

Singular sectors of the 1-layer Benney and dToda systems and their interrelations.

B. Konopelchenko¹, L. Martínez Alonso² and E. Medina³

¹ *Dipartimento di Fisica, Università del Salento and Sezione INFN
73100 Lecce, Italy*

² *Departamento de Física Teórica II, Universidad Complutense
E28040 Madrid, Spain*

³ *Departamento de Matemáticas, Universidad de Cádiz
E11510 Puerto Real, Cádiz, Spain*

November 6, 2013

Abstract

Complete description of the singular sectors of the 1-layer Benney system (classical long wave equation) and dToda system is presented. Associated Euler-Poisson-Darboux equations $E(1/2, 1/2)$ and $E(-1/2, -1/2)$ are the main tool in the analysis. A complete list of solutions of the 1-layer Benney system depending on two parameters and belonging to the singular sector is given. Relation between Euler-Poisson-Darboux equations $E(\varepsilon, \varepsilon)$ with opposite sign of ε is discussed.

1 Introduction

The 1-layer Benney system (classical long wave equation)

$$\begin{cases} u_t + u u_x + v_x = 0, \\ v_t + (u v)_x = 0 \end{cases} \quad (1)$$

and dToda equation $v_{xx} = (\log v)_{tt}$ or equivalently the system

$$\begin{cases} u_t + v_x = 0, \\ v_t + v u_x = 0 \end{cases} \quad (2)$$

are the two distinguished integrable systems of hydrodynamical type (see e.g. [1, 2]). 1-layer Benney system describes long waves in shallow water with free

surface in gravitational field. It is the dispersionless limit of the nonlinear Schrödinger equation [3]. Recently, the 1-layer Benney ($B(1)$) system became a crucial ingredient in the analysis of the universality of critical behavior for nonlinear equations [4]. The dToda equation is the 1+1-dimensional version of the Boyer-Finley equation from the general relativity [5]. It shows up in various problems of fluid mechanics (see e.g. [6]). It is known also (see e.g.[7]) that the hodograph equations of the dToda hierarchy determine the large N -limit of the Hermitian model in random matrix theory. In general, these two systems are an excellent laboratory for studying properties of integrable hydrodynamical type systems.

In the present work we analyze the structures of the set of hodograph equations of the $B(1)$ hierarchy and dToda hierarchy in terms of its Riemann invariants. These hodograph solutions describe the critical points

$$\frac{\partial W}{\partial \beta_i} = 0, \quad i = 1, 2, \quad (3)$$

of a function $W = W(\mathbf{t}, \beta_1, \beta_2)$ which depend linearly on the coordinates \mathbf{t} , where \mathbf{t} denotes the flow parameters of the Benney hierarchy and dToda hierarchy respectively and obey an Euler-Poisson-Darboux equations $E(\varepsilon, \varepsilon)$ [8]

$$(\beta_1 - \beta_2) \frac{\partial^2 W}{\partial \beta_1 \partial \beta_2} = \varepsilon \left(\frac{\partial W}{\partial \beta_1} - \frac{\partial W}{\partial \beta_2} \right). \quad (4)$$

where for the Benney system one has $\varepsilon = 1/2$ and for the dToda system $\varepsilon = -1/2$. The equation (4) and its multidimensional version are well known for a long time in classical geometry [8]. Its relevance to the theory of Whitham equations has been demonstrated recently in the papers [9]-[11]. Here we will use classical notation $E(\varepsilon, \mu)$ for the Euler-Poisson-Darboux equation proposed in [8] where such equations with different ε and μ have been studied too.

If we denote by \mathcal{M} the set of solutions $(\mathbf{t}, \boldsymbol{\beta})$ ($\beta_1 \neq \beta_2$) of the hodograph equations (3), we may distinguish a regular and a singular sector in \mathcal{M}

$$\mathcal{M} = \mathcal{M}^{\text{reg}} \cup \mathcal{M}^{\text{sing}},$$

such that given $(\mathbf{t}, \boldsymbol{\beta}) \in \mathcal{M}$

$$(\mathbf{t}, \boldsymbol{\beta}) \in \mathcal{M}^{\text{reg}} \text{ if } \det \left(\frac{\partial^2 W(\mathbf{t}, \boldsymbol{\beta})}{\partial \beta_i \partial \beta_j} \right) \neq 0, \quad (\mathbf{t}, \boldsymbol{\beta}) \in \mathcal{M}^{\text{sing}} \text{ if } \det \left(\frac{\partial^2 W(\mathbf{t}, \boldsymbol{\beta})}{\partial \beta_i \partial \beta_j} \right) = 0.$$

