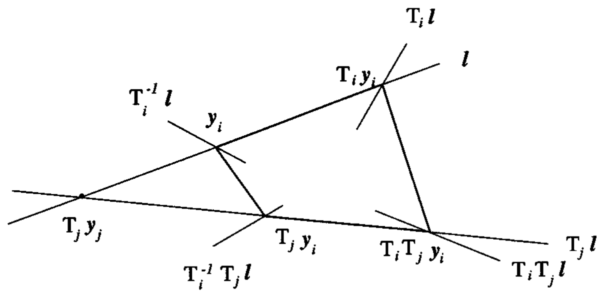

FIG. 1. Planarity of (ij) quadrilaterals.

congruence iff it has focal surfaces. One-parameter families of straight lines tangent to a curve are called *developable surfaces*; one can consider developable surfaces as one-dimensional congruences. A three-parameter family of lines in $\mathbb{R}^3$ is sometimes also called line complex.

Our goal is to construct the theory of *N-dimensional* congruences of straight lines within the *discrete geometry approach*. In doing this, we use the idea of *constructability* of discrete integrable geometries presented in Refs. 15, 20, and 32.

A. Congruences and their focal lattices

Definition 2.1: An N-dimensional rectilinear congruence (or, simply, congruence) is a mapping $\mathfrak{l}\colon \mathbb{Z}^N \to \mathbb{L}(M)$ *from the integer lattice to the space of lines in* $\mathbb{R}^M$ *such that every two neighboring lines* $\mathfrak{l}$ *and* $T_i\mathfrak{l}$, $i=1,...,N$, *are coplanar*.

Let us make a trivial, but important, remark: the planarity of two neighboring lines of the congruence allows for their intersection. When the lines are parallel, we consider their intersection in the hyperplane at infinity. In fact, as it was observed in Refs. 25 and 15, the quadrilateral lattices should be considered within the projective geometry approach; i.e., the ambient space should be the M-dimensional projective space $\mathbb{P}^M$. Accordingly, the space of lines in the affine space modeled on $\mathbb{R}^M$ should be then replaced by the space of lines in $\mathbb{P}^M$; that is to say, by the Grassmannian $\mathrm{Gr}(2,M+1)$.

One can associate with any N-dimensional congruence in a canonical way N lattices defined as follows.

Definition 2.2: The ith *focal lattice* $\mathbf{y}_i(\mathfrak{l})$ *of a congruence* $\mathfrak{l}$ *is the lattice constructed out of the intersection points of the lines* $\mathfrak{l}$ *with* $T_i^{-1}\mathfrak{l}$.

In our paper we study the interplay between congruences of lines and quadrilateral lattices, and we shall show that the focal lattices of a ''generic'' congruence are indeed quadrilateral. To explain what a generic congruence is, let us consider any four lines:

$$\mathfrak{l},\ T_i\mathfrak{l},\ T_j\mathfrak{l},\ T_k\mathfrak{l},\quad i\neq j\neq k\neq i;$$

the congruence is *generic* if the linear space $\mathbb{V}_{ijk}(\mathfrak{l})$ generated by these lines is of the maximal possible dimension: $\dim \mathbb{V}_{ijk}(\mathfrak{l})=4$. The congruence is called *weakly generic* if the linear space $\mathbb{V}_{ij}(\mathfrak{l})$ generated by any three lines $\mathfrak{l}$, $T_i\mathfrak{l}$, $T_j\mathfrak{l}$, $i\neq j$, is of maximal possible dimension: $\dim \mathbb{V}_{ij}(\mathfrak{l})=3$.

Obviously, any generic congruence is also a weakly generic one. In our studies we may violate the genericity assumption, but we *we always assume we deal with weakly generic congruences*.

Theorem 2.1: *Focal lattices of a generic congruence are quadrilateral lattices*.

Proof: The proof splits naturally into two parts. In the first part, illustrated in Fig. 1, we show the planarity of the elementary quadrilaterals with vertices $\mathbf{y}_i$, $T_i\mathbf{y}_i$, $T_j\mathbf{y}_i$, $T_iT_j\mathbf{y}_i$, where $j\neq i$. In the second part, illustrated in Fig. 2, we prove the same for the elementary quadrilaterals with vertices $\mathbf{y}_i$, $T_j\mathbf{y}_i$, $T_k\mathbf{y}_i$, $T_jT_k\mathbf{y}_i$, where $j,k\neq i$, $j\neq k$.

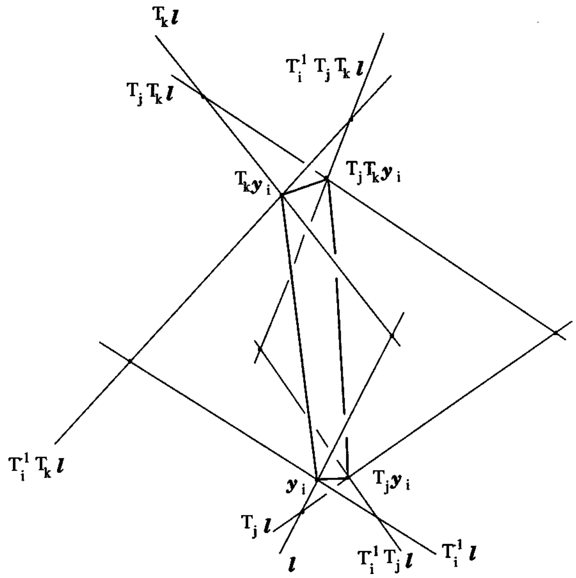

FIG. 2. Planarity of (jk) quadrilaterals.

(i) Let us observe that the vertices $\mathbf{y}_i$, and $T_i\mathbf{y}_i$ are points of the line $\mathfrak{l}$. Similarly, the vertices $T_j\mathbf{y}_i$ and $T_iT_j\mathbf{y}_i$ belong to the line $T_j\mathfrak{l}$. But the lines $\mathfrak{l}$ and $T_j\mathfrak{l}$ are coplanar, which concludes the first part of the proof.

(ii) Consider the configuration of the four lines:

$$\mathfrak{l}, \quad T_j\mathfrak{l}, \quad T_k\mathfrak{l}, \quad T_jT_k\mathfrak{l},$$

contained in the three-dimensional space $\mathbb{V}_{jk}(\mathfrak{l})$, and the similar configuration of four lines:

$$T_i^{-1}\mathfrak{l}, \quad T_i^{-1}T_j\mathfrak{l}, \quad T_i^{-1}T_k\mathfrak{l}, \quad T_i^{-1}T_jT_k\mathfrak{l},$$

contained in a three-dimensional subspace $\mathbb{V}_{jk}(T_i^{-1}\mathfrak{l})$. We remark that $\mathbb{V}_{ijk}(T_i^{-1}\mathfrak{l}) = \mathbb{V}_{jk}(T_i^{-1}\mathfrak{l}) + \mathbb{V}_{jk}(\mathfrak{l})$.

Let us notice that corresponding lines of the two configurations have one point in common,

$$\mathbf{y}_i = (T_i^{-1}\mathfrak{l}) \cap \mathfrak{l}, \quad T_j\mathbf{y}_i = (T_i^{-1}T_j\mathfrak{l}) \cap (T_j\mathfrak{l}),$$

$$T_k\mathbf{y}_i = (T_i^{-1}T_k\mathfrak{l}) \cap (T_k\mathfrak{l}), \quad T_jT_k\mathbf{y}_i = (T_i^{-1}T_jT_k\mathfrak{l}) \cap (T_jT_k\mathfrak{l});$$

these points are vertices of the quadrilateral whose planarity we would like to show. The points $\mathbf{y}_i$, $T_j\mathbf{y}_i$, $T_k\mathbf{y}_i$ define a plane $\mathbb{V}_{jk}(\mathbf{y}_i)$, which is contained in both subspaces $\mathbb{V}_{jk}(\mathfrak{l})$ and $\mathbb{V}_{jk}(T_i^{-1}\mathfrak{l})$. Since, for a generic congruence,

$$\dim(\mathbb{V}_{jk}(\mathfrak{l}) \cap \mathbb{V}_{jk}(T_i^{-1}\mathfrak{l})) = \dim \mathbb{V}_{jk}(T_i^{-1}\mathfrak{l}) + \dim \mathbb{V}_{jk}(\mathfrak{l}) - \dim \mathbb{V}_{ijk}(T_i^{-1}\mathfrak{l}) = 2,$$

then

$$\mathbb{V}_{jk}(\mathbf{y}_i) = \mathbb{V}_{jk}(\mathfrak{l}) \cap \mathbb{V}_{jk}(T_i^{-1}\mathfrak{l}),$$

and, therefore, also $T_jT_k\mathbf{y}_i \in \mathbb{V}_{jk}(\mathbf{y}_i)$; this proves the planarity of the quadrilateral under consideration. □

It turns out that even in the nongeneric case, if one of the focal lattices is quadrilateral, then all the others are quadrilateral as well; to show it we need the following simple but basic fact.

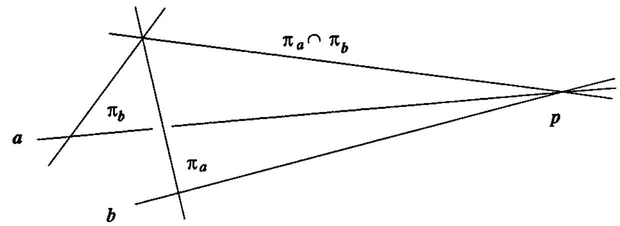

FIG. 3. The intersection point of two lines.

Lemma 2.1: Consider, in the three-dimensional space, two different coplanar lines a and b and two different planes π_a and π_b that contain the lines a and b, correspondingly: $a \subset \pi_a$, $b \subset \pi_b$. Then the common line (it exists and is unique) of the two planes contains the intersection point $\mathbf{p}$ *of the two lines:* $\mathbf{p} = (a \cap b) \in \pi_a \cap \pi_b$ (see Fig. 3).

Proposition 2.1: If one of the focal lattices of the congruence is quadrilateral, then the other focal lattices are quadrilateral as well.

Proof: Let us assume that the ith focal lattice is planar. Therefore the lines $a = \langle T_j\mathbf{y}_i, T_jT_k\mathbf{y}_i\rangle$ and $b = \langle \mathbf{y}_i, T_k\mathbf{y}_i\rangle$ intersect at $\mathbf{p}$. From Lemma 2.1, the intersection line $\langle T_j\mathbf{y}_j, T_jT_k\mathbf{y}_j\rangle$ of the planes $\pi_a = \langle T_j\mathfrak{l}, T_jT_k\mathfrak{l}\rangle$ and $\pi_b = \langle \mathfrak{l}, T_k\mathfrak{l}\rangle$ passes through the point $\mathbf{p}$. Analogously, also the line $\langle T_i^{-1}T_j\mathbf{y}_j, T_i^{-1}T_jT_k\mathbf{y}_j\rangle$ passes through $\mathbf{p}$. This proves the planarity of the quadrilateral $T_i^{-1}T_j\{\mathbf{y}_j, T_i\mathbf{y}_j, T_k\mathbf{y}_j, T_iT_k\mathbf{y}_j\}$. □

Corollary 2.1: The intersection points of the pairs of lines $\langle T_i\mathbf{y}_i, T_iT_k\mathbf{y}_i\rangle$ *with* $\langle T_iT_j\mathbf{y}_i, T_iT_jT_k\mathbf{y}_i\rangle$ *and* $\langle T_j\mathbf{y}_j, T_jT_k\mathbf{y}_j\rangle$ *with* $\langle T_iT_j\mathbf{y}_j, T_iT_jT_k\mathbf{y}_j\rangle$ *coincide.*

B. Constructability of congruences

In this section we look at the congruences from the point of view of their constructability. We recall that, in the case of quadrilateral lattices,[15] given the points $\mathbf{x}$, $T_i\mathbf{x}$, $T_j\mathbf{x}$, $T_k\mathbf{x}$ in the general position, and points $T_iT_j\mathbf{x} \in \mathbb{V}_{ij}(\mathbf{x})$, $T_iT_k\mathbf{x} \in \mathbb{V}_{ik}(\mathbf{x})$, and $T_jT_k\mathbf{x} \in \mathbb{V}_{jk}(\mathbf{x})$, then the point $T_iT_jT_k\mathbf{x}$ is uniquely determined as the intersection point of the three planes $\mathbb{V}_{jk}(T_i\mathbf{x})$, $\mathbb{V}_{ik}(T_j\mathbf{x})$, and $\mathbb{V}_{ij}(T_k\mathbf{x})$ in the three-dimensional space $\mathbb{V}_{ijk}(\mathbf{x})$.

A similar procedure is valid also for congruences. Given the lines $\mathfrak{l}$, $T_i\mathfrak{l}$, and $T_j\mathfrak{l}$, the admissible lines $T_iT_j\mathfrak{l}$ form a two-parameter space (any pair of points of $T_i\mathfrak{l}$ and $T_j\mathfrak{l}$ may be connected by a line), like for the lattice case. This is actually another reason why one can view congruences of lines as dual objects to quadrilateral lattices.

In a generic situation, the ''initial'' lines $\mathfrak{l}$, $T_i\mathfrak{l}$, $T_j\mathfrak{l}$, $T_k\mathfrak{l}$, $T_iT_j\mathfrak{l}$, $T_iT_k\mathfrak{l}$, and $T_jT_k\mathfrak{l}$ are contained in the four-dimensional space $\mathbb{V}_{ijk}(\mathfrak{l})$. The line $T_iT_jT_k\mathfrak{l}$ is therefore the *unique* line that intersects the three lines $T_iT_j\mathfrak{l}$, $T_iT_k\mathfrak{l}$, and $T_jT_k\mathfrak{l}$ [or, equivalently, the intersection line of the three spaces $\mathbb{V}_{ij}(T_k\mathfrak{l})$, $\mathbb{V}_{ik}(T_j\mathfrak{l})$, and $\mathbb{V}_{jk}(T_i\mathfrak{l})$]. Therefore genericity of the congruence and uniqueness of the construction are sinonimous, implying that the focal lattices are quadrilateral.

In the nongeneric case, when the lines $T_iT_j\mathfrak{l}$, $T_iT_k\mathfrak{l}$, and $T_jT_k\mathfrak{l}$ are contained in a three-dimensional space, there exists a one-parameter family of lines intersecting the three given lines and the construction is not unique. We remark that, in this situation, for any point of the line $T_jT_k\mathfrak{l}$, say, there exists a unique line passing through the other two lines $T_iT_k\mathfrak{l}$ and $T_iT_j\mathfrak{l}$; such family of lines forms a one-sheeted hyperboloid. Any element of this family is admissible, but may not give rise to quadrilateral focal lattices.

However, in this nongeneric case, we may single out the line $T_iT_jT_k\mathfrak{l}$ from the above one-parameter family of lines by requiring that the intersection point $T_iT_jT_k\mathbf{y}_\mathbf{i}$ of $T_iT_jT_k\mathfrak{l}$ with the line $T_jT_k\mathfrak{l}$ belong to the plane $\mathbb{V}_{jk}(T_i\mathbf{y}_i) = \langle T_i\mathbf{y}_i, T_iT_j\mathbf{y}_i, T_iT_k\mathbf{y}_i\rangle$ or, equivalently, that the *focal lattice* $\mathbf{y}_i$ be quadrilateral. We remark that this procedure does not depend on the focal lattice we consider (from Proposition 2.1).

We have seen that, given an N-dimensional congruence, one can associate with it N focal (quadrilateral, in general) lattices. There is, of course, a dual picture, and one can associate with a lattice that is quadrilateral N (tangent) congruences.

Definition 2.3: Given an N-dimensional quadrilateral lattice $\mathbf{x}$, *its* ith *tangent congruence* $t_i(\mathbf{x})$ *consists of the lines passing through the points* $\mathbf{x}$ *of the lattice and directed along the tangent vectors* $\Delta_i\mathbf{x}$.

We remark that the planarity of the elementary quadrilaterals of $\mathbf{x}$ implies that the tangent congruence is a congruence of lines in the sense of Definition 2.1. Obviously, excluding degenerations, any congruence $\mathfrak{l}$ can be viewed as the ith tangent congruence of its ith focal lattice $\mathbf{y}_i(\mathfrak{l})$.

In the previous section we have shown that, for nongeneric congruences, the focal lattices may not be quadrilateral. However, for tangent congruences, due to Proposition 2.1, we have the following.

Theorem 2.2: *Focal lattices of tangent congruences are quadrilateral lattices.*

C. Conjugacy of quadrilateral lattices and rectilinear congruences

The following mutual relation between a congruence and a quadrilateral lattice is of particular importance in our theory.

Definition 2.4: An N-dimensional quadrilateral lattice $\mathbf{x}$ *and an N-dimensional congruence* $\mathfrak{l}$ *are called conjugate if* $\mathbf{x}(\mathbf{n})\in\mathfrak{l}(\mathbf{n})$, *for all* $\mathbf{n}\in\mathbb{Z}^N$.

In the definition of conjugate net (on a surface) conjugate to a congruence, first given by Guichard,[3] the developables of the congruence intersect the net in conjugate-parameter lines; the focal nets of the congruence were excluded *a priori* from the definition.

In our approach, instead, we include focal lattices (and focal manifolds) in a natural way as special limiting cases of generic lattices (manifolds) conjugate to the congruence; this observation will be used in Sec. VIII.

We will show now that a quadrilateral lattice conjugate to a congruence may be conveniently used to improve the construction of the congruence itself making it unique in the nongeneric case.

We first show that, for a generic congruence $\mathfrak{l}$, the construction of a quadrilateral lattice $\mathbf{x}$ conjugate to the congruence is compatible with the construction of the congruence itself. We assume, for simplicity, that the points of the lattice are not the focal ones. We observe that, given three points $\mathbf{x}$, $T_i\mathbf{x}$, and $T_j\mathbf{x}$, $i\neq j$, marked on the lines $\mathfrak{l}$, $T_i\mathfrak{l}$, and $T_j\mathfrak{l}$, the point $T_iT_j\mathbf{x}$ is then uniquely determined as the intersection point of the plane $\mathbb{V}_{ij}(\mathbf{x})=\langle\mathbf{x},T_i\mathbf{x},T_j\mathbf{x}\rangle$ with the line $T_iT_j\mathfrak{l}$ in the three-dimensional space $\mathbb{V}_{ij}(\mathfrak{l})$. In the dual picture, given the point $T_iT_j\mathbf{x}$, then the line $T_iT_j\mathfrak{l}$ is the intersection line of the planes $\langle T_i\mathfrak{l},T_iT_j\mathbf{x}\rangle$ and $\langle T_j\mathfrak{l},T_iT_j\mathbf{x}\rangle$.

If we also give the point $T_k\mathbf{x}$ on $T_k\mathfrak{l}$, then the lines $T_iT_k\mathfrak{l}$ and $T_jT_k\mathfrak{l}$ allow us to find the points $T_iT_k\mathbf{x}$ and $T_jT_k\mathbf{x}$, and vice versa.

Now we can use the standard construction of the MQL lattice to find the eight points $T_iT_jT_k\mathbf{x}$ from the seven points $\mathbf{x},\ldots,T_jT_k\mathbf{x}$, and we can use the above presented construction of the nondegenerate congruence to find the line $T_iT_jT_k\mathfrak{l}$ from the seven lines $\mathfrak{l},..,T_jT_k\mathfrak{l}$. At this point a natural and important question arises: *does the point* $T_iT_jT_k\mathbf{x}$ *belong to the line* $T_iT_jT_k\mathfrak{l}$? If it does not, then the notion of quadrilateral lattice conjugate to congruence would not be a very relevant one.

To show that the answer is positive let us proceed as follows. Denote by z the *unique* intersection point of the line $T_iT_jT_k\mathfrak{l}$ with the three-dimensional subspace $\mathbb{V}_{ijk}(\mathbf{x})=\langle\mathbf{x},T_i\mathbf{x},T_j\mathbf{x},T_k\mathbf{x}\rangle$ (our congruence is a generic one). Since $\mathbb{V}_{jk}(T_i\mathbf{x})\subset\mathbb{V}_{ijk}(\mathbf{x})$ and $\mathbb{V}_{jk}(T_i\mathbf{x})\cap T_iT_jT_k\mathfrak{l}\neq\emptyset$, then $z\in\mathbb{V}_{jk}(T_i\mathbf{x})$. Similarly, $z\in\mathbb{V}_{ik}(T_j\mathbf{x})$ and $z\in\mathbb{V}_{ij}(T_k\mathbf{x})$; which implies that $\mathbf{z}=T_iT_jT_k\mathbf{x}$.

Remark: The above construction properties imply that, for a given generic congruence, a quadrilateral lattice conjugate to it is uniquely defined, assigning its initial curves.

In the nongeneric case, we may again single out the line $T_iT_jT_k\mathfrak{l}$ from the one-parameter family of lines by the following requirement, which has been proved to hold in the generic situation.

