

SIMPLY CONNECTED K-CONTACT AND SASAKIAN MANIFOLDS OF DIMENSION 7

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ABSTRACT. We construct a compact simply connected 7-dimensional manifold admitting a K-contact structure but not a Sasakian structure. We also study rational homotopy properties of such manifolds, proving in particular that a simply connected 7-dimensional Sasakian manifold has vanishing cup product $H^2 \times H^2 \rightarrow H^4$ and that it is formal if and only if all its triple Massey products vanish.

1. INTRODUCTION

Sasakian geometry has become an important and active subject, especially after the appearance of the fundamental treatise of Boyer and Galicki [3]. Chapter 7 of this book contains an extended discussion of the topological problems in the theory of Sasakian, and, more generally, K-contact manifolds. These are odd-dimensional analogues to Kähler and symplectic manifolds, respectively.

The precise definition is as follows. Let (M, η) be a co-oriented contact manifold with a contact form $\eta \in \Omega^1(M)$, that is $\eta \wedge (d\eta)^n > 0$ everywhere, with $\dim M = 2n + 1$. We say that (M, η) is *K-contact* if there is an endomorphism Φ of TM such that:

- $\Phi^2 = -\text{Id} + \xi \otimes \eta$, where ξ is the Reeb vector field of η (that is $i_\xi \eta = 1$, $i_\xi(d\eta) = 0$),
- the contact form η is compatible with Φ in the sense that $d\eta(\Phi X, \Phi Y) = d\eta(X, Y)$, for all vector fields X, Y ,
- $d\eta(\Phi X, X) > 0$ for all nonzero $X \in \ker \eta$, and
- the Reeb field ξ is Killing with respect to the Riemannian metric defined by the formula $g(X, Y) = d\eta(\Phi X, Y) + \eta(X)\eta(Y)$.

In other words, the endomorphism Φ defines a complex structure on $\mathcal{D} = \ker \eta$ compatible with $d\eta$, hence Φ is orthogonal with respect to the metric $g|_{\mathcal{D}}$. By definition, the Reeb vector field ξ is orthogonal to $\ker \eta$, and it is a Killing vector field.

Let (M, η, g, Φ) be a K-contact manifold. Consider the contact cone as the Riemannian manifold $C(M) = (M \times \mathbb{R}^{>0}, t^2g + dt^2)$. One defines the almost complex structure I on $C(M)$ by:

- $I(X) = \Phi(X)$ on $\ker \eta$,
- $I(\xi) = t \frac{\partial}{\partial t}$, $I(t \frac{\partial}{\partial t}) = -\xi$, for the Killing vector field ξ of η .

We say that (M, η, Φ, g, I) is *Sasakian* if I is integrable. Thus, by definition, any Sasakian manifold is K-contact.

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There are several topological obstructions to the existence of the aforementioned structures on a compact manifold M of dimension $2n + 1$, for example:

- (1) the evenness of the p -th Betti number for p odd with $1 \leq p \leq n$, of a Sasakian manifold,
- (2) some torsion obstructions in dimension 5 discovered by Kollár [17],
- (3) the fundamental group of Sasakian manifolds are special,
- (4) the cohomology algebra of a Sasakian manifold satisfies the hard Lefschetz property,
- (5) formality properties of the rational homotopy type.

An early result [13] establishes that the odd Betti numbers up to the middle dimension of Sasakian manifolds must be even. The parity of b_1 was used to produce the first examples of K-contact manifolds with no Sasakian structure [3, example 7.4.16]. More refined tools are needed in the case of even Betti numbers. The cohomology algebra of a Sasakian manifold satisfies a hard Lefschetz property [4]. Using it examples of K-contact non-Sasakian manifolds are produced in [5] in dimensions 5 and 7. These examples are nilmanifolds with even Betti numbers, so in particular they are not simply connected.

The fundamental group can also be used to construct K-contact non-Sasakian manifolds. Fundamental groups of Sasakian manifolds are called Sasaki groups, and satisfy strong restrictions. Using this it is possible to construct (non-simply connected) compact manifolds which are K-contact but not Sasakian [8].

When one moves to the case of simply connected manifolds, K-contact non-Sasakian examples of any dimension ≥ 9 were constructed in [16] using the evenness of the third Betti number of a compact Sasakian manifold. Alternatively, using the hard Lefschetz property for Sasakian manifolds there are examples [19] of simply connected K-contact non-Sasakian manifolds of any dimension ≥ 9 .

In [23] and in [2] the rational homotopy type of Sasakian manifolds is studied. In [2] it is proved that all higher order Massey products for simply connected Sasakian manifolds vanish, although there are Sasakian manifolds with non-vanishing triple Massey products. This yields examples of simply connected K-contact non-Sasakian manifolds in dimensions ≥ 17 . However, Massey products are not suitable for the analysis of lower dimensional manifolds.

Hence, the problem of the existence of simply connected K-contact non-Sasakian compact manifolds (open problem 7.4.1 in [3]) is still open in dimensions 5 and 7. Dimension 5 is the most difficult one, and it is treated in [3] separately. Here one has to use the obstructions of [17] which are very subtle torsion obstructions associated to the classification of Kähler surfaces. By definition, a simply connected compact oriented 5-manifold is called a *Smale-Barden manifold*. These manifolds are classified topologically by $H_2(M, \mathbb{Z})$ and the second Stiefel-Whitney class. Chapter 10 of the book by Boyer and Galicki is devoted to a description of some Smale-Barden manifolds which carry Sasakian structures. The following problem is still open (open problem 10.2.1 in [3]).

