AKNS Hierarchy, Self–Similarity, String Equations and the Grassmannian*

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Abstract

In this paper the Galilean, scaling and translational self–similarity conditions for the AKNS hierarchy are analysed geometrically in terms of the infinite dimensional Grassmannian. The string equations found recently by non–scaling limit analysis of the one–matrix model are shown to correspond to the Galilean self–similarity condition for this hierarchy. We describe, in terms of the initial data for the zero–curvature 1–form of the AKNS hierarchy, the moduli space of these self–similar solutions in the Sato Grassmannian. As a byproduct we characterize the points in the Segal–Wilson Grassmannian corresponding to the Sachs rational solutions of the AKNS equation and to the Nakamura–Hirota rational solutions of the NLS equation. An explicit 1–parameter family of Galilean self–similar solutions of the AKNS equation and the associated solution to the NLS equation is determined.

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1 Introduction

Matrix models have been extensively used as a non–perturbative formulation of string theory. In the Hermitian one–matrix model with even potentials [4], the double scaling limit implies for the specific heat the Korteweg–de Vries hierarchy and an additional constraint, the so called string equation. This is relevant in the Witten–Kontsevich [27] description of the the intersection theory of the moduli space of complex curves. Motivated by some anomalous behaviour of the solutions to the string equation, a modification of it was proposed in [6], the 2D–stable quantum gravity. The former string equation corresponds to invariance under Galilean transformations and the later to invariance under scaling transformations. Further, it was shown [20] that for the symmetric unitary matrix model with even potentials and some boundary terms in the double scaling limit the specific heat satisfies the modified Korteweg–de Vries hierarchy and a string equation, corresponding to the self–similarity condition under scaling transformations.

The infinite–dimensional Grassmannian model of Sato [24] for the Korteweg–de Vries hierarchy and the associated periodic flag manifold have been extensively used in the analysis of these string equations, [15, 14].

The interplay of matrix models with different integrable systems is of great interest. Very recently a non–scaling limit analysis of the Hermitian one–matrix model not restricted to the even potential case has been given, [5]. It is found that the Ablowitz–Kaup–Newell–Segur (AKNS) hierarchy appears naturally in the model besides a string equation. The Korteweg–de Vries hierarchy is contained in the AKNS hierarchy as a reduction, and therefore appears in the model before one takes the double scaling limit. The AKNS hierarchy is a complexified version of the Nonlinear Schrödinger (NLS) hierarchy and also contains the modified Korteweg–de Vries hierarchy. In [5] one can find a discussion of the topological field theory associated with the AKNS hierarchy. The mentioned string equation corresponds, as we shall show, to a Galilean self–similarity condition for the AKNS hierarchy.

Because the AKNS hierarchy is not so well–known as the Korteweg–de Vries and that we shall deal with it along this paper, let us present now some facts about it. In [2] this hierarchy was first used implicitly to solve a number of equations by a multicomponent inverse scattering method or inverse spectral transform [1]. But the hierarchy appeared explicitly in [12] where it was extensively studied [11, 18]. In [10] the finite gap solutions were analysed and for the real version, the NLS hierarchy, this was done in [22]. One can express these solutions in terms of theta functions for the corresponding hiperelliptic curve. In the papers [7, 26] a detailed account of the Grassmannian model, Baker and τ–functions can be found. In addition, in [3] the Toda–AKNS hierarchy and its τ–functions were introduced from the point of view of representation theory for affine Lie algebras and the Birkhoff factorization method! . Notice also the similarity with the hierarchy appearing in [5].

In this paper we analyse the Sato Grassmannian geometry of the moduli space of solutions to the string equation of the non–scaling limit of the one–matrix model, and more generally of self–similar solutions under any of the local symmetries of the AKNS hierarchy. These are Galilean, scaling and translational transformations. We give a parametrization of this moduli space in terms of the initial condition for the zero–curvature 1–form of the AKNS hierarchy. As a byproduct we obtain the points in the Segal–Wilson Grassmannian corresponding to the weighted scaling self–similar rational solutions of [23, 17], and we find a 1–parameter family of Galilean self–similar solutions to the AKNS hierarchy.

In the second section we introduce the AKNS and $\mathrm{NLS}^{\pm}$ hierarchies. We prove that a Galilean self–similar solution is self–similar under certain weighted scaling, with the scaling weights determined by the initial data for some associated conserved densities. We present also a zero–curvature type formulation of the string equations.

In the following section we consider the Birkhoff factorization problem for the AKNS hierarchy and its relation with the Grassmannian. There we formulate the two main results of the paper. The first one determines the stucture of the initial conditions for which the Birkhoff factorization problem implies self–similar solutions, and the second giving the structure of the set of points in the Grassmannian associated with solutions to the string equations. That is, we analyse the moduli space in the Grassmannian.

Finally, in section 4 we examine several examples. We consider the mixed Galilean and translational self–similar condition, which corresponds to Galilean self–similarity in appropriate shifted coordinates. We obtain points that do not belong to the Segal–Wilson Grassmannian but to the Sato Grassmannian and can be expressed in terms of Gaussian and Weber's parabolic cylinder functions. We also give a family of Galilean self–similar solutions of the AKNS equation and the corresponding reduction to the NLS^{+} equation, an explode–decay non–localized wave. The scaling case with different weights is also considered in shifted coordinates. Now, there are some points that belong to the Segal–Wilson Grassmannian, they correspond to the rational solutions of [23] for the AKNS equation, and some of them reduces to the NLS^{+} equation giving the rational solutions of [17]. The subspaces in the Sato Grassmannian can be expressed in terms of Tricomi–Kumm! er's hipergeometric confluent func tions that, in the mentioned rational case, are Laurent polynomials.

2 AKNS hierarchy and string equations

We begin this section with the definition of the integrable equations known as the AKNS hierarchy, which is a complexified version of the NLS hierarchy. It is defined in terms of a couple of scalar functions p, q that depend on an infinite number of variables $\mathbf{t} := \{t_n\}_{n\geq 0} \in \mathbb{C}^\infty$ which are local coordinates for the time manifold $\mathcal{T}$. In this convention we adopted t_1 to be the space coordinate, usually denoted by x, and t_n with $n > 1$ corresponds to a time variable. The coordinate t_0 as we shall see below is associated to a symmetry of the standard AKNS hierarchy ($n > 1$).

Definition 2.1 *The AKNS hierarchy for p, q is the following collection of com-*

patible equations

$$\begin{cases} \partial_n p = 2p_{n+1}, \\ \partial_n q = -2q_{n+1}, \end{cases}$$

where $n \geq 0$, $\partial_n := \partial/\partial t_n$ *and* p_n, q_n *and* h_n *are defined recursively by the relations*

$$\begin{aligned} p_n &= \frac{1}{2}\partial_1 p_{n-1} + p h_{n-1}, \\ q_n &= -\frac{1}{2}\partial_1 q_{n-1} + q h_{n-1}, \\ \partial_1 h_n &= p q_n - q p_n, \quad n \geq 1 \end{aligned}$$

with the initial data

$$p_0 = q_0 = 0,\ h_0 = 1.$$

From the recurrence relations one gets for example

$$\begin{aligned} & p_1 = p,\ q_1 = q,\ h_1 = 0 \\ & p_2 = \frac{1}{2}\partial_1 p,\ q_2 = -\frac{1}{2}\partial_1 q,\ h_2 = -\frac{1}{2}pq \\ & p_3 = \frac{1}{4}\partial_1^2 p - \frac{1}{2}p^2 q,\ q_3 = \frac{1}{4}\partial_1^2 q - \frac{1}{2}pq^2,\ h_3 = \frac{1}{4}(p\partial_1 q - q\partial_1 p). \end{aligned}$$

The $n = 0$ flow is usually not considered in the standard AKNS hierarchy, but its inclusion will prove quite convenient. The equations for that flow are

$$\begin{cases} \partial_0 p = 2p, \\ \partial_0 q = -2q, \end{cases}$$

which means that

$$p(t_0, t_1, \dots) = \exp(2t_0)\tilde{p}(t_1, \dots),\ \ q(t_0, t_1, \dots) = \exp(-2t_0)\tilde{q}(t_1, \dots).$$

The functions $(\tilde{p}, \tilde{q})$ satisfy the standard AKNS hierarchy, and this t_0–flow reflects the fact that given a solution $(\tilde{p}, \tilde{q})$ to the standard AKNS hierarchy $(n > 0)$ then $(e^c\tilde{p}, e^{-c}\tilde{q})$ is a solution as well for any $c \in \mathbb{C}$. The $n = 1$ flow is an identity.

