Expansion of matter waves in static and driven periodic potentials

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We study the nonequilibrium dynamics of cold atoms held in an optical lattice subjected to a periodic driving potential. The expansion of an initially confined atom cloud occurs in two phases: an initial quadratic expansion followed by a ballistic behavior at long times. Accounting for this gives a good description of recent experimental results and provides a robust method to extract the effective intersite tunneling from time-of-flight measurements.

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I. Introduction. Experimental advances in confining ultracold atoms in optical lattices have undergone spectacular progress in recent years. Optical lattice potentials are extremely clean and controllable, and the excellent coherence properties of atomic condensates hold out the prospect of controlling their dynamics using quantum coherent methods. This is interesting from the point of view of fundamental physics and has many potential applications to quantum information processing, where it is vital that the coherence of the system is preserved during its time evolution. One such scheme is to use a time-periodic driving potential to induce the effect termed "dynamical localization" [1] or "coherent destruction of tunneling" [2]. This is a quantum interference effect in which a particle acquires a phase from its interaction with the driving potential, which leads to a renormalization of the single-particle tunneling probability. For specific values of the driving parameters the effective tunneling probability can be highly suppressed, providing a sensitive means of coherently controlling the localization of the atoms [3]. This renormalization has recently been directly observed in cold atom experiments [4–6].

A convenient way of measuring the effective tunneling is to observe the rate of expansion of a condensate once a harmonic potential trapping the atoms along the direction of the optical lattice has been switched off [4,5]. Although the results of these experiments agreed well with the theoretically expected scaling of the renormalized tunneling probability with the Bessel function of the driving strength, for particular initial conditions the scaling seemed to be *quadratic* rather than linear in the Bessel function. A number of explanations of this phenomenon have since been put forward, including a possible crossover from coherent to sequential tunneling [5,7] induced by phase scrambling arising from dynamical instabilities, a time-averaging effect produced by finite time-resolution of the measurement [8], and driving-induced atom pairing [9]. A recent theoretical stability analysis [10] has shown, however, that phase scrambling is unlikely to occur for the experimental parameters of Refs. [4,5], and the pairing mechanism would require rather stronger interactions than were present in those experiments. In this brief report we propose an alternative explanation for the observed quadratic scaling of the renormalized tunneling probability based on the exact form of the expansion of the condensate, which is linear in the long time limit but quadratic for short times [11]. Using this complete time dependence in order to extract the tunneling probability from the experimental data gives quantitatively accurate agreement with theory, with no adjustable parameters.

II. Model and analysis. The Bose-Hubbard model is described by the Hamiltonian

$$H_{\mathrm{BH}} = -J \sum_{\langle i,j \rangle} [a_i^\dagger a_j + \mathrm{H.c.}] + \frac{U}{2} \sum_j n_j (n_j - 1) + \sum_j V(r_j) n_j, \quad (1)$$

where a_j and $a_j^\dagger$ are annihilation and creation operators for a boson on lattice site j, J (taken to be positive) describes the hopping amplitude between nearest-neighbor sites $\langle i,j \rangle$, and U is the repulsive energy between two bosons occupying the same site. The operator $n_j = a_j^\dagger a_j$ is the standard number operator, and $V(r)$ is the external trap potential, which is usually considered to be parabolic, $V(r) = m\omega_T^2 r^2/2$, where ω_T is the trap frequency. Although simple in appearance, the Bose-Hubbard model can provide an excellent description [12] of ultracold atoms held in optical lattice potentials.