Do there exist Smale-Barden manifolds which carry K-contact but do not carry Sasakian structures?

In this note we solve the described problem in the easier case of dimension 7 (the solution is still possible by means of homotopy theory combined with symplectic surgery).

Theorem 1. *There exist 7-dimensional compact simply connected K-contact manifolds which do not admit a Sasakian structure.*

We then turn around to the study of the rational homotopy type of K-contact and Sasakian simply connected manifolds of dimension 7. In particular, we prove:

Corollary 2. *Let M be a simply connected compact K-contact 7-dimensional manifold. Suppose that the cup product map $H^2(M) \times H^2(M) \rightarrow H^4(M)$ is non-zero. Then M does not admit a Sasakian structure.*

Formality is a very useful rational homotopy property that has been widely used to distinguish between symplectic and Kähler manifolds [21] (see Section 6 for definitions and details). Simply connected compact manifolds of dimension ≤ 6 are always formal, so formality becomes interesting in dimension 7. We study this property in detail giving a precise characterisation for Sasakian manifolds (see Theorem 14). In particular, we have the following:

Corollary 3. *Let M be a simply connected compact Sasakian 7-dimensional manifold. Then M is formal if and only if all triple Massey products are zero.*

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2. GOMPF-CAVALCANTI MANIFOLD

Let (M, ω) be a symplectic manifold of dimension $2n$. For every $0 \leq k \leq n$, we define the Lefschetz map as $L_\omega : H^{n-k}(M) \rightarrow H^{n+k}(M)$, $L_\omega([\beta]) = [\beta \wedge \omega^{n-k}]$. We say that M satisfies the hard Lefschetz property if L_ω is an isomorphism for every $0 \leq k \leq n$.

Proposition 4. *There exists a simply connected 6-dimensional symplectic manifold (M, ω) such that $\dim \ker(L_\omega : H^2(M) \rightarrow H^4(M))$ is odd.*

Proof. Gompf constructs in [14, Theorem 7.1] an example of a simply connected 6-dimensional symplectic manifold (M, ω) which does not satisfy the hard Lefschetz property, that is, the Lefschetz map $L_\omega : H^2(M) \rightarrow H^4(M)$ is not an isomorphism. If $\dim \ker L_\omega$ is already odd then we have finished.

So let us suppose that $\dim \ker L_\omega$ is even. Take a cohomology class $a \in H^2(M)$ which belongs to the kernel of L_ω . In [7, Lemma 2.4] Cavalcanti proves that given a symplectic manifold (M, ω) as above satisfying that there exists a symplectic surface $S \hookrightarrow M$ with $\langle a, [S] \rangle \neq 0$, then there is another 6-dimensional symplectic manifold (M', ω') (the symplectic blow-up of M along S) satisfying

$$\dim \ker(L_{\omega'} : H^2(M') \rightarrow H^4(M')) = \dim \ker(L_\omega : H^2(M) \rightarrow H^4(M)) - 1.$$

The symplectic blow-up of M along S is constructed in [20], where it is proved that the fundamental groups $\pi_1(M') \cong \pi_1(M)$, hence M' is simply connected. This means that the simply connected 6-dimensional symplectic manifold M' satisfies that $\dim \ker(L_{\omega'} : H^2(M') \rightarrow H^4(M'))$ is odd, as required.

It remains to find $S \hookrightarrow M$ as required. The cohomology class a is non-zero, so there is some $b \in H^4(M, \mathbb{Z})$ such that $a \cup b \neq 0$. It is easy to see that there is a rank 2 complex vector bundle $E \rightarrow M$ with $c_1(E) = 0$, $c_2(E) = 2b$. This corresponds to the fact that the map $[M, BSU(2)] \rightarrow H^4(M, \mathbb{Z})$ given by the second Chern class exhausts $2H^4(M, \mathbb{Z})$. A short proof runs as follows: $BSU(2)$ has trivial 3-skeleton and it has $\pi_4(BSU(2)) = \mathbb{Z}$ and $\pi_5(BSU(2)) = \mathbb{Z}_2$. Represent the cohomology class b by a cocycle $\varphi_b : C_4(M) \rightarrow \mathbb{Z}$, where $C_4(M)$ is the space of cellular chains. Given b , we define $f : M \rightarrow BSU(2)$ inductively on the skeleta (in what follows we denote by $X[k]$ the k -skeleton of a space X). It is trivial on the 3-skeleton of M . For every 4-cell c , we define $f : c \rightarrow BSU(2)[4] = S^4$ to have degree $\varphi_b(c) \in \mathbb{Z}$. As M is simply connected there are no 5-cells, so it only remains to attach the 6-cell c_6 to the 4-skeleton $M[4]$. The attaching map is given by some $g : S^5 \rightarrow M[4]$. When composed with f , we have a map $f \circ g : S^5 \rightarrow BSU(2)$, which gives an *obstruction* element $o_f \in \pi_5(BSU(2)) = \mathbb{Z}_2$. If we multiply b by two, then the map φ_b gets multiplied by 2. The corresponding f is given by composing f with a double cover of S^4 , hence the obstruction element is $2o_f = 0$. This means that the map f associated to $2b$ can be extended to $M \rightarrow BSU(2)$.