For $n = 2$ the equations are

$$\begin{cases} 2\partial_2 p = \partial_1^2 p - 2p^2 q, \\ 2\partial_2 q = -\partial_1^2 q + 2pq^2, \end{cases}$$

and for $n = 3$ one has

$$\begin{cases} 4\partial_3 p = \partial_1^3 p - 6pq\partial_1 p, \\ 4\partial_3 q = \partial_1^3 q - 6pq\partial_1 q. \end{cases}$$

The principal reduction $p = q = v$ implies the modified Korteweg–de Vries equation $4\partial_3 v = \partial_1^3 v - 6v^2\partial_1 v$, and the reduction defined by $p = 1$ and $q = -u$ determines the Korteweg–de Vries equation $\partial_3 u = \partial_1^3 u + 6u\partial_1 u$. Observe also that when $p = 0$ one obtains the heat equation $2\partial_2 p = \partial_1^2 p$, which is a particular case of the heat hierarchy $2^{n-1}\partial_n p = \partial_1^n p$.

From the recurrence relations one easily deduces that

$$\partial_1 h_{n+1} = \partial_n h_2,$$

from where it follows that $h_{n+1} = \partial_n \partial_1^{-1} h_2$ and so

$$\partial_m h_{n+1} = \partial_n h_{m+1}, \tag{2.1}$$

giving an infinity set of non–trivial local conservation laws [8].

Notice that the real reduction $q = \mp p^*$ and $t_n \mapsto it_n$ produces the NLS$^\pm$ hierarchy for which the t_2–flow is $2i\partial_2 p = -\partial_1^2 p \pm 2|p|^2 p$, the NLS$^\pm$ equation, and the t_3–flow is $4\partial_3 p = -\partial_1^3 p \pm 6|p|^2\partial_1 p$.

Definition 2.2 *The NLS$^\pm$ hierarchy*

$$i\partial_n p = 2p_{n+1}$$

is defined in terms of the recursion relations

$$\begin{aligned} p_n &= \frac{i}{2}\partial_1 p_{n-1} + p h_{n-1} \\ \partial_1 h_n &= \mp 2\operatorname{Im} p p_n^*, \end{aligned}$$

and $p_0 = 0, h_0 = 1$.

An essential feature of the AKNS hierarchy relies in its zero–curvature formulation [2, 12, 18]. If $\{E, H, F\}$ is the standard Weyl–Cartan basis for the simple Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ of 2×2 complex, traceless matrices we define

$$Q_n := p_n E + h_n H + q_n F,$$

and denote by

$$L_n(\lambda) := \sum_{m=0}^{n} \lambda^m Q_{n-m},$$

where λ is a complex spectral parameter. Introducing the differential 1–form

$$\chi = \sum_{n\geq 0} L_n dt_n,$$

one is allowed to formulate the AKNS hierarchy as the zero–curvature condition

$$[d - \chi, d - \chi] = 0,$$

where d is the exterior derivative operator on the differential forms $\Lambda\mathcal{T}$. This aspect of the AKNS hierarchy is connected with the spectral problem

$$\partial_1\Psi = \begin{pmatrix} \lambda & p \\ q & -\lambda \end{pmatrix}\Psi,$$

where

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}.$$

For the NLS$^{\pm}$ hierarchy one has also a zero–curvature formulation. Now the $Q_n = p_n E + ih_n H \mp p_n^* F$ are maps from $\mathcal{T}$ into the real Lie algebras $\mathfrak{su}(2)$ and $\mathfrak{su}(1,1)$ respectively.

Let us now describe the local symmetries of the hierarchy. First we have the shifts in the time variables, the infinite set of translational symmetries are isospectral symmetries of the hierarchy in the sense that they preserve the associated spectral problem. Let ϑ be

$$\vartheta(\mathbf{t}) := \mathbf{t} + \boldsymbol{\theta},$$

the action of translations, where

$$\boldsymbol{\theta} := \{\theta_n\}_{n\geq 0} \in \mathbb{C}^\infty,$$

are the shifts of the time variables.

We have a local action of the abelian group $\mathbb{C}^\infty$ on the time manifold $\mathcal{T}$, then it follows

PROPOSITION 2.1 *If (p,q) is a solution to the AKNS hierarchy then so is $(\vartheta^* p, \vartheta^* q)$.*

But there are also two local non–isospectral symmetries. One is the scaling symmetry, and the other is the Galilean symmetry. Next we define both of them

DEFINITION 2.3 *The Galilean transformation $\mathbf{t} \mapsto \gamma_a(\mathbf{t})$ is given by*

$$\gamma_a(\mathbf{t})_n := \sum_{m\geq 0} \binom{n+m}{m} a^m t_{n+m}$$

where $a \in \mathbb{C}$.

The scaling transformation $\mathbf{t} \mapsto \varsigma_b(\mathbf{t})$ is represented by the relations

$$\varsigma_b(\mathbf{t})_n := e^{nb} t_n$$

where $b \in \mathbb{C}$.

We have two additive local actions of $\mathbb{C}$ over $\mathcal{T}$. One can show that

PROPOSITION 2.2 *If (p,q) is a solution of the AKNS hierarchy then so are $(\gamma_a^* p, \gamma_a^* q)$ and $(e^b \varsigma_b^* p, e^b \varsigma_b^* q)$.*

It proves convenient to define

$$t(\lambda) := \sum_{n\geq 0} \lambda^n t_n.$$

Observe that for the isospectral symmetries we have

$$\vartheta^* t(\lambda) = t(\lambda) + \theta(\lambda)$$

where

$$\theta(\lambda) := \sum_{n\geq 0} \theta_n \lambda^n,$$

and that for the non–isospectral symmetries one has

$$\gamma_a^* t(\lambda) = t(\lambda + a), \;\; \varsigma_b^* t(\lambda) = t(e^b \lambda).$$

Notice that for the corresponding solutions $(\tilde{p}, \tilde{q})$ of the standard AKNS hierarchy the Galilean action is $(\exp(2t(a))\gamma_a^*\tilde{p}, (\exp(-2t(a))\gamma_a^*\tilde{q})$, the exponential factors are a result of the flow in t_0 induced by the Galilean transformation. The related fundamental vector fields, infinitesimal generators of the action of translation, Galilean and scaling transformations are given by

$$\partial_n, \; n \geq 0, \;\; \boldsymbol{\gamma} = \sum_{n\geq 0} (n+1) t_{n+1} \partial_n, \;\; \boldsymbol{\varsigma} = \sum_{n\geq 1} n t_n \partial_n,$$

respectively. They generate the linear space $\mathbb{C}\{\partial_n, \boldsymbol{\varsigma}, \boldsymbol{\gamma}\}_{n\geq 0}$ which is the Lie algebra of local symmetries of the AKNS hierarchy, the non–trivial Lie brackets are

$$[\partial_n, \boldsymbol{\varsigma}] = n\partial_n, \;\; [\partial_{n+1}, \boldsymbol{\gamma}] = (n+1)\partial_n, \;\; [\boldsymbol{\varsigma}, \boldsymbol{\gamma}] = 2\boldsymbol{\gamma}.$$

Consider the following vector field belonging to this Lie algebra,

$$X := \boldsymbol{\vartheta} + a\boldsymbol{\gamma} + b\boldsymbol{\varsigma},$$

with

$$\boldsymbol{\vartheta} = \sum_{n\geq 0} \theta_n \partial_n,$$

defining a superposition of translations, Galilean and scaling transformations.