Adding a static and a sinusoidally varying force to the system leads to the general time-dependent potential

$$H(t) = H_{\mathrm{BH}} + \sum_j n_j \, j \, (\Delta + K \cos \omega t), \quad (2)$$

where Δ is the static tilt applied to the lattice, and K and ω are the amplitude and frequency, respectively, of the oscillating component. Experimentally, the two forces are introduced into the rest frame of the optical lattice by applying appropriate frequency differences to the acousto-optic modulators creating the lattice beams, as described in detail in [4]. A time-periodic system of this type can be analyzed using Floquet theory, revealing [13] that the effect of the driving can be described by the static Hamiltonian (1) with a *renormalized tunneling* J_{eff}. For an untilted lattice ($\Delta = 0$), this renormalization is given by the zeroth-order Bessel function $J_{\mathrm{eff}} = J \mathcal{J}_0(K_0)$, where for convenience we define $K_0 \equiv K/\hbar\omega$. Thus at the values $K_0 = 2.404, 5.52, \ldots$, at which the Bessel function vanishes, the effective tunneling is suppressed. This effect thus provides a means to coherently control the dynamics of trapped atoms, without altering any of the parameters of the optical lattice.

When the trap potential along the lattice direction is removed, the atom cloud will expand in time, at a rate determined by J_{eff}. To quantify this process we calculate the spread of the wave function:

$$\sigma(t) = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}, \tag{3}$$

aligning the optical lattice with the x axis. We begin by setting the Hubbard interaction, U, to zero. In the continuum approximation, valid when the kinetic energy of the condensate is much less than the width of the first Bloch band, the ground state of a parabolic trap is simply given by a Gaussian, $\psi(x) = N \exp[-x^2/(2a^2)]$, where N is the normalization such that $\int_{-\infty}^{\infty} |\psi(x)|^2 dx = 1$ and $a = \sqrt{\hbar/(m\omega_T)}$ is the harmonic trap-length. If the trap potential is now removed, this initial state will expand while *remaining* Gaussian. This expansion can be calculated analytically [11] yielding the result

$$\sigma(t) = \sigma_0 \sqrt{1 + 4(Jt/\hbar)^2 (d_L/a)^4}, \tag{4}$$

where d_L is the spacing of the optical lattice and $\sigma_0 = a/\sqrt{2}$. Similar, but more complicated, expressions that coincide with this result were obtained by Korsch *et al.* [14] using a lattice representation instead of the continuum approximation. The expansion clearly occurs in two different phases, separated by a crossover time, $t_c = (\hbar/2|J|)(a/d_L)^2$. For long expansion times, $t \gg t_c$, the wave function spreads linearly with time, $\sigma(t) \propto |J|t$, reproducing the expected ballistic expansion of a released wave packet. For short times, however, $t < t_c$, the expansion is instead *quadratic*, $\sigma(t) - \sigma_0 \propto J^2 t^2$. It is important to note that the expansion depends on both the magnitude of the tunneling, J, and the size of the initial wave packet a. In particular, a tightly confined wave packet will have a higher spread in momentum and so will enter the regime of linear expansion more quickly. The extreme case where only a *single* lattice site is filled was considered in Ref. [1], where it was found that the expansion was always linear with time. This result is exactly reproduced by Eq. (4) by appropriately taking the limit $a \to 0$.

III. Results. A. Undriven lattice. We first consider the case of a static lattice, with $\Delta, K_0 = 0$. In Fig. 1 we show the time dependence of the expansion of an initial Gaussian wave packet, numerically evolved in time under the time-dependent Hamiltonian (2). Using $J = 0.1E_{\text{rec}}$, where the recoil energy $E_{\text{rec}} = \hbar^2\pi^2/2md_L^2$, we see that the analytic expression (4) accords exactly with the numerical result, indicating the validity of the continuum approximation. The transition from the initial quadratic expansion to the ballistic regime is clearly visible. Halving the value of J used produces the expected result of reducing the rate of expansion and moving the crossover from the quadratic to the ballistic regime to a later time.