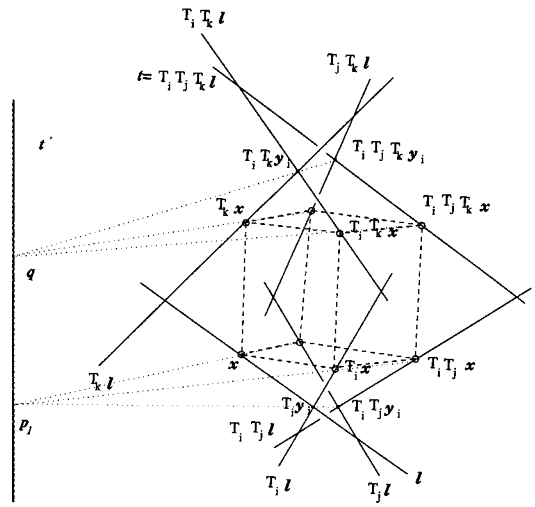

FIG. 4. Construction of a quadrilateral lattice conjugate to a congruence.

(i) *The line passes through the point $T_iT_jT_k\mathbf{x}$ and meets the lines $T_iT_j\mathfrak{l}$, $T_iT_k\mathfrak{l}$, and $T_jT_k\mathfrak{l}$. If such a line exists, for the construction to be the canonical one we would like also two additional conditions to be satisfied.*

(ii) *The line does not depend on the particular positions of the initial points $\mathbf{x}$, $T_i\mathbf{x}$, $T_j\mathbf{x}$, and $T_k\mathbf{x}$.*

(iii) *The new construction gives the same result as the previous one; i.e., the focal lattices are quadrilateral.*

To check that the above construction is the canonical one, we first show that there exists a unique line that satisfies conditions (i) and (iii); due to the uniqueness of the line satisfying condition (iii), the condition (ii) will be also proven.

Assume we have points $\mathbf{x},\ldots,T_jT_k\mathbf{x}$ satisfying the planarity conditions and belonging to the corresponding lines $\mathfrak{l},\ldots,T_jT_k\mathfrak{l}$. Using the standard MQL construction, we find the point $T_iT_jT_k\mathbf{x}$; the point $T_iT_jT_k\mathbf{y}_i$ is the intersection point of the plane $\mathbb{V}_{jk}(T_i\mathbf{y}_i)$ with the line $T_jT_k\mathfrak{l}$.

Denote by $\mathfrak{t}$ the line passing through $T_iT_jT_k\mathbf{x}$ and $T_iT_jT_k\mathbf{y}_i$ (see Fig. 4). Our goal is to demonstrate that the quadrilaterals $\{T_iT_k\mathbf{y}_i, T_iT_jT_k\mathbf{y}_i, T_iT_k\mathbf{x}, T_iT_jT_k\mathbf{x}\}$ and $\{T_iT_j\mathbf{y}_i, T_iT_jT_k\mathbf{y}_i, T_iT_j\mathbf{x}, T_iT_jT_k\mathbf{x}\}$ are planar; this would show that the line $\mathfrak{t}$ meets lines $T_iT_k\mathfrak{l}$ and $T_iT_j\mathfrak{l}$, which would imply that the line $T_iT_jT_k\mathfrak{l}=\mathfrak{t}$ satisfying condition (i) does exist.

Denote by $\mathfrak{t}'$ the intersection line of the planes $\mathbb{V}_{jk}(\mathbf{x})$ and $\mathbb{V}_{jk}(T_i\mathbf{x})$. Obviously, the points $\mathbf{p}_1=\langle\mathbf{x},T_j\mathbf{x}\rangle\cap\langle T_i\mathbf{x},T_iT_j\mathbf{x}\rangle=\mathbb{V}_j(\mathbf{x})\cap\mathbb{V}_j(T_i\mathbf{x})$ and $\mathbf{p}_2=\mathbb{V}_k(\mathbf{x})\cap\mathbb{V}_k(T_i\mathbf{x})$ belong to $\mathfrak{t}'$. The application of Lemma 2.1 gives $\mathbf{p}_1\in\mathbb{V}_j(T_i\mathbf{y}_i)$ and $\mathbf{p}_2\in\mathbb{V}_k(T_i\mathbf{y}_i)$, which implies that the line $\mathfrak{t}'$ is contained in the plane $\mathbb{V}_{jk}(T_i\mathbf{y}_i)$.

Since the quadrilateral $\{T_k\mathbf{x}, T_kT_j\mathbf{x}, T_iT_k\mathbf{y}_i, T_iT_jT_k\mathbf{y}_i\}$ is planar then the lines $\mathbb{V}_j(T_iT_k\mathbf{y}_i)$ and $\mathbb{V}_j(T_k\mathbf{x})$ intersect in a point $\mathbf{q}$, which, according to the reasoning above, must belong to the line $\mathfrak{t}'$. Since the lines $\mathbb{V}_j(T_k\mathbf{x})$ and $\mathbb{V}_j(T_iT_k\mathbf{x})$ intersect also in a point of $\mathfrak{t}'$, then the point $\mathbf{q}$ is the intersection point of all the three lines. This implies that the quadrilateral $\{T_iT_k\mathbf{y}_i, T_iT_jT_k\mathbf{y}_i, T_iT_k\mathbf{x}, T_iT_jT_k\mathbf{x}\}$ is planar. Similar reasonings show that the quadrilateral $\{T_iT_j\mathbf{y}_i, T_iT_jT_k\mathbf{y}_i, T_iT_j\mathbf{x}, T_iT_jT_k\mathbf{x}\}$ is planar as well, which shows that the new construction of the congruence is indeed the canonical one.

The above reasoning allows us to formulate the following.

Proposition 2.2: If, for a nongeneric congruence, there exists a quadrilateral lattice conjugate to it, then the focal lattices of the congruence are quadrilateral.

This result, together with Proposition 2.1, implies the following important corollary.

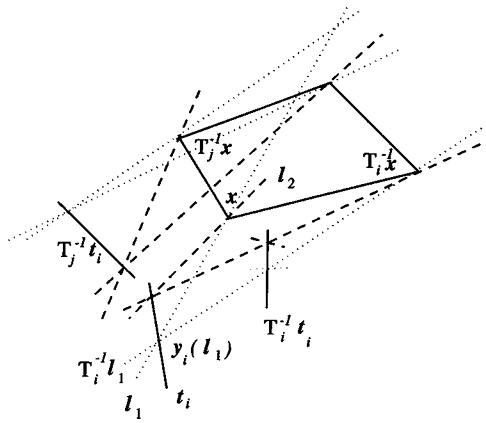

FIG. 5. Two congruences conjugate to the same lattice.

Corollary 2.2: Focal lattices of congruences conjugate to quadrilateral lattices are quadrilateral lattices.

In the sequel we will need also the following result.

Proposition 2.3: Given two congruences $\mathfrak{l}_1$, $\mathfrak{l}_2$ *conjugate to the same quadrilateral lattice* $\mathbf{x}$, *then the lines defined by joining corresponding points of two focal lattices* $\mathbf{y}_i(\mathfrak{l}_1)$ *and* $\mathbf{y}_i(\mathfrak{l}_2)$ *form a congruence* $\mathfrak{t}_i$ *conjugate to both focal lattices.*

Proof: In Fig. 5, two congruences $\mathfrak{l}_1$ and $\mathfrak{l}_2$ are represented, respectively, by dotted and dashed lines. We have to prove that the lines $\mathfrak{t}_i$ form a congruence. The lines $\mathfrak{t}_i$ and $T_i^{-1}\mathfrak{t}_i$ are coplanar because they belong to the plane of the two intersecting (in $T_i^{-1}\mathbf{x}$) lines $T_i^{-1}\mathfrak{l}_1$ and $T_i^{-1}\mathfrak{l}_2$.

To show that the lines $\mathfrak{t}_i$ and $T_j^{-1}\mathfrak{t}_i$, $j\neq i$, are coplanar, let us consider the quadrilateral with vertices $\mathbf{y}_i(\mathfrak{l}_1)$, $\mathbf{y}_i(\mathfrak{l}_2)$, $T_j^{-1}\mathbf{y}_i(\mathfrak{l}_1)$, and $T_j^{-1}\mathbf{y}_i(\mathfrak{l}_2)$. Due to Lemma 2.1 the lines $\langle \mathbf{y}_i(\mathfrak{l}_1),T_j^{-1}\mathbf{y}_i(\mathfrak{l}_1)\rangle$ and $\langle \mathbf{y}_i(\mathfrak{l}_2),T_j^{-1}\mathbf{y}_i(\mathfrak{l}_2)\rangle$ intersect in the point $\langle \mathbf{x}, T_j^{-1}\mathbf{x}\rangle\cup\langle T_i^{-1}\mathbf{x}, T_i^{-1}T_j^{-1}\mathbf{x}\rangle$, which proves the planarity of the quadrilateral and, therefore, the coplanarity of the lines $\mathfrak{t}_i=\langle \mathbf{y}_i(\mathfrak{l}_2),\mathbf{y}_i(\mathfrak{l}_2)\rangle$ and $T_j^{-1}\mathfrak{t}_i=\langle T_j^{-1}\mathbf{y}_i(\mathfrak{l}_2),T_j^{-1}\mathbf{y}_i(\mathfrak{l}_2)\rangle$ (see Fig. 5).

III. LAPLACE TRANSFORMATIONS

In Sec. II we considered congruences of lines and their focal lattices. In this section we are interested, in particular, in the relations between two focal lattices of the same congruence; these relations are described by the Laplace transformations.

The Laplace transformations of conjugate nets were introduced by Darboux (see Refs. 1, 3, and 33). For $N=2$ this transformation provides the geometric meaning of the transformation (known already to Laplace) connecting solutions of two Laplace equations.

Definition 3.1 The Laplace transform $\mathcal{L}_{ij}(\mathbf{x})$ *of the quadrilateral lattice* $\mathbf{x}$ *is the* jth *focal lattice of its* ith *tangent congruence,*

$$\mathcal{L}_{ij}(\mathbf{x})=\mathbf{y}_j(t_i(\mathbf{x})). \tag{3.1}$$

In simple terms, $\mathcal{L}_{ij}(\mathbf{x})$ is the intersection point of the line passing through $T_j^{-1}\mathbf{x}$ and $T_j^{-1}T_i\mathbf{x}$ with the line passing through $\mathbf{x}$ and $T_i\mathbf{x}$[25] (see Fig. 6).

The points of the first line are of the form

$$\mathbf{p}(t)=T_j^{-1}\mathbf{x}+tT_j^{-1}\mathbf{X}_i, \tag{3.2}$$

which can be transformed, using (1.5) and (1.6), into

$$\mathbf{p}(t)=\mathbf{x}+t\mathbf{X}_i-(H_j+tQ_{ij})T_j^{-1}\mathbf{X}_j; \tag{3.3}$$

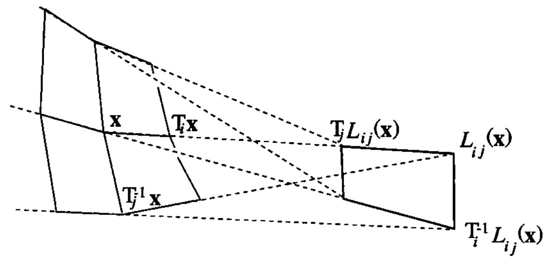

FIG. 6. The Laplace transformation.

the intersection point of the two lines is therefore given by

$$t=-\frac{H_j}{Q_{ij}}. \tag{3.4}$$

Therefore we have the following.

Proposition 3.1: The Laplace transformation of the quadrilateral lattice $\mathbf{x}$ *is given by*

$$\mathcal{L}_{ij}(\mathbf{x})=\mathbf{x}-\frac{H_j}{Q_{ij}}\mathbf{X}_i=\mathbf{x}-\frac{1}{A_{ji}}\Delta_i\mathbf{x}. \tag{3.5}$$

By direct calculations, one has the following.

Corollary 3.1: (i) The Laplace transformed A coefficients are of the form

$$\mathcal{L}_{ij}(A_{ij})=\frac{A_{ji}}{T_jA_{ji}}(T_iA_{ij}+1)-1, \tag{3.6}$$

$$\mathcal{L}_{ij}(A_{jk})=T_j^{-1}\left(\frac{T_k\mathcal{L}_{ij}(A_{ij})}{\mathcal{L}_{ij}(A_{ij})}(A_{jk}+1)\right)-1, \tag{3.7}$$

$$\mathcal{L}_{ij}(A_{ik})=A_{jk}T_k\left(1-\frac{A_{ki}}{A_{ji}}\right),\quad k\neq i,j, \tag{3.8}$$

$$\mathcal{L}_{ij}(A_{kl})=(A_{kl}+1)\frac{T_k(1-A_{ki}/A_{ji})}{(1-A_{ki}/A_{ji})}-1,\quad k\neq j,i,\quad l\neq k. \tag{3.9}$$

(ii) The Lamé coefficients of the transformed lattice read as

$$\mathcal{L}_{ij}(H_i)=\frac{T_iH_i}{A_{ji}}=\frac{H_j}{Q_{ij}}, \tag{3.10}$$

$$\mathcal{L}_{ij}(H_j)=T_j^{-1}(H_j\mathcal{L}_{ij}(A_{ij}))=T_j^{-1}\left(Q_{ij}\Delta_j\left(\frac{H_j}{Q_{ij}}\right)\right), \tag{3.11}$$

$$\mathcal{L}_{ij}(H_k)=H_k\left(1-\frac{A_{ki}}{A_{ji}}\right)=H_k-\frac{Q_{ik}}{Q_{ij}}H_j,\quad k\neq i,j. \tag{3.12}$$

(iii) The tangent vectors of the new lattice read as

$$\mathcal{L}_{ij}(\mathbf{X}_i)=-\Delta_i\mathbf{X}_i+\frac{\Delta_iQ_{ij}}{Q_{ij}}\mathbf{X}_i, \tag{3.13}$$

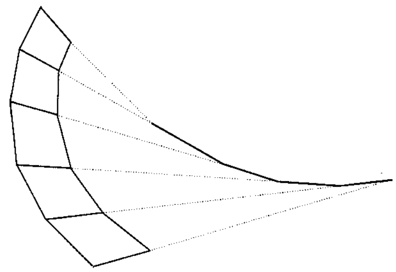

FIG. 7. The Laplace transformation of a quadrilateral.

$$\mathcal{L}_{ij}(\mathbf{X}_j) = -\frac{1}{Q_{ij}}\mathbf{X}_i, \tag{3.14}$$

$$\mathcal{L}_{ij}(\mathbf{X}_k) = \mathbf{X}_k - \frac{Q_{kj}}{Q_{ij}}\mathbf{X}_i, \quad k \neq i,j. \tag{3.15}$$

Finally, we remark that, apart from the identity

$$\mathcal{L}_{ij} \circ \mathcal{L}_{ji} = \mathrm{id}, \tag{3.16}$$

which follows just from the definition of the Laplace transformation (see also Ref. 25), there are two other identities:

$$\mathcal{L}_{jk} \circ \mathcal{L}_{ij} = \mathcal{L}_{ik}, \tag{3.17}$$

$$\mathcal{L}_{ki} \circ \mathcal{L}_{ij} = \mathcal{L}_{kj}; \tag{3.18}$$

which follow from Corollary 2.1, or may be verified directly from the above equations.

Notice that, to construct a line of the new lattice, one needs a quadrilateral strip of the old lattice (see Fig. 7). Similarly, one $(N-1)$-dimensional level of the new lattice can be constructed out of two $(N-1)$-dimensional levels of the original lattice [i.e., out of a quadrilateral strip with an $(N-1)$-dimensional basis]. In fact, we may define the Laplace transform of a quadrilateral strip; this last observation will be used in the next sections.

IV. COMBESCURE TRANSFORMATIONS

In this section we study quadrilateral lattices related by parallelism of the tangent vectors. Basically, we generalize to a discrete level the results about the Combescure transformations of the conjugate nets, as presented in the monograph.[3] Definition 4.1 and Proposition 4.1 of Sec. IV A is also contained in Ref. 22.

A. Combescure transformations of quadrilateral lattices

Definition 4.1: A lattice $\mathcal{C}(\mathbf{x}):\mathbb{Z}^N \to \mathbb{R}^M$ is called the Combescure transform of (or parallel to) the quadrilateral lattice $\mathbf{x}:\mathbb{Z}^N \to \mathbb{R}^M$ if the tangent vectors of both lattices in the corresponding points are proportional:

$$\Delta_i \mathcal{C}(\mathbf{x}) = (T_i C_i)\Delta_i x, \quad i = 1,...,N. \tag{4.1}$$

We mention that the definition of the Combescure transformation makes use of the notion of parallelism, which has an affine geometry origin and comes from fixing the hyperplane at infinity.[34]

The following results can be verified by direct calculation.

Proposition 4.1: (i) The proportionality factors C_i satisfy the equations

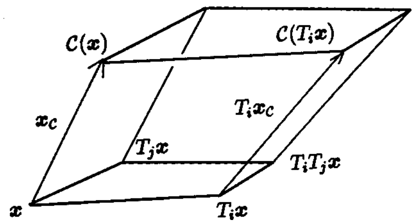

FIG. 8. The Combescure transformation.

$$\Delta_j C_i = A_{ij} T_j (C_j - C_i), \quad i \neq j. \tag{4.2}$$

(ii) The transformed lattice is a quadrilateral lattice with Combescure-transformed functions of the form

$$\mathcal{C}(A_{ij}) = A_{ij} \frac{T_j C_j}{C_i}, \quad i \neq j,$$

$$\mathcal{C}(\mathbf{X}_i) = \mathbf{X}_i,$$

$$\mathcal{C}(H_i) = C_i H_i.$$

(iii) All the quadrilaterals with vertices $\{\mathbf{x}, T_i\mathbf{x}, \mathcal{C}(\mathbf{x}), \mathcal{C}(T_i\mathbf{x})\}$ *are planar.*

From the last property of Proposition 4.1, it follows that the lattices $\mathbf{x}$ and $\mathcal{C}(\mathbf{x})$ form a quadrilateral strip with the N-dimensional basis $\mathbf{x}$ and the transversal direction given by the Combescure transform $\mathcal{C}$ (direction $\mathcal{C}$); see Fig. 8.

Therefore the recursive application of a Combescure transformation to the N-dimensional quadrilateral lattice $\mathbf{x}$ can be viewed as generating a new dimension (say, the $N+1$st) of the lattice. The corresponding data are simply

$$H_{N+1} = 1,$$

$$\mathbf{X}_{N+1} = \mathbf{x}_\mathcal{C},$$

up to an arbitrary function of n_{N+1}, always present in the definition of H and $\mathbf{X}$ (see Ref. 15).

We observe that the transversal vector $\mathbf{x}_\mathcal{C}$, given by

$$\mathbf{x}_\mathcal{C} = \mathcal{C}(\mathbf{x}) - \mathbf{x}, \tag{4.3}$$

satisfies the equations

$$\Delta_i \mathbf{x}_\mathcal{C} = (T_i \sigma_i) \Delta_i \mathbf{x} = (T_i \nu_i^*) \mathbf{X}_i, \tag{4.4}$$

where the functions σ_i and ν_i^*, $i = 1, \ldots, N$ are given by

$$\sigma_i = C_i - 1, \quad \nu_i^* = (C_i - 1) H_i. \tag{4.5}$$

The following facts are easy to verify.

Corollary 4.1: (i) Functions ν_i^* *satisfy the adjoint linear system (1.8).*

(ii) Functions σ_i *satisfy the equation*

$$\Delta_j \sigma_i = \frac{\Delta_j H_i}{H_i} T_j (\sigma_j - \sigma_i), \quad i \neq j. \tag{4.6}$$

(iii) In the notation of Theorem 1.1, the vector $\mathbf{x}_\mathcal{C}$ *can be rewritten as*

$$\mathbf{x}_{\mathcal{C}} = \mathbf{\Omega}[\mathbf{X}, v^*]; \tag{4.7}$$

i.e., the function $\mathbf{x}_{\mathcal{C}}: \mathbb{Z}^N \rightarrow \mathbb{R}^M$ *is a solution of the Laplace equation,*

$$\Delta_i \Delta_j \mathbf{x}_{\mathcal{C}} = \left(T_i \frac{\Delta_j v_i^*}{v_i^*} \right) \Delta_i \mathbf{x}_{\mathcal{C}} + \left(T_j \frac{\Delta_i v_j^*}{v_j^*} \right) \Delta_j \mathbf{x}_{\mathcal{C}}. \tag{4.8}$$

(iv) The lattice $\mathbf{x}_{\mathcal{C}}$ *is also a Combescure transform of* $\mathbf{x}$.

From the above considerations we can extract the following construction of the Combescure transform, which will be used in the next sections.

Proposition 4.2: In order to construct a Combescure transform of the lattice $\mathbf{x}$, *we*
(i) find a scalar solution v_i^* *of the adjoint linear problem,*

$$\Delta_j v_i^* = (T_j v_j^*) Q_{ji};$$

(ii) the Combescure transform of $\mathbf{x}$ *is then given by*

$$\mathcal{C}(\mathbf{x}) = \mathbf{x} + \mathbf{\Omega}[\mathbf{X}, v^*] = \mathbf{\Omega}[\mathbf{X}, H + v^*]. \tag{4.9}$$

Given any scalar solution ϕ of the Laplace equation (1.3), we define its Combescure transformed function $\phi_{\mathcal{C}}$ in terms of ϕ in the same way in which $\mathbf{x}_{\mathcal{C}}$ follows from $\mathbf{x}$:

$$\Delta_i \phi_{\mathcal{C}} = (T_i \sigma_i) \Delta_i \phi. \tag{4.10}$$

Equivalently, since ϕ defines a scalar solution v_i, $i = 1,\ldots,N$ of the linear problem (1.6) via

$$\Delta_i \phi = (T_i H_i) v_i, \tag{4.11}$$

we have

$$\phi_{\mathcal{C}} = \mathbf{\Omega}[v, v^*]. \tag{4.12}$$

B. Combescure congruences

Let us consider an important example of congruence obtained from a quadrilateral lattice and its Combescure-transformed lattice.