If (p, q) is a solution of the AKNS hierarchy then there is a 1–parameter family of solutions (p_τ, q_τ) generated by the vector field X. We have the important notion

Definition 2.4 *A self–similar solution under any of the mentioned symmetries is a solution which remains invariant under the corresponding transformation.*

Then we have,

Proposition 2.3 *A solution (p,q) of the AKNS hierarchy is self–similar under the action of the vector field X if and only if it satisfies the generalized string equations*

$$\begin{cases} Xp + bp = 0, \\ Xq + bq = 0, \end{cases} \tag{2.2}$$

Notice that when $X = \boldsymbol{\gamma} + \sum_{n\geq 0} \theta_n \partial_n$ one can perform the coordinate transformation $t_{n+1} \mapsto t_{n+1} + \theta_n/(n+1)$. Thus, the coefficient θ_n is equivalent to a shift in the time coordinate t_{n+1}.

Now, if $X = \boldsymbol{\varsigma} + \sum_{n\geq 0} \theta_n \partial_n$ we can define the transformation $t_{n+1} \mapsto t_{n+1} + \theta_{n+1}/(b(n+1))$ and obtain in the new coordinates a vector field corresponding to scaling and a term of type $\theta_0 \partial_0$. This last term can be understood as follows. Given a solution (p,q) to the AKNS hierarchy then $(\exp(b(1+2\theta_0))\varsigma_b^* p, (\exp(b(1-2\theta_0))\varsigma_b^* q)$ is a solution as well. So solutions self–similar under the vector field X correspond in adequate coordinates, to self–similarity under this particular scaling, that we shall call $(1+2\theta_0, 1-2\theta_0)$ weighted scaling.

Now we shall prove that Galilean self–similarity implies scaling self–similarity. We have,

Proposition 2.4 *If (p,q) is a solution to the AKNS hierarchy self–similar under the action of the vector field*

$$\boldsymbol{\gamma} + \sum_{n\geq 0} \theta_n \partial_n,$$

then it is also self–similar under the action of the vector field

$$\boldsymbol{\varsigma} + \sum_{n\geq 0} \theta_n \partial_{n+1} - (\sum_{n\geq 1} \theta_n h_{n+1}\,|_{\mathbf{t}=0})\partial_0.$$

This proposition simply says that the L_{-1}–Virasoro constraint implies the L_0–Virasoro constraint.

Proof: We have

$$(\boldsymbol{\gamma} + \sum_{n\geq 0} \theta_n \partial_n)p = (\boldsymbol{\gamma} + \sum_{n\geq 0} \theta_n \partial_n)q = 0.$$

Therefore, we obtain the relations

$$(\boldsymbol{\gamma} + \sum_{n\geq 0} \theta_n \partial_n)p_{n+1} = -np_n, \;\; (\boldsymbol{\gamma} + \sum_{n\geq 0} \theta_n \partial_n)q_{n+1} = -nq_n,$$

where, for example, we have used the fact that $2p_{n+1} = \partial_n p$, p is killed by $\boldsymbol{\gamma} + \sum_{n\geq 0} \theta_n \partial_n$ and the commutation relation of this vector field and ∂_n. One can equally deduce

$$(\boldsymbol{\gamma} + \sum_{n\geq 0} \theta_n \partial_n)h_{n+1} = -nh_n.$$

Because

$$\begin{cases}\partial_{n+1}p = (\frac{1}{2}\partial_n + 2h_{n+1})p\\ \partial_{n+1}q = -(\frac{1}{2}\partial_n + 2h_{n+1})q\end{cases}$$

it follows

$$\begin{cases}(\boldsymbol{\varsigma} + \sum_{n\geq 0}\theta_n\partial_{n+1})p = \frac{1}{2}(\boldsymbol{\gamma} + \sum_{n\geq 0}\theta_n\partial_n)p + 2(\sum_{n\geq 1}(nt_n + \theta_{n-1})h_n)p\\ (\boldsymbol{\varsigma} + \sum_{n\geq 0}\theta_n\partial_{n+1})q = -\frac{1}{2}(\boldsymbol{\gamma} + \sum_{n\geq 0}\theta_n\partial_n)q - 2(\sum_{n\geq 1}(nt_n + \theta_{n-1})h_n)q.\end{cases}$$

Observe that

$$\partial_n \sum_{m\geq 1}(mt_m + \theta_{m-1})h_m = nh_n + (\boldsymbol{\gamma} + \sum_{m\geq 0}\theta_m\partial_m)h_{n+1},$$

as follows from (2.1). Hence, when (p, q) is self–similar under $\boldsymbol{\gamma} + \sum_{m\geq 0}\theta_m\partial_m$ we have

$$\sum_{n\geq 1}(nt_n + \theta_{n-1})h_n = \sum_{n\geq 0}\theta_n h_{n+1}\,|_{\mathbf{t}=0}\ .$$

This implies

$$\begin{cases}(\boldsymbol{\varsigma} + \sum_{n\geq 0}\theta_n\partial_{n+1})p - 2(\sum_{n\geq 0}\theta_n h_{n+1}\,|_{\mathbf{t}=0})p = 0\\ (\boldsymbol{\varsigma} + \sum_{n\geq 0}\theta_n\partial_{n+1})q + 2(\sum_{n\geq 0}\theta_n h_{n+1}\,|_{\mathbf{t}=0})q = 0,\end{cases}$$

and the proposition follows.□

If we denote by $p = \exp(s)$ and $q = -u\exp(-s)$ then the AKNS hierarchy transforms in the hierarchy appearing in [5] for the fields $u = R$ and $S = \partial_1 s$, and the string equation is the one above with $t_0 = 0$ and $b = \theta_n = 0$. This hierarchy appears in that papers as a result of a non–scaling limit analysis of the Hermitian one–matrix model. The first conserved density of the AKNS hierarchy is proportional to the specific heat

$$2h_2 = -pq = \partial_1^2 \ln Z.$$

If $a = b = 0$ one is led to the translational self–similar solutions of the AKNS hierarchy, that is, the finite–gap solutions of the integrable equation in the spirit of Novikov. The solutions of that type can be constructed in terms of Riemann surfaces, in particular hiperelliptic curves, and the corresponding τ and Baker functions can be expressed in terms of theta functions of such curves (see [10, 7] for the AKNS equation and [22] for the NLS equation). The Galilean self–similarity condition in the KdV case is considered by Novikov [19] as a quantized version of the finite gap solutions.

In general the self–similarity condition can be reformulated as a zero–curvature type condition. We define the outer derivative

$$\delta := (a + b\lambda)\frac{d}{d\lambda}, \qquad (2.3)$$

and

$$M := \langle \chi, X \rangle, \tag{2.4}$$

Here $\langle \cdot, \cdot \rangle$ is the standard pairing between 1–forms and vector fields. Then one has,

Theorem 2.1 *The zero–curvature type condition*

$$[d - \chi, \delta - M] = 0 \tag{2.5}$$

is equivalent to the string equation (2.2).

Proof: This follows from the condition

$$\delta \chi = L_X \chi,$$

where L_X denotes the Lie derivative along the vector field X. But

$$L_X \chi = (i_X d + d i_X) \chi,$$

and recalling the zero–curvature condition for χ, we obtain the desired result. □

This theorem plays a key rôle for the analysis of the moduli space of the string equation and it is associated with the isomonodromy method.

All results regarding symmetries can be reduced to the $\mathrm{NLS}^{\pm}$ hierarchy with $\theta_n = i\tilde{\theta}_n$ and $\tilde{\theta}_n, a, b \in \mathbb{R}$.

3 Grassmannians and the moduli space for the string equations

In this section we use the Grassmannian manifold $\mathrm{Gr}^{(2)}$ to describe the AKNS flows, and to characterize geometrically the string equations for the self–similar solutions of the AKNS hierarchy. This manifold appears when one considers the Birkhoff factorization problem.