The elements of \mathcal{M}^{reg} , correspond to the case when the system (3) is uniquely solvable and hence, it defines a unique solution $\boldsymbol{\beta}(\mathbf{t})$. The singular class $\mathcal{M}^{\text{sing}}$ represents degenerate critical points of the function W and are the points on which the implicit solutions $\boldsymbol{\beta}(\mathbf{t})$ of the hodograph equations exhibit “gradient catastrophe” behaviour. As we will see in this paper, the Euler-Poisson-Darboux equation is of great help to analyze the structure of $\mathcal{M}^{\text{sing}}$. As the illustration of the general result a complete list of solutions of the 1-layer Benney hierarchy

from $\mathcal{M}^{\text{sing}}$ depending on two parameters is presented. We also discuss the relation between Euler-Poisson-Darboux equations with opposite a and Euler-Poisson-Darboux equations for symmetries and densities of integrals of motion for integrable hydrodynamical type systems.

2 1-layer Benney hierarchy and its singular sector

The $B(1)$ system (1) is a member of a dispersionless integrable hierarchy of deformations of the curve (see e.g. [14, 15]).

$$p^2 = (\lambda - \beta_1)(\lambda - \beta_2). \quad (5)$$

where $u = -(\beta_1 + \beta_2)$, $v = \frac{1}{4}(\beta_1 - \beta_2)^2$. The flows $\boldsymbol{\beta}(t)$ are characterized by the following condition: There exists a family of functions $S(\lambda, t, \boldsymbol{\beta})$ satisfying

$$\partial_{t_n} S(\lambda, t, \boldsymbol{\beta}(t)) = \Omega_n(\lambda, \boldsymbol{\beta}(t)), \quad n \geq 1. \quad (6)$$

where

$$\Omega_n(\lambda, \boldsymbol{\beta}) = \left(\frac{\lambda^n}{\sqrt{(\lambda - \beta_1)(\lambda - \beta_2)}} \right)_{\oplus} \sqrt{(\lambda - \beta_1)(\lambda - \beta_2)}. \quad (7)$$

where \oplus denotes the standard projection on the positive powers of λ . Functions S which satisfy (6) are referred to as *action functions* in the theory of dispersionless integrable systems (see e.g. [16]). Notice that for $n = 1$ (6) reads

$$p = \frac{\partial S}{\partial x}, \quad x := t_1,$$

so that the system (6) is equivalent to

$$\partial_{t_n} p = \partial_x \Omega_n, \quad (8)$$

and, in terms of Riemann invariants $\boldsymbol{\beta}$, it can be rewritten in the hydrodynamical form

$$\partial_{t_n} \beta_i = \left(\frac{\lambda^n}{\sqrt{(\lambda - \beta_1)(\lambda - \beta_2)}} \right)_{\oplus} \Big|_{\lambda=\beta_i} \partial_x \beta_i, \quad i = 1, 2. \quad (9)$$

The t_2 -flow of this hierarchy is the $B(1)$ system (1) ($t := t_2$)

$$\begin{cases} \partial_t \beta_1 = \frac{1}{2} (3\beta_1 + \beta_2) \beta_{1x}, \\ \partial_t \beta_2 = \frac{1}{2} (3\beta_2 + \beta_1) \beta_{2x}. \end{cases} \quad (10)$$

For $v > 0$ the $B(1)$ system is hyperbolic while for $v < 0$ it is elliptic.

It was proved in [12] that the system (3) for the critical points of the function

$$W(\mathbf{t}, \boldsymbol{\beta}) := \oint_{\gamma} \frac{d\lambda}{2i\pi} \frac{V(\lambda, \mathbf{t})}{\sqrt{(\lambda - \beta_1)(\lambda - \beta_2)}}, \quad (11)$$

where $V(\lambda, \mathbf{t}) = \sum_{n \geq 1} t_n \lambda^n$, is a system of hodograph equations for the Benney hierarchy. Moreover, the action function for the corresponding solutions is given by

$$S(\lambda, \mathbf{t}, \boldsymbol{\beta}) = \sum_{n \geq 1} t_n \Omega_n(\lambda, \boldsymbol{\beta}) = h(\lambda, \mathbf{t}, \boldsymbol{\beta}) \sqrt{(\lambda - \beta_1)(\lambda - \beta_2)}. \quad (12)$$

where

$$h(\lambda, \mathbf{t}, \boldsymbol{\beta}) := \left(\frac{V(\lambda, \mathbf{t})}{\sqrt{(\lambda - \beta_1)(\lambda - \beta_2)}} \right)_{\oplus}.$$

Obviously, the function W satisfies the Euler-Poisson-Darboux equation $E(1/2, 1/2)$.