B. Untilted driven lattice. We now consider the effect of including the time-dependent driving potential $V(t) = K\cos\omega t$. We choose a high driving frequency, $\hbar\omega = 4J$, and tune the amplitude of the driving so that $J_{\text{eff}} = 0.5J$. In accordance with the predictions of the Floquet analysis, we see that the expansion of the wave packet in the driven system with a bare tunneling of $J = 0.1E_{\text{rec}}$ closely follows the result for $J = 0.05E_{\text{rec}}$, indicating that the driving field indeed renormalizes the tunneling as expected. Looking in detail at the expansion (inset of Fig. 1) we see that, although on average the expansion of the driven condensate closely matches that of the static case with $J = 0.05E_{\text{rec}}$, the driven result contains small amplitude oscillations with the same frequency as the driving. These oscillations arise from the intrinsic time dependence of the Floquet states themselves. Their amplitude reduces as the driving frequency becomes larger, indicating that the approximation of modeling the driven system with a renormalized static Hamiltonian becomes increasingly good. Finally we also show in Fig. 1 the most dramatic effect of the renormalization of tunneling. Since $J_{\text{eff}} = J\mathcal{J}_0(K_0)$, tuning K_0 to a zero of the Bessel function should result in the complete suppression of tunneling (neglecting next-nearest-neighbor tunneling [15]). We indeed see that setting $K_0 = 2.404$ results in the condensate not expanding with time, due to the vanishing of J_{eff}. Similar to the $K_0 = 1.22$ case, this curve again displays

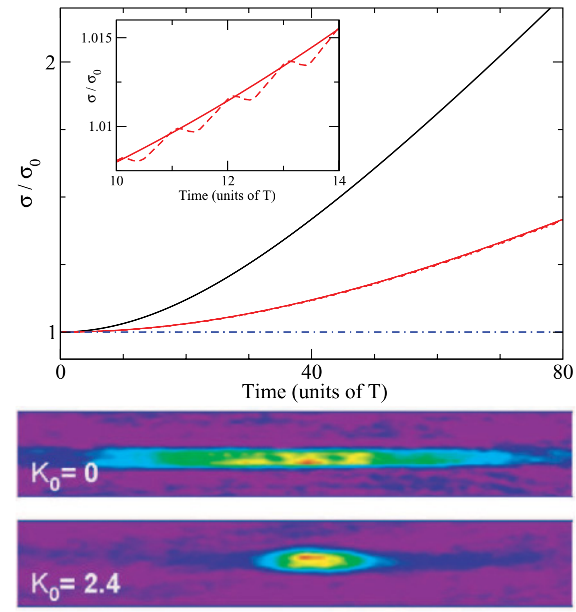

FIG. 1. (Color online) Top: Expansion of an initial Gaussian wave packet in a flat lattice potential, obtained by the numerical propagation under Hamiltonian (2). When no driving potential is applied ($K = 0$), $\sigma(t)$ increases following Eq. (4). For $J = 0.1E_{\text{rec}}$ (black solid line) the expansion is clearly quadratic initially and becomes linear at long times. Setting $J = 0.05E_{\text{rec}}$ [red (gray) solid line] reduces the expansion rate as expected. By applying a periodic driving potential the tunneling can be renormalized to an effective value J_{eff}. Tuning $K_0 = 1.22$ reduces J_{eff} so that the expansion of the condensate (dashed red line) reproduces the $J = 0.05E_{\text{rec}}$ result. Setting $K_0 = 2.404$—first zero of the Bessel function—produces coherent destruction of tunneling (CDT), and the condensate no longer expands with time (blue dash-dotted line). Inset: Detail of the periodically driven result ($K_0 = 1.22$). The driven expansion *on average* reproduces the $J = 0.05E_{\text{rec}}$ result, but shows small oscillations with the same frequency of the driving. The amplitude of these oscillations decreases with increasing driving frequency. Bottom: Experimental comparison of the free expansion of a condensate ($K_0 = 0$) with a condensate experiencing CDT ($K_0 = 2.4$). As predicted, the expansion of the second condensate is strongly suppressed.