Recall that χ defines a 1–form with values in the loop algebra $L\mathfrak{sl}(2, \mathbb{C})$ of smooth maps from the circle $S^1 := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ to the simple Lie algebra $\mathfrak{sl}(2, \mathbb{C})$. We define an infinite set of commuting flows in the corresponding loop group $LSL(2, \mathbb{C})$

$$\psi(\mathbf{t}, \lambda) := \exp(t(\lambda) H) \cdot g(\lambda)$$

where g is the initial condition. Denote by $L^+SL(2, \mathbb{C})$ those loops which have a holomorphic extension to the interior of S^1 [21], and by $L_1^-SL(2, \mathbb{C})$ those which extend analitically to the exterior of the circle and are normalized by the identity at ∞.

The Birkhoff factorization problem for a given $\psi(\mathbf{t})$ consists in finding the representation

$$\psi = \psi_-^{-1} \cdot \psi_+, \tag{3.1}$$

where $\psi_- \in L_1^- SL(2,\mathbb{C})$ and $\psi_+ \in L^+ SL(2,\mathbb{C})$, and is connected with the AKNS hierarchy. The element ψ_- can be parametrized by functions (p,q) in such a way that ψ_- is a solution to the factorization problem if and only if (p,q) is a solution to the AKNS hierarchy [13]. To this end one factorizes ψ_- as follows

$$\psi_- = u \cdot \phi$$

where

$$\ln u = \sum_{n\geq 1} \lambda^{-n} U_n, \ \ \phi = \exp(\sum_{n\geq 1} \Phi_n \lambda^{-n} H)$$

here $U_n(\mathbf{t}) \in \operatorname{Im} \operatorname{ad} H$ and $\partial_m \Phi_n$ can be expressed as polynomials on (p,q) and its ∂_1–derivatives. For the elements of Sec. 2 we have the relation

$$Q_n = \sum_{\substack{i_1+\cdots+i_m=n \\ 1\leq m\leq n}} \frac{1}{m!} \operatorname{ad} U_{i_1} \cdots \operatorname{ad} U_{i_m} H,$$

and an infinite set of non–trivial local conservation laws given by

$$\partial_n(\partial_1 \Phi_m) = \partial_1(\partial_n \Phi_m),$$

for the evolution generated by the vector field ∂_n. This conservation laws where first found in [9] and are equivalent to the h_n of [8] as is shown in [3].

The $n = 0$ flow is trivial, $\partial_0 \Phi_n = 0$ and $(\partial_0 - \operatorname{ad} H) U_n = 0$.

One also has that

$$\chi := d\psi_+ \cdot \psi_+^{-1} = P_+ \operatorname{Ad} \psi_- \left(H dt(\lambda)\right) \tag{3.2}$$

is the zero–curvature 1–form for the AKNS hierarchy [13]. Here $\mathrm{id} = P_+ + P_-$ is the resolution of the identity related to the spliting

$$L\mathfrak{sl}(2,\mathbb{C}) = L^+\mathfrak{sl}(2,\mathbb{C}) \oplus L_1^- \mathfrak{sl}(2,\mathbb{C}).$$

Observe that

$$\operatorname{Ad} \psi_- H = \sum_{n\geq 0} \lambda^{-n} Q_n. \tag{3.3}$$

One can conclude from these considerations that the projection of the commuting flows $\psi(\mathbf{t})$ on the Grassmannian manifold [21, 25]

$$LSL(2,\mathbb{C})/L^+SL(2,\mathbb{C}) \cong \mathrm{Gr}^{(2)},$$

can be described in terms of the AKNS hierarchy [13, 26].

The element g determines a point in the Grassmannian manifold up to the gauge freedom $g \mapsto g \cdot h$, where $h \in L^+SL(2,\mathbb{C})$. A solution of the AKNS hierarchy does not change when $g(\lambda) \mapsto \exp(\beta(\lambda)H) \cdot g(\lambda)$ if $\exp(\beta H) \in L_1^- SL(2,\mathbb{C})$. We can say that the moduli space for the AKNS hierarchy contains the double coset space

$$\mathcal{M} := \Gamma_- \backslash LSL(2,\mathbb{C})/L^+SL(2,\mathbb{C})$$

where Γ_- is the abelian subgroup with Lie algebra $\mathbb{C}\{\lambda^n H\}_{n<0}$.

This makes a connection with the Grassmannian description for the AKNS hierarchy given in [7, 26]. The Baker function $w(\mathbf{t}) \in LSL(2,\mathbb{C})$ corresponds to

$$w = \psi_- \cdot \exp(tH) = \psi_+ \cdot g^{-1}.$$

If we introduce the notation

$$g = \begin{pmatrix} \varphi_1 & \tilde{\varphi}_1 \\ \varphi_2 & \tilde{\varphi}_2 \end{pmatrix},$$

then we have the associated subspace

$$W = \mathbb{C}\left\{\lambda^n\left(\tilde{\varphi}_2,\ -\tilde{\varphi}_1\right), \lambda^n\left(\varphi_2,\ -\varphi_1\right)\right\}_{n\geq 0},$$

with $\lambda W \subset W$, in the Grassmannian $\mathrm{Gr}^{(2)}$, [21, 25]. The Baker function is the unique function with its rows taking its values in W such that $P_+(w\cdot\exp(-tH)) = 1$. Obviously we have

$$\partial_1 w = L_1 w$$

and also

$$\partial_n w = L_n w.$$

The rows of the adjoint Baker function $w^* = (w^{-1})^t$ are maps into the subspace

$$W^* = \mathbb{C}\left\{\lambda^n\Phi, \lambda^n\tilde{\Phi}\right\}_{n\geq 0} \in \mathrm{Gr}^{(2)},$$

where

$$\Phi := (\varphi_1,\ \ \varphi_2),\ \ \tilde{\Phi} := (\tilde{\varphi}_1,\ \ \tilde{\varphi}_2).$$

We shall adopt this subspace as a representative of the coset $g \cdot L^+SL(2,\mathbb{C})$.

Let us now try to find for which initial conditions g one gets self–similar solutions, i.e. points in the Grassmannian that are connected to self–similar solutions of the AKNS hierarchy.

Recall that we have the derivation $\delta \in \mathrm{Der}L^+\mathfrak{sl}(2,\mathbb{C})$ defined in (2.3) and the vector $M(\mathbf{t}) \in L^+\mathfrak{sl}(2,\mathbb{C})$ defined in (2.4). One has the

THEOREM 3.1 *If the initial condition g satisfies the equation*

$$\delta g \cdot g^{-1} + \mathrm{Ad} g K = (\theta + f)H, \tag{3.4}$$

for some $K \in L^+\mathfrak{sl}(2,\mathbb{C})$ and some $f \in L_1^-\mathbb{C}$, then the corresponding solution to the AKNS hierarchy satisfies the string equation (2.2).

Proof: For $\chi = d\psi_+ \cdot \psi_+^{-1}$ we observe that the equation (2.5) holds if and only if

$$M = \delta\psi_+ \cdot \psi_+^{-1} + \mathrm{Ad}\psi_+ K, \tag{3.5}$$

for some $K \in L^+\mathfrak{sl}(2,\mathbb{C})$. This, together with the factorization problem (3.1), implies the relation

$$M = \delta\psi_- \cdot \psi_-^{-1} + \mathrm{Ad}\psi_- \left((a+b\lambda)\frac{dt}{d\lambda}H + \mathrm{Ad}\exp(tH)(\delta g \cdot g^{-1} + \mathrm{Ad}gK)\right).$$

Now, $M(\mathbf{t}) \in L^+\mathfrak{sl}(2,\mathbb{C})$ and Eq.(3.4) gives

$$M = P_+ \mathrm{Ad}\psi_- \left((a+b\lambda)\frac{dt}{d\lambda} + \theta H\right).$$

Taking into account Eq.(3.2) we recover (2.4) and therefore the string equation is satisfied. □

Notice that the function f can be transformed into

$$f(\lambda) \mapsto f(\lambda) + (a+\lambda b)\frac{d\beta}{d\lambda}(\lambda),$$

where $\beta \in L_1^-\mathbb{C}$. If $b \neq 0$ then one transforms $f \mapsto 0$, but when $b = 0, a \neq 0$ one is only allowed to do $f \mapsto c\lambda^{-1}$, finally if $a = b = 0$ we can not remove f.