Written explicitly, W represents itself the series

$$\begin{aligned} W &= \frac{x}{2}(\beta_1 + \beta_2) + \frac{t_2}{8}(3\beta_1^2 + 2\beta_1\beta_2 + 3\beta_2^2) + \frac{t_3}{16}(5\beta_1^3 + 3\beta_1^2\beta_2 + 3\beta_1\beta_2^2 + 5\beta_2^3) \\ &\quad + \frac{t_4}{128}(35\beta_1^4 + 20\beta_1^3\beta_2 + 18\beta_1^2\beta_2^2 + 20\beta_1\beta_2^3 + 35\beta_2^4) + \dots \end{aligned} \quad (13)$$

The hodograph equations (3) with $t_n = 0$ for $n \geq 5$ take the form

$$\begin{cases} 8x + 4t_2(3\beta_1 + \beta_2) + 3t_3(5\beta_1^2 + 2\beta_1\beta_2 + \beta_2^2) + \frac{t_4}{8}(140\beta_1^3 + 60\beta_1^2\beta_2 + 18\beta_1\beta_2^2 + 20\beta_2^3) = 0, \\ 8x + 4t_2(\beta_1 + 3\beta_2) + 3t_3(\beta_1^2 + 2\beta_1\beta_2 + 5\beta_2^2) + \frac{t_4}{8}(140\beta_2^3 + 60\beta_2^2\beta_1 + 18\beta_2\beta_1^2 + 20\beta_1^3) = 0. \end{cases} \quad (14)$$

Detailed analysis of equations (14) will be performed in section 3. Here, we would like to make two observations. First, one is that the formulae (14) point out on the possible alternative interpretation of the times t_2, t_3, t_4, \dots of the $B(1)$ hierarchy. Namely, taking $t_2 = 0$ in the formulae (14), we see that t_3 and t_4 are parameters appearing in the initial data $\beta_1(x, t_2 = 0)$ and $\beta_2(x, t_2 = 0)$. Thus, one can view hodograph equations (3) (in particular, equations (14)) as equations describing the time evolution of the family of initial data for the $B(1)$ system, parametrized by the variables t_3, t_4, t_5, \dots

Second observation concerns with the elliptic version of the $B(1)$ system. In this case $\beta_2 = \bar{\beta}_1$ and the system (10) reduces to the single equation

$$\partial_t \beta = \frac{1}{2}(3\beta + \bar{\beta})\beta_x, \quad t := t_2, \quad \beta := \beta_1. \quad (15)$$

This equation is equivalent to the nonlinear Beltrami equation

$$\beta_{\bar{z}} = \frac{2i - 3\beta - \bar{\beta}}{2i + 3\beta + \bar{\beta}} \beta_z, \quad (16)$$

where $z = x + it$. This fact indicates that the theory of quasi-conformal mappings (see e.g. [17]) can be relevant for the analysis of properties of the elliptic $B(1)$ system (1) ($v < 0$). Hence, since the elliptic $B(1)$ system is the quasiclassical limit [3] of the focusing nonlinear Schrödinger (NLS) equation

$$i \epsilon \psi_t + \frac{\epsilon^2}{2} \psi_{xx} + |\psi|^2 \psi = 0,$$

with $\psi = A \exp\left(\frac{i}{\epsilon} S\right)$, $u = \frac{\epsilon}{2i} \left(\frac{\psi_x}{\psi} - \frac{\overline{\psi}_x}{\overline{\psi}}\right)$, $v = -|\psi|^2$, $\epsilon \rightarrow 0$, the quasiconformal mapping can be useful also in the study of the small dispersion limit of the focusing NLS equation (compare with [4]).