From Proposition 4.1 (iii), it follows that, given a pair of parallel lattices, the lines passing through $\mathbf{x}$ and $\mathcal{C}(\mathbf{x})$ define a congruence that we call Combescure congruence.

The focal lattices of this congruence can be found in the following way. Given a real function $t: \mathbb{Z}^N \rightarrow \mathbb{R}$, define a new lattice $\mathbf{y}$ with points on the lines of the congruence,

$$\mathbf{y} = \mathbf{x} + t \mathbf{x}_{\mathcal{C}}; \tag{4.13}$$

the tangent vectors of the new lattice are given by

$$\Delta_i \mathbf{y} = (1 + T_i(\sigma_i t)) \Delta_i \mathbf{x} + (\Delta_i t) \mathbf{x}_{\mathcal{C}}. \tag{4.14}$$

When

$$t = -\frac{1}{\sigma_i}, \tag{4.15}$$

then the line of the ith tangent vector $\Delta_i \mathbf{y}$ is the line of the congruence and therefore the lattice

$$\mathbf{y}_i = \mathbf{x} - \frac{1}{\sigma_i} \mathbf{x}_{\mathcal{C}} \tag{4.16}$$

is the ith focal lattice of the Combescure congruence.

Corollary 4.2: All the lattices $\mathbf{x}$, $\mathcal{C}(\mathbf{x})$, $\mathbf{y}_i$, $i=1,...,N$, *are conjugate to the same (Combescure) congruence.*

The Combescure congruences will be used extensively throughout the paper due to the following result.

Proposition 4.3: Any congruence conjugate and transversal to a quadrilateral lattice $\mathbf{x}$ *(i.e., not tangent to the lattice in the corresponding points) comes from a Combescure transform* $\mathcal{C}(\mathbf{x})$.

Proof: Geometrically, the construction of such a lattice $\mathcal{C}(\mathbf{x})$ is as follows. Mark on the line $\mathfrak{l}(\mathbf{0})$, $\mathbf{0}\in\mathbb{Z}^N$, of the congruence a point $\mathcal{C}(\mathbf{x}(\mathbf{0}))$ different from $\mathbf{x}(\mathbf{0})$. The point $T_i\mathcal{C}(\mathbf{x}(\mathbf{0}))$ is the intersection of the line $T_i\mathfrak{l}(\mathbf{0})$ with the line passing through $\mathcal{C}(\mathbf{x}(\mathbf{0}))$ and parallel to the line $\langle \mathbf{x}(\mathbf{0}), T_i\mathbf{x}(\mathbf{0})\rangle$. The compatibility of this construction, i.e., $T_iT_j\mathcal{C}(\mathbf{x})=T_jT_i\mathcal{C}(\mathbf{x})$, follows from the fact that $T_iT_j\mathcal{C}(\mathbf{x})$ is the intersection point of $T_iT_j\mathfrak{l}$ with the plane $\langle \mathcal{C}(\mathbf{x}), T_i\mathcal{C}(\mathbf{x}), T_j\mathcal{C}(\mathbf{x})\rangle$.

Since this proposition is one of the most important in our paper, we give an alternative algebraic proof. A congruence $\mathfrak{l}$ conjugate to $\mathbf{x}$ can be described by giving the vector function $\mathbf{X}:\mathbb{Z}^N\to\mathbb{R}^M$ in the direction of the line of the congruence, which passes through the corresponding point $\mathbf{x}$ of the lattice. Our goal is to rescale the direction vector of the congruence by a function t, such that the lattice $\mathbf{x}+t\mathbf{X}$ is parallel to $\mathbf{x}$.

The coplanarity of the neighboring lines of the congruence implies that, if $\Delta_i\mathbf{X}\neq 0$, then $\Delta_i\mathbf{x}$ can be decomposed into a linear combination of $\mathbf{X}$ and $\Delta_i\mathbf{X}$, i.e.,

$$\Delta_i\mathbf{x}\in \mathrm{Span}\{\mathbf{X},\Delta_i\mathbf{X}\}. \tag{4.17}$$

This implies that $\Delta_i\Delta_j\mathbf{x}$ is a linear combination of $\mathbf{X}$, $\Delta_i\mathbf{X}$, $\Delta_j\mathbf{X}$, and $\Delta_i\Delta_j\mathbf{X}$. But since

$$\Delta_i\Delta_j\mathbf{x}\in \mathrm{Span}\{\Delta_i\mathbf{x},\Delta_j\mathbf{x}\}\subset \mathrm{Span}\{\mathbf{X},\Delta_i\mathbf{X},\Delta_j\mathbf{X}\},\quad i\neq j, \tag{4.18}$$

then, there must exist a linear relation between $\mathbf{X}$, $\Delta_i\mathbf{X}$, $\Delta_j\mathbf{X}$, and $\Delta_i\Delta_j\mathbf{X}$, which can be written in the form of the generalized Laplace equation,

$$\Delta_i\Delta_j\mathbf{X}=(T_iB_{ij})\Delta_i\mathbf{X}+(T_jB_{ji})\Delta_j\mathbf{X}+C_{(ij)}\mathbf{X},\quad i\neq j. \tag{4.19}$$

The compatibility condition between (4.19) implies the existence of the logarithmic potentials F_i (see also the discussion in Ref. 15), such that

$$B_{ij}=\frac{\Delta_jF_i}{F_i},\quad i\neq j. \tag{4.20}$$

Let us consider functions $\lambda_i:\ \mathbb{Z}^N\to\mathbb{R}$ that describe the focal lattices $\mathbf{y}_i$ of the congruence in terms of the reference lattice $\mathbf{x}$ and of the direction vectors $\mathbf{X}$,

$$\mathbf{y}_i=\mathbf{x}-\lambda_i\mathbf{X}; \tag{4.21}$$

note that, due to the transversality of the congruence, the functions λ_i never vanish. Since $\mathbf{y}_i$ are the focal lattices of $\mathfrak{l}$, then the vectors $\Delta_i\mathbf{y}_i$ are directed along $\mathbf{X}$:

$$\Delta_i\mathbf{y}_i=\rho_i\mathbf{X}, \tag{4.22}$$

and this equation can be rewritten, using Eq. (4.21), as

$$\Delta_i\mathbf{x}=(T_i\lambda_i)\Delta_i\mathbf{X}+\mu_i\mathbf{X}, \tag{4.23}$$

where $\mu_i=\rho_i+\Delta_i\lambda_i$.

The application of the partial difference operator Δ_j to Eq. (4.23) and the Laplace equation (1.9) with Eq. (4.23) give

$$(T_jT_i\lambda_i)\Delta_i\Delta_j\mathbf{X}+(\Delta_jT_i\lambda_i)\Delta_i\mathbf{X}+(T_j\mu_i)\Delta_j\mathbf{X}+(\Delta_j\mu_i)\mathbf{X}$$
$$=\left(T_i\frac{\Delta_jH_i}{H_i}\right)((T_i\lambda_i)\Delta_i\mathbf{X}+\mu_i\mathbf{X})+\left(T_j\frac{\Delta_iH_j}{H_j}\right)((T_j\lambda_j)\Delta_j\mathbf{X}+\mu_j\mathbf{X}).$$

Rewriting this equation in the form of the generalized Laplace equations (4.19) allows us to calculate the coefficients B_{ij}:

$$B_{ij}=\frac{\Delta_j(H_i/\lambda_i)}{H_i/\lambda_i}\Rightarrow F_i=\frac{H_i}{\lambda_i}. \tag{4.24}$$

Comparing both expressions for B_{ji}, one obtains the following identity:

$$\frac{T_iH_j}{H_j}=\frac{(T_i\lambda_i)(T_i\lambda_j)}{\lambda_j(T_i\lambda_i-T_i\lambda_j)}\left(\frac{\lambda_j+\mu_i}{T_i\lambda_i}-1\right). \tag{4.25}$$

Since $C_{(ij)}$ should be symmetric with respect to the change of indices (see Ref. 15), then, using Eq. (4.25), one arrives at

$$\frac{\mu_i}{T_i\lambda_i}\left(T_i\frac{\mu_j}{T_j\lambda_j}-1\right)=\frac{\mu_j}{T_j\lambda_j}\left(T_j\frac{\mu_i}{T_i\lambda_i}-1\right), \tag{4.26}$$

which implies the existence of a potential function t: $\mathbb{Z}^n\rightarrow\mathbb{R}$, such that

$$\frac{T_it}{t}=\left(1-\frac{\mu_i}{T_i\lambda_i}\right)^{-1}. \tag{4.27}$$

Now, we can scale the direction vector $\mathbf{X}$ of the congruence multiplying it by the potential t, and check that

$$\Delta_i(t\mathbf{X})=\left(T_i\frac{t}{\lambda_i}\right)\Delta_i\mathbf{x}, \tag{4.28}$$

which asserts that the lattice with points given by $\mathbf{x}+t\mathbf{X}$ is a Combescure transform of $\mathbf{x}$. We only remark that an arbitrary scalar constant in the potential t corresponds to the freedom in choosing the initial point $\mathcal{C}(\mathbf{x}(\mathbf{0}))$. □

V. LÉVY TRANSFORMATIONS AND THEIR ADJOINT

In this section we are interested in the relations between two quadrilateral lattices, in which one of the lattices is a focal lattice of the congruence conjugate to the other. In the continuous context, these transformations are called Lévy transformations[35] and are studied in detail in Refs. 3 and 30. We remark that, in the limiting case when also the second lattice (net) is focal, we arrive at the Laplace transformations considered in Sec. III.

A. Adjoint Lévy transformations

Definition 5.1: The ith *adjoint Lévy transform* $\mathcal{L}_i^*(\mathbf{x})$ *of the quadrilateral lattice* $\mathbf{x}$ *is the* ith *focal lattice of a congruence conjugate to* $\mathbf{x}$ (see Fig. 9)

Remark: Adjoint Lévy transformations are usually called in soliton theory *adjoint elementary Darboux transformations*.[36–38]

Assuming that we deal with a generic case, i.e., the congruence conjugate to $\mathbf{x}$ is transversal to it, we construct this congruence via a Combescure transformation vector $\mathbf{x}_{\mathcal{C}}$ of the lattice $\mathbf{x}$. Combining Propositions 4.2 and 4.3 with formula (4.16) for the focal lattices of the Combescure congruence, we obtain the following.

Proposition 5.1: (i) The adjoint Lévy transform of the lattice $\mathbf{x}$ *is given by*

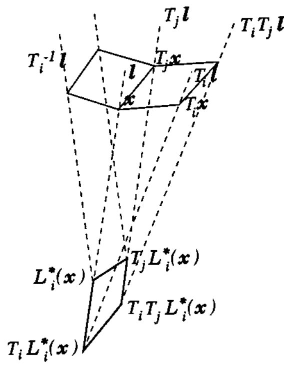

FIG. 9. The adjoint Lévy transformation.

$$\mathcal{L}_i^*(\mathbf{x}) = \mathbf{x} - \frac{1}{\sigma_i}\mathbf{x}_{\mathcal{C}}, \tag{5.1}$$

where the functions σ_i are solutions of Eq. (4.6).

(ii) The Lamé coefficients of the new lattice are of the form

$$\mathcal{L}_i^*(H_i) = T_i^{-1}\left(H_i \frac{\Delta_i \sigma_i}{T_i \sigma_i}\right), \quad \mathcal{L}_i^*(H_j) = H_j\left(1 - \frac{\sigma_j}{\sigma_i}\right). \tag{5.2}$$

Since $\Delta_i \mathcal{L}_i^*(\mathbf{x})$ is, by definition, proportional to $\mathbf{x}_{\mathcal{C}}$, it is easy to check that

$$\mathcal{L}_i^*(\mathbf{x}) = \mathbf{x} + \frac{1/\sigma_i}{\Delta_i(1/\sigma_i)}\Delta_i \mathcal{L}_i^*(\mathbf{x}). \tag{5.3}$$

At this point we can also verify the result that we will use in the next section.

Lemma 5.1: The function $1/\sigma_i$ satisfies the point equation of the lattice $\mathcal{L}_i^*(\mathbf{x})$.

It is convenient to reformulate our results in the notation of Theorem 1.1. Using the functions v_i^* defined in (4.5), we have the following algebraic formulation of the adjoint Lévy transformation.

Proposition 5.2: To construct the adjoint Lévy transform $\mathcal{L}_i^*(\mathbf{x})$ of the quadrilateral lattice $\mathbf{x}$.

(i) Find a scalar solution v_i^* of the adjoint linear problem,

$$\Delta_j v_i^* = (T_j v_j^*) Q_{ji},$$

which defines the direction vectors $\mathbf{x}_{\mathcal{C}} = \mathbf{\Omega}[\mathbf{X}, v^*]$ of a congruence conjugate to $\mathbf{x}$.

(ii) Its ith focal lattice is the adjoint Lévy transform:

$$\mathcal{L}_i^*(\mathbf{x}) = \mathbf{x} - \frac{H_i}{v_i^*}\mathbf{\Omega}[\mathbf{X}, v^*]. \tag{5.4}$$

(iii) The Lamé coefficients and the tangent vectors of the new lattice are of the form

$$\mathcal{L}_i^*(H_i) = -T_i^{-1}\left(v_i^* \Delta_i \left(\frac{H_i}{v_i^*}\right)\right), \tag{5.5}$$

$$\mathcal{L}_i^*(H_j) = H_j - \frac{v_j^*}{v_i^*} H_i, \tag{5.6}$$

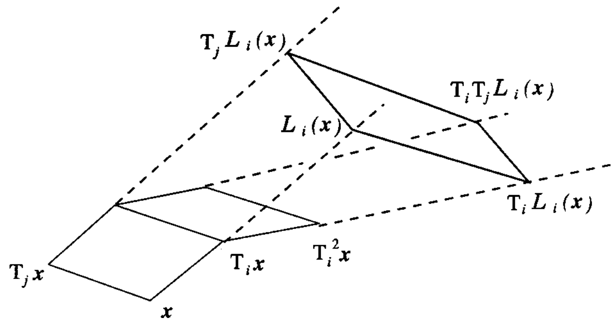

FIG. 10. The Lévy transformation.

$$\mathcal{L}_i^*(\mathbf{X}_i) = \frac{1}{v_i^*}\boldsymbol{\Omega}[\mathbf{X}, v^*], \tag{5.7}$$

$$\mathcal{L}_i^*(\mathbf{X}_j) = \mathbf{X}_j - \frac{Q_{ji}}{v_i^*}\boldsymbol{\Omega}[\mathbf{X}, v^*]. \tag{5.8}$$

Let us observe that the lattices $\mathbf{x}$ and $\mathbf{x}+\mathbf{x}_{\mathcal{C}}$ form a quadrilateral strip with the N-dimensional basis $\mathbf{x}$ and one transversal direction $\mathbf{x}_{\mathcal{C}}$. The adjoint Lévy transformation $\mathcal{L}_i^*$ of the lattice $\mathbf{x}$ can be interpreted as the Laplace transformation $\mathcal{L}_{\mathcal{C}i}$ of the strip.

We also remark that Proposition 2.3 can be formulated in the following way.

Proposition 5.3: Two lattices that have been obtained by the ith *adjoint Lévy transformation of the same quadrilateral lattice are conjugate to the same congruence.*

B. Lévy transformations

Definition 5.2: The ith *Lévy transform* $\mathcal{L}_i(\mathbf{x})$ *of the quadrilateral lattice* $\mathbf{x}$ *is a quadrilateral lattice conjugate to the* ith *tangent congruence of* $\mathbf{x}$ (see Fig. 10).

Remark: In the soliton theory, Lévy transformations of multiconjugate systems are usually called *elementary Darboux transformations*.[36–38]

It is evident from Definitions 5.2 and 5.1 that the Lévy transform is in a sense the inverse of the adjoint Lévy transform. Therefore, in the notation of this section, formula (5.3) can be rewritten as

$$\mathbf{x} = \mathcal{L}_i(\mathbf{x}) + \frac{1/\sigma_i}{\Delta_i(1/\sigma_i)}\Delta_i\mathbf{x}. \tag{5.9}$$

Finally, making use of Lemma 5.1, we may formulate the following result.

Proposition 5.4: (i) The Lévy transform $\mathcal{L}_i(\mathbf{x})$ *of the quadrilateral lattice* $\mathbf{x}$ *is given by*

$$\mathcal{L}_i(\mathbf{x}) = \mathbf{x} - \frac{\phi}{\Delta_i\phi}\Delta_i\mathbf{x}, \tag{5.10}$$

where the function $\phi\colon \mathbb{Z}^N \to \mathbb{R}$ *is a solution of the Laplace equation (1.3) of the lattice* $\mathbf{x}$.

(ii) The Lamé coefficients of the new lattice read as

$$\mathcal{L}_i(H_i) = (T_iH_i)\frac{\phi}{\Delta_i\phi}, \quad \mathcal{L}_i(H_j) = H_j - \frac{\phi}{\Delta_i\phi}\Delta_iH_j. \tag{5.11}$$

Formula (5.10), presented in the form coming from the $\bar{\partial}$ approach, was first written in Ref. 16.

The geometric meaning of the function ϕ entering into formula (5.10) can be explained as follows. Given an additional scalar solution ϕ: $\mathbb{Z}^N \to \mathbb{R}$ of the Laplace equation (1.3), we define a new quadrilateral lattice $\tilde{\mathbf{x}}$: $\mathbb{Z}^N \to \mathbb{R}^{M+1}$ as

$$\tilde{\mathbf{x}}: \mathbb{Z}^N \to \begin{pmatrix} \mathbf{x} \\ \phi \end{pmatrix}. \tag{5.12}$$

The point $\mathcal{L}_i(\mathbf{x})$ is the intersection point of the line $\tilde{\mathbf{x}} + t\,\Delta_i \tilde{\mathbf{x}}$ with its projection $\mathbf{x} + t\,\Delta_i \mathbf{x}$ on the $\mathbb{R}^M$ space, therefore for the intersection parameter t_0 we have

$$\begin{pmatrix} \mathbf{x} \\ \phi \end{pmatrix} + t_0 \begin{pmatrix} \Delta_i \mathbf{x} \\ \Delta_i \phi \end{pmatrix} = \begin{pmatrix} \mathcal{L}_i(\mathbf{x}) \\ 0 \end{pmatrix}, \tag{5.13}$$

which implies formula (5.10).

Let us observe that the direction of the transversal vector $\tilde{\mathbf{x}} - \mathbf{x}$ is fixed; this implies that the quadrilaterals with vertices $\mathbf{x}$, $T_i \mathbf{x}$, $\tilde{\mathbf{x}}$, $T_i \tilde{\mathbf{x}}$ are planar. Then both lattices form a quadrilateral strip with an N-dimensional basis and one transversal direction $\mathcal{L}$. The Lévy transformation $\mathcal{L}_i$ of the lattice $\mathbf{x}$ can be interpreted as the Laplace transformation $\mathcal{L}_{i\mathcal{L}}$ of this strip. Therefore the Lévy transformed lattice $\mathcal{L}_i(x)$ is quadrilateral.

As we mentioned in Sec. IV, given a solution ϕ of the Laplace equation (1.3), we have automatically, via the formula (4.11), the solution ν_i of the linear problem (1.6). Therefore we may conclude this section with the following corollary.

Corollary 5.1: To construct a Lévy transform of the lattice $\mathbf{x}$: *(i) find a scalar solution* ν_i *of the linear problem (1.6); i.e.,*

$$\Delta_j \nu_i = (T_j Q_{ij}) \nu_j .$$

(ii) The Lévy transform is then given by

$$\mathcal{L}_i(\mathbf{x}) = \mathbf{x} - \frac{\mathbf{\Omega}[\nu, H]}{\nu_i} \mathbf{X}_i . \tag{5.14}$$

(iii) The Lamé coefficients and the tangent vectors of the new lattice are of the form

$$\mathcal{L}_i(H_i) = \frac{1}{\nu_i} \mathbf{\Omega}[\nu, H], \tag{5.15}$$

$$\mathcal{L}_i(H_j) = H_j - \frac{Q_{ij}}{\nu_i} \mathbf{\Omega}[\nu, H], \tag{5.16}$$

$$\mathcal{L}_i(\mathbf{X}_i) = -\Delta_i \mathbf{X}_i + \frac{\Delta_i \nu_i}{\nu_i} \mathbf{X}_i , \tag{5.17}$$

$$\mathcal{L}_i(\mathbf{X}_j) = \mathbf{X}_j - \frac{\nu_j}{\nu_i} \mathbf{X}_i . \tag{5.18}$$

VI. RADIAL TRANSFORMATIONS

Given a quadrilateral lattice $\mathbf{x}$ and a point $\mathbf{p} \in \mathbb{R}^M$, consider lines passing through that point and the points of the lattice. The conditions of Definition 2.1 are obviously satisfied. In this way we obtain a special type of congruence, which we call radial congruence. Such congruence is of a very degenerate type—its focal lattices consist of the point $\mathbf{p}$ only.

Without loss of generality we may assume that the point $\mathbf{p}$ is the coordinate center, and we define the radial congruence $\mathfrak{r}(\mathbf{x})$ of $\mathbf{x}$ with respect to that point.