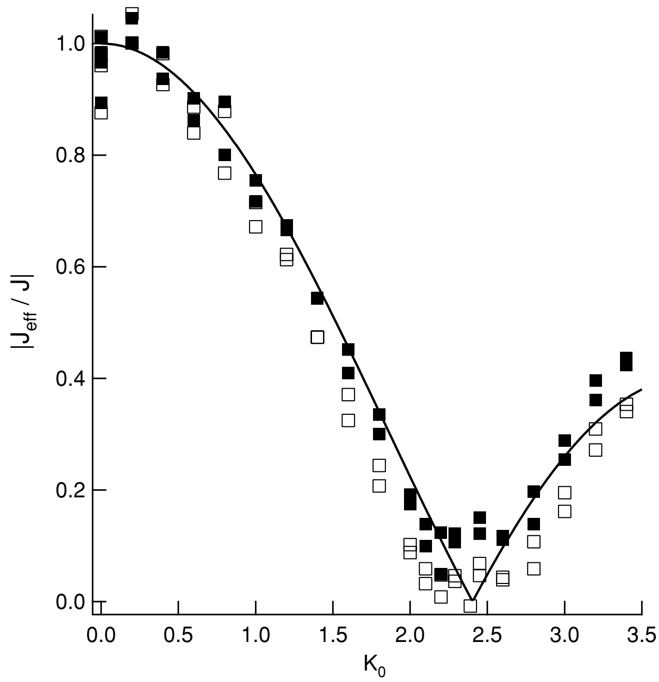

FIG. 2. Dynamical suppression of tunneling in a periodically driven lattice. The effective tunneling parameter was calculated assuming a simple linear expansion (open symbols) and by taking into account the exact expansion dynamics (solid symbols). In the latter case the agreement with the theoretically expected scaling (solid line) is clearly better. The experimental parameters were $J/h = 240$ Hz and $\omega/2\pi = 4$ kHz, and the expansion time was 150 ms.

small oscillations, which become larger at low values of ω. This low-frequency behavior would correspond to the "dynamical localization" regime [1], where the wave packet periodically returns to its initial state at stroboscopic times $t = nT = n2\pi/\omega$, but between these times can exhibit large excursions.

In Ref. [4] the effective tunneling was deduced by measuring the expansion rate of the condensate at a fixed time and assuming this rate was directly proportional to J_{eff}. Accordingly the ratio between the tunneling parameters in the static and the driven lattice was calculated as $|J_{\text{eff}}/J| = (\sigma(t) - \sigma_0)/(\sigma_{\text{stat}} - \sigma_0)$, where σ_{stat} is the size of the condensate after expansion in the static lattice. For the experimental parameters ($d_L = 426$ nm and $J/h = 270$ Hz), we calculate a crossover time of $t_c \simeq 9.7$ ms for a weakly interacting condensate released from a 20-Hz harmonic trap. As the experiment employed an expansion time of 100 ms, the results can thus be expected to be reliable only as long as $|J_{\text{eff}}/J| \geqslant 0.1$. In order to get better agreement with theory, we now use Eq. (4) containing the full expansion dynamics, giving

$$|J_{\text{eff}}/J| = \sqrt{\frac{\sigma(t)^2 - \sigma_0^2}{\sigma_{\text{stat}}^2 - \sigma_0^2}}. \tag{5}$$

Figure 2 shows that, as expected, using Eq. (5) to calculate the renormalized tunneling gives better agreement with theory.

It is interesting to note that although J_{eff} is strongly suppressed near $K_0 = 2.4$, it does not actually reach zero when the Bessel function vanishes. This is due to the effect of higher-order hopping terms present in the system's dynamics. Although they too are renormalized by the driving potential, for sinusoidal driving they will not vanish at the same driving parameters as for the nearest-neighbor hopping. The residual value of J_{eff}, visible in Fig. 2, is in reasonable agreement with the value of the next-to-nearest-neighbor hopping (around 5%) calculated for a similar system in Ref. [15].