In order to analyse singular sector of the 1-layer Benney hierarchy we first observe that due to the Euler-Poisson-Darboux equation for given $(\mathbf{t}, \boldsymbol{\beta}) \in \mathcal{M}$, as a consequence of (4) one has

$$\frac{\partial^2 W}{\partial \beta_1 \partial \beta_2} = 0. \quad (17)$$

Consequently

$$\det\left(\frac{\partial^2 W}{\partial \beta_i \partial \beta_j}\right) = \frac{\partial^2 W}{\partial \beta_1^2} \cdot \frac{\partial^2 W}{\partial \beta_2^2}. \quad (18)$$

Thus, we have

Proposition 1. *Given $(\mathbf{t}, \boldsymbol{\beta}) \in \mathcal{M}$ then*

1. $(\mathbf{t}, \boldsymbol{\beta}) \in \mathcal{M}^{reg}$ if and only if $\frac{\partial^2 W}{\partial \beta_1^2} \neq 0$ and $\frac{\partial^2 W}{\partial \beta_2^2} \neq 0$.
2. $(\mathbf{t}, \boldsymbol{\beta}) \in \mathcal{M}^{sing}$ if and only at least one of the derivatives $\frac{\partial^2 W}{\partial \beta_1^2}$, $\frac{\partial^2 W}{\partial \beta_2^2}$, vanishes.

Furthermore, using (4) it follows easily that at any point $(\mathbf{t}, \boldsymbol{\beta}) \in \mathcal{M}$ all mixed derivatives $\partial_{\beta_1}^i \partial_{\beta_2}^j W$ can be expressed in terms of linear combination of derivatives $\partial_{\beta_1}^n W$ and $\partial_{\beta_2}^m W$. Hence if we define $\mathcal{M}_{n_1, n_2}^{sing}$ as the set of points $(\mathbf{t}, \boldsymbol{\beta}) \in \mathcal{M}$ such that

$$\frac{\partial^{n_i+2} W}{\partial \beta_i^{n_i+2}} \neq 0, \quad \frac{\partial^k W}{\partial \beta_i^k} = 0, \quad \forall 1 \leq k \leq n_i + 1, \quad (i = 1, 2), \quad (19)$$

it follows that

$$\mathcal{M}^{sing} = \bigcup_{n_1+n_2 \geq 1} \mathcal{M}_{n_1, n_2}^{sing},$$

where

$$\mathcal{M}_{n_1, n_2}^{sing} \cap \mathcal{M}_{n'_1, n'_2}^{sing} = \emptyset, \quad \text{for } (n_1, n_2) \neq (n'_1, n'_2)$$

We may characterize the classes $\mathcal{M}_{n_1, n_2}^{\text{sing}}$ of the singular sector in terms of the behaviour of $S(\lambda)$ at $\lambda = \beta_i$ ($i = 1, 2$). Indeed the derivative $\partial_{\beta_1}^{k+1} W$ with $k \geq 1$ is proportional to the integral

$$\oint_{\gamma} \frac{d\lambda}{2i\pi} \frac{V(\lambda, \mathbf{t})}{(\lambda - \beta_1)^{k+1} \sqrt{(\lambda - \beta_1)(\lambda - \beta_2)}} = \oint_{\gamma} \frac{d\lambda}{2i\pi} \frac{h(\lambda)}{(\lambda - \beta_1)^{k+1}} = \frac{1}{k!} \left(\partial_{\lambda}^k h(\lambda) \right) \Big|_{\lambda=\beta_1},$$

and a similar result follows for the derivatives $\partial_{\beta_2}^{k+1} W$ with $k \geq 2$. As a consequence we have

Proposition 2. *A point $(\mathbf{t}, \boldsymbol{\beta}) \in \mathcal{M}$ belongs to the singularity class $\mathcal{M}_{n_1, n_2}^{\text{sing}}$ if and only if*

$$S(\lambda, \mathbf{t}, \boldsymbol{\beta}) \sim (\lambda - \beta_i)^{\frac{2n_i+3}{2}} \quad \text{as } \lambda \rightarrow \beta_i, \quad (i = 1, 2) \quad (20)$$

3 Explicit determination of singular sectors

It is easy to see that the singular classes $\mathcal{M}_{n_1, n_2}^{\text{sing}}$ can be determined by means of a system of $n_1 + n_2$ constraints for the coordinates \mathbf{t} . Indeed, the points $(\mathbf{t}, \boldsymbol{\beta})$ of $\mathcal{M}_{n_1, n_2}^{\text{sing}}$ are characterized by the equations

$$\frac{\partial^k W}{\partial \beta_i^k} = 0, \quad \forall 1 \leq k \leq n_i + 1, \quad i = 1, 2, \quad (21)$$

and

$$\frac{\partial^{n_i+2} W}{\partial \beta_i^{n_i+2}} \neq 0, \quad i = 1, 2. \quad (22)$$