Now take the rank 2 bundle $E \rightarrow M$ just constructed. Assume that $[\omega]$ is an integral cohomology class (which can always be done by perturbing ω slightly to make it rational and multiplying it by a large integer). Let $L \rightarrow M$ be the line bundle with first Chern class $c_1(L) = [\omega]$. We now use the asymptotically holomorphic techniques introduced by Donaldson in [10]. Specifically, the result of [1] guarantees the existence of a suitable large $k \gg 0$ and a section of $E \otimes L^{\otimes k}$ whose zero locus is a symplectic manifold (an asymptotically holomorphic manifold in fact). This zero locus $S \subset M$ is a symplectic surface, and the cohomology class defined by S is $c_2(E \otimes L^{\otimes k}) = c_2(E) + 2kc_1(L) = 2b + 2k[\omega]$. Therefore $\langle a, [S] \rangle = \langle a, 2b + 2k[\omega] \rangle = 2\langle a, b \rangle \neq 0$, as required. \square

We will call the manifold produced in Proposition 4 the *Gompf-Cavalcanti manifold*, because it is constructed by the surgery technique of Gompf [14] together with the symplectic blow-up of Cavalcanti [7]. Note however that this is not a unique one but a family of manifolds.

3. SIMPLY-CONNECTED K-CONTACT NON-SASAKIAN MANIFOLDS IN DIMENSION 7

We show the existence of simply connected compact K-contact non-Sasakian manifolds in dimension 7 by proving that the Boothby-Wang fibration over the Gompf-Cavalcanti manifold is K-contact but non-Sasakian. The existence of a K-contact structure on such fibration is shown in [2] and [16]. For the convenience of the reader we briefly recall these constructions.

Let (B, ω) be a symplectic manifold such that the cohomology class $[\omega]$ is integral. Consider the principal S^1 -bundle $\pi : M \rightarrow B$ given by the cohomology class $[\omega] \in H^2(B, \mathbb{Z})$. Fibrations of this kind were first considered by Boothby and Wang and are called *Boothby-Wang fibrations*. By [24], the total space M carries an S^1 -invariant contact form η such that η is a connection form whose curvature is $d\eta = \pi^*\omega$. We have the following result.

Theorem 5. *Any Boothby-Wang fibration admits a K-contact structure on the total space.*

Proof. To prove this theorem we need to introduce a certain tool, called the universal contact moment map in the sense of Lerman [18]. Recall that by our assumption the given contact distribution \mathcal{D} is determined by the contact form η , that is $\mathcal{D} = \ker \eta$. Consider its annihilator $\mathcal{D}^0 \subset T^*M$. Clearly, \mathcal{D}^0 is a line bundle, and, therefore, it has two components after the removal of the zero section,

$$\mathcal{D}^0 \setminus M = \mathcal{D}_+^0 \sqcup \mathcal{D}_-^0.$$

Single out one of these components, say \mathcal{D}_+^0 . Consider the Lie algebra of contact vector fields $\chi(M, \eta)$ on M . It is known that this Lie algebra can be identified with a space of sections of the vector bundle TM/\mathcal{D} , that is $\chi(M, \eta) \cong \Gamma(M, TM/\mathcal{D})$. Because of that there is a natural pairing between points of the line bundle \mathcal{D}^0 and contact vector fields given by the formula

$$\mathcal{D}^0 \times \chi(M, \eta) \rightarrow \mathbb{R}, \quad ((p, \beta), X) \mapsto \langle \beta, X_p \rangle$$

where $\beta \in \mathcal{D}_+^0$, $X_p \in T_pM$, $p \in M$. Suppose that a Lie algebra \mathfrak{g} acts on M by contact vector fields, that is, there exists a representation $\rho : \mathfrak{g} \rightarrow \chi(M, \eta)$. Define the *universal moment map* as the map

$$\psi : \mathcal{D}_+^0 \rightarrow \mathfrak{g}^*$$

by the formula

$$\langle \psi(p, \beta), X \rangle = \langle (p, \beta), \rho(X) \rangle = \langle \beta, \rho(X)_p \rangle,$$

where $(p, \beta) \in (\mathcal{D}_+^0)_p \subset T_p^*M$, $X \in \mathfrak{g}$. Now the proof becomes a consequence of the following criterion proved by Lerman in [18].

Proposition 6. *A compact co-orientable contact manifold (M, η) admits a K-contact metric g if and only if there exists an action of a torus T on M preserving the contact structure \mathcal{D} and a vector $X \in \mathfrak{t} = L(T)$ so that the function $\langle \psi, X \rangle : \mathcal{D}_+^0 \rightarrow \mathbb{R}$ is strictly positive. \square*

We continue with the proof of Theorem 5. Consider the S^1 -action on M given by the Reeb vector field. Let $\mathfrak{g} = L(S^1)$, and $\rho : \mathfrak{g} \rightarrow \chi(M, \eta)$ be the homomorphism of Lie algebras determined by this action (thus, $\mathfrak{g} = \mathfrak{t} = L(S^1)$ in this particular situation). Since the S^1 -action is free, $\rho(X)_p \neq 0$ for any $p \in M$. Now,

$$\langle \psi, X \rangle(p, \beta) = \langle \psi(p, \beta), X \rangle = \langle \beta, \rho(X)_p \rangle.$$

Note that in the considered case $\beta \in (\mathcal{D}_+^0)_p \subset T_p^*M$, and, therefore, $\beta \neq 0$. Also (p, β) belongs to the annihilator of the distribution \mathcal{D} , while $\rho(X)$ is transversal to \mathcal{D} , since it is given by the Reeb vector field. Thus, for any point p , $\langle (p, \beta), \rho(X)_p \rangle \neq 0$. Hence, X may be chosen to yield positive sign everywhere, and we complete the proof by applying Proposition 6. \square

The following gives a proof of Theorem 1.