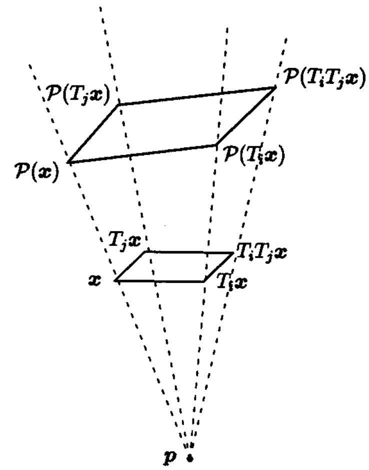

FIG. 11. The radial transformation.

Definition 6.1: The radial (or projective) transform $\mathcal{P}(\mathbf{x})$ *of the quadrilateral lattice* $\mathbf{x}$ *is a quadrilateral lattice conjugate to the radial congruence* $\mathfrak{r}(\mathbf{x})$ *of* $\mathbf{x}$ (see Fig. 11).

Proposition 6.1: (i) The radial transform $\mathcal{P}(\mathbf{x})$ *is given by*

$$\mathcal{P}(\mathbf{x}) = \frac{1}{\phi}\mathbf{x}, \tag{6.1}$$

where ϕ*:* $\mathbb{Z}^N \to \mathbb{R}$ *is a solution of the Laplace equation (1.3) of the lattice* $\mathbf{x}$.

(ii) The Lamé coefficients of the new lattice read as

$$\mathcal{P}(H_i) = \frac{H_i}{\phi}. \tag{6.2}$$

Proof: We first notice that the transformed lattice should consist of the points of the form given by (6.1), where ϕ must be such that the new lattice is quadrilateral. For an arbitrary ϕ the new lattice $\tilde{\mathbf{x}} = (1/\phi)\mathbf{x}$ satisfies the equation

$$\Delta_i \Delta_j \tilde{\mathbf{x}} = (T_i \tilde{A}_{ij}) \Delta_i \tilde{\mathbf{x}} + (T_j \tilde{A}_{ji}) \Delta_j \tilde{\mathbf{x}} + \tilde{C}_{ij} \tilde{\mathbf{x}}, \quad i \neq j, \tag{6.3}$$

with the coefficients

$$\tilde{A}_{ij} = (T_j \phi)^{-1} (A_{ij}\phi - \Delta_j \phi) = \frac{\Delta_i (H_i/\phi)}{H_i/\phi}, \quad i \neq j, \tag{6.4}$$

$$\tilde{C}_{ij} = (T_i T_j \phi)^{-1} (-\Delta_i \Delta_j \phi + (T_i A_{ij}) \Delta_i \phi + (T_j A_{ji}) \Delta_j \phi). \tag{6.5}$$

Formula (6.5) precises the form of ϕ, whereas (6.4) implies the form of the new Lamé coefficients. □

VII. FUNDAMENTAL TRANSFORMATIONS OF THE MQL

The transformations studied in this section were introduced, in the continuous context, by Jonas[39] as the most general transformations of conjugate nets on a surface satisfying the permutability property. Eisenhart, who discovered these transformations independently, but a little bit later, called them fundamental transformations.[3] The content of Proposition 7.1 and Corollary 7.1 can also be found in Ref. 22.

A. Fundamental transformations

In the previous sections we considered transformations between multidimensional quadrilateral lattices conjugate to the same congruence. We studied four particular cases: (1) both lattices are focal lattices of the congruence (Laplace transformation); (2) one of the lattices is a focal lattice (Lévy transformation and its adjoint); (3) parallel lattices (Combescure transformation); and (4) lattices conjugate to a radial congruence (radial transformation).

In this section we study the most general transformation between multidimensional quadrilateral lattices conjugate to the same congruence, which contains the above ones as particular reductions.

Definition 7.1: Two quadrilateral lattices are related by the fundamental transformation when they are conjugate to the same congruence, which is called the congruence of the transformation.

Consider a generic case, when the congruence of the transformation can be constructed via a Combescure transformation vector $\mathbf{x}_{\mathcal{C}}$ of the lattice $\mathbf{x}$. Since the same congruence should be constructed also via a Combescure transformation vector $\mathcal{F}(\mathbf{x})_{\mathcal{C}}$ of the lattice $\mathcal{F}(\mathbf{x})$, we have

$$\mathcal{F}(\mathbf{x})_{\mathcal{C}}=\frac{1}{\theta}\mathbf{x}_{\mathcal{C}}; \tag{7.1}$$

i.e., both vectors are related by a radial transformation, where, by Proposition 6.1, the function θ satisfies the point equation of the lattice $\mathbf{x}_{\mathcal{C}}$.

The transformed lattice $\mathcal{F}(\mathbf{x})$ is therefore necessarily of the form

$$\mathcal{F}(\mathbf{x})=\mathbf{x}-\phi\mathcal{F}(\mathbf{x})_{\mathcal{C}}=\mathbf{x}-\frac{\phi}{\theta}\mathbf{x}_{\mathcal{C}}, \tag{7.2}$$

where the function ϕ is to be determined.

The first derivatives of $\mathcal{F}(\mathbf{x})$ are reducible, due to Eqs. (4.3), (7.1), and (7.2), to the form

$$\Delta_i\mathcal{F}(\mathbf{x})=\left(\frac{T_i\theta}{T_i\sigma_i}-T_i\phi\right)\Delta_i\mathcal{F}(\mathbf{x})_{\mathcal{C}}+\left(\frac{\Delta_i\theta}{T_i\sigma_i}-\Delta_i\phi\right)\mathcal{F}(\mathbf{x})_{\mathcal{C}}. \tag{7.3}$$

From these expressions it follows that $\mathcal{F}(\mathbf{x})_{\mathcal{C}}$ is a Combescure transformation vector of $\mathcal{F}(\mathbf{x})$ if and only if θ and ϕ satisfy

$$\Delta_i\theta=(T_i\sigma_i)\Delta_i\phi. \tag{7.4}$$

The above equations imply that ϕ is a solution of the point equation of the lattice $\mathbf{x}$, whereas $\theta=\phi_{\mathcal{C}}$ is the Combescure transformed function of ϕ.

Proposition 7.1: (i) The fundamental transform $\mathcal{F}(\mathbf{x})$ of the quadrilateral lattice $\mathbf{x}$ is given by

$$\mathcal{F}(\mathbf{x})=\mathbf{x}-\frac{\phi}{\phi_{\mathcal{C}}}\mathbf{x}_{\mathcal{C}}, \tag{7.5}$$

where (i) ϕ: $\mathbb{Z}^N\to\mathbb{R}$ is a solution of the Laplace equation (1.3) of the lattice $\mathbf{x}$, (ii) $\mathbf{x}_{\mathcal{C}}$ is the vector of the Combescure transformation of $\mathbf{x}$, and *(iii) $\phi_{\mathcal{C}}$: $\mathbb{Z}^N\to\mathbb{R}$ is the corresponding Combescure transformed function of ϕ.*

Corollary 7.1: In the notation of Theorem 1.1, the fundamental transformation can be written in the form

$$\mathcal{F}(\mathbf{x})=\mathbf{x}-\mathbf{\Omega}[\mathbf{X},\nu^*]\frac{\mathbf{\Omega}[\nu,H]}{\mathbf{\Omega}[\nu,\nu^*]}, \tag{7.6}$$

where ν_i, and ν_i^*, $i=1,...,N$, are solutions of the linear problem (1.6) and its adjoint (1.8). The Lamé coefficients and the tangent vectors are transformed in the following way:

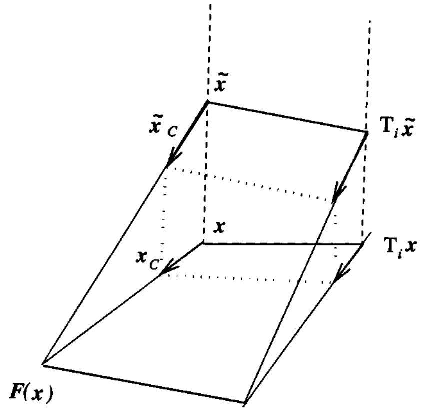

FIG. 12. Construction of the fundamental transformation.

$$\mathcal{F}(H_i)=H_i-v_i^*\frac{\mathbf{\Omega}[v,H]}{\mathbf{\Omega}[v,v^*]}, \tag{7.7}$$

$$\mathcal{F}(\mathbf{X}_i)=\mathbf{X}_i-\mathbf{\Omega}[\mathbf{X},v^*]\frac{v_i}{\mathbf{\Omega}[v,v^*]}, \tag{7.8}$$

and the corresponding transformation of the fields Q_{ij} *reads as*

$$\mathcal{F}(Q_{ij})=Q_{ij}-\frac{v_j^*v_i}{\mathbf{\Omega}[v,v^*]}. \tag{7.9}$$

The geometric meaning of the formula (7.5) can be explained as follows. Given an additional scalar solution ϕ: $\mathbb{Z}^N\to\mathbb{R}$ of the Laplace equation (1.3), we define, like in the case of the Levy transformation, a new quadrilateral lattice $\tilde{\mathbf{x}}$: $\mathbb{Z}^N\to\mathbb{R}^{M+1}$ as

$$\tilde{\mathbf{x}}:\mathbb{Z}^N\to\begin{pmatrix}\mathbf{x}\\ \phi\end{pmatrix}. \tag{7.10}$$

We construct then a Combescure transform of the lattice $\tilde{\mathbf{x}}$; i.e., we find the corresponding vector $\tilde{\mathbf{x}}_{\mathcal{C}}$,

$$\tilde{\mathbf{x}}_{\mathcal{C}}=\mathbf{\Omega}\left[\begin{pmatrix}\mathbf{x}\\ v\end{pmatrix},v^*\right]=\begin{pmatrix}\mathbf{\Omega}[\mathbf{X},v^*]\\ \mathbf{\Omega}[v,v^*]\end{pmatrix}=\begin{pmatrix}\mathbf{x}_{\mathcal{C}}\\ \phi_{\mathcal{C}}\end{pmatrix}. \tag{7.11}$$

The point $\mathcal{F}(\mathbf{x})$ is the intersection point of the line $\tilde{\mathbf{x}}+t\tilde{\mathbf{x}}_{\mathcal{C}}$ with its projection $\mathbf{x}+t\mathbf{x}_{\mathcal{C}}$ on the $\mathbb{R}^M$ space; therefore, for the intersection parameter t_0 we have

$$\begin{pmatrix}\mathbf{x}\\ \phi\end{pmatrix}+t_0\begin{pmatrix}\mathbf{x}_{\mathcal{C}}\\ \phi_{\mathcal{C}}\end{pmatrix}=\begin{pmatrix}\mathcal{F}(\mathbf{x})\\ 0\end{pmatrix}. \tag{7.12}$$

Let us observe that the quadrilaterals with vertices $\mathbf{x},\tilde{\mathbf{x}},\mathbf{x}+\mathbf{x}_{\mathcal{C}},\tilde{\mathbf{x}}+\tilde{\mathbf{x}}_{\mathcal{C}}$ are planar. All the lattices form a quadrilateral strip with the N-dimensional basis and two transversal directions $\mathcal{L}$ and $\mathcal{C}$. The fundamental transformation $\mathcal{F}$ of the lattice $\mathbf{x}$ can be interpreted as the Laplace transformation $\mathcal{L}_{\mathcal{C}\mathcal{L}}$ of the strip; see Fig. 12. Therefore the new lattice $\mathcal{F}(\mathbf{x})$ is quadrilateral.

Given a quadrilateral lattice $\mathbf{x}$ and its fundamental transform $\mathcal{F}(\mathbf{x})$ conjugate to the congruence I, we are automatically given also N focal lattices $\mathbf{y}_i$ of the congruence. Obviously, $\mathbf{y}_i$ is the ith adjoint Lévy transform of both lattices $\mathbf{x}$ and $\mathcal{F}(\mathbf{x})$; moreover, the lattices $\mathbf{x}$ and $\mathcal{F}(\mathbf{x})$ are two

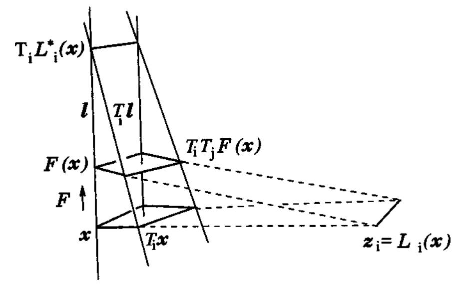

FIG. 13. The fundamental transformation as a binary transformation.

different ith Lévy transforms of $\mathbf{y}_i$. This implies that the fundamental transformation can be considered as the superposition of an adjoint Lévy and a Lévy transformations.

Corollary 7.2: In order to construct a fundamental transform $\mathcal{F}(\mathbf{x})$ *of the quadrilateral lattice* $\mathbf{x}$ *we may proceed in the following way:*
(i) construct a congruence $\mathfrak{l}$ *conjugate to* $\mathbf{x}$;
(ii) find the ith *focal lattice* $\mathbf{y}_i = \mathcal{L}_i^*(\mathbf{x})$ *of the congruence* $\mathfrak{l}$;
(iii) construct its ith *Lévy transform,*

$$\mathcal{L}_i(\mathbf{y}_i) = \mathcal{L}_i(\mathbf{L}_i^*(\mathbf{x})) = \mathcal{F}(\mathbf{x}). \tag{7.13}$$

Let us observe also that the transformation $\mathcal{F}(\mathbf{x})$ builds, from the lattice $\mathbf{x}$, a quadrilateral strip with basis $\mathbf{x}$ and transversal direction $\mathcal{F}$. If we define the lattice $\mathbf{z}_i$ as the $\mathcal{L}_{i\mathcal{F}}$th Laplace transform of this strip, then $\mathbf{z}_i$ is the ith Lévy transform of both lattices $\mathbf{x}$ and $\mathcal{F}(\mathbf{x})$, while the lattices $\mathbf{x}$ and $\mathcal{F}(\mathbf{x})$ are different ith adjoint Lévy transforms of $\mathbf{y}_i$. This observation, together with Proposition 5.3, provides a third way to construct the fundamental transform $\mathcal{F}(\mathbf{x})$ (see Fig. 13).

Corollary 7.3: In order to construct a fundamental transform $\mathcal{F}(\mathbf{x})$ *of the quadrilateral lattice* $\mathbf{x}$, *we may proceed in the following way:*
(i) we find the ith *Lévy transform* $\mathbf{z}_i = \mathcal{L}_i(\mathbf{x})$ *of* $\mathbf{x}$;
(ii) we construct a congruence conjugate to $\mathbf{z}_i$;
(iii) we find the ith *focal lattice of the congruence,*

$$\mathcal{L}_i^*(\mathbf{z}_i) = \mathcal{L}_i^*(\mathcal{L}_i(\mathbf{x})) = \mathcal{F}(\mathbf{x}). \tag{7.14}$$

Remark: The fundamental transformation, superposition of Lévy, and adjoint Lévy transformations, is usually called, in the soliton theory, *binary Darboux transformation*.[36–38]

We end this section remarking that, from the previous observations, it is possible to interpret the transformation $\mathbf{x} \to \mathcal{F}(\mathbf{x})$ as a generic addition of a new dimension [the $(N+1)$st] to the original lattice $\mathbf{x}$. We will discuss this interesting aspect of the fundamental transformations in Sec. IX.

B. Superposition of fundamental transformations

In this section we consider vectorial fundamental transformations, which are nothing else but superpositions of the fundamental transformations. Generalizing the procedure of the previous section, we consider $K \geqslant 1$ solutions ϕ^k, $k = 1,...,K$ of the Laplace equation of the lattice $\mathbf{x}$, which we arrange in the K component vector $\phi = (\phi^1, \ldots, \phi^K)^t$; this allows us to introduce the quadrilateral lattice $\tilde{\mathbf{x}} = \binom{\mathbf{x}}{\phi}$ in the space $\mathbb{R}^{M+K}$. We also consider K Combescure transformation vectors $\mathbf{x}_{\mathcal{C},k}$; also, the Combescure transformation vectors $\mathbf{x}_{\mathcal{C},k}$ can be extended (this procedure involves K arbitrary constants) to the Combescure transformations vectors $\tilde{\mathbf{x}}_{\mathcal{C},k} = \binom{\mathbf{x}_{\mathcal{C},k}}{\phi_{\mathcal{C},k}}$ of the lattice $\tilde{\mathbf{x}}$, where the K component vector $\phi_{\mathcal{C},k} = (\phi^1_{\mathcal{C},k}, \ldots, \phi^K_{\mathcal{C},k})^t$ consists of the Combescure transformed functions $\phi^l_{\mathcal{C},k}$ of ϕ^l; each of the vectors $\tilde{\mathbf{x}}_{\mathcal{C},k}$ defines a Combescure transform of the lattice $\tilde{\mathbf{x}}$. The K vectors $\tilde{\mathbf{x}}_{\mathcal{C},k}$ define the K-dimensional subspace,

$$\tilde{\mathbf{x}}+\sum_{k=1}^{K}\tilde{\mathbf{x}}_{\mathcal{C},k}t^k=\tilde{\mathbf{x}}+(\tilde{\mathbf{x}}_{\mathcal{C},1},\ldots,\tilde{\mathbf{x}}_{\mathcal{C},N})\begin{pmatrix}t^1\\ \vdots\\ t^K\end{pmatrix}=\tilde{\mathbf{x}}+\tilde{\mathbf{x}}_{\mathcal{C}}t=\begin{pmatrix}\mathbf{x}\\ \boldsymbol{\phi}\end{pmatrix}+\begin{pmatrix}\mathbf{x}_{\mathcal{C}}\\ \boldsymbol{\phi}_{\mathcal{C}}\end{pmatrix}\mathbf{t}. \tag{7.15}$$

The intersection point of this subspace with $\mathbb{R}^M$ [in general, a K-dimensional and an M-dimensional subspaces of the $(M+K)$-dimensional space intersect in a single point] defines the new lattice $\mathcal{T}(\mathbf{x})$

$$\begin{pmatrix}\mathcal{T}(\mathbf{x})\\ \mathbf{0}\end{pmatrix}=\begin{pmatrix}\mathbf{x}\\ \boldsymbol{\phi}\end{pmatrix}+\begin{pmatrix}\mathbf{x}_{\mathcal{C}}\\ \boldsymbol{\phi}_{\mathcal{C}}\end{pmatrix}\mathbf{t}_0. \tag{7.16}$$

The corresponding values of the parameters t_0^k can be found from the lower part of the above equation,

$$\mathbf{0}=\boldsymbol{\phi}+\boldsymbol{\phi}_{\mathcal{C}}\mathbf{t}_0, \tag{7.17}$$

and then inserted into the upper part, giving

$$\mathcal{T}(\mathbf{x})=\mathbf{x}-\mathbf{x}_{\mathcal{C}}\boldsymbol{\phi}_{\mathcal{C}}^{-1}\boldsymbol{\phi}. \tag{7.18}$$

In the notation of Theorem 1.1, we have

$$\boldsymbol{\phi}=\boldsymbol{\Omega}[\mathbf{v},H],\quad \boldsymbol{\phi}_{\mathcal{C}}=\boldsymbol{\Omega}[\mathbf{v},\mathbf{v}^*],\quad \mathbf{x}_{\mathcal{C}}=\boldsymbol{\Omega}[\mathbf{X},\mathbf{v}^*]$$

and

$$\mathcal{T}(\mathbf{x})=\Omega[\mathbf{X},H]-\boldsymbol{\Omega}[\mathbf{X},\mathbf{v}^*]\boldsymbol{\Omega}[\mathbf{v},\mathbf{v}^*]^{-1}\boldsymbol{\Omega}[\mathbf{v},H]. \tag{7.19}$$

One can prove that the new lattice $\mathcal{T}(\mathbf{x})$ is also a quadrilateral one. This is a consequence of Theorem 1.1 and the proof can be found in Sec. IX. In that section it will also be shown that the vectorial fundamental transformation is the superposition of K fundamental transformations.

In this section we consider only the simplest case $K=2$, emphasizing the geometric meaning of all the steps involved in the construction.

Proposition 7.2: (i) The two component vectorial fundamental transformation is equivalent to the superposition of two fundamental transformations:
(1) the transformation $\mathcal{F}_1$ of the lattice $\mathbf{x}$, with parameters ϕ^1 and $\mathbf{x}_{\mathcal{C},1}$:

$$\mathcal{F}_1(\mathbf{x})=\mathbf{x}-\frac{\phi^1}{\phi^1_{\mathcal{C},1}}\mathbf{x}_{\mathcal{C},1}; \tag{7.20}$$

(2) the transformation $\mathcal{F}_2$ of the lattice $\mathcal{F}_1(\mathbf{x})$ with parameters $\phi^{2\prime}$, $\mathbf{x}'_{\mathcal{C},2}$:

$$\mathcal{T}(\mathbf{x})=\mathcal{F}_2(\mathcal{F}_1(\mathbf{x}))=\mathcal{F}_1(\mathbf{x})-\frac{\phi^{2\prime}}{\phi^{2\prime}_{\mathcal{C},2}}\mathbf{x}'_{\mathcal{C},2}, \tag{7.21}$$

where $\phi^{2\prime}$, $\mathbf{x}'_{\mathcal{C},2}$ are nothing but the parameters ϕ^2 and $\mathbf{x}_{\mathcal{C},2}$ transformed by the first transformation,

$$\phi^{2\prime}=\mathcal{F}_1(\phi^2)=\phi^2-\frac{\phi^1}{\phi^1_{\mathcal{C},1}}\phi^2_{\mathcal{C},1},$$

$$\mathbf{x}'_{\mathcal{C},2}=\mathcal{F}_1(\mathbf{x}_{\mathcal{C},2})=\mathbf{x}_{\mathcal{C},2}-\frac{\phi^1_{\mathcal{C},2}}{\phi^1_{\mathcal{C},1}}\mathbf{x}_{\mathcal{C},1},$$

and, correspondingly,

$$\phi^{2\prime}_{\mathcal{C},2}=\mathcal{F}_1(\phi^2_{\mathcal{C},2})=\phi^2_{\mathcal{C},2}-\frac{\phi^1_{\mathcal{C},2}}{\phi^1_{\mathcal{C},1}}\phi^2_{\mathcal{C},1}. \tag{7.22}$$

(ii) The result of the superposition of $\mathcal{F}_1$ and $\mathcal{F}_2$ is independent of the order.