Now the observation is that the jacobian matrix of the the system of two equations

$$\frac{\partial^{n_i+1} W}{\partial \beta_i^{n_i+1}} = 0, \quad i = 1, 2 \quad (23)$$

is not singular as

$$\Delta := \begin{vmatrix} \frac{\partial^{n_1+2} W}{\partial \beta_1^{n_1+2}} & \frac{\partial^{n_2+2} W}{\partial \beta_1 \partial \beta_2^{n_2+1}} \\ \frac{\partial^{n_1+2} W}{\partial \beta_1^{n_1+1} \partial \beta_2} & \frac{\partial^{n_2+2} W}{\partial \beta_2^{n_2+2}} \end{vmatrix} \neq 0. \quad (24)$$

Indeed, we notice that as a consequence of (4) the derivatives outside the diagonal of Δ are linear combinations of the derivatives $\{\partial_{\beta_i}^k W, 1 \leq k \leq n_i + 1, i = 1, 2\}$, so that from (21)-(22) we have

$$\Delta = \frac{\partial^{n_1+2} W}{\partial \beta_1^{n_1+2}} \cdot \frac{\partial^{n_2+2} W}{\partial \beta_2^{n_2+2}} \neq 0.$$

Therefore, one can solve (23) and get a solution $\beta(\mathbf{t})$. Substituting this solution in the remaining equations (21) gives $n_1 + n_2$ constraints of the form

$$f_k(\mathbf{t}) = 0, \quad k = 1, \dots, n_1 + n_2.$$

It is not difficult to determine the solutions of (21)-(22) in two simple cases: with one parameter t_3 ($t_4 = t_5 = \dots = 0$), and with two parameters t_3, t_4 ($t_5 = t_6 = \dots = 0$). We have that in this case

$$\mathcal{M}^{\text{sing}} = \mathcal{M}_{10}^{\text{sing}} \cup \mathcal{M}_{01}^{\text{sing}}$$

with $\mathcal{M}_{10}^{\text{sing}}$ defined by

1.
$$x = \frac{-45t_4t_3^3 + 180t_2t_4^2t_3 + \sqrt{15}(8t_2t_4 - 3t_3^2)\sqrt{t_4^2(3t_3^2 - 8t_2t_4)}}{360t_4^3},$$

$$\beta_1 = -\frac{5t_3t_4 + \sqrt{15}\sqrt{t_4^2(3t_3^2 - 8t_2t_4)}}{20t_4^2}, \quad \beta_2 = \frac{-3t_3t_4 + \sqrt{15}\sqrt{t_3^2(3t_3^2 - 8t_2t_4)}}{12t_4^2},$$
2.
$$x = \frac{-45t_4t_3^3 + 180t_2t_3^2t_3 - \sqrt{15}(8t_2t_4 - 3t_3^2)\sqrt{t_4^2(3t_3^2 - 8t_2t_4)}}{360t_4^3},$$

$$\beta_1 = \frac{-5t_3t_4 + \sqrt{15}\sqrt{t_4^2(3t_3^2 - 8t_2t_4)}}{20t_4^2}, \quad \beta_2 = -\frac{3t_3t_4 + \sqrt{15}\sqrt{t_4^2(3t_3^2 - 8t_2t_4)}}{12t_4^2},$$

and $\mathcal{M}_{01}^{\text{sing}}$ by

3.
$$x = \frac{-45t_4t_3^3 + 180t_2t_4^2t_3 - \sqrt{15}(8t_2t_4 - 3t_3^2)\sqrt{t_4^2(3t_3^2 - 8t_2t_4)}}{360t_4^3},$$

$$\beta_1 = -\frac{3t_3t_4 + \sqrt{15}\sqrt{t_4^2(3t_3^2 - 8t_2t_4)}}{12t_4^2}, \quad \beta_2 = \frac{-5t_3t_4 + \sqrt{15}\sqrt{t_4^2(3t_3^2 - 8t_2t_4)}}{20t_4^2},$$
4.
$$x = \frac{-45t_4t_3^3 + 180t_2t_4^2t_3 + \sqrt{15}(8t_2t_4 - 3t_3^2)\sqrt{t_4^2(3t_3^2 - 8t_2t_4)}}{360t_4^3},$$

$$\beta_1 = \frac{-3t_3t_4 + \sqrt{15}\sqrt{t_4^2(3t_3^2 - 8t_2t_4)}}{12t_4^2}, \quad \beta_2 = -\frac{5t_3t_4 + \sqrt{15}\sqrt{t_4^2(3t_3^2 - 8t_2t_4)}}{20t_4^2}.$$