Theorem 7. *The total space of the Boothby-Wang fibration over the Gompf-Cavalcanti manifold is a simply connected K-contact non-Sasakian manifold of dimension 7.*

Proof. Let (M, ω) be a Gompf-Cavalcanti manifold as given by Proposition 4. We can assume that $[\omega]$ is an integral cohomology class. Let

$$S^1 \rightarrow E \rightarrow M \tag{1}$$

be the associated Boothby-Wang fibration. By Theorem 5, E has a K-contact structure. Now we need to prove that E cannot carry Sasakian structures.

There is an exact sequence

$$H_2(M) \rightarrow H_1(S^1) = \mathbb{Z} \rightarrow H_1(E) \rightarrow 0$$

from the Serre spectral sequence. The map $H_2(M) \rightarrow \mathbb{Z}$ is cupping with $[\omega] \in H^2(M)$. Taking $[\omega]$ integral cohomology class and primitive, we have that $H_2(M) \rightarrow \mathbb{Z}$ is surjective and hence $H_1(E) = 0$. The long homotopy exact sequence gives $\pi_1(S^1) = \mathbb{Z} \rightarrow \pi_1(E) \rightarrow \pi_1(M) = 0$, hence $\pi_1(E)$ is abelian. Therefore E is simply connected.

The Gysin exact sequence associated to (1) is

$$H^1(M) = 0 \xrightarrow{\wedge\omega} H^3(M) \longrightarrow H^3(E) \longrightarrow H^2(M) \xrightarrow{\wedge\omega} H^4(M).$$

Thus

$$b^3(E) = b^3(M) + \dim(\ker L_\omega : H^2(M) \rightarrow H^4(M)).$$

As M is a 6-manifold, we have that $b^3(M)$ is even (by Poincaré duality, the intersection pairing on $H^3(M)$ is an antisymmetric non-degenerate bilinear form, hence the dimension of $H^3(M)$ is even). By construction, $\dim(\ker L_\omega : H^2(M) \rightarrow H^4(M))$ is odd, so $b^3(E)$ is odd. As the third Betti number of a 7-dimensional Sasakian manifold has to be even [13], we have that E cannot admit a Sasakian structure. \square

4. REGULARITY AND QUASI-REGULARITY

A Sasakian or a K-contact structure on a compact manifold M is called *quasi-regular* if there is a positive integer δ satisfying the condition that each point of M has a foliated coordinate chart (U, t) with respect to ξ (the coordinate t is in the direction of ξ) such that each leaf for ξ passes through U at most δ times. If $\delta = 1$, then the Sasakian or K-contact structure is called *regular*. (See [3, p. 188].)

If N is a Kähler manifold whose Kähler form ω defines an integral cohomology class, then the total space of the circle bundle $S^1 \hookrightarrow M \xrightarrow{\pi} N$ with Euler class $[\omega] \in H^2(M, \mathbb{Z})$ is a regular Sasakian manifold with contact form η such that $d\eta = \pi^*(\omega)$. The converse also holds: if M is a regular Sasakian structure then the space of leaves N is a Kähler manifold, and we have a circle bundle $S^1 \rightarrow M \rightarrow N$ as above. If M has a quasi-regular Sasakian structure, then the space of leaves N is a Kähler orbifold with cyclic quotient singularities, and there is an orbifold circle bundle $S^1 \rightarrow M \rightarrow N$ such that the contact form η satisfies $d\eta = \pi^*(\omega)$, where ω is the orbifold Kähler form.

Similar properties hold in the K-contact case, substituting Kähler by symplectic (actually almost Kähler). If M has a regular K-contact structure, then it is the total space of a circle bundle $S^1 \hookrightarrow M \xrightarrow{\pi} N$, where (N, ω) is a symplectic manifold, with Euler class $[\omega] \in H^2(M, \mathbb{Z})$ and $d\eta = \pi^*(\omega)$. If M has a quasi-regular K-contact structure, then it is the total space of an orbifold circle bundle $S^1 \rightarrow M \rightarrow N$ over a symplectic orbifold N with cyclic quotient singularities and Euler class $[\omega] \in H^2(M, \mathbb{Z})$, where ω is the orbifold symplectic form.

A result of [22] says that if M admits a Sasakian structure, then it admits also a quasi-regular Sasakian structure. This also extends to the case of K-contact structures.

Proposition 8. *If a compact manifold M admits a K-contact structure, it admits a quasi-regular contact structure.*

Proof. Assume that there is a K-contact structure on M . By Proposition 6, there exists a torus action $T \times M \rightarrow M$ preserving the contact distribution and a vector $X \in \mathfrak{t}$ such that $\langle \psi, X \rangle > 0$. Choose a vector $Y \in \mathfrak{t}$ with the property that it is tangent to an embedding $T' = S^1 \hookrightarrow T$. Clearly, the corresponding fundamental vector field Y_M has the property that the leaves of the corresponding foliations are compact. The set of such Y is dense in \mathfrak{t} . Therefore, for vectors Y which are sufficiently close to X , the condition $\langle \psi, Y \rangle > 0$ is still satisfied.