C. Tilted lattice, resonant driving. Applying Eq. (5) to the experimental data on photon-assisted tunneling [5] leads to an even more striking improvement in the agreement between theory and experiment. In those experiments a tilt was applied to the lattice through a constant acceleration, leading to a suppression of tunneling by Wannier-Stark localization. Periodic driving of the lattice at a frequency ω matching the energy offset between two adjacent lattice wells then led to partial restoration of the tunneling probability, with the effective tunneling probability given by $|J_{\text{eff}}/J| = \mathcal{J}_1(K_0)$; that is, one expects a scaling with the first-order Bessel function. As shown in Fig. 3(b), assuming linear expansion in order to extract J_{eff}/J led to a scaling that interpolated between a linear and a quadratic dependence on $\mathcal{J}_1(K_0)$, depending on the initial size of the condensate (which in the experiment was varied through the nonlinearity by changing the atom number). If the full expansion dynamics is taken into account through

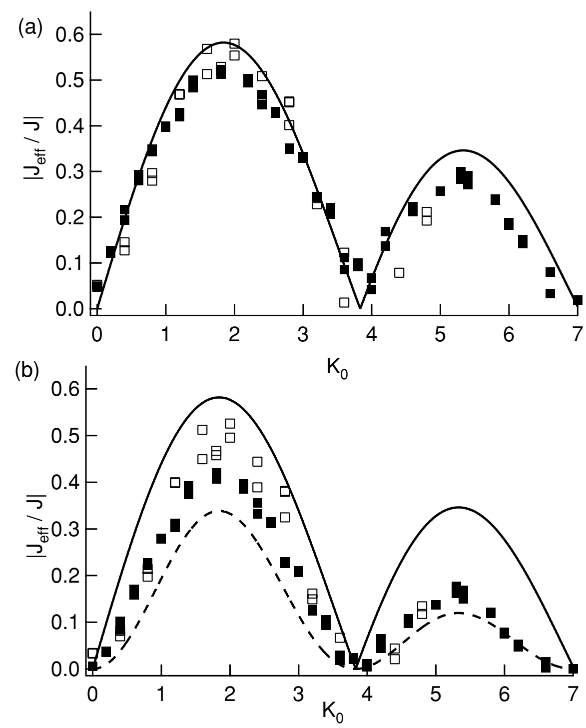

FIG. 3. (a) Effective tunneling for resonant driving in a tilted lattice (photon-assisted tunneling). The effective tunneling parameters were calculated for two different initial condensate sizes [around 15 μm (open symbols) and around 17 μm (solid symbols)] using Eq. (4). For comparison, (b) shows the same experimental data with the renormalized tunneling parameter calculated assuming linear expansion. One clearly sees that in this case the experimental data interpolate between a linear Bessel scaling (solid line) and a quadratic scaling (dashed line).

Eq. (5), however, both data sets give the same dependence on K_0, which is very close to the theoretical prediction.

IV. Conclusions. We have shown that in order to extract the effective tunneling from expansion measurements it is important to account for the detailed time dependence of the condensate expansion. When the effective tunneling is small, or the initial width of the condensate is large, the crossover to ballistic expansion will not be reached until very long times. Measurements made at earlier times will thus underestimate the effective tunneling rate, which gives a quantitatively accurate interpretation of the "squared Bessel function" behavior noted in Refs. [4,5]. Although we have not included the effects of interactions, this is a reasonable approximation for the systems studied in Refs. [4,5] where the interactions were fairly small ($U/h \simeq 10$ Hz) and their effect rapidly became negligible as the condensate expanded and became more dilute. Using the correct expansion formula, Eq. (4), not only provides an accurate means of deducing the value of the effective tunneling but also is essential to study subtle effects, such as the transition to diffusive tunneling and the influence of higher-order tunneling terms, which would otherwise be masked by this behavior when the effective tunneling is small.

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