Proof: The proof is by direct calculation; we only remark that, by construction, $\phi^{2\prime}$ is a solution of the Laplace equation of the lattice $\mathcal{F}_1(\mathbf{x})$, and $\mathbf{x}'_{\mathcal{C},2}$ is a vector of the Combescure transformation of the same lattice.

One can look at the above superposition of the fundamental transformations as follows.

(a) The fundamental transformation of the lattice

$$\begin{pmatrix} \mathbf{x} \\ 0 \\ \phi^2 \end{pmatrix}$$

using the solution ϕ^1 of the Laplace equation and the Combescure transformation vector

$$\begin{pmatrix} \mathbf{x}_{\mathcal{C},1} \\ 0 \\ \phi^2_{\mathcal{C},1} \end{pmatrix},$$

which gives

$$\begin{pmatrix} \mathcal{F}_1(\mathbf{x}) \\ 0 \\ \phi^{2\prime} \end{pmatrix}.$$

(b) The simultaneous transformation of the Combescure vector

$$\begin{pmatrix} \mathbf{x}_{\mathcal{C},2} \\ 0 \\ \phi^2_{\mathcal{C},2} \end{pmatrix},$$

which gives the Combescure transformation vector

$$\begin{pmatrix} \mathbf{x}'_{\mathcal{C},2} \\ 0 \\ \phi^2_{\mathcal{C},2} \end{pmatrix}$$

of the lattice obtained in point (a).

(c) The combination of the lattice in $\mathbb{R}^{M+1}$ constructed in point (a) with the Combescure transformation vector constructed in point (b) gives the lattice $\mathcal{T}(\mathbf{x})$ in $\mathbb{R}^M$. □

Corollary 7.4: The points $\mathbf{x}$, $\mathcal{F}_1(\mathbf{x})$, $\mathcal{F}_2(\mathbf{x})$, and $\mathcal{T}(\mathbf{x})=\mathcal{F}_1(\mathcal{F}_2(\mathbf{x}))=\mathcal{F}_2(\mathcal{F}_1(\mathbf{x}))$ are coplanar.

VIII. ARE THE FUNDAMENTAL TRANSFORMATIONS REALLY FUNDAMENTAL?

The main goal of this section is to show explicitly that all the transformations discussed in the previous sections are special cases of the fundamental transformations. Since focal lattices can be viewed as limiting cases of generic lattices conjugate to the congruence, this statement is rather obvious, from a geometrical point of view. Nevertheless, due to the fact that the Combescure transformation vector $\mathbf{x}_{\mathcal{C}}$ is not suited well to describe tangent congruences, the consequent subtleties associated with the analytic limits require a detailed study.

A. Reduction to the Combescure and radial transformations

We first illustrate the straightforward reduction from the fundamental transformations to the Combescure and radial transformations.

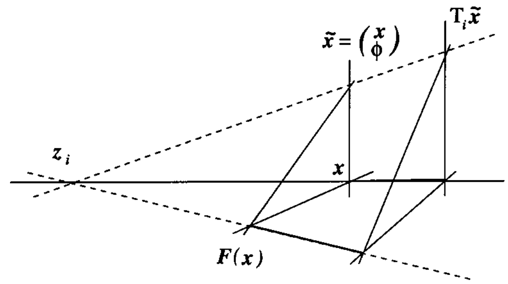

FIG. 14. The singular limit from the fundamental transformation to the adjoint Lévy transformation.

To obtain the Combescure transformation from the fundamental one we put $\nu_i=0$, $i=1,...,N$, in Corollary 7.1. This implies that both ϕ and $\phi_{\mathcal{C}}$ are constants. The constant $\phi/\phi_{\mathcal{C}}$ can always be absorbed by the corresponding rescaling of ν_i^*.

In looking for the reduction of the fundamental transformation to the radial one, we may notice that, in the radial transformation, the Combescure vector $\mathbf{x}_{\mathcal{C}}$ of the congruence must be proportional to the lattice vector $\mathbf{x}$. This gives $\nu_i^*=H_i$, $\mathbf{x}_{\mathcal{C}}=\mathbf{x}$ and, therefore, $\phi_{\mathcal{C}}$ is a solution of the Laplace equation of the points of the lattice $\mathbf{x}$. This implies that $\phi_{\mathcal{C}}-\phi$ must be a constant c:

$$\mathcal{F}(\mathbf{x})\to\frac{c}{\phi+c}\mathbf{x}, \tag{8.1}$$

and this formula is obviously equivalent to formula (6.1).

B. Singular limit to the adjoint Lévy transformation

From Secs. V A and VII, it follows that the adjoint Lévy transformation $\mathcal{L}_i^*(\mathbf{x})$ can be viewed as the limiting case of the fundamental transformation $\mathcal{F}(\mathbf{x})$ in which the transformed lattice becomes the ith focal lattice of the associated congruence.

As it was shown in Sec. VII, the construction of $\mathcal{F}(\mathbf{x})$ is the following sequence of three geometric processes: (i) the extension of the lattice $\mathbf{x}\subset\mathbb{R}^M$ to the lattice $\binom{\mathbf{x}}{\phi}\subset\mathbb{R}^{M+1}$; (ii) the Combescure transformation,

$$\mathcal{C}\begin{pmatrix}\mathbf{x}\\ \phi\end{pmatrix}=\begin{pmatrix}\mathbf{x}+\mathbf{x}_{\mathcal{C}}\\ \phi+\phi_{\mathcal{C}}\end{pmatrix},$$

which gives the quadrilateral strip with N-dimensional basis $\mathbf{x}$ and two transversal directions, called $\mathcal{L}$ and $\mathcal{C}$; and (iii) the Laplace transformation $\mathcal{L}_{\mathcal{CL}}$ of the strip (see Fig. 14).

In order to investigate the nature of the limit $\mathcal{F}(\mathbf{x})\to\mathcal{L}_i^*(\mathbf{x})$, it is convenient to study the properties of ϕ when $\mathbf{x}$ and $\mathcal{F}(\mathbf{x})$ are given. If ϕ is given in the initial point, then $T_i\phi$ is obtained from the intersection point $\binom{T_i\mathbf{x}}{T_i\phi}$ of the line passing through $\binom{T_i\mathbf{x}}{0}$ in the $(M+1)$th direction with the line passing through the points $\binom{\mathbf{z}_i}{0}$ and $\binom{\mathbf{x}}{\phi}$, where $\mathbf{z}_i\in\mathbb{R}^M$ was defined in Sec. VII as the intersection of the ith tangent line of the lattice $\mathbf{x}$ with the corresponding tangent line of $\mathcal{F}(\mathbf{x})$.

By construction, the vector $\mathbf{x}-\mathbf{z}_i$ is proportional to $\Delta_i\mathbf{x}$:

$$\mathbf{x}-\mathbf{z}_i=\nu\,\Delta_i\mathbf{x},\quad \nu\in\mathbb{R}; \tag{8.2}$$

consequently,

$$T_i\phi=\frac{1+\nu}{\nu}\phi. \tag{8.3}$$

In the limit in which $T_i\mathcal{F}(\mathbf{x})\to T_i\mathcal{L}_i^*(\mathbf{x})$ we have also $\mathbf{z}_i\to\mathbf{x}$ and $\nu\to0$. Therefore

$$T_i \phi \simeq \nu^{-1} \phi, \quad |\nu| \ll 1. \tag{8.4}$$

We remark that, in formula (8.4), the lattice function ν in the uniform limit $\mathcal{F}(\mathbf{x}) \to \mathcal{L}_i^*(\mathbf{x})$ is of order ϵ, $|\epsilon| \ll 1$. This suggests the following ansatz for the asymptotics of ϕ:

$$\phi = \epsilon^{-n_i} \alpha(1 + O(\epsilon)); \tag{8.5}$$

substituting (8.5) into the Laplace equations (1.9), we obtain

$$\Delta_j \alpha = \frac{\Delta_j H_i}{H_i} \alpha, \quad \Delta_j \Delta_k \alpha = \frac{\Delta_j H_k}{H_k} \Delta_j \alpha + \frac{\Delta_k H_j}{H_j} \Delta_k \alpha, \quad i \neq j \neq k \neq i,$$

which imply that $\alpha = H_i$. From similar considerations we also obtain that

$$\phi_{\mathcal{C}} = \epsilon^{-n_i} v_i^* (1 + O(\epsilon)); \tag{8.6}$$

for completeness we also write down the asymptotics of v_i:

$$v_i = \epsilon^{-n_i} (\epsilon^{-1} + Q_{ii} + O(\epsilon)),$$

$$v_j = \epsilon^{-n_i} Q_{ji} (1 + O(\epsilon)),$$

where

$$\Delta_j Q_{ii} = (T_j Q_{ij}) Q_{ji}.$$

Therefore, in the limit $\epsilon \to 0$, the asymptotics of the lattice points, the Lamé coefficients and the tangent vectors read as

$$\mathcal{F}(\mathbf{x}) = \mathbf{x} - \frac{H_i}{v_i^*} \mathbf{\Omega}[\mathbf{X}, v^*] + O(\epsilon) = \mathcal{L}_i^*(\mathbf{x}) + O(\epsilon),$$

$$\mathcal{F}(\mathbf{X}_i) = -\epsilon^{-1} \frac{\mathbf{\Omega}[\mathbf{X}, v^*]}{v_i^*} + O(1),$$

$$\mathcal{F}(\mathbf{X}_j) = \mathbf{X}_j - \frac{\mathbf{\Omega}[\mathbf{X}, v^*]}{v_i^*} Q_{ji} + O(\epsilon),$$

$$\mathcal{F}(H_i) = \epsilon T_i^{-1} \left(v_i^* \Delta_i \left(\frac{H_i}{v_i^*} \right) \right) + O(\epsilon^2),$$

$$\mathcal{F}(H_j) = H_j - \frac{v_j^*}{v_i^*} H_i + O(\epsilon),$$

and agree (up to possible ϵ scalings) with the formulas of Sec. V A.

C. Singular limit to the Lévy transformation

In the limit when the fundamental transformation $\mathcal{F}(\mathbf{x})$ reduces to the Lévy transformation $\mathcal{L}_i(\mathbf{x})$, the congruence of the transformation becomes the ith tangent congruence of the lattice $\mathbf{x}$; i.e., $\mathbf{x}_{\mathcal{C}}$ becomes proportional to $\Delta_i \mathbf{x}$.

On the other hand, iterating Eq. (4.3), we obtain the formal series

$$\mathbf{x}_{\mathcal{C}} = -(T_i \sigma_i) \Delta_i \mathbf{x} - (T_i^2 \sigma_i) T_i \Delta_i \mathbf{x} - \cdots, \tag{8.7}$$

which, in the above limit, becomes asymptotic in some small parameter ϵ. This suggests the following ansatz:

$$\sigma_i(\mathbf{n}) \sim \epsilon^{n_i-1}\beta(\mathbf{n})(1+O(\epsilon)), \tag{8.8}$$

which gives

$$\mathbf{x}_{\mathcal{C}} \sim -\epsilon^{n_i}(T_i\beta)\Delta_i\mathbf{x} = -\epsilon^{n_i}T_i(\beta H_i)\mathbf{X}_i\,. \tag{8.9}$$

Applying the difference operator Δ_j to Eq. (8.9) and using Eqs. (4.3) and (1.7), we infer that

$$\beta = \frac{1}{H_i}, \quad \sigma_j \sim \epsilon^{n_i}\frac{Q_{ij}}{H_j}; \tag{8.10}$$

i.e.,

$$\mathbf{x}_{\mathcal{C}} = -\epsilon^{n_i}(\mathbf{X}_i + O(\epsilon)), \quad \sigma_i = \epsilon^{n_i-1}\left(\frac{1}{H_i} + O(\epsilon)\right), \quad \sigma_j = -\epsilon^{n_i}\left(\frac{Q_{ij}}{H_j} + O(\epsilon)\right),$$

which allow us to calculate the asymptotics of the other relevant objects:

$$v_i^* = \epsilon^{n_i-1}(1-\epsilon Q_{ii} + O(\epsilon)), \quad v_j^* = -\epsilon^{n_i}(Q_{ij} + O(\epsilon)), \quad \phi_{\mathcal{C}} = -\epsilon^{n_i}(v_i + O(\epsilon)).$$

Therefore, in the limit $\epsilon \to 0$, the asymptotics of the lattice points, of the Lamé coefficients and of the tangent vectors, read as

$$\mathcal{F}(\mathbf{x}) = \mathbf{x} - \frac{\phi}{v_i}\mathbf{X}_i + O(\epsilon) = \mathcal{L}_i(\mathbf{x}) + O(\epsilon),$$

$$\mathcal{F}(\mathbf{X}_i) = -\epsilon\left(\Delta_i\mathbf{X}_i - \frac{\Delta_i v_i}{v_i}\mathbf{X}_i\right) + O(\epsilon^2),$$

$$\mathcal{F}(\mathbf{X}_j) = \mathbf{X}_j - \frac{v_j}{v_i}\mathbf{X}_i + O(\epsilon),$$

$$\mathcal{F}(H_i) = \frac{1}{\epsilon}\frac{\phi}{v_i} + O(1),$$

$$\mathcal{F}(H_j) = H_j - \frac{Q_{ij}}{v_i}\phi + O(\epsilon),$$

and agree with the formulas of Sec. V B.

D. Singular limit to the Laplace transformations

The Laplace transformation can be considered as the special limit of the fundamental transformation such that both lattices are focal lattices of the congruence of the transformation. Therefore it can be obtained combining the asymptotics presented in the previous Secs. VIII B and VIII C. The corresponding asymptotics read as follows:

$$\mathcal{F}(\mathbf{x})=\mathbf{x}-\frac{H_j}{Q_{ij}}\mathbf{X}_i+O(\epsilon)=\mathcal{L}_{ij}(\mathbf{x})+O(\epsilon),$$

$$\mathcal{F}(H_i)=-\frac{1}{\epsilon}\frac{H_j}{Q_{ij}}+O(1),$$

$$\mathcal{F}(H_j)=\epsilon T_j^{-1}\left(Q_{ij}\Delta_j\left(\frac{H_j}{Q_{ij}}\right)\right)+O(\epsilon^2),$$

$$\mathcal{F}(H_k)=H_k-\frac{Q_{ik}}{Q_{ij}}H_j+O(\epsilon),$$

$$\mathcal{F}(\mathbf{X}_i)=\epsilon\left(\Delta_i\mathbf{X}_i-\frac{\Delta_i Q_{ij}}{Q_{ij}}\mathbf{X}_i\right)+O(\epsilon^2),$$

$$\mathcal{F}(\mathbf{X}_j)=-\frac{1}{\epsilon}\frac{\mathbf{X}_i}{Q_{ij}}+O(1),$$

$$\mathcal{F}(\mathbf{X}_k)=\mathbf{X}_k-\frac{Q_{kj}}{Q_{ij}}\mathbf{X}_i+O(\epsilon).$$

IX. CONNECTION WITH VECTORIAL DARBOUX TRANSFORMATIONS AND PERMUTABILITY THEOREMS

A. Fundamental transformations from the vectorial formalism

Our main goal in this section is to show that the fundamental transformations and, therefore, all the particular transformations discussed in the previous sections, are special cases of the vectorial transformation described in Theorem 1.1 and introduced in Ref. 27.

Consider the following splitting of the vector space $\mathbf{W}$ of Theorem 1.1:

$$\mathbf{W}=\mathbb{E}\oplus\mathbf{V}\oplus\mathbb{F},\quad \mathbf{W}^*=\mathbb{E}^*\oplus\mathbf{V}^*\oplus\mathbb{F}^*; \tag{9.1}$$

if

$$\mathbf{Y}_i=(\mathbf{X}_i,\mathbf{v}_i,0)^T,\quad \mathbf{Y}_i^*=(0,\mathbf{v}_i^*,\mathbf{X}^*), \tag{9.2}$$

then, the corresponding potential matrix is of the form

$$\mathbf{\Omega}[\mathbf{Y},\mathbf{Y}^*]=\begin{pmatrix}\mathbb{I}_{\mathbb{E}} & \mathbf{\Omega}[\mathbf{X},\mathbf{v}^*] & \mathbf{\Omega}[\mathbf{X},\mathbf{X}^*]\\ 0 & \mathbf{\Omega}[\mathbf{v},\mathbf{v}^*] & \mathbf{\Omega}[\mathbf{v},\mathbf{X}^*]\\ 0 & 0 & \mathbb{I}_{\mathbb{F}}\end{pmatrix}, \tag{9.3}$$

and its inverse is

$$\mathbf{\Omega}[\mathbf{Y},\mathbf{Y}^*]^{-1}=\begin{pmatrix}\mathbb{I}_{\mathbb{E}} & -\mathbf{\Omega}[\mathbf{X},\mathbf{v}^*]\mathbf{\Omega}[\mathbf{v},\mathbf{v}^*]^{-1} & -\mathbf{\Omega}[\mathbf{X},\mathbf{X}^*]+\mathbf{\Omega}[\mathbf{X},\mathbf{v}^*]\mathbf{\Omega}[\mathbf{v},\mathbf{v}^*]^{-1}\mathbf{\Omega}[\mathbf{v},\mathbf{X}^*]\\ 0 & \mathbf{\Omega}[\mathbf{v},\mathbf{v}^*]^{-1} & -\mathbf{\Omega}[\mathbf{v},\mathbf{v}^*]^{-1}\mathbf{\Omega}[\mathbf{v},\mathbf{X}^*]\\ 0 & 0 & \mathbb{I}_{\mathbb{F}}\end{pmatrix}. \tag{9.4}$$

This implies that

$$\hat{\mathbf{Y}}_i=\begin{pmatrix}\hat{\mathbf{X}}_i\\ \hat{\mathbf{v}}_i\\ 0\end{pmatrix}=\begin{pmatrix}\mathbf{X}_i-\boldsymbol{\Omega}[\mathbf{X},\mathbf{v}^*]\boldsymbol{\Omega}[\mathbf{v},\mathbf{v}^*]^{-1}\mathbf{v}_i\\ \boldsymbol{\Omega}[\mathbf{v},\mathbf{v}^*]^{-1}\mathbf{v}_i\\ 0\end{pmatrix},$$

$$\hat{\mathbf{Y}}_i^*=(0,\hat{\mathbf{v}}_i^*,\hat{\mathbf{X}}_i^*)=(0,\mathbf{v}_i^*\boldsymbol{\Omega}[\mathbf{v},\mathbf{v}^*]^{-1},\mathbf{X}_i^*-\mathbf{v}_i^*\boldsymbol{\Omega}[\mathbf{v},\mathbf{v}^*]^{-1}\boldsymbol{\Omega}[\mathbf{v},\mathbf{X}^*])$$

and

$$\hat{Q}_{ij}=Q_{ij}-\mathbf{v}_j^*\boldsymbol{\Omega}[\mathbf{v},\mathbf{v}^*]^{-1}\mathbf{v}_i\,. \tag{9.5}$$

Theorem 1.1 implies, in particular, that, up to a constant operator,

$$\boldsymbol{\Omega}[\hat{\mathbf{X}},\hat{\mathbf{X}}^*]=\boldsymbol{\Omega}[\mathbf{X},\mathbf{X}^*]-\boldsymbol{\Omega}[\mathbf{X},\mathbf{v}^*]\boldsymbol{\Omega}[\mathbf{v},\mathbf{v}^*]^{-1}\boldsymbol{\Omega}[\mathbf{v},\mathbf{X}^*]. \tag{9.6}$$

The fundamental transformation can be obtained in the simplest case, by putting $\mathbb{F}=\mathbb{V}=\mathbb{R}$, $\mathbb{E}=\mathbb{R}^M$, $\mathbf{w}=(0,0,1)^T$ and choosing the projection operator on the space $\mathbb{E}$ along $\mathbb{V}\oplus\mathbb{F}$. Then $\mathbf{X}_i^*=H_i$, the scaled tangent vectors are just $\mathbf{X}_i$ and $\mathbf{x}=\boldsymbol{\Omega}[\mathbf{X},H]$; the transformation data $\mathbf{v}_i$ and $\mathbf{v}_i^*$ are scalar functions. The transformed lattice points and the transformed functions Q_{ij} are given then by formulas (9.6) and (9.5), which coincide with (7.6) and (7.9).