So it remains to see that there is K-contact structure whose Reeb vector field is Y_M , since this will be quasi-regular because the leaves of the characteristic foliation are all compact. We follow the notations of the proof of Theorem 5. The action of the circle T' on M preserves \mathcal{D} , hence the lifted action of T' on T^*M preserves \mathcal{D}^0 . Since T' is connected, the lifted action preserves the connected component \mathcal{D}_+^0 as well. It follows that for any 1-form β on M with $\ker \beta = \mathcal{D}$, the average $\bar{\beta}$ of β over T' still satisfies $\ker \bar{\beta} = \mathcal{D}$. So $\bar{\beta} \in \mathcal{D}^0$. Now use the formula (derived in [18], formulae (3.4) and (3.5)),

$$i_{Y_M} \bar{\beta} = \langle \psi \circ \bar{\beta}, Y \rangle > 0.$$

Now let

$$\eta = (\langle \psi \circ \bar{\beta}, Y \rangle)^{-1} \bar{\beta},$$

which satisfies $i_{Y_M} \eta = 1$. Hence η defines the contact structure and Y_M is its Killing vector field. Then $TM = \mathcal{D} \oplus \langle Y_M \rangle$, and the splitting is T' -invariant. We use the splitting to define the desired Riemannian metric g . Declare \mathcal{D} and $\langle Y_M \rangle$ to be orthogonal and define $g(Y_M, Y_M) = 1$, thus Y_M becomes a unit normal to \mathcal{D} . On \mathcal{D} we choose a T' -invariant complex structure compatible with $d\eta|_{\mathcal{D}}$ and define $g|_{\mathcal{D}}(\cdot, \cdot) = d\eta|_{\mathcal{D}}(\cdot, \Phi \cdot)$. Then g is T' -invariant and hence $L_{Y_M} g = 0$. Thus we have obtained a K-contact structure on M . \square

5. MINIMAL MODELS AND FORMALITY

Now we want to analyse the rational homotopy type of K-contact and Sasakian simply connected 7-manifolds, in particular the property of formality. Simply connected compact manifolds of dimension ≤ 6 are always formal [12], so dimension 7 is the first instance in which formality is an issue.

We start by reviewing concepts about minimal models and formality from [12, 11, 15]. A *differential graded algebra* (or DGA) over the real numbers \mathbb{R} , is a pair (A, d) consisting of a graded commutative algebra $A = \bigoplus_{k \geq 0} A^k$ over \mathbb{R} , and a differential d satisfying the Leibnitz rule $d(a \cdot b) = (da) \cdot b + (-1)^{|a|} a \cdot (db)$, where $|a|$ is the degree of a . Given a differential graded commutative algebra (A, d) , we denote its cohomology by $H^*(A)$. The cohomology of a differential graded algebra $H^*(A)$ is naturally a DGA with the product inherited from that on A and with the differential being identically zero. The DGA (A, d) is *connected* if $H^0(A) = \mathbb{R}$, and A is *1-connected* if, in addition, $H^1(A) = 0$. Henceforth we shall assume that all our DGAs are connected. In our context, the main example of DGA is the de Rham complex $(\Omega^*(M), d)$ of a connected differentiable manifold M , where d is the exterior differential.

Morphisms between DGAs are required to preserve the degree and to commute with the differential. A morphism $f : (A, d) \rightarrow (B, d)$ is a quasi-isomorphism if the map induced in cohomology $f^* : H^*(A, d) \rightarrow H^*(B, d)$ is an isomorphism. Quasi-isomorphism produces an equivalence relation in the category of DGAs.

A DGA (\mathcal{M}, d) is *minimal* if

- (1) \mathcal{M} is free as an algebra, that is, \mathcal{M} is the free algebra $\bigwedge V$ over a graded vector space $V = \bigoplus_i V^i$, and
- (2) there is a collection of generators $\{x_\tau\}_{\tau \in I}$ indexed by some well ordered set I , such that $|x_\mu| \leq |x_\tau|$ if $\mu < \tau$ and each dx_τ is expressed in terms of preceding x_μ , $\mu < \tau$.

We say that $(\bigwedge V, d)$ is a *minimal model* of the differential graded commutative algebra (A, d) if $(\bigwedge V, d)$ is minimal and there exists a quasi-isomorphism $\rho: (\bigwedge V, d) \rightarrow (A, d)$. A connected DGA (A, d) has a minimal model unique up to isomorphism. For 1-connected DGAs, this is proved in [9]. In this case, the minimal model satisfies that $V^1 = 0$ and the condition (2) above is equivalent to dx_τ not having a linear part.

A *minimal model* of a connected differentiable manifold M is a minimal model $(\bigwedge V, d)$ for the de Rham complex $(\Omega^*(M), d)$ of differential forms on M . If M is a simply connected manifold, then the dual of the real homotopy vector space $\pi_i(M) \otimes \mathbb{R}$ is isomorphic to V^i for any i (see [9]).

A *model* of a DGA (A, d) is any DGA (B, d) with the same minimal model (that is, they are equivalent with respect to the equivalence relation determined by the quasi-isomorphisms).

A minimal algebra $(\bigwedge V, d)$ is called *formal* if there exists a morphism of differential algebras $\psi: (\bigwedge V, d) \rightarrow (H^*(\bigwedge V), 0)$ inducing the identity map on cohomology. Also a differentiable manifold M is called formal if its minimal model is formal. The formality of a minimal algebra is characterized as follows.

Proposition 9 ([9]). *A minimal algebra $(\bigwedge V, d)$ is formal if and only if the space V can be decomposed into a direct sum $V = C \oplus N$ with $d(C) = 0$ and d injective on N , such that every closed element in the ideal $I(N)$ in $\bigwedge V$ generated by N is exact.*

This characterization of formality can be weakened using the concept of s -formality introduced in [12].

Definition 10. *A minimal algebra $(\bigwedge V, d)$ is s -formal ($s > 0$) if for each $i \leq s$ the space V^i of generators of degree i decomposes as a direct sum $V^i = C^i \oplus N^i$, where the spaces C^i and N^i satisfy the three following conditions:*

- (1) $d(C^i) = 0$,
- (2) the differential map $d: N^i \rightarrow \bigwedge V$ is injective, and
- (3) any closed element in the ideal $I_s = I(\bigoplus_{i \leq s} N^i)$, generated by the space $\bigoplus_{i \leq s} N^i$ in the free algebra $\bigwedge(\bigoplus_{i \leq s} V^i)$, is exact in $\bigwedge V$.