The Sato Grassmannian [24] contains much more self–similar solutions than the Segal–Wilson one [25]. In fact, only the finite gap solutions —pure translational self–similarity— and the scaling self–similar rational solutions of Sachs [23] for the AKNS equation, and the corresponding Nakamura–Hirota solutions for NLS^+ equation [17], are found in this Grassmannian. Therefore, we shall consider the Sato Grassmannian $\mathrm{Gr}^{(2)}$. The statements above, which are rigorous in the Segal–Wilson case, can be extended to the Sato frame if the formal group $L_1^- SL(2,\mathbb{C})$ is considered only when acting by its adjoint action or by gauge transformations in the Lie algebra $\mathfrak{sl}(2,\mathbb{C})[[\lambda^{-1},\lambda]$. In this context Eqs.(2.5,3.5,3.4) still hold.

Notice that for each equivalence class in $\mathcal{M}$ an element g can be taken such that $\ln g \in \mathfrak{sl}(2,\mathbb{C})[[\lambda^{-1})$, and that any element in the coset $g \cdot L^+SL(2,\mathbb{C})$ gives the same point in the Grassmannian. One has the

Theorem 3.2 *The subspace*

$$W^* = \mathbb{C}\left\{\lambda^n \Phi, \lambda^n \tilde{\Phi}\right\}_{n\geq 0},$$

with $\Phi(\lambda), \tilde{\Phi}(\lambda) \in \mathbb{C}^2$, corresponds to a self–similar solution of the AKNS hierarchy under the action of the vector field $X = a\boldsymbol{\gamma} + b\boldsymbol{\varsigma} + \sum_{n\geq 0}\theta_n\partial_n$, if $\Phi, \tilde{\Phi}$ have the asymptotic expansion

$$\Phi(\lambda) \sim \left(\ 1 + \varphi_{11}\lambda^{-1} + \cdots,\ \ \varphi_{21}\lambda^{-1} + \varphi_{22}\lambda^{-2} + \cdots\ \right),\ \lambda \to \infty$$
$$\tilde{\Phi}(\lambda) \sim \left(\ \tilde{\varphi}_{11}\lambda^{-1} + \tilde{\varphi}_{12}\lambda^{-2} + \cdots,\ \ 1 + \tilde{\varphi}_{21}\lambda^{-1} + \cdots\ \right),\ \lambda \to \infty,$$

and satisfy

1. *When $b \neq 0$ the ordinary differential equations*

$$(a+b\lambda)\frac{d\Phi}{d\lambda} + (\sum_{n,m\geq 0} \lambda^n \theta_{n+m} h_{m,0})\Phi + (\sum_{n,m\geq 0} \lambda^n \theta_{n+m} q_{m,0})\tilde{\Phi} = \theta(\lambda)\Phi H,$$
$$(a+b\lambda)\frac{d\tilde{\Phi}}{d\lambda} - (\sum_{n,m\geq 0} \lambda^n \theta_{n+m} h_{m,0})\tilde{\Phi} + (\sum_{n,m\geq 0} \lambda^n \theta_{n+m} p_{m,0})\Phi = \theta(\lambda)\tilde{\Phi} H.$$

2. *When $b = 0, a \neq 0$ the ordinary differential equations*

$$a\frac{d\Phi}{d\lambda} + (\sum_{n,m\geq 0} \lambda^n \theta_{n+m} h_{m,0})\Phi + (\sum_{n,m\geq 0} \lambda^n \theta_{n+m} q_{m,0})\tilde{\Phi} = (\theta(\lambda) - \lambda^{-1}\sum_{n\geq 0}\theta_n h_{n+1,0})\Phi H,$$
$$a\frac{d\tilde{\Phi}}{d\lambda} - (\sum_{n,m\geq 0} \lambda^n \theta_{n+m} h_{m,0})\tilde{\Phi} + (\sum_{n,m\geq 0} \lambda^n \theta_{n+m} p_{m,0})\Phi = (\theta(\lambda) - \lambda^{-1}\sum_{n\geq 0}\theta_n h_{n+1,0})\tilde{\Phi} H,$$

3. *And when $a = b = 0$ the algebraic relations*

$$(\sum_{n,m\geq 0} \lambda^n \theta_{n+m} h_{m,0})\Phi + (\sum_{n,m\geq 0} \lambda^n \theta_{n+m} q_{m,0})\tilde{\Phi} = (\theta(\lambda) + f(\lambda))\Phi H,$$
$$-(\sum_{n,m\geq 0} \lambda^n \theta_{n+m} h_{m,0})\tilde{\Phi} + (\sum_{n,m\geq 0} \lambda^n \theta_{n+m} p_{m,0})\Phi = (\theta(\lambda) + f(\lambda))\tilde{\Phi} H,$$

where

$$f(\lambda) = \sqrt{-\det(\sum_{n\geq 0}\theta_n L_{n,0}(\lambda))} - \theta(\lambda) = \sqrt{-\det(\sum_{\substack{n\geq 0\\ m\geq 0}}\theta_m Q_{n+m,0}\lambda^{-n})}, \tag{3.6}$$

has the asymptotic expansion

$$f(\lambda) \sim \sum_{n>0} f_n \lambda^{-n}, \;\; \lambda \to \infty$$

with the recursion relation

$$f_n = -\sum_{m=1}^{n-2} h_{n-m,0} f_m - \sum_{m\geq 0}\theta_m h_{n+m,0}.$$

Here we denote $F_0 = F\,|_{\mathbf{t}=0}$.

Proof: Since $\exp(tH)\,|_{\mathbf{t}=0} = \mathrm{id}$ it follows from (3.1) that $\psi_+\,|_{\mathbf{t}=0} = \mathrm{id}$ (formally $g^{-1} = \psi_-\,|_{\mathbf{t}=0}$) and Eq.(3.5) gives

$$K = M\,|_{\mathbf{t}=0}\;.$$

But, from (2.4) we have

$$K = \langle \chi\,|_{\mathbf{t}=0}\,, \boldsymbol{\vartheta}\rangle,$$

where we have taken into account that

$$X\,|_{\mathbf{t}=0} = \boldsymbol{\vartheta}.$$

Observe that

$$K = \sum_{n\geq 0} \theta_n L_n\,|_{\mathbf{t}=0} = \mathrm{Ad}g^{-1}(\theta H) - P_{-}\mathrm{Ad}g^{-1}\theta H, \tag{3.7}$$

where we have used $(\psi_{-}\,|_{\mathbf{t}=0})^{-1} = g$. Therefore, we have

$$\mathrm{Ad}gK = \theta H - \mathrm{Ad}gP_{-}\mathrm{Ad}g^{-1}\theta H, \tag{3.8}$$

When $b \neq 0$ we can remove the function f, and from (3.4) one gets the desired result. When $b = 0, a \neq 0$ we have a contribution from f of type $c\lambda^{-1}$. This can be handled as follows. With the aid of Eq.(3.8) the equation (3.4) can be written as

$$a\frac{dg}{d\lambda} \cdot g^{-1} - \mathrm{Ad}gP_{-}\mathrm{Ad}g^{-1}\theta H = c\lambda^{-1}H.$$

Now, because the residue at $\lambda = 0$ of the first term on the left hand side of the equation above vanishes we have

$$-\mathrm{res}_{\lambda=0}\mathrm{Ad}g^{-1}\theta H = cH,$$

or

$$-\sum_{n\geq 0} \theta_n Q_{n+1,0} = cH,$$

thus

$$c = -\sum_{n\geq 0} \theta_n h_{n+1,0}.$$

When $a = b = 0$ the Eqs.(3.4,3.7) implies the form of f in the first equality of (3.6), the second expression follows from (3.8,3.3). With this the proof is completed.□

This theorem provides us with a parametrization of the moduli space of self–similar solutions of the AKNS hierarchy under the action of the vector field X in terms of initial conditions for the zero–curvature 1–form χ. Notice that the equation characterizing g depends on $K = \sum_{n\geq 0} \theta_n L_n\,|_{\mathbf{t}=0}$. Thus, if θ is a polynomial of degree N the matrix K depends on $3N$ constants $\{p_n, q_n, h_n\}_{n=1}^{N}$, but the h_n can be expressed as polynomials of $\{p_m, q_m\}_{m=1}^{n-1}$. When a or b do not vanish we have an inclusion of this $2N$–dimensional algebraic variety into the Sato Grassmannian, but one of the parameters can be supressed because the freedom $(p,q) \mapsto (e^c p, e^{-c} q)$. Thus, there is an inclusion of a $2N-1$–dimensional algebraic variety into the Sato Grassmannian providing us with a description of the moduli space. When $a = b = 0$ one has the additional dependence on f which is a function of K only, and therefore one has an inclusion of that algebraic variety into the Segal–Wilson Grassmannian, the finite–gap solutions associated with hiperelliptic curves.