We recall that, in Sec. VII B, the geometric meaning of Eq. (9.6) was given in the case in which $\mathbb{F}=\mathbb{R}$, $\mathbb{V}=\mathbb{R}^K$, $\mathbb{E}=\mathbb{R}^M$.

B. Permutability of the fundamental transformations

Let us assume that the transformation datum space split as $\mathbb{V}=\mathbb{V}_1\oplus\mathbb{V}_2$, so that we write

$$\boldsymbol{\Omega}[\mathbf{v},\mathbf{v}^*]=\begin{pmatrix}m_{11} & m_{12}\\ m_{21} & m_{22}\end{pmatrix},$$

with $m_{ij}=\boldsymbol{\Omega}[\mathbf{v}_{(i)},\mathbf{v}_{(j)}^*]$: $\mathbb{V}_j\to\mathbb{V}_i$. Correspondingly, we have the following decompositions:

$$\begin{gathered}\mathbf{v}_i=\begin{pmatrix}\mathbf{v}_{(1),i}\\ \mathbf{v}_{(2),i}\end{pmatrix},\\ \mathbf{v}_i^*=(\mathbf{v}_{(1),i}^*,\mathbf{v}_{(2),i}^*),\\ \boldsymbol{\Omega}[\mathbf{X},\mathbf{v}^*]=(M_{(1)},M_{(2)}),\quad M_{(i)}=\boldsymbol{\Omega}[\mathbf{X},\mathbf{v}_{(i)}^*],\\ \boldsymbol{\Omega}[\mathbf{v},\mathbf{X}^*]=\begin{pmatrix}M_{(1)}^*\\ M_{(2)}^*\end{pmatrix},\quad M_{(i)}^*=\boldsymbol{\Omega}[\mathbf{v}_{(i)},\mathbf{X}^*].\end{gathered} \tag{9.7}$$

If $m_{22}\in \mathrm{GL}(\mathbf{V}_2)$, we have the factorizations

$$\begin{aligned}\boldsymbol{\Omega}[\mathbf{v},\mathbf{v}^*]&=\begin{pmatrix}1 & m_{12}m_{22}^{-1}\\ 0 & 1\end{pmatrix}\begin{pmatrix}m_{11}-m_{12}m_{22}^{-1}m_{21} & 0\\ m_{21} & m_{22}\end{pmatrix}\\ &=\begin{pmatrix}m_{11}-m_{12}m_{22}^{-1}m_{21} & m_{12}\\ 0 & m_{22}\end{pmatrix}\begin{pmatrix}1 & 0\\ m_{22}^{-1}m_{21} & 1\end{pmatrix}.\end{aligned} \tag{9.8}$$

Using the formulas (9.7) and (9.8), together with (9.6), we obtain

$$\begin{aligned}\hat{Q}_{ij}=Q_{ij}&-\langle\mathbf{v}_{(2),j}^*,m_{22}^{-1}\mathbf{v}_{(2),i}\rangle\\ &-\langle\mathbf{v}_{(1),j}^*-\mathbf{v}_{(2),j}^*m_{22}^{-1}m_{21},\quad (m_{11}-m_{12}m_{22}^{-1}m_{21})^{-1}(\mathbf{v}_{(1),j}-m_{12}m_{22}^{-1}\mathbf{v}_{(2),j})\rangle,\end{aligned}$$

$$\hat{\mathbf{X}}_i=\mathbf{X}_i-M_{(2)}m_{22}^{-1}\mathbf{v}_{(2),i}-(M_{(1)}-M_{(2)}m_{22}^{-1}m_{21})(m_{11}-m_{12}m_{22}^{-1}m_{21})^{-1}(\mathbf{v}_{(1),i}-m_{12}m_{22}^{-1}\mathbf{v}_{(2),i}),$$

$$\hat{\mathbf{X}}_i^*=\mathbf{X}_i^*-\mathbf{v}_{(2),i}^*m_{22}^{-1}M_{(2)}^*-(\mathbf{v}_{(1),i}-m_{12}m_{22}^{-1}\mathbf{v}_{(2),i})(m_{11}-m_{12}m_{22}^{-1}m_{21})^{-1}(M_{(1)}^*-m_{22}^{-1}m_{21}M_{(2)}^*). \tag{9.9}$$

As we shall see, these formulas coincide with those coming from performing first a fundamental transformation with the transformation data $(\mathbb{V}_2,\mathbf{v}_{(2)},\mathbf{v}_{(2)}^*)$:

$$Q_{ij}'=Q_{ij}-\langle\mathbf{v}_{(2),j}^*,m_{22}^{-1}\mathbf{v}_{(2),i}\rangle,$$

$$\mathbf{X}_i'=\mathbf{X}_i-M_{(2)}m_{22}^{-1}\mathbf{v}_{(2),i},$$

$$(\mathbf{X}_i^*)'=\mathbf{X}_i^*-\mathbf{v}_{(2),i}^*m_{22}^{-1}M_{(2)}^*,$$

and then transforming with the data $(\mathbb{V}_1,\mathbf{v}_{(1)}',\mathbf{v}_{(1)}^*)')$, where $\mathbf{v}_{(1)}'$, $\mathbf{v}_{(1)}^{*\prime}$ are the data $\mathbf{v}_{(1)}$, $\mathbf{v}_{(1)}^*$ after the first fundamental transform indicated by $'$. Therefore the resulting functions are

$$Q_{ij}''=Q_{ij}'-\langle(\mathbf{v}_{(1),j}^*)',M(\mathbf{v}',(\mathbf{v}^*)')^{-1}\mathbf{v}_{(1),i}'\rangle,$$

$$\mathbf{X}_i''=\mathbf{X}_i'-M(\mathbf{X}',(\mathbf{v}^*)')M(\mathbf{v}',(\mathbf{v}^*)')^{-1}\mathbf{v}',$$

$$(\mathbf{X}_i^*)''=(\mathbf{X}_i^*)'-(\mathbf{v}_{(1),i}^*)'M(\mathbf{v}',(\mathbf{v}^*)')^{-1}M(\mathbf{v}',(\mathbf{X}^*)').$$

To show this, it is important to use the relations (9.6) to realize that

$$\boldsymbol{\Omega}(\mathbf{X}',(\mathbf{v}^*)')=M_{(1)}-M_{(2)}m_{22}^{-1}m_{21},$$

$$\boldsymbol{\Omega}(\mathbf{v}',(\mathbf{X}^*)')=M_{(1)}^*-m_{(12)}m_{22}^{-1}M_{(2)}^*,$$

$$\boldsymbol{\Omega}(\mathbf{v}',(\mathbf{v}^*)')=m_{11}-m_{12}m_{22}^{-1}m_{21},$$

so that the above equations for the second fundamental transformation are just (9.9):

$$Q_{ij}''=\hat{Q}_{ij},$$

$$\mathbf{X}_i''=\hat{\mathbf{X}}_i,$$

$$(\mathbf{X}_i^*)''=\hat{\mathbf{X}}_i^*.$$

Proposition 9.1: The vectorial Darboux transformation (9.9) with the transformation data $(\mathbb{V}_1\oplus\mathbb{V}_2,\binom{\mathbf{v}_{(1)}}{\mathbf{v}_{(2)}},(\mathbf{v}_{(1)}^*,\mathbf{v}_{(2)}^*))$ *coincides with the following composition of fundamental transformations: (1) First transform with data* $(\mathbb{V}_2,\mathbf{v}_{(2)},\mathbf{v}_{(2)}^*)$, *and denote the transformation by* $'$. *(2) On the result of this transformation apply a second one with data* $(\mathbb{V}_1,\mathbf{v}_{(1)}',(\mathbf{v}_{(1)}^*)')$.

Corollary 9.1: Assuming that $m_{11}\in\mathrm{GL}(\mathbb{V}_1)$ *and following the above steps, it is easy to show that this composition does not depend on the order of the two transformations.*

Corollary 9.2: Applying the mathematical induction to Proposition 9.1, it is possible to show that, assuming a general splitting $\mathbb{V}=\oplus_{i=1}^K\mathbb{V}_i$ *of the transformation space, the final result does not depend on the order in which the K transformations are made.*

C. Fundamental transformations as integrable discretization

In Sec. VII B we have observed that the fundamental transformation $\mathcal{F}$ can be interpreted as generating a new dimension [the $(N+1)$st] of the lattice $\mathbf{x}$; more precisely, a single fundamental transformation can be interpreted as an elementary translation in this new dimension. Moreover,

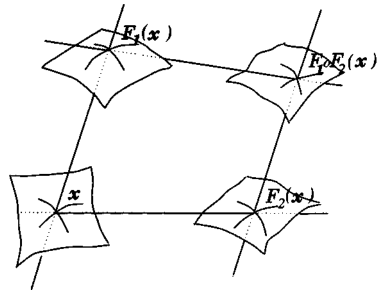

FIG. 15. Superposition of fundamental transformations of conjugate nets.

the Combescure vector $\mathbf{x}_{\mathcal{C}}$ of the transformation can be viewed as the corresponding normalized tangent vector $\mathbf{X}_{N+1}$. Obviously, in order to have an $(N+1)$-dimensional quadrilateral lattice, we have to apply recursively fundamental transformations.

The application of two fundamental transformations $\mathcal{F}_1$ and $\mathcal{F}_2$ to the quadrilateral lattice $\mathbf{x}$ can be viewed as one step in the generation of two new dimensions; the permutability theorem (Proposition 7.2) guarantees that these translations commute. Moreover, the elementary quadrilateral,

$$\{\mathbf{x},\mathcal{F}_1(\mathbf{x}),\mathcal{F}_2(\mathbf{x}),\mathcal{F}_1(\mathcal{F}_2(\mathbf{x}))=\mathcal{F}_2(\mathcal{F}_1(\mathbf{x}))\}, \tag{9.10}$$

is planar (see Corollary 7.4), which makes the theory self-consistent.

The statements about the permutability of the fundamental transformations $\mathcal{F}_1$ and $\mathcal{F}_2$, and about the planarity of the elementary quadrilateral (9.10) are also valid in the limiting case in which $\mathbf{x}$ represents a submanifold parametrized by conjugate coordinates (see Fig. 15); this last result, which was known to Jonas[39] and Eisenhart.[3]

Therefore, the Darboux-type transformations of conjugate nets generate quadrilateral lattices, which are their natural discrete generalization, from both points of view of the integrability and geometric properties. Similar results have been obtained for many other geometrically relevant integrable systems.[28,40,10,41,20,13,22]

For discrete integrable systems there is obviously no essential difference between ''finite transformations'' and new dimensions. This shows once more that, from the point of view of the theory of integrable systems, the discrete ones are more basic.

We finally remark that all the basic transformations we considered here: the Lévy, adjoint Lévy, and fundamental transformations can be considered as Laplace transformations of quadrilateral strips. This observation shows that, although the Laplace transformations are of a very special type, they can be considered as the basic objects of the theory of transformations of lattices. This interpretation provides, for example, a very transparent geometric meaning to the additional solution ϕ of the Laplace equation entering into the Lévy transformation.

This formulation in terms of the Laplace transformations remains also valid in the limit from the ''quadrilateral lattice $\mathbf{x}$'' to the ''conjugate net $\mathbf{x}$,'' but, since the intermediate steps of the transformation involve ''differential-difference'' nets, it was unknown to the geometers who studied conjugate nets only.

X. $\bar{\partial}$ FORMALISM AND TRANSFORMATIONS

A. The $\bar{\partial}$ dressing for the Darboux and MQL equations

The central role of the $\bar{\partial}$ problem in the study of integrable multidimensional systems was established in Ref. 42; soon after that, the $\bar{\partial}$ problem was incorporated successfully in the dressing method, giving rise to the $\bar{\partial}$ dressing method,[5] which is a very general and convenient inverse method, based on the theory of complex analysis, introduced to construct: (i) integrable nonlinear systems of partial differential equations, together with large classes of solutions; (ii) the finite transformations (of a Bäcklund and Darboux type) between different solutions of these integrable systems; and (iii) the integrable discrete analogs of these integrable systems.

The Darboux equations (1.2) and their integrable discrete analogs, the MQL equations (1.10), provide a very precious illustrative example of the power and elegance of the $\bar{\partial}$ dressing method. Our goal in this section is to reconsider the main results of the previous sections, investigated so far from geometric and algebraic points of view, in the framework of the $\bar{\partial}$ formalism. More precisely, we shall present the $\bar{\partial}$ formulation of the radial, Combescure, and fundamental transformations, together with their limiting cases: the Lévy, adjoint Lévy, and Laplace transformations; we shall also discuss the permutability theorem and the essential equivalence between integrable discretizations of integrable PDEs and finite transformations of them. We shall find that the main results of the previous sections have a very elementary interpretation in the framework of the $\bar{\partial}$ formalism. Although the $\bar{\partial}$ formalism associated with the Darboux and MQL equations is scalar, we have decided to consider its matrix generalization because we expect that the matrix analog of the Darboux and MQL equations will find a geometric meaning.

Let us consider the following nonlocal $\bar{\partial}$ problem:[5]

$$\partial_{\bar{\lambda}}\chi(\lambda)=\partial_{\bar{\lambda}}\eta(\lambda)+\int_{\mathbb{C}}\chi(\lambda')R(\lambda',\lambda)d\lambda'\wedge d\bar{\lambda}', \quad \lambda,\lambda'\in\mathbb{C}, \tag{10.1}$$

for square $D\times D$ matrices, where $R(\lambda',\lambda)$ is a given $\bar{\partial}$ datum, which decreases quickly enough at ∞ in λ and λ', and the function $\eta(\lambda)$, the *normalization* of the unknown $\chi(\lambda)$, is a given function of λ and $\bar{\lambda}$, which describes, in particular, the polar behavior of $\chi(\lambda)$ in $\mathbb{C}$ and its behavior at ∞: $\chi-\eta\to 0$ as $\lambda\to\infty$. Therefore the $\bar{\partial}$ problem (10.1) is equivalent to the following Fredholm integral equation of the second type:

$$\chi(\lambda)=\eta(\lambda)+\frac{1}{2\pi i}\int_{\mathbb{C}}\frac{d\lambda'\wedge\lambda\bar{\lambda}'}{\lambda'-\lambda}\int_{\mathbb{C}}\chi(\lambda'')R(\lambda'',\lambda')d\lambda''\wedge d\bar{\lambda}''. \tag{10.2}$$

We remark that the dependence of $\chi(\lambda)$ and $R(\lambda,\lambda')$ on $\bar{\lambda}$ and $\bar{\lambda}'$ will be systematically omitted, for notational convenience, throughout this section. Furthermore, it will be assumed that the $\bar{\partial}$ problem (10.1) be uniquely solvable; i.e., if $\xi(\lambda)$ solves the homogeneous version of the $\bar{\partial}$ problem (10.1) and $\xi(\lambda)\to 0$ as $|\lambda|\to\infty$, then $\xi(\lambda)=0$.

The dependence of $R(\lambda',\lambda)$ [and, consequently, of $\chi(\lambda)$] on the continuous $\mathbf{u}\in R^N$ and discrete $\mathbf{n}\in Z^N$ space coordinates is assigned, respectively, through the following compatible equations:

$$\partial_i R(\lambda,\lambda')=K_i(\lambda)R(\lambda,\lambda')-R(\lambda,\lambda')K_i(\lambda'), \quad i=1,...,N, \tag{10.3}$$

$$T_i R(\lambda,\lambda')=(1+K_i(\lambda))R(\lambda,\lambda')(1+K_i(\lambda'))^{-1}, \quad i=1,...,N, \tag{10.4}$$

where $\partial_i=\partial/\partial u_i$, $i=1,...,N$ and $K_i(\lambda)$, $i=1,...,N$ are given commuting matrices constant in $\mathbf{u}$ and $\mathbf{n}$; in the following, for simplicity, the matrices $K_i(\lambda)$ will be assumed to be diagonal. If we are interested in the construction of continuous (discrete) systems, we concentrate on (10.3) [on (10.4)] only; but, in general, both dependences can be considered at the same time. Equations (10.3) and (10.4) admit the general solution

$$R(\lambda,\lambda';\mathbf{u},\mathbf{n})=G(\lambda)R_0(\lambda,\lambda')(G(\lambda))^{-1},$$

where

$$G(\lambda)=\exp\left(\sum_{i=1}^{N} u_i K_i(\lambda)\right)\prod_{j=1}^{N}(1+K_j(\lambda))^{n_j}.$$

We finally assume that $R_0(\lambda,\lambda')$ be identically zero in both variables in a neighborhood of the following points: the poles (λ_i) and the zeros of $\det(1+K_i(\lambda))$, $i=1,...,N$ and the poles of $\eta(\lambda)$. This restriction ensures the analyticity of $\chi-\eta$ at these points.[43,16] We briefly recall that, in the $\bar{\partial}$ dressing method, a crucial role is played by the *long derivatives* $\mathcal{D}_{u_i}$, $\mathcal{D}_{n_i}$, $i=1,...,N$, defined, respectively, by

$$\mathcal{D}_{u_i}\chi(\lambda):=\partial_i\chi(\lambda)+\chi(\lambda)K_i(\lambda),\quad i=1,...,N, \tag{10.5}$$

$$\mathcal{D}_{n_i}\chi(\lambda):=\Delta_i\chi(\lambda)+(T_i\chi(\lambda))K_i(\lambda),\quad i=1,...,N, \tag{10.6}$$

which are the generators of the Zakharov–Manakov ring of operators;[5] i.e., any linear combination, with coefficients depending only on **u** and **n**, of the operators

$$\prod_k \mathcal{D}_{u_k}^{l_k},\quad \prod_k \mathcal{D}_{n_k}^{l_k},\quad l_k\in N, \tag{10.7}$$

transforms solutions of (10.1) into solutions of (10.1) (corresponding, in general, to different normalizations). For instance,

$$\partial_{\bar{\lambda}}(\mathcal{D}_{n_i}\chi(\lambda))=(T_i\chi(\lambda))\partial_{\bar{\lambda}}K_i(\lambda)+\mathcal{D}_{n_i}(\partial_{\bar{\lambda}}\eta)+\int_{\mathbb{C}}d\lambda'\wedge d\overline{\lambda'}(\mathcal{D}_{n_i}\chi(\lambda'))R(\lambda',\lambda), \tag{10.8}$$

$$\begin{aligned}\partial_{\bar{\lambda}}(\mathcal{D}_{n_i}\mathcal{D}_{n_j}\chi(\lambda))=&(T_i\mathcal{D}_{n_j}\chi(\lambda))\partial_{\bar{\lambda}}K_i(\lambda)+(T_j\mathcal{D}_{n_i}\chi(\lambda))\partial_{\bar{\lambda}}K_j(\lambda)+\mathcal{D}_{n_i}\mathcal{D}_{n_j}(\partial_{\bar{\lambda}}\eta)\\&+\int_{\mathbb{C}}d\lambda'\wedge d\overline{\lambda'}(\mathcal{D}_{n_i}\mathcal{D}_{n_j}\chi(\lambda'))R(\lambda',\lambda).\end{aligned} \tag{10.9}$$

The goal of the method is to use this ring of operators to construct a set of solutions $\{\xi(\lambda)\}$ of (10.1) such that $\xi(\lambda)\to 0$ as $\lambda\to\infty$ and use uniqueness to infer the set of equations: $\{\xi(\lambda)=0\}$, which are equivalent to the integrable nonlinear system.

A given choice of the rational functions $K_i(\lambda)$ gives rise to solutions of a particular integrable nonlinear system; for instance, the Darboux and MQL equations (1.2) and (1.4) correspond to the following choice[5,16] (see Proposition 10.1 below):

$$K_i(\lambda):=\frac{\alpha_i}{\lambda-\lambda_i},\quad i=1,...,N, \tag{10.10}$$

where α_i are the constant diagonal matrices.

Different normalizations are associated instead with different solutions of such a nonlinear system. As it was observed in Ref. 43, the richness of this mechanism of constructing solutions is typical of multidimensional problems, since, in the case of the *local* $\bar{\partial}$ problem, arising in $1+1$ dimensions, different normalizations are all gauge equivalent. In this paper we shall limit our considerations to bounded (in λ and $\bar{\lambda}$) normalizations, which give rise to bounded (in λ and $\bar{\lambda}$) solutions of the $\bar{\partial}$ problem (10.1).

We first recall the basic results concerning the $\bar{\partial}$ integrability of the Darboux and MQL equations, obtained, respectively, in Refs. 5 and 16.

Proposition 10.1: Let $\varphi(\lambda)$ *be the solution of (10.1) corresponding to the canonical normalization* $\eta=1$. *Then the complex function,*

$$\psi(\lambda)=\varphi(\lambda)G(\lambda), \tag{10.11}$$

solves the continuous and discrete Laplace equations:

$$L_{ij}[H]\psi(\lambda)=\Lambda_{ij}[H]\psi(\lambda)=0, \quad i,j=1,...,N, \quad i\neq j, \tag{10.12}$$

where

$$L_{ij}[H]:=\partial_i\partial_j-(\partial_j H_i)H_i^{-1}\partial_i-(\partial_i H_j)H_j^{-1}\partial_j, \tag{10.13}$$

$$\Lambda_{ij}[H]:=\Delta_i\Delta_j-T_i((\Delta_j H_i)H_i^{-1})\Delta_i-T_j((\Delta_i H_j)H_j^{-1})\Delta_j, \tag{10.14}$$

and the set of functions H_i, $i=1,...,N$, defined by

$$H_i:=\varphi(\lambda_i)G_i, \tag{10.15}$$

$$G_i:=\exp\left(\sum_{k=1,k\neq i}^{N} u_k K_k(\lambda_i)\right)\prod_{k=1,k\neq i}^{N}(1+K_k(\lambda_i))^{n_k}, \tag{10.16}$$

solve the matrix analogs of the Darboux (1.2) and MQL equations (1.10).