A differentiable manifold M is s -formal if its minimal model is s -formal. Clearly, if M is formal then M is s -formal, for any $s > 0$. The main result of [12] shows that sometimes the weaker condition of s -formality implies formality.

Theorem 11 ([12]). *Let M be a connected and orientable compact manifold of dimension $2n$ or $(2n - 1)$. Then M is formal if and only if it is $(n - 1)$ -formal.*

By Corollary 3.3 in [12] a simply connected compact manifold is always 2-formal. Therefore, Theorem 11 implies that any simply connected compact manifold of dimension

not more than six is formal. For simply connected 7-dimensional compact manifolds, we have that M is formal if and only if M is 3-formal.

Theorem 11 also holds for compact connected orientable orbifolds, since the proof of [12] only uses that the cohomology $H^*(M)$ is a Poincaré duality algebra.

6. HOMOTOPY PROPERTIES OF SIMPLY CONNECTED SASAKIAN 7-MANIFOLDS

Proposition 12. *Let M be a simply connected compact K-contact 7-dimensional manifold. Then a model for M is $(H \otimes \wedge(x), d)$, where H is the cohomology algebra of a simply connected symplectic 6-dimensional orbifold and $dx = \omega \in H^2$ is the class of the symplectic form.*

If M is Sasakian, then H is the cohomology algebra of a simply connected 6-dimensional Kähler orbifold.

Proof. Suppose M admits a Sasakian structure. Then M admits a quasi-regular Sasakian structure [22]. Therefore, there is an orbifold circle bundle $S^1 \rightarrow M \rightarrow B$, where B is a compact Kähler orbifold of dimension 6, with Euler class given by the Kähler form $\omega \in H^2(B)$. We note that B is simply connected because M is so (see [3, Theorem 4.3.18]). In particular, $S^1 \rightarrow M \rightarrow B$ is a rational fibration, hence if \mathcal{M} is a model for B , then $\mathcal{M} \otimes \wedge(x)$, with $|x| = 1$, $dx = \omega$, is a model for M .

Now B is a simply connected compact orbifold of dimension 6. So it is 2-formal. Theorem 11 also holds for orbifolds, hence B is formal. Therefore $\mathcal{M} \sim (H, 0)$, where $H = H^*(B)$ is the cohomology algebra of B . So a model for M is of the form $(H \otimes \wedge(x), d)$, $dx = \omega \in H^2$.

The case where M admits a K-contact structure is similar. By Proposition 8, it admits a quasi-regular K-contact structure. Therefore, M is an orbifold S^1 -bundle over a symplectic orbifold $S^1 \rightarrow M \rightarrow B$, with Euler class given by the orbifold symplectic form $\omega \in H^2(B)$. As above, a model for M is $(H \otimes \wedge(x), d)$, $dx = \omega \in H^2$, where $H = H^*(B)$. \square

We prove now Corollary 2.

Corollary 13. *Let M be a simply connected compact K-contact 7-dimensional manifold. Suppose that the cup product map $H^2(M) \times H^2(M) \rightarrow H^4(M)$ is non-zero. Then M does not admit a Sasakian structure.*

Proof. Let us compute the cohomology of M from its model $(\mathcal{M}, d) = (H \otimes \wedge(x), d)$, $dx = \omega$, where $H = H^*(B)$ is the cohomology algebra of a 6-dimensional simply connected symplectic manifold. Note that $\omega \in H^2$ is a non-zero element with $\omega^3 \in H^6$ generating the top cohomology.

Consider the Lefschetz map $L_\omega : H^* \rightarrow H^{*+2}$, and let $K^* = \ker L_\omega$, $Q^* = \text{coker } L_\omega$. We have a (non-canonical) isomorphism $H^i(M) \cong Q^i \oplus K^{i-1}$. Note that $Q^3 = K^3 = H^3$ and $H^6 = \mathbb{R}$. Also $Q^2 = H^2 / \langle \omega \rangle$, and $K^4 = \ker(L_\omega : H^4 \rightarrow \mathbb{R})$ are vector spaces of

codimension one. We have the following:

$$\begin{aligned}
H^0(M) &= \mathbb{R}, \\
H^1(M) &= 0, \\
H^2(M) &= Q^2, \\
H^3(M) &= H^3 \oplus K^2x, \\
H^4(M) &= Q^4 \oplus H^3x, \\
H^5(M) &= K^4x, \\
H^6(M) &= 0, \\
H^7(M) &= \langle \omega^3x \rangle.
\end{aligned}$$

The map $H^2(M) \times H^2(M) \rightarrow H^4(M)$ factors through $Q^2 \times Q^2 \rightarrow Q^4$. Hence if it is non-zero then $Q^4 \neq 0$. In particular, the Lefschetz map $L_\omega : H^2 \rightarrow H^4$ is not an isomorphism, so B is not hard Lefschetz.