4 Examples

We give in this section a concrete analysis of the ODE's characterizing the points in the Grassmannian associated with self–similar solutions. We start with the Galilean case and then we study the weighted scaling case. For the Galilean case we see that the points corresponding to self–similar solutions can be expressed in terms of Gaussian and Weber's parabolic cylinder functions, and that they never belong to the Segal–Wilson Grassmannian but to the Sato Grassmannian. We give the analytic expression of the solution for the AKNS system when $t_n = 0$ for $n > 2$. In the weighted scaling case we find that the points in the Grassmannian can be constructed with the aid of Tricomi–Kummer's confluent hipergeometric functions. We see that for certain cases, when the rows of g are Laurent polynomials of different degrees and therefore define points in the Segal–Wilson Grassmannian, these points are associated to the rational solutions of the AKNS equation found in [23] and! to the corresponding rational sol utions of the NLS^+ equation of [17].

4.1 Galilean self–similarity

We are going to consider the string equation defined by the vector field $X = \boldsymbol{\gamma} + \theta_1\partial_1$. As we have already discussed this corresponds to self–similar solutions under the Galilean symmetry in the shifted coordinates $t_2 \mapsto t_2 + \theta_1/2$ and $t_n \mapsto t_n$ for $n \neq 2$. This shift allows us to avoid the singularities of the solution at $t_2 = 0$.

The form of the initial condition is

$$g = \mathrm{id} + \lambda^{-1}X_1 + \cdots,$$

which corresponds to a self–similar solution under the vector field X if it satisfies

$$\frac{dg}{d\lambda} + g\theta_1(p_0E + \lambda H + q_0F) = \theta_1\left(\lambda + \frac{p_0q_0}{2}\lambda^{-1}\right)Hg \tag{4.9}$$

that for X_n reads

$$-\left(n + \theta_1\frac{p_0q_0}{2}H\right)X_n + \theta_1X_{n+1}(p_0E + q_0F) = \theta_1[H, X_{n+2}].$$

If we introduce the notation

$$X_n = \begin{pmatrix} A_n & X_n^+ \\ X_n^- & B_n \end{pmatrix} \tag{4.10}$$

it results

$$X_{n+2}^+ = -\frac{1}{2\theta_1}\frac{\left(n + \frac{\theta_1}{2}p_0q_0\right)\left(n + 1 + \frac{\theta_1}{2}p_0q_0\right)}{n+1}X_n^+$$
$$X_{n+2}^- = \frac{1}{2\theta_1}\frac{\left(n - \frac{\theta_1}{2}p_0q_0\right)\left(n + 1 - \frac{\theta_1}{2}p_0q_0\right)}{n+1}X_n^-$$

and

$$A_n = \frac{\theta_1 q_0}{n + \frac{\theta_1}{2} p_0 q_0} X_n^+$$
$$B_n = \frac{\theta_1 p_0}{n - \frac{\theta_1}{2} p_0 q_0} X_n^-$$

that together with $X_1^+ = p_0/2$ and $X_1^- = -q_0/2$ gives us the matrix g. Observe that $X_{2n}^+ = X_{2n}^- = 0$ and $A_{2n+1} = B_{2n+1} = 0$. The expansion never converges, we can choose $p_0 q_0$ such that the first row of g is polynomial in λ^{-1} but the the second row does not converge. We conclude that this solution belongs to the Sato Grassmannian and not to the Segal–Wilson one.

Now, writing

$$g = \begin{pmatrix} A & X^+ \\ X^- & B \end{pmatrix} \tag{4.11}$$

Eq.(4.9) for A, B reads

$$\lambda^2 \frac{d^2 A}{d\lambda^2} - 2\theta_1 \lambda^2 \left(\lambda + \frac{p_0 q_0}{2}\right) \frac{dA}{d\lambda} + \frac{\theta_1}{2} p_0 q_0 \left(1 + \frac{\theta_1}{2} p_0 q_0\right) A = 0, \tag{4.12}$$
$$\lambda^2 \frac{d^2 A}{d\lambda^2} + 2\theta_1 \lambda^2 \left(\lambda + \frac{p_0 q_0}{2}\right) \frac{dA}{d\lambda} - \frac{\theta_1}{2} p_0 q_0 \left(1 - \frac{\theta_1}{2} p_0 q_0\right) A = 0, \tag{4.13}$$

and for X^+, X^- gives

$$X^+ = -\frac{1}{\theta_1 q_0} \left(\frac{dA}{d\lambda} - \frac{\theta_1}{2} p_0 q_0 \lambda^{-1} A\right), \tag{4.14}$$
$$X^- = -\frac{1}{\theta_1 p_0} \left(\frac{dB}{d\lambda} + \frac{\theta_1}{2} p_0 q_0 \lambda^{-1} B\right). \tag{4.15}$$

Equations (4.12,4.13) can be transformed into confluent hipergeometric equations. Recall that the Tricomi–Kummer's confluent hipergeometric function $U(a, c, z)$, [16], is a solution of

$$z \frac{d^2 U}{dz^2} + (c - z) \frac{dU}{dz} - aU = 0$$

and has the asymptotic expansion [16]

$$U(a, c, z) \sim z^{-a} \sum_{n \geq 0} (-1)^n \frac{(a)_n (a + 1 - c)_n}{n!} z^{-n}, \;\; z \to \infty, \; -\frac{3}{2}\pi < \arg z < \frac{3}{2}\pi,$$

where $(\alpha)_n = \Gamma(\alpha + n)/\Gamma(n)$. One can show that

$$A(\lambda) = (\theta_1 \lambda^2)^\mu U\left(\frac{\mu}{2}, \frac{1}{2}, \theta_1 \lambda^2\right)$$

where

$$\mu := \frac{\theta_1}{2} p_0 q_0.$$

Thus,

$$A(\lambda) \sim \sum_{n\geq 0} (-1)^n \frac{(\frac{\mu}{2})_n (\frac{\mu+1}{2})_n}{n!} (\theta_1 \lambda^2)^{-n}, \ \ \lambda \to \infty.$$

For B one only needs to replace in the expression for A the parameters $\theta_1 \mapsto -\theta_1$ and $\mu \mapsto -\mu$. Hence

$$B(\lambda) \sim \sum_{n\geq 0} \frac{(-\frac{\mu}{2})_n (\frac{-\mu+1}{2})_n}{n!} (\theta_1 \lambda^2)^{-n}, \ \ \lambda \to \infty.$$

From (4.14,4.15) one gets the corresponding asymptotic expansions for X^+, X^-. In terms of the Weber's parabolic cylinder functions [16] one has for example

$$A(\lambda) = 2^{-\frac{\mu}{2}} \left(\sqrt{2\theta_1}\lambda\right)^{2\mu} \exp\left(\frac{\theta_1}{2}\lambda^2\right) D_{-\mu}\left(\sqrt{2\theta_1}\lambda\right),$$

and an analogous expression for B is obtained once θ_1 and μ are multiplied by -1. Notice the appearence of the Hermite polynomials H_n and the error function Erf, [16], when $\mu \in \mathbb{Z}$. For example when $\mu + 1 = -N$ with $N \in \mathbb{N}$ we have

$$A(\lambda) = \left(\sqrt{2\theta_1}\lambda\right)^{-2(N+1)} H_{N+1}\left(\sqrt{\theta_1}\lambda\right),$$

so that the first row of g is a polynomial, but the second is not as we already observed. For example, we have

$$B(\lambda) = K_N \exp(-\theta_1\lambda^2) \frac{d^N}{d\lambda^N}\left(\exp(\theta_1\lambda^2) \mathrm{Erfc}\left(\sqrt{-\theta_1}\lambda\right)\right)$$

where K_N is some normalization constant and $\mathrm{Erfc} = 1 - \mathrm{Erf}$ is the complement to the error function,

$$\mathrm{Erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z dt \exp(-t^2)$$

which is not a polynomial. As we have remarked before, the Galilean self–similar solutions are always associated to subspaces in the Sato Grassmannian which never belongs to the Segal–Wilson Grassmannian.