Proof: In the philosophy of the $\bar{\partial}$ method, one shows that the solutions $\tilde{L}_{ij}\varphi(\lambda)$, $\tilde{\Lambda}_{ij}\varphi(\lambda)$ of the homogeneous version of the $\bar{\partial}$ problem (10.1) go to zero as $\lambda\to\infty$, where

$$\tilde{L}_{ij}\varphi(\lambda):=\mathcal{D}_{u_i}\mathcal{D}_{u_j}\varphi(\lambda)-(\mathcal{D}_{u_j}\varphi(\lambda_i))\varphi(\lambda_i)^{-1}\mathcal{D}_{u_i}\varphi(\lambda)-(\mathcal{D}_{u_i}\varphi(\lambda_j))\varphi(\lambda_j)^{-1}\mathcal{D}_{u_j}\varphi(\lambda),$$

$$\tilde{\Lambda}_{ij}\varphi(\lambda):=\mathcal{D}_{n_i}\mathcal{D}_{n_j}\varphi(\lambda)-T_i((\mathcal{D}_{n_j}\varphi(\lambda_i))\varphi(\lambda_i)^{-1})\mathcal{D}_{n_i}\varphi(\lambda)-T_j((\mathcal{D}_{n_i}\varphi(\lambda_j))\varphi(\lambda_j)^{-1})\mathcal{D}_{n_j}\varphi(\lambda). \tag{10.17}$$

Therefore, uniqueness implies that

$$\tilde{L}_{ij}\varphi(\lambda)=\tilde{\Lambda}_{ij}\varphi(\lambda)=0,$$

or, equivalently, Eqs. (10.12). Finally, evaluating Eq. (10.17) at $\lambda=\lambda_k$, $k\neq i\neq j\neq k$ and using (10.15), we obtain the Darboux and MQL equations, respectively. □

The above function $\psi(\lambda)$ allows one to construct the $D\times M$ matrix solution $\mathbf{x}$:

$$\mathbf{x}(\mathbf{u},\mathbf{n})=\int \psi(\lambda)h(\lambda)d\lambda\wedge d\bar{\lambda},$$

of the Laplace equations (10.12), where $h(\lambda)$ is an arbitrary localized $D\times M$ matrix function of λ and $\bar{\lambda}$ (but independent of the coordinates). If the $\bar{\partial}$ problem (10.1) is scalar, i.e., $D=1$, $\mathbf{x}$ is an M-dimensional vector solution of the Laplace equations. Therefore, keeping $\mathbf{n}$ fixed, $\mathbf{x}$ describes an N-dimensional manifold in $\mathbb{R}^M$, parametrized by the conjugate coordinates $\mathbf{u}$ (a conjugate net). Different values of $\mathbf{n}$ can therefore be interpreted as defining an N-dimensional (quadrilateral) sequence of conjugate nets. In the second interpretation we privilege, instead, the discrete aspect of the problem: keeping $\mathbf{u}$ fixed, $\mathbf{x}$ describes an N-dimensional quadrilateral lattice in $\mathbb{R}^M$, while the continuous coordinates $\mathbf{u}$ describe ''isoconjugate'' deformations of this lattice.

We finally remark that Eq. (10.8) can be viewed as the continuous limit $\epsilon\to 0$ of (10.9), in which $\epsilon n_i\to u_i$ and $T_i\sim 1+\epsilon\partial_i$ (replacing α_i by $\epsilon\alpha_i$).

Exploiting completely the possible normalizations of the $\bar{\partial}$ problem, one obtains more solutions of the Laplace equations, together with the relations between them. The radial (or projective) and the Combescure transformations can be obtained in this way.

B. Radial transformations

Proposition 10.2: Let $\varphi_{\mathcal{P}}(\lambda)$ *be the solution of (10.1) corresponding to the normalization* $\eta=\phi^{-1}$, *where* ϕ *is any solution of the continuous and discrete Laplace equations (10.12). Define the function*

$$\psi_{\mathcal{P}}(\lambda) := \varphi_{\mathcal{P}}(\lambda) G(\lambda); \tag{10.18}$$

then we have the following.

(i) $\psi_{\mathcal{P}}(\lambda)$ *is related to the function* $\psi(\lambda)$, *defined in (10.11), through the radial (gauge) transformation:*

$$\psi_{\mathcal{P}}(\lambda) = \phi^{-1} \psi(\lambda). \tag{10.19}$$

(ii) $\psi_{\mathcal{P}}(\lambda)$ *solves the Laplace equations,*

$$L_{ij}[\mathcal{P}(H)] \psi_{\mathcal{P}}(\lambda) = \Lambda_{ij}[\mathcal{P}(H)] \psi_{\mathcal{P}}(\lambda) = 0, \quad i,j=1,...,N, \quad i \neq j, \tag{10.20}$$

where the functions

$$\mathcal{P}(H_i) = \varphi_{\mathcal{P}}(\lambda_i) G_i = \phi^{-1} H_i \tag{10.21}$$

solve the matrix Darboux and MQL equations.

Proof: The proof goes as in Proposition 10.1. The uniqueness of the $\bar{\partial}$ problem implies the following equations:

$$\varphi_{\mathcal{P}}(\lambda) - \phi^{-1} \varphi(\lambda) = 0,$$

$$\tilde{L}_{ij} \varphi_{\mathcal{P}}(\lambda) + \phi^{-1}(L_{ij}[H]\phi) \phi^{-1} \varphi(\lambda) = 0,$$

$$\tilde{\Lambda}_{ij} \varphi_{\mathcal{P}}(\lambda) + (T_i T_j \phi^{-1})(\Lambda_{ij}[H]\phi) \phi^{-1} \varphi(\lambda) = 0,$$

equivalent, respectively, to (10.3) and (10.4).

Therefore the $D \times M$ matrix,

$$\mathcal{P}(\mathbf{x}) = \int_{\mathbb{C}} \psi_{\mathcal{P}}(\lambda) h(\lambda) d\lambda \wedge d\bar{\lambda}, \tag{10.22}$$

satisfies the equations

$$L_{ij}[\mathcal{P}(H)] \mathcal{P}(\mathbf{x}) = \Lambda_{ij}[\mathcal{P}(H)] \mathcal{P}(\mathbf{x}) = 0, \quad i,j=1,...,N, \quad i \neq j,$$

$$\mathcal{P}(\mathbf{x}) = \phi^{-1} \mathbf{x},$$

and, if the $\bar{\partial}$ problem (10.1) is scalar ($D=1$), it defines the radial transform $\mathcal{P}(\mathbf{x})$ of $\mathbf{x}$ (see Sec. VI). □

C. Combescure transformations

We first introduce the basic, localized in λ and $\bar{\lambda}$, solutions of the $\bar{\partial}$ problem, corresponding to the simple pole normalization $\eta = (\lambda - \mu)^{-1}$. These solutions were first used in a multidimensional context in Ref. 43 and used extensively in Ref. 44. The following proposition can be found in Ref. 16.

Proposition 10.3: Let $\varphi(\lambda,\mu)$ *be the solution of (10.1) corresponding to the simple pole normalization* $\eta = (\lambda - \mu)^{-1}$, $\mu \neq \lambda_i$, $i=1,...,N$. *Define the function*

$$\psi(\lambda,\mu) := G(\mu)^{-1}\varphi(\lambda,\mu)G(\lambda); \tag{10.23}$$

then we have the following.

(i) $\psi(\lambda,\mu)$ solves the Laplace equations

$$L_{ij}[H(\mu)]\psi(\lambda,\mu) = \Lambda_{ij}[H(\mu)](\lambda,\mu) = 0, \tag{10.24}$$

and the functions

$$H_i(\mu) = G(\mu)^{-1}\varphi(\lambda_1,\mu)G_i$$

solve the Darboux and MQL equations.

(ii) $\psi(\lambda,\mu)$ is a Combescure transform of $\psi(\lambda)$, i.e., the following formulas hold:

$$\partial_i\psi(\lambda,\mu) = C_i(\mu)\partial_i\psi(\lambda), \quad \Delta_i\psi(\lambda,\mu) = (T_iC_i(\mu))\Delta_i\psi(\lambda), \tag{10.25}$$

where

$$C_i(\mu) = H_i(\mu)H_i^{-1} \tag{10.26}$$

and

$$\partial_i H_j(\mu) = C_i(\mu)\partial_i H_j, \quad \Delta_i H_j(\mu_j) = (T_iC_i(\mu))\Delta_i H_j, \quad i \neq j, \tag{10.27}$$

$$\begin{gathered}\partial_i C_j(\mu) + (C_j(\mu) - C_i(\mu))(\partial_i H_j)H_j^{-1} = 0,\\ \Delta_i C_j(\mu) + (T_iC_j(\mu) - T_iC_i(\mu))(\Delta_i H_j)H_j^{-1} = 0.\end{gathered} \tag{10.28}$$

Proof: The uniqueness of the $\bar{\partial}$ problem implies the following equations:

$$\begin{gathered}\tilde{\Lambda}'_{ij}\varphi(\lambda,\mu) = 0,\\ \mathcal{D}'_{n_i}\varphi(\lambda,\mu) - T_i(\varphi(\lambda_i,\mu)(\varphi(\lambda_i))^{-1})\mathcal{D}_{n_i}\varphi(\lambda) = 0,\end{gathered} \tag{10.29}$$

and their continuous analogs, equivalent, respectively, to (10.24) and (10.25), where $\tilde{\Lambda}'_{ij}$ is obtained from $\tilde{\Lambda}_{ij}$ replacing $\mathcal{D}_{n_i}$ by

$$\mathcal{D}'_{n_i}f := \mathcal{D}_{n_i}f - \frac{\alpha_i}{\mu - \lambda_i}f, \quad i = 1,\ldots,N.$$

Equations (10.27) follow by multiplying Eq. (10.29) by $(1 + K_j(\lambda))^{-1}$ and then setting $\lambda = \lambda_j$; Eqs. (10.28) are direct consequences of (10.27) and (10.26). □

Remark: The formula (10.25) suggests that one could start with the solution of (10.1) normalized by $\eta = (\lambda - \mu)^{-1}G(\mu)^{-1}$, avoiding in this way the introduction of the generalized operators $\mathcal{D}'_{u_i}, \mathcal{D}'_{n_i}$ and simplifying the proof. This is actually a key observation in the following construction of more general solutions, bounded in λ, of the Laplace equations.

The canonical and simple pole normalizations allow one to construct the prototype examples of, respectively, bounded and localized solutions of the Laplace equations. This is due to the fact that the corresponding normalizations: $\eta = 1$ and $\eta = (\lambda - \mu)^{-1}G(\mu)^{-1}$ satisfy the equations

$$\mathcal{D}_{n_i}(\partial_{\bar{\lambda}}\eta) = \mathcal{D}_{u_i}(\partial_{\bar{\lambda}}\eta) = 0,$$

implying that the forcings of Eqs. (10.8), (10.9) do not depend on η. Observing that

$$\mathcal{D}_{n_i} f = \mathcal{D}_{u_i} f = 0, \quad i = 1,...,N \Leftrightarrow f = \gamma(\lambda) G(\lambda)^{-1},$$

where $\gamma(\lambda)$ is an arbitrary function of $\lambda, \overline{\lambda}$, but constant in the coordinates, we infer that a general, bounded in $\lambda, \overline{\lambda}$, solution of the Laplace equations is obtained considering the solution $\Phi(\lambda)$ of the $\overline{\partial}$ problem (10.1) corresponding to the normalization

$$\eta = a + \frac{i}{2} \int_{\mathbb{C}} \frac{d\mu' \wedge d\overline{\mu}'}{\lambda - \mu'} \gamma(\mu') G(\mu')^{-1} \Rightarrow \partial_{\overline{\lambda}} \eta = \gamma(\lambda) G(\lambda)^{-1}, \tag{10.30}$$

where γ is any localized function of $\lambda, \overline{\lambda}$, constant in the coordinates and a is any constant (in λ and in the coordinates) matrix. The general solution $\Psi(\lambda) = \Phi(\lambda) G(\lambda)$ of the Laplace equations reduces to the solutions $\psi(\lambda)$ and $\psi(\lambda,\mu)$, corresponding to the canonical and simple pole normalizations, through the following obvious specifications:

$$a = 1, \quad \gamma(\lambda) = 0 \Rightarrow \Psi(\lambda) = \psi(\lambda),$$

$$a = 0, \quad \gamma(\lambda) = \delta(\lambda - \mu) \Rightarrow \Psi(\lambda) = \psi(\lambda, \mu).$$

Proposition 10.4: Let $\Phi(\lambda)$ be the solution of (10.1) corresponding to the normalization (10.30). Define the function $\Psi(\lambda)$ in the usual way:

$$\Psi(\lambda) = \Phi(\lambda) G(\lambda);$$

then we have the following.

(i) $\Psi(\lambda)$ solves the Laplace equations

$$L_{ij}[H] \Psi(\lambda) = \Lambda_{ij}[H] \Psi(\lambda) = 0, \quad i,j = 1,...,N, \quad i \neq j, \tag{10.31}$$

and the functions

$$H_i = \Phi(\lambda_i) G_i,$$

solve the Darboux and MQL equations.

(ii) If $\Psi^{(l)}(\lambda) = \Phi^{(l)}(\lambda) G(\lambda)$, $l = 1,2$ are two different solutions of (10.31) corresponding to the different normalizations $a^{(l)}$, $\gamma^{(l)}(\lambda)$, $l = 1,2$, then these solutions are related by the Combescure transformation, i.e.,

$$\partial_i \Psi^{(2)}(\lambda) = C_i^{(2,1)} \partial_i \Psi^{(1)}(\lambda), \quad \Delta_i \Psi^{(2)}(\lambda) = (T_i C_i^{(2,1)}) \Delta_i \Psi^{(1)}(\lambda), \quad i = 1,...,N, \tag{10.32}$$

where the functions

$$\begin{gathered} C_i^{(2,1)} = H_i^{(2)} (H_i^{(1)})^{-1}, \\ H_i^{(l)} = \Phi^{(l)}(\lambda_i) G_i, \quad l = 1,2, \end{gathered} \tag{10.33}$$

satisfy the equations

$$\partial_i H_j^{(2)} = C_i^{(2,1)} \partial_i H_j^{(1)}, \quad \Delta_i H_j^{(2)} = (T_i C_i^{(2,1)}) \Delta_i H_j^{(1)}, \quad i \neq j,$$

$$\partial_i C_j^{(2,1)} + (C_j^{(2,1)} - C_i^{(2,1)}) (\partial_i H_j^{(1)}) (H_j^{(1)})^{-1} = 0,$$

$$\Delta_i C_j^{(2,1)} + (T_i C_j^{(2,1)} - T_i C_i^{(2,1)}) (\Delta_i H_j^{(1)}) (H_j^{(1)})^{-1} = 0.$$

(iii) The following relations hold:

$$\mathcal{D}_{n_i}\Phi(\lambda)=\Phi(\lambda_i)\alpha_i\varphi(\lambda,\lambda_i),\quad \mathcal{D}_{u_i}\Phi(\lambda)=(T_i\Phi(\lambda_i))\alpha_i\varphi(\lambda,\lambda_i),\quad i=1,...,N. \tag{10.34}$$

(iv) If $\lambda_0\neq\lambda_i$, $i=1,...,N$ is an additional complex parameter associated with the additional coordinates u_0 and n_0:

$$\begin{aligned}\partial_{u_0}R(\lambda,\lambda')&=\frac{\alpha_0}{\lambda-\lambda_0}R(\lambda,\lambda')-R(\lambda,\lambda')\frac{\alpha_0}{\lambda'-\lambda_0},\quad i=1,...,N,\\ T_0R(\lambda,\lambda')&=\left(1+\frac{\alpha_0}{\lambda-\lambda_0}\right)R(\lambda,\lambda')\left(1+\frac{\alpha_0}{\lambda'-\lambda_0}\right)^{-1},\quad i=1,...,N,\end{aligned} \tag{10.35}$$

where α_0 is a diagonal matrix and $R_0(\lambda',\lambda)$ is zero in a neighborhood of $\lambda=\lambda_0$ and $\lambda'=\lambda_0$, then $\varphi(\lambda,\lambda_0)$ and $\Phi(\lambda)$ are connected through the analogs of Eqs. (10.34):

$$\mathcal{D}_{n_0}\Phi(\lambda)=\Phi(\lambda_0)\alpha_0\varphi(\lambda,\lambda_0),\quad \mathcal{D}_{u_0}\Phi(\lambda)=(T_0\Phi(\lambda_0))\alpha_0\varphi(\lambda,\lambda_0),$$

equivalent to equations

$$\mathcal{D}_{u_0}\Psi(\lambda)=\Psi(\lambda_0)\alpha_0\psi(\lambda,\lambda_0),\quad \mathcal{D}_{n_0}\Psi(\lambda)=(T_0\Psi(\lambda_0))\alpha_0\psi(\lambda,\lambda_0). \tag{10.36}$$

Proof: As before, the uniqueness of the $\bar\partial$ problem implies equations

$$\tilde\Lambda_{ij}[\Phi(\lambda)]=0,$$

$$\mathcal{D}_{n_i}\Phi^{(2)}(\lambda)-T_i(\Phi^{(2)}(\lambda_i)(\Phi^{(1)}(\lambda_i))^{-1})\mathcal{D}_{n_i}\Phi^{(1)}(\lambda)=0,$$

$$\mathcal{D}_{n_i}\Phi(\lambda)-(T_i\Phi(\lambda_i)\alpha_i\varphi(\lambda,\lambda_i))=0,$$

and their continuous analogs, equivalent, respectively, to Eqs. (10.31), (10.32), and (10.34). The rest of the proof is as in the previous propositions. □

Remark: We remark that the localized solutions Φ of (10.1), corresponding to the normalization (10.30) with $a=0$, can be obtained, integrating the simple pole solutions with an arbitrary measure:

$$\Phi(\lambda)=\frac{i}{2\pi}\int_{\mathbb{C}}d\mu\wedge d\bar\mu\,\gamma(\mu)G(\mu)^{-1}\varphi(\lambda,\mu).$$

This formula establishes a contact with the class of Combescure related solutions of the Laplace equation obtained in Refs. 44 and 45.

Remark: The Combescure solutions introduced in this proposition form a linear space. For instance, the solution $\Psi(\lambda)$, corresponding to the normalization,

$$1+\frac{i}{2}\int_{\mathbb{C}}\frac{d\lambda'\wedge d\bar\lambda'}{\lambda-\lambda'}\gamma(\lambda')G(\lambda')^{-1},$$

is the linear combination

$$\Psi(\lambda)=\psi(\lambda)+\psi_{\mathcal{C}}(\lambda), \tag{10.37}$$

of the solution $\psi(\lambda)$, corresponding to the canonical normalization, and of the solution $\psi_{\mathcal{C}}(\lambda)$, corresponding to the normalization

$$\frac{i}{2}\int_{\mathbb{C}}\frac{d\lambda'\wedge d\overline{\lambda'}}{\lambda-\lambda'}\gamma(\lambda')G(\lambda')^{-1}.$$

Therefore, the $D\times M$ matrix solutions

$$\mathbf{x}^{(l)}(\mathbf{u},\mathbf{n})=\int_{\mathbb{C}}\Psi^{(l)}(\lambda)h(\lambda)d\lambda\wedge d\overline{\lambda},\quad l=1,2,$$

of the Laplace equations,

$$L_{ij}[H^{(l)}]\mathbf{x}^{(l)}=\Lambda_{ij}[H^{(l)}]\mathbf{x}^{(l)}=0,\quad l=1,2,\quad i,j=1,...,N,\quad i\neq j,$$

satisfy the Combescure relations

$$\partial_i\mathbf{x}^{(2)}=C_i^{(2,1)}\,\partial_i\mathbf{x}^{(1)},\quad \Delta_i\mathbf{x}^{(2)}=(T_iC_i^{(2,1)})\Delta_i\mathbf{x}^{(1)},\quad i=1,...,N.$$

At last, from Eq. (10.37) we have the relation

$$\mathcal{C}(\mathbf{x})=\mathbf{x}+\mathbf{x}_{\mathcal{C}},$$

where

$$\mathcal{C}(\mathbf{x})=\int_{\mathbb{C}}\Psi(\lambda)h(\lambda)d\lambda\wedge d\overline{\lambda},$$

$$\mathbf{x}_{\mathcal{C}}=\int_{\mathbb{C}}\psi_{\mathcal{C}}(\lambda)h(\lambda)d\lambda\wedge d\overline{\lambda}.$$

In the scalar case $D=1$, the M-dimensional vectors $\mathbf{x}^{(l)}$, $l=1,2$, $\mathcal{C}(\mathbf{x})$, $\mathbf{x}$ and $\mathbf{x}_{\mathcal{C}}$ are related by the Combescure transformation formulas of Sec. IV.

D. Fundamental transformations and their composition

So far we have used only different normalizations of the $\overline{\partial}$ problem. In order to generate more solutions of the Laplace equation, this mechanism must be combined with a more classical one, discovered long ago[46] in the context of $1+1$-dimensional problems.