If M admits a Sasakian structure, then there is a quasi-regular fibration $S^1 \rightarrow M \rightarrow B$ with B satisfying the hard Lefschetz property (it is a Kähler orbifold, so [25] is applicable). This contradicts the above. \square

Now we shall study the case of Sasakian 7-manifolds in more detail. Let M be a simply connected compact Sasakian 7-dimensional manifold. Then

$$\mathcal{M} = (H \otimes \bigwedge(x), d)$$

is a model for M , by Proposition 12, where $H = H^*(B)$ is the cohomology algebra of a simply connected compact 6-dimensional Kähler orbifold. This algebra H has a very rich structure:

- (1) there is a canonical isomorphism $H^6 \cong \mathbb{R}$, which is given by integration $\int_M : H^6 \rightarrow \mathbb{R}$;
- (2) H is a Poincaré duality algebra, hence $H^3 \otimes H^3 \rightarrow \mathbb{R}$ is an antisymmetric bilinear pairing;
- (3) there is a scalar product on each H^j . This is given by the Hodge star operator $*$: $H^j \rightarrow H^{6-j}$ combined with wedge and integration;
- (4) there is a distinguished element $\omega \in H^2$. This defines the space of primitive forms $P = \langle \omega \rangle^\perp \subset H^2$. Hence $H^2 = \langle \omega \rangle \oplus P$;
- (5) the Lefschetz map $L_\omega : H^2 \rightarrow H^4$ is an isomorphism. Therefore $H^4 = \langle \omega^2 \rangle \oplus \omega P$. On the other hand, for $\alpha \in P$ we have $*\alpha = \alpha \wedge \omega$, and $*\omega = \frac{1}{2}\omega^2$. This implies that $\omega P = \langle \omega^2 \rangle^\perp$, and $L_\omega : \langle \omega \rangle \oplus P \rightarrow \langle \omega^2 \rangle \oplus \omega P$ is of the form $L_\omega(\alpha) = L_\omega(\alpha_0 + \alpha_1) = \frac{1}{2}*\alpha_0 + *\alpha_1$, where $\alpha = \alpha_0 + \alpha_1$ is the decomposition according to $H^2 = \langle \omega \rangle \oplus P$.

The Lefschetz map $L_\omega : H^2 \rightarrow H^4$ is an isomorphism so there is an inverse $L_\omega^{-1} : H^4 \rightarrow H^2$. Using it, we can define a map $\mathcal{F} : P \times P \times P \times P \rightarrow \mathbb{R}$ by

$$\mathcal{F}(\alpha, \beta, \gamma, \delta) = \int_M L_\omega^{-1}(\alpha \wedge \beta) \wedge \gamma \wedge \delta.$$

This clearly factors through $\text{Sym}^2 P \times \text{Sym}^2 P$. Using (5) above, we have the alternative description

$$\mathcal{F}(\alpha, \beta, \gamma, \delta) = 2\langle(\alpha \wedge \beta)_0, (\gamma \wedge \delta)_0\rangle + \langle(\alpha \wedge \beta)_1, (\gamma \wedge \delta)_1\rangle,$$

from where it follows that \mathcal{F} factors as a map $\text{Sym}^2(\text{Sym}^2 P) \rightarrow \mathbb{R}$.

Let \mathcal{K}_M be the kernel of the map $\text{Sym}^2(\text{Sym}^2 P) \rightarrow \text{Sym}^4 P$. Then we define a map

$$\mathcal{F}_M = \mathcal{F}|_{\mathcal{K}_M} : \mathcal{K}_M \rightarrow \mathbb{R}.$$

We have the following result.

Theorem 14. *Let M be a simply connected compact Sasakian 7-dimensional manifold. Then M is formal if and only if $\mathcal{F}_M = 0$.*

Proof. Using Theorem 11, we only have to check whether M is 3-formal. For this we have to construct the minimal model $\rho : (\bigwedge V, d) \rightarrow \mathcal{M} = (H \otimes \bigwedge(x), d)$ up to degree 3. This is easy:

$$\begin{aligned} V^1 &= 0, \\ V^2 &= P, \\ V^3 &= H^3 \oplus N^3, \quad \text{where } N^3 = \text{Sym}^2 P, \end{aligned}$$

where the differential is given by $d = 0$ on P and H^3 , and $d : N^3 \rightarrow \bigwedge V^2$ is the isomorphism $\text{Sym}^2 P \rightarrow \bigwedge^2 P$. The map ρ is given as follows. $\rho : V^2 = P \rightarrow \mathcal{M}^2 = H^2$ is defined as the obvious (inclusion) map, $\rho : H^3 \rightarrow \mathcal{M}^3 = H^3 \oplus H^2x$ is the inclusion on the first summand, and $\rho : N^3 = \text{Sym}^2 P \rightarrow \mathcal{M}^3 = H^3 \oplus H^2x$ is defined as $\rho(\alpha \cdot \beta) = L_\omega^{-1}(\alpha \wedge \beta)x$. Note that

$$d(\rho(\alpha \cdot \beta)) = L_\omega^{-1}(\alpha \wedge \beta) dx = L_\omega^{-1}(\alpha \wedge \beta) \omega = \alpha \wedge \beta = \rho(\alpha) \wedge \rho(\beta) = \rho(\alpha \wedge \beta) = \rho(d(\alpha \cdot \beta)),$$

so ρ is a DGA map. Clearly it is a 3-equivalence (it induces an isomorphism on cohomology up to degree 3 and an inclusion on degree 4).

The space of closed elements is $C^3 = H^3$. Now let us check when the elements $z \in I(N^3)$ with $dz = 0$ satisfy $[\rho(z)] = 0 \in H^*(M)$. The only cases to check is when z has degree 5 or 7. If z has degree 5, then $[\rho(z)] \neq 0$ if and only if there exists some $\beta \in P$, $[\rho(\beta)] \in H^2(M)$, such that $[\rho(z)] \wedge [\rho(\beta)] \neq 0$, by Poincaré duality. Hence $[\rho(z\beta)] \neq 0$. This means that we can restrict to elements z of degree 7, that is $z \in N^3 \cdot \bigwedge^2 P$.