For the NLS$^\pm$ reduction we need $q = \mp p^*$, therefore

$$\mathcal{J}^\pm g(\lambda^*)^\dagger \mathcal{J}^\pm = g(\lambda)^{-1}$$

where $\mathcal{J}^+ = \mathrm{id}$ and $\mathcal{J}^- = H$. Taking into account the Eqs.(4.12,4.13,4.14,4.15) this is fulfilled when $\theta_n = i\tilde{\theta}_n, \tilde{\theta}_n \in \mathbb{R}$, the initial condition $q_0 = \mp p^*$ and $A(\lambda^*)^* = B(\lambda)$. Therefore, $\mu = \mp i\tilde{\theta}_1/2\, |p_0|^2 \in \mathbb{R}$.

Now, we analyse the Galilean invariant solutions of the AKNS equation, thus we suppose $t_n = 0$ for $n > 2$, and $b = \theta_n = 0$ for $n \geq 0$. This corresponds to a Galilean self–similar solution of the AKNS hierarchy evaluated at $t_n = 0$ for $n > 2$. It will turn out to be singular in $t_2 = 0$. So we need to shift t_2 in order to avoid it.

The string equation is

$$\begin{cases} t_1 \partial_0 p + 2t_2 \partial_1 p = 0 \\ t_1 \partial_0 q + 2t_2 \partial_1 q = 0. \end{cases}$$

Now

$$p(t_0, t_1, t_2) = \exp(2t_0)\tilde{p}(t_1, t_2), \;\; q(t_0, t_1, t_2) = \exp(-2t_0)\tilde{q}(t_1, t_2),$$

with

$$\begin{cases} 2t_1 \tilde{p} + 2t_2 \partial_1 \tilde{p} = 0 \\ -2t_1 \tilde{q} + 2t_2 \partial_1 \tilde{q} = 0. \end{cases}$$

The solutions to these equations are

$$\tilde{p}(t_1, t_2) = \exp\left(-\frac{t_1^2}{2t_2}\right) \hat{p}(t_2), \;\; \tilde{q}(t_1, t_2) = \exp\left(\frac{t_1^2}{2t_2}\right) \hat{q}(t_2)$$

where the functions $\hat{p}, \hat{q}$ must be fixed in order to have solutions to the AKNS equation, thus

$$\begin{cases} 2\partial_2 \hat{p} = -\frac{1}{t_2}\hat{p} - 2\hat{p}^2 \hat{q} \\ 2\partial_2 \hat{q} = -\frac{1}{t_2}\hat{q} + 2\hat{p}\hat{q}^2. \end{cases}$$

Finally, one finds

$$\tilde{p}(t_1, t_2) = \exp\left(-\frac{t_1^2}{2t_2}\right) a t_2^{-ab-1/2}, \;\; \tilde{q}(t_1, t_2) = \exp\left(\frac{t_1^2}{2t_2}\right) b t_2^{ab-1/2}.$$

This is a two parameter family of Galilean self–similar solutions to the AKNS hierarchy. In fact, when one performs the shift $t_2 \to t_2 + \theta_1/2$ one finds $\mu = ab$, that together with p_0 (or q_0) parametrizes the solution. One can see that μ is the unique non trivial parameter recalling that if (p, q) is a solution of the AKNS then so is any $(e^c p, e^{-c} q)$. We have for the specific heat

$$\partial_1^2 \ln Z(t_1, t_2, 0, \dots) = -\frac{\mu}{t_2}.$$

This is the solution corresponding to the point in the Grassmannian we have found above.

The corresponding Galilean self–similar solution to the NLS$^{\pm}$ is of the form

$$\tilde{p}(t_1, t_2) = \exp\left(i\frac{t_1^2}{2t_2}\right) \hat{p}(t_2),$$

where $\hat{p}$ satisfies

$$2\partial_2\hat{p} = -\frac{1}{t_2}\hat{p} \mp 2i|\hat{p}|^2\hat{p}.$$

Writing $\hat{p} = |\hat{p}|\exp(i\arg\hat{p})$ one obtains the equations

$$\partial_2|\hat{p}| = -\frac{1}{2t_2}|\hat{p}|,$$
$$\partial_2\arg\hat{p} = \mp|\hat{p}|^2.$$

Therefore

$$\tilde{p}(t_1,t_2) = e^{ia}\sqrt{\left|\frac{\mu}{t_2}\right|}\exp\left(i\left(\frac{t_1^2}{2t_2} \mp |\mu|\mathrm{sgn}t_2\ln|t_2|\right)\right).$$

Here $a \in \mathbb{R}$ is an arbitrary phase that can be removed. We have a 1–parameter family of Galilean self–similar solutions to the NLS$^\pm$ defined for $t_2 \neq 0$, with

$$|p| = \sqrt{\left|\frac{\mu}{t_2}\right|},$$

so that it vanishes at $t_2 \to \pm\infty$ and generates a singular behaviour at $t_2 = 0$, an explode–decay phenomena for a non–localized wave.

4.2 Scaling self–similarity

We are going now to consider the string equation corresponding to the vector field $X = \varsigma + \theta_0\partial_0 + \theta_1\partial_1$. As we have already discussed this corresponds to self–similar solutions under a $(1+2\theta_0, 1-2\theta_0)$ weighted scaling in the shifted coordinates $t_1 \mapsto t_1 + \theta_1$ and $t_n \mapsto t_n$ for $n > 1$. This last shift allows us to avoid possible singularities of the solution at $t_1 = 0$.

Let

$$g = \mathrm{id} + \lambda^{-1}X_1 + \cdots$$

be the initial condition for the commuting flows $\psi(\mathbf{t})$. In order to have self–similar solutions under the vector field X, it must satisfy

$$\lambda\frac{dg}{d\lambda} + g(\theta_1 p_0 E + (\theta_0 + \theta_1\lambda)H + \theta_1 q_0 F) = (\theta_0 + \theta_1\lambda)Hg \tag{4.16}$$

which implies for the matrix coefficients X_n of the Laurent expansion of g

$$-nX_n - \theta_0[H, X_n] + \theta_1 X_n(p_0E + q_0F) = \theta_1[H, X_{n+1}].$$

With the use of (4.10) one finds the recurrence laws

$$X^+_{n+1} = \frac{1}{2\theta_1}\left(-n - 2\theta_0 + \frac{\theta_1^2 p_0 q_0}{n}\right)X^+_n$$
$$X^-_{n+1} = -\frac{1}{2\theta_1}\left(-n + 2\theta_0 + \frac{\theta_1^2 p_0 q_0}{n}\right)X^-_n$$

and

$$A_n = \frac{\theta_1 q_0}{n} X_n^+$$
$$B_n = \frac{\theta_1 p_0}{n} X_n^-$$

that together with $X_1^+ = p_0/2$ and $X_1^- = -q_0/2$ give us the matrix g. There are cases for which this expansion is a polynomial in λ^{-1} and represents therefore not only an asymptotic expansion but also a well defined function. We require

$$\theta_1^2 p_0 q_0 = (N^+ + 2\theta_0)N^+ = (N^- - 2\theta_0)N^- \tag{4.17}$$

with $N^\pm \in \mathbb{N} \cup \{0\}$, so that

$$X_n^+, A_n = 0, \;\; n > N^+$$

and

$$X_n^-, B_n = 0, \;\; n > N^-.$$

Hence, we get a polynomial g in λ^{-1} of degree N^+ in the first row and degree N^- in the second one. Eqs.(4.17) imply

$$2\theta_0 = N^- - N^+ \in \mathbb{Z}$$
$$\theta_1^2 p_0 q_0 = N^+ N^- \in \mathbb{N} \cup \{0\}.$$

This gives points in Segal–Wilson Grassmannian associated with solutions of the AKNS hierarchy (p, q) which are self–similar under the $(1+N^+-N^-, 1-N^++N^-)$ weighted scaling symmetry.