Proposition 10.5: Let us consider the (by assumption uniquely solvable) $\overline{\partial}$ problem,

$$\partial_{\overline{\lambda}}\tilde{\chi}(\lambda)=\partial_{\overline{\lambda}}\tilde{\eta}(\lambda)+\int_{\mathbb{C}}\tilde{\chi}(\lambda')\tilde{R}(\lambda',\lambda)d\lambda'\wedge d\overline{\lambda'},\quad \lambda,\lambda'\in\mathbb{C},\tag{10.38}$$

where the $\overline{\partial}$ datum $\tilde{R}(\lambda',\lambda)$ is related to $R(\lambda',\lambda)$ through the transformation

$$\tilde{R}(\lambda',\lambda)=g(\lambda')R(\lambda',\lambda)g(\lambda)^{-1},\tag{10.39}$$

where $g(\lambda)$ is a diagonal matrix [more generally—commuting with $K_i(\lambda)$] and independent of $\mathbf{n}$, $\mathbf{u}$, *and $R_0(\lambda',\lambda)$ is assumed to be zero in a neighborhood of the zeros and poles of* $\det g(\lambda)$. *Then we have the following.*

(i) If $\tilde{\eta}$ satisfies the equation $\mathcal{D}_{n_i}(\partial_{\overline{\lambda}}\tilde{\eta})=0$, then the corresponding solutions of (10.38), (10.39) give rise to solutions of the Laplace equations.

(ii) If $\chi(\lambda)$ solves the $\overline{\partial}$ problem (10.1), then the function $\chi(\lambda)g(\lambda)^{-1}$ solves the $\overline{\partial}$ problem (10.38), corresponding to the inhomogeneous term:

$$\partial_{\overline{\lambda}}\tilde{\eta}=(\partial_{\overline{\lambda}}\eta)g(\lambda)^{-1}+\chi(\lambda)\partial_{\overline{\lambda}}g(\lambda)^{-1}.\tag{10.40}$$

Proof: Since $\tilde{R}(\lambda',\lambda)$ satisfies Eqs. (10.3) and (10.4), then the results of Propositions (10.1)–(10.4) apply also to this case. (ii) follows from taking the $\partial_{\bar{\lambda}}$ derivative of $\chi(\lambda)g(\lambda)^{-1}$ and using (10.1). □

The matrix function $g(\lambda)$ appearing in this proposition is usually chosen to be a rational function of λ, such that $g(\lambda)\to 1$ as $\lambda\to\infty$, in order to preserve the properties at ∞ of the $\bar{\partial}$ problem. We shall show now that the simplest nontrivial example of this type,

$$g(\lambda)=1+\frac{\beta}{\lambda-\mu}, \tag{10.41}$$

corresponds to the Fundamental Transformation of a quadrilateral lattice and conjugate net.

Proposition 10.6: Let $\Phi(\lambda)$ and $\tilde{\Phi}(\lambda)$ be the solutions of, respectively, the $\bar{\partial}$ problems (10.1) and (10.38) [with $g(\lambda)$ defined in (10.41)], corresponding to the normalizations

$$\eta=a+\frac{i}{2}\int_{\mathbb{C}}\frac{d\mu\wedge d\bar{\mu}}{\lambda-\mu}\gamma(\mu)G(\mu)^{-1},\quad \tilde{\eta}=a+\frac{i}{2}\int_{\mathbb{C}}\frac{d\mu\wedge d\bar{\mu}}{\lambda-\mu}\gamma(\mu)g(\mu)^{-1}G(\mu)^{-1}.$$

Let $\varphi(\lambda,\mu)$ be the solution of the $\bar{\partial}$ problem (10.1) corresponding to the normalization $\eta=(\lambda-\mu)^{-1}$. Define the function

$$\tilde{\Psi}(\lambda):=\tilde{\Phi}(\lambda)G(\lambda); \tag{10.42}$$

then we have the following.

(i) $\tilde{\Psi}(\lambda)$ satisfies the Laplace equations,

$$L_{ij}[\mathcal{F}(H)]\tilde{\Psi}(\lambda)=0,\quad \Lambda_{ij}[\mathcal{F}(H)]\tilde{\Psi}(\lambda)=0,\quad i,j=1,...,N,\quad i\neq j, \tag{10.43}$$

and the functions

$$\mathcal{F}(H_i):=\tilde{\Phi}(\lambda_i)G_i,\quad i-1,...,N, \tag{10.44}$$

satisfy the matrix Darboux and MQL equations.

(ii) $\tilde{\Psi}(\lambda)$ is the fundamental transform of $\Psi(\lambda)$, i.e.,

$$\tilde{\Psi}(\lambda)=[\Psi(\lambda)+A\psi(\lambda,\mu)]\left(1+\frac{\beta}{\lambda-\mu}\right)^{-1}, \tag{10.45}$$

where the matrix A is defined in the following two ways:

$$\begin{gathered}A=-\Psi(\boldsymbol{\nu})(\psi(\boldsymbol{\nu},\mu))^{-1},\quad A=\tilde{\Psi}(\mu)\beta,\\ (\Psi(\boldsymbol{\nu}))_{lm}:=\Psi_{lm}(\boldsymbol{\nu}_m),\quad (\psi(\boldsymbol{\nu},\mu))_{lm}:=\psi_{lm}(\nu_m,\mu),\quad l,m=1,...,D,\end{gathered} \tag{10.46}$$

and ν_m, $m=1,...,D$ are the zeros of $\det g(\lambda)$.

Proof: (i) is an immediate consequence of part (i) of Proposition 10.5. To prove part (ii), first remark that

$$\partial_{\bar{\lambda}}(g(\lambda)^{-1})=\pi\sum_{k=1}^{L}(\nu_k-\mu)\delta(\lambda-\nu_k)P_k,$$

where P_k, $k=1,...,D$ are the usual matrix projectors: $(P_k)_{lm}=\delta_{lk}\delta_{km}$. Then observe that the matrix B, defined by the following generalized equation:

$$[\Phi(\lambda)+B\varphi(\lambda,\mu)]g(\lambda)^{-1}=0,$$

is given by

$$B = -\Phi(\boldsymbol{\nu})(\varphi(\boldsymbol{\nu},\mu))^{-1},$$

where

$$(\Phi(\boldsymbol{\nu}))_{lm} = \Phi_{lm}(\nu_m), \quad (\varphi(\boldsymbol{\nu},\mu))_{lm} = \varphi_{lm}(\nu_m,\mu), \quad l,m = 1,...,D.$$

The uniqueness of the $\bar{\partial}$ problem (10.1) implies that

$$\tilde{\Phi}(\lambda) - [\Phi(\lambda) + B\varphi(\lambda,\mu)]\left(1 + \frac{\beta}{\lambda - \mu}\right)^{-1} = 0. \tag{10.47}$$

In addition, since $\tilde{\Phi}(\lambda)$ is analytic in $\lambda = \mu$, it follows that $B = \tilde{\Phi}(\mu)\beta$. Multiplying (10.47) by $G(\lambda)$ one obtains Eq. (10.45), with $A = BG(\mu)$. □

If the $\bar{\partial}$ problem is scalar $(D=1)$, then

$$\tilde{\Psi}(\lambda) = \left[\Psi(\lambda) - \frac{\Psi(\nu)}{\psi(\nu,\mu)}\psi(\lambda,\mu)\right]\frac{\lambda - \mu}{\lambda - \nu}, \quad \nu = \mu - \beta,$$

and the quadrilateral lattices (and conjugate nets),

$$\mathbf{x} = \int_{\mathbb{C}} \Psi(\lambda)h(\lambda)d\lambda \wedge d\bar{\lambda}, \quad \mathbf{x}_{\mathcal{C}}(\mu) = \int_{\mathbb{C}} \psi(\lambda,\mu)h(\lambda)d\lambda \wedge d\bar{\lambda},$$

$$\mathcal{F}(\mathbf{x}) = \int_{\mathbb{C}} \tilde{\Psi}(\lambda)\frac{\lambda - \nu}{\lambda - \mu}h(\lambda)d\lambda \wedge d\bar{\lambda},$$

are related through the Fundamental Transformation (see Sec. VII),

$$\mathcal{F}(\mathbf{x}) = \mathbf{x} - \frac{\Psi(\nu)}{\psi(\nu,\mu)}\mathbf{x}_{\mathcal{C}}(\mu).$$

This result can be generalized in a straightforward way to the case of the composition of several fundamental transformations. In terms of the $\bar{\partial}$ datum, the sequence of transformations reads as

$$\begin{aligned}
R(\lambda,\lambda') \to R_1(\lambda,\lambda') &= g_1(\lambda)R(\lambda,\lambda')g_1(\lambda')^{-1} \to R_{12}(\lambda,\lambda') \\
&= g_2(\lambda)R_1(\lambda,\lambda')g_2(\lambda')^{-1} \\
&= g_1(\lambda)g_2(\lambda)R(\lambda,\lambda')(g_1(\lambda')g_2(\lambda'))^{-1} \to \cdots \to R_{12\cdots L} \\
&= \prod_{k=1}^{L} g_k(\lambda)R(\lambda,\lambda')\prod_{k=1}^{L}(g_k(\lambda'))^{-1},
\end{aligned}$$

where

$$g_i(\lambda) = 1 + \frac{\beta_i}{\lambda - \mu_i}, \quad i = 1,...,L. \tag{10.48}$$

Therefore the sequence of L fundamental transformations $g_i(\lambda)$ is equivalent to a single transformation, in which

$$g(\lambda)=\prod_{k=1}^{L}\left(1+\frac{\beta_k}{\lambda-\mu_k}\right). \tag{10.49}$$

Furthermore, the commutation of the diagonal matrices $g_i(\lambda)$, $i=1,...,L$ implies that the sequence of fundamental transformations does not depend on the order in which it is obtained (the famous permutability theorem therefore has a very elementary interpretation in the $\bar{\partial}$ formalism). The corresponding transformation in configuration space is described by the following.

Proposition 10.7: Let $\Phi(\lambda)$ *and* $\tilde{\Phi}(\lambda)$ *be the solutions, respectively, of the* $\bar{\partial}$ *problems (10.1) and (10.38), (10.39), (10.49), with* $\mu_k\neq\mu_j$, $k\neq j$, *corresponding to the normalizations,*

$$\eta=a+\frac{i}{2}\int_{\mathbb{C}}\frac{d\mu\wedge d\bar{\mu}}{\lambda-\mu}\gamma(\mu)G(\mu)^{-1},\quad \tilde{\eta}=a+\frac{i}{2}\int_{\mathbb{C}}\frac{d\mu\wedge d\bar{\mu}}{\lambda-\mu}\gamma(\mu)g(\mu)^{-1}G(\mu)^{-1}.$$

Let $\varphi(\lambda,\mu_k)$, $k=1,...,L$ be the solutions of the $\bar{\partial}$ problem (10.1) corresponding to the normalizations $\eta=(\lambda-\mu_k)^{-1}$, $k=1,...,L$. Define the function $\tilde{\Psi}(\lambda)$ as in (10.42); then we have the following.

(i) The function $\tilde{\Psi}(\lambda)$ *satisfies the Laplace equations,*

$$L_{ij}[\tilde{H}]\tilde{\Psi}(\lambda)=0,\quad \Lambda_{ij}[\tilde{H}]\tilde{\Psi}(\lambda)=0,\quad i,j=1,...,N,\quad i\neq j, \tag{10.50}$$

and

$$H_i=\Phi(\lambda_i)G_i,\quad \tilde{H}_i=\tilde{\Phi}(\lambda_i)G_i,\quad i=1,...,N.$$

(ii) The following relation holds:

$$\tilde{\Psi}(\lambda)=\left[\Psi(\lambda)+\sum_{k=1}^{L}A^{(k)}\psi(\lambda,\mu_k)\right]\prod_{k=1}^{L}\left(1+\frac{\beta_k}{\lambda-\mu_k}\right)^{-1},$$

where the $D\times D$ *matrices* $A^{(k)}$, $k=1,...,L$ *are defined in two independent ways; through the following linear system of L equations for* $D\times D$ *matrices:*

$$\sum_{k=1}^{M}A^{(k)}\psi(\boldsymbol{\nu}_i,\mu_k)=-\Psi(\boldsymbol{\nu}_i),\quad i=1,...,L,$$

where

$$(\Psi(\boldsymbol{\nu}_i))_{lm}=\Psi_{lm}(\nu_{im}),\quad (\psi(\boldsymbol{\nu}_i,\mu_k))_{lm}=\psi_{lm}(\nu_{im},\mu_k),\quad l,m=1,...,D,$$

and ν_{im}, $m=1,...,D$ *are the zeros of* $\det g_i(\lambda)$, *and through the equations*

$$A^{(k)}=\tilde{\Psi}(\mu_k)\beta_k\prod_{l=1,l\neq k}^{L}\left(1+\frac{\beta}{\mu_k-\mu_l}\right),\quad k=1,...,L.$$

Proof: The proof is a straightforward generalization of that of Proposition 10.6. □

In the scalar case, the above equations simplify to

$$\tilde{\Psi}(\lambda)=\left[\Psi(\lambda)+\sum_{k=1}^{L}A^{(k)}\psi(\lambda,\mu_k)\right]\prod_{k=1}^{L}\frac{\lambda-\mu_k}{\lambda-\nu_k},\quad \nu_k=\mu_k-\beta_k,$$

$$\sum_{k=1}^{L}A^{(k)}\psi(\boldsymbol{\nu}_i,\mu_k)=-\Psi(\nu_i),\quad i=1,...,L.$$

Therefore, the M-dimensional vector,

$$\tilde{\mathbf{x}}=\int_{\mathbb{C}}\tilde{\Psi}(\lambda)\prod_{k=1}^{L}\frac{\lambda-\nu_k}{\lambda-\mu_k}h(\lambda)d\lambda\wedge d\bar{\lambda},$$

obtained combining in an arbitrary order L fundamental transformations described by the Combescure vectors,

$$\mathbf{x}_{\mathcal{C}}^{(k)}=\int_{\mathbb{C}}\psi(\lambda,\mu_k)h(\lambda)d\lambda\wedge d\bar{\lambda},\quad k=1,...,L,$$

satisfies the following equation:

$$\tilde{\mathbf{x}}=\mathbf{x}+\sum_{k=1}^{L}A^{(k)}\mathbf{x}_{\mathcal{C}}^{(k)},$$

which agrees with Eq. (7.19).

E. Lévy, adjoint Lévy, and Laplace transformations

As we have seen in Sec. VIII, the fundamental transformation contains, as significant geometric limits, the Lévy, adjoint Lévy, and Laplace transformations. Here we shall briefly discuss the analytic counterpart of these geometric limits, limiting our considerations to the scalar case.

Proposition 10.8: Let $\Phi(\lambda)$, $\tilde{\Phi}(\lambda)$, *and* $\varphi(\lambda,\mu)$ *be the solutions of the* scalar $\bar{\partial}$ *problems (10.1) and (10.38), (10.39) considered in Proposition (10.6) and therefore connected by the fundamental transformation (10.45). Then we have the following.*

(1) If $\nu\to\lambda_i$, *the fundamental transformation* $\mathcal{F}$ *reduces to the adjoint Lévy transformation* $\mathcal{L}_i^*$:

$$\tilde{\Psi}(\lambda)\to\left[\Psi(\lambda)-\frac{H_i}{H_i(\mu)}\psi(\lambda,\mu)\right]\frac{\lambda-\mu}{\lambda-\lambda_i},\quad \nu\to\lambda_i,$$

$$\Rightarrow\mathcal{F}(\mathbf{x})\to\mathbf{x}-\frac{H_i}{H_i(\mu)}\mathbf{x}(\mu)=\mathcal{L}_i^*(\mathbf{x}),\quad \nu\to\lambda_i. \tag{10.51}$$

(2) If $\mu\to\lambda_j$, *then the fundamental transformation* $\mathcal{F}$ *reduces to the Lévy transformation* $\mathcal{L}_j$:

$$\tilde{\Psi}(\lambda)\to\left[\Psi(\lambda)-\frac{\Psi(\nu)}{\Delta_j\Psi(\nu)}\Delta_j\Psi(\lambda)\right]\frac{\lambda-\lambda_j}{\lambda-\nu},\quad \mu\to\lambda_j,$$

$$\Rightarrow\mathcal{F}(\mathbf{x})\to\mathbf{x}-\frac{\Psi(\nu)}{\Delta_j\Psi(\nu)}\Delta_j\mathbf{x}=\mathcal{L}_j(\mathbf{x}),\quad \mu\to\lambda_j. \tag{10.52}$$

(3) If $\nu\to\lambda_i$ *and* $\mu\to\lambda_j$, *then the Fundamental Transformation* $\mathcal{F}$ *reduces to the Laplace Transformation* $\mathcal{L}_{ji}$:

$$\tilde{\Psi}(\lambda)\to\left[\Psi(\lambda)-\frac{H_i}{\Delta_jH_i}\Delta_j\Psi(\lambda)\right]\frac{\lambda-\lambda_j}{\lambda-\lambda_i},\quad \nu\to\lambda_i,\quad \mu\to\lambda_j,$$

$$\Rightarrow\mathcal{F}(\mathbf{x})\to\mathbf{x}-\frac{H_i}{\Delta_jH_i}\Delta_j\mathbf{x}=\mathcal{L}_{ji}(\mathbf{x}),\quad \nu\to\lambda_i,\quad \mu\to\lambda_j. \tag{10.53}$$

Proof: We first observe that

$$\frac{\Psi(\nu)}{\psi(\nu,\mu)} = \frac{\Phi(\nu)}{G^{-1}(\mu)\varphi(\nu,\mu)} \xrightarrow{\nu\to\lambda_i} \frac{\Phi(\lambda_i)}{G^{-1}(\mu)\varphi(\lambda_i,\mu)} = \frac{H_i}{H_i(\mu)},$$

implying Eq. (10.51). Equation (10.52) follows from

$$\frac{\psi(\lambda,\mu)}{\psi(\nu,\mu)} = \frac{\varphi(\lambda,\mu)G(\lambda)}{\varphi(\nu,\mu)G(\nu)} \xrightarrow{\mu\to\lambda_i} \frac{\varphi(\lambda,\lambda_i)G(\lambda)}{\varphi(\nu,\lambda_i)G(\nu)} \overset{*}{=} \frac{(\mathcal{D}_{n_i}\Phi(\lambda))G(\lambda)}{(\mathcal{D}_{n_i}\Phi(\nu))G(\nu)} = \frac{\Delta_i\Psi(\lambda)}{\Delta_i\Psi(\nu)},$$

where the equality $\overset{*}{=}$ is a consequence of Eq. (10.34). Finally, Eq. (10.53) follows from (10.52), observing that

$$\frac{\Psi(\nu)}{\Delta_j\Psi(\nu)} = \frac{\Phi(\nu)G(\nu)}{\Delta_j(\Phi(\nu)G(\nu))} \xrightarrow{\nu\to\lambda_i} \frac{H_i}{\Delta_j H_i}.$$

□

Remark: (i) The Lévy transformation was first derived, in the $\bar{\partial}$ context, in Ref. 16, in the particular case in which $\Psi(\lambda)$ is canonically normalized.

(ii) We have seen that the limits of the fundamental transformation have a very elementary interpretation in the $\bar{\partial}$ formalism as limits on the zeros and poles of the corresponding transformation function $g(\lambda)$; this is another indication of the power of this approach. As a consequence of that, the basic identities (3.16)–(3.18) associated with the Laplace transformations have the following elementary interpretation in terms of multiplications of rational functions:

$$\frac{\lambda-\lambda_i}{\lambda-\lambda_j}\frac{\lambda-\lambda_j}{\lambda-\lambda_i} = 1 \Rightarrow \mathcal{L}_{ij}\circ\mathcal{L}_{ji} = \mathrm{id},$$

$$\frac{\lambda-\lambda_j}{\lambda-\lambda_k}\frac{\lambda-\lambda_i}{\lambda-\lambda_j} = \frac{\lambda-\lambda_i}{\lambda-\lambda_k} \Rightarrow \mathcal{L}_{jk}\circ\mathcal{L}_{ij} = \mathcal{L}_{ik},$$

$$\frac{\lambda-\lambda_k}{\lambda-\lambda_i}\frac{\lambda-\lambda_i}{\lambda-\lambda_j} = \frac{\lambda-\lambda_k}{\lambda-\lambda_j} \Rightarrow \mathcal{L}_{ki}\circ\mathcal{L}_{ij} = \mathcal{L}_{kj}.$$

F. Finite transformations versus discretization

We finally conclude this section with a short discussion, in the framework of the $\bar{\partial}$ formalism, on the connections[28] between finite transformations of integrable continuous systems and the integrable discrete analogs of such continuous systems. It is enough to observe that the fundamental transformation of a conjugate net, which, on the $\bar{\partial}$ level, reads as

$$\tilde{R}(\lambda',\lambda) = \left(1+\frac{\beta}{\lambda'-\mu}\right)R(\lambda',\lambda)\left(1+\frac{\beta}{\lambda-\mu}\right)^{-1},$$

can be formally interpreted as the shift in the additional discrete variable n_0 described in Eq. (10.35), after the identifications: $\beta=\alpha_0$, $\mu=\lambda_0$. It is a simple exercise to verify that the fundamental transformation (10.45) is equivalent to the relation (10.36).

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