Let $z \in N^3 \cdot \bigwedge^2 P \cong \text{Sym}^2 P \times \text{Sym}^2 P$. Then the map $d : N^3 \cdot \bigwedge^2 P \rightarrow \bigwedge^4 P$ coincides the full symmetrization map $\text{Sym}^2 P \times \text{Sym}^2 P \rightarrow \text{Sym}^4 P$. So

$$Z = \ker d|_{I(N^3)^7} = \mathcal{K}_M \oplus \text{Ant}^2(\text{Sym}^2 P),$$

where $\text{Ant}^2(W)$ denotes the antisymmetric 2-power of a vector space W .

Now we have to study the map

$$\rho : Z \rightarrow H^7(M) = H^6x,$$

and see if this is non-zero. This is given (on the basis elements) by

$$\rho((\alpha \cdot \beta) \cdot (\gamma \cdot \delta)) = (L_\omega^{-1}(\alpha \wedge \beta) \wedge \gamma \wedge \delta) x,$$

so $\mathcal{F}_M = \rho|_{\mathcal{K}_M}$. Note that ρ automatically vanishes on $\text{Ant}^2(\text{Sym}^2 P)$, hence M is formal if and only if ρ vanishes on \mathcal{K}_M if and only if $\mathcal{F}_M = 0$.

According to Theorem 11, to check non-formality we have to test the relevant property (2) on *any* splitting $V^3 = C^3 + N'^3$. If we take another splitting $V^3 = C^3 + N'^3$, then the projection $\pi : V^3 \rightarrow N^3$ gives an isomorphism $\pi : N'^3 \rightarrow N^3$, and so an isomorphism $N'^3 \cdot \text{Sym}^2 P \cong N^3 \cdot \text{Sym}^2 P$. Clearly, $d \circ \pi = d$ on N'^3 , so the spaces of cycles correspond $\mathcal{K}' \cong \mathcal{K}$. On the other hand $H^3 \cdot H^2 \cdot H^2 = 0$, so the maps $\rho : \mathcal{K} \rightarrow H^6 x$ and $\rho : \mathcal{K}' \rightarrow H^6 x$ also correspond. This means that the corresponding \mathcal{F} and \mathcal{F}' coincide under the isomorphism $\mathcal{K} \cong \mathcal{K}'$. This means that the choice of splitting is not relevant. \square

This result means that the formality or non-formality of M only depends on the cohomology algebra H . Theorem 14 can be applied to the examples in Section 5.3 of [2]. For instance for $B = \mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$, we have a simply connected Sasakian 7-manifold which is non-formal (Theorem 12 of [2]). For $B = \mathbb{C}P^3$, we have obviously $P = 0$ and hence M is formal.

The element \mathcal{F}_M of Theorem 14 is the *principal Massey product* defined by Crowley and Nordström in [6] for simply connected compact 7-manifolds in general. The principal Massey product is the full obstruction to formality for simply connected compact 7-manifolds.

Now we deduce Corollary 3.

Corollary 15. *Let M be a simply connected compact Sasakian 7-dimensional manifold. Then M is formal if and only if all triple Massey products are zero.*

Proof. Suppose that $\mathcal{F}_M \neq 0$. We choose an orthonormal basis for $H^2 = \langle e_0, e_1, \dots, e_m \rangle$, where $e_0 = \frac{1}{\sqrt{3}}\omega$, and $\langle e_1, \dots, e_m \rangle = P$. The vector space \mathcal{K}_M is generated by elements of the form

$$a_{ijkl} = (e_i \cdot e_j) \cdot (e_k \cdot e_l) - (e_k \cdot e_j) \cdot (e_i \cdot e_l),$$

for $1 \leq i, j, k, l \leq m$ (here, as usual, the dot product means symmetric product). Now define the numbers

$$\lambda_{ijk} = \int_M e_i \wedge e_j \wedge e_k \in \mathbb{R},$$

for $1 \leq i, j, k \leq m$. Note that these numbers are fully symmetric on i, j, k . Also $\lambda_{000} = \frac{2}{\sqrt{3}}$ and $\lambda_{ij0} = \frac{1}{\sqrt{3}}\delta_{ij}$, for $(i, j) \neq (0, 0)$. Then

$$L_\omega^{-1}(e_i \wedge e_j) = 2 * (e_i \wedge e_j)_0 + (e_i \wedge e_j)_1 = 2\lambda_{ij0}e_0 + \sum_{t>0} \lambda_{ijt}e_t.$$

So

$$\mathcal{F}_M((e_i \cdot e_j) \cdot (e_k \cdot e_l)) = 2\lambda_{ij0}\lambda_{kl0} + \sum_{t>0} \lambda_{ijt}\lambda_{klt}.$$

Evaluating \mathcal{F}_M on a_{ijkl} gives a set of equations to determine the formality of M . M is non-formal when there exists some a_{ijkl} with $\mathcal{F}_M(a_{ijkl}) \neq 0$. By [6], we have that the triple Massey product $\langle e_i, e_j, e_k \rangle$ is a well-defined element of $H^5(M)$ and it satisfies

$$\mathcal{F}_M(a_{ijkl}) = \langle e_i, e_j, e_k \rangle \cup e_l.$$

So $\langle e_i, e_j, e_k \rangle \neq 0$, as required. \square

This result is of relevance since it is not known if for general simply connected compact 7-dimensional manifolds there are obstructions to formality different from triple Massey products, as remarked in [6]. It is true that for higher dimensional manifolds, there are

obstructions to formality even when all Massey products (triple and higher order) can be zero.

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