Using (4.11), Eq.(4.16) for A, B reads

$$\lambda^2 \frac{d^2 A}{d\lambda^2} + ((1-2\theta_0)\lambda - 2\theta_1 \lambda^2)\frac{dA}{d\lambda} - \theta_1^2 p_0 q_0 A = 0, \tag{4.18}$$
$$\lambda^2 \frac{d^2 B}{d\lambda^2} + ((1+2\theta_0)\lambda + 2\theta_1 \lambda^2)\frac{dB}{d\lambda} - \theta_1^2 p_0 q_0 B = 0, \tag{4.19}$$

and for X^+, X^- we obtain the expressions

$$X^+ = -\frac{\lambda}{\theta_1 q_0}\frac{dA}{d\lambda}, \tag{4.20}$$
$$X^- = -\frac{\lambda}{\theta_1 p_0}\frac{dB}{d\lambda}. \tag{4.21}$$

Equations (4.18,4.19) are equivalent to confluent hipergeometric equations. Consider the roots (μ_+, μ_-) of

$$\mu^2 - 2\theta_0 \mu - \theta_1^2 p_0 q_0 = 0,$$

we get for θ_0 the value

$$2\theta_0 = \mu_+ + \mu_-, \;\; \mu_+\mu_- = -\theta_1^2 p_0 q_0.$$

If we define

$$A(\lambda) = \lambda^{\mu_+} U(2\theta_1\lambda),$$

then $U(z)$ satisfies

$$z\frac{d^2U}{dz^2} + (1+\mu_+ - \mu_- - z)\frac{dU}{dz} - \mu_+ U = 0,$$

thus we are dealing with the Tricomi–Kummer's confluent hipergeometric function $U(a,c,z)$ with $a=\mu_+$ and $c = 1+\mu_+-\mu_-$, and we deduce for $A(\lambda)$ the behaviour

$$A(\lambda) \sim \sum_{n\geq 0}(-1)^n\frac{(\mu_+)_n(\mu_-)_n}{n!}(2\theta_1\lambda)^{-n}, \;\; \lambda\to\infty.$$

For B the analysis is the same, we only need to replace $2\theta_0$ and $2\theta_1$ by $-2\theta_0$ and $-2\theta_1$ respectively in the formulas above. So the asymptotic expansion for B is

$$B(\lambda) \sim \sum_{n\geq 0}\frac{(-\mu_+)_n(-\mu_-)_n}{n!}(2\theta_1\lambda)^{-n}, \;\; \lambda\to\infty.$$

From formulas (4.20,4.21) we obtain the asymptotic expansions for X^+ and X^-. We have

$$X^+(\lambda) \sim \frac{1}{\theta_1 q_0}\sum_{n\geq 1}(-1)^n\frac{(\mu_+)_n(\mu_-)_n}{(n-1)!}(2\theta_1\lambda)^{-n}, \;\; \lambda\to\infty$$
$$X^-(\lambda) \sim \frac{1}{\theta_1 p_0}\sum_{n\geq 1}\frac{(-\mu_+)_n(-\mu_-)_n}{(n-1)!}(2\theta_1\lambda)^{-n}, \;\; \lambda\to\infty.$$

Let us notice that when $\mu_+ + \mu_- = 0$ the function U can be expressed in terms of the Macdonalds–Basset function [16], for example if $z = 2\theta_1\lambda$ we have

$$A(\lambda) = \left(1+\mu_+ - \frac{d}{dz}\right)\sqrt{\frac{z}{\pi}}\exp(z/2)K_{\mu_+-1/2}(z/2).$$

For the NLS$^\pm$ reduction we need that μ_+,μ_- be solutions of

$$\mu^2 - 2i\tilde{\theta}_0 \pm |\tilde{\theta}_1 p_0|^2 = 0.$$

In the polynomial case of the AKNS hierarchy we must have (or the other way around)

$$\mu_+ = -N^+, \;\; \mu_- = N^-.$$

Again, from the asymptotic expansions, we see that A, X^+ and B, X^- are a polynomials in λ^{-1} of degree N^+ and N^- respectively. The solutions in the polynomial case are the rational solutions of the AKNS hierarchy appearing in [23]. To connect with the notation of that paper we notice that $1 + \mathrm{p} - \mathrm{q} = N^+ - N^-$ and that $\mathrm{p} = N^+N^-$ where p,q are the degree of the polynomials corresponding to the tau functions σ, τ for the AKNS hierarchy defined in that paper. This implies that $\mathrm{q} = (N^+ - 1)(N^- + 1)$, and so $n - k = N^+$ and $k + 1 = N^-$ or viceversa $(n + 1 = N^+ + N^-)$, where n, k are those of [23].

One can easily see that the polynomial case described above is the only case for which the asymptotic series converges and defines a function in a neighbourhood of $\lambda = \infty$. Therefore, they are the only points in the Segal–Wilson Grassmannian corresponding to weighted scaling self–similar solutions, generically we have points in the Sato Grassmannian. Observe that for the NLS$^\pm$ hierarchies one arrives to the condition $2\theta_0 = N^- - N^+$ with $\theta_0 \in i\mathbb{R}$, so $\theta_0 = 0$. Then $\mu_\pm = \pm|\tilde{\theta}_1 p_0|$ in the NLS$^+$ case and $\mu_\pm = \pm i|\tilde{\theta}_1 p_0|$ for the NLS$^-$ case. So that none of the Sachs rational solutions for the AKNS system reduces to the NLS$^-$ equation, furthermore it is known that this equation does not have rational solutions. Only for the NLS$^+$ hierarchy we have points in the Segal–Wilson Grassmannian corresponding to the reduced Sachs solutions, the Nakamura–Hirota rational solutions for NLS$^+$ equation, [17]. Now, $N^+ = N^-$ and $n = 2k + 1$. Notice that in [17] it is considered not only $n = 2k$, when they analyse the Boussinesq system, as was claimed in [23] but also $n = 2k + 1$, when they study the NLS$^+$ equation.

Summing, for the Segal–Wilson case we have

Proposition 4.1 *The (n, k) rational solution for the AKNS hierarchy found in [23] corresponds to the point in the Segal–Wilson Grassmannian associated to the coset $g \cdot L^+SL(2, \mathbb{C})$ where $g \in L_1^-SL(2, \mathbb{C})$ is given by the following Laurent polynomial*

$$g(\lambda/2\theta_1) = \begin{pmatrix} \sum_{n=0}^{N^+} \frac{(-N^+)_n (N^-)_n}{n!} (-\lambda)^{-n} & \frac{1}{q_0} \sum_{n=1}^{N^+} \frac{(-N^+)_n (N^-)_n}{(n-1)!} (-\lambda)^{-n} \\ \frac{1}{p_0} \sum_{n=1}^{N^-} \frac{(N^+)_n (-N^-)_n}{(n-1)!} (\lambda)^{-n} & \sum_{n=0}^{N^-} \frac{(N^+)_n (-N^-)_n}{n!} (\lambda)^{-n} \end{pmatrix},$$

where $n + 1 = N^+ + N^-$ and $k + 1 = N^-$. These are the only weighted scaling self–similar solutions with a corresponding point in the Segal–Wilson Grassmannian. None of these reduce to the NLS$^-$ hierarchy and only when $N^+ = N^-$, $(n = 2k+1)$, $-p_0^* = q_0$ they reduce to solutions of the NLS$^+$ hierarchy.

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