4 Singular sector of the elliptic $B(1)$ system

Now, we will consider the elliptic $B(1)$ system (1). Singular sector $\mathcal{M}^{\text{sing}}$ has in this case a structure which is quite different from that of the hyperbolic system.

Indeed, since $\beta_1 = \bar{\beta}_2$, the function W for real x, t_2, t_3, \dots is the real valued function

$$\overline{W(\mathbf{t}, \beta, \bar{\beta})} = W(\mathbf{t}, \beta, \bar{\beta}),$$

and the hodograph equation (3) has the form of the Cauchy-Riemann condition ($\beta = \beta_1$)

$$\frac{\partial W}{\partial \bar{\beta}} = 0. \quad (25)$$

Regular and singular sectors \mathcal{M}^{reg} and $\mathcal{M}^{\text{sing}}$ are defined as the sets $(\mathbf{t}, \beta, \bar{\beta})$ of solutions of equation (25) such that the hermitian form

$$d^2 W = \frac{\partial^2 W}{\partial \beta^2} d\beta^2 + 2 \frac{\partial^2 W}{\partial \beta \partial \bar{\beta}} d\beta d\bar{\beta} + \frac{\partial^2 W}{\partial \bar{\beta}^2} d\bar{\beta}^2,$$

is nondegenerate or degenerate, respectively. For unreduced solutions ($\beta \neq \bar{\beta}$), the corresponding Euler-Poisson-Darboux equation implies that

$$\frac{\partial^2 W}{\partial \beta \partial \bar{\beta}} = 0,$$

and, hence

$$\begin{vmatrix} \frac{\partial^2 W}{\partial \beta^2} & \frac{\partial^2 W}{\partial \beta \partial \bar{\beta}} \\ \frac{\partial^2 W}{\partial \beta \partial \bar{\beta}} & \frac{\partial^2 W}{\partial \bar{\beta}^2} \end{vmatrix} = \left| \frac{\partial^2 W}{\partial \bar{\beta}^2} \right|^2. \quad (26)$$

Thus, one has

Proposition 3. *For unreduced solutions of the elliptic $B(1)$ system, the regular sector \mathcal{M}^{reg} is defined by the condition*

$$\frac{\partial W}{\partial \bar{\beta}} = 0, \quad \frac{\partial^2 W}{\partial \bar{\beta}^2} \neq 0. \quad (27)$$

A similar analysis to that of the hyperbolic case readily leads to

Proposition 4. *Singular sector $\mathcal{M}^{\text{sing}}$ of the elliptic $B(1)$ system (1) is the union of the subspaces $\mathcal{M}_n^{\text{sing}}$, ($n = 1, 2, 3, \dots$) defined as*

$$\mathcal{M}_n^{\text{sing}} = \left\{ (\mathbf{t}, \beta, \bar{\beta}) \in \mathcal{M}^{\text{sing}} : \frac{\partial^k W}{\partial \bar{\beta}^k} = 0, k = 1, \dots, n+1; \frac{\partial^{(n+2)} W}{\partial \bar{\beta}^{(n+2)}} \neq 0 \right\} \quad (28)$$

Solutions belonging to $\mathcal{M}_n^{\text{sing}}$ are defined on a subspace of codimension $2n$ in the space of parameters x, t_2, t_3, \dots .

So, in the elliptic case, gradient catastrophe happens in the point (x, t) at fixed parameters t_2, t_3, \dots . Similar to the hyperbolic case, the subspace $\mathcal{M}_n^{\text{sing}}$ is not empty if at least n parameters t_2, t_3, \dots, t_{n+1} are different from zero in the formula (13).

It is instructive to rewrite the formula (13) for the function W in terms of the real and imaginary part of β_1 , i.e. $\beta_1 = U + iV$:

$$\begin{aligned} W = & xU + t_2(U^2 - \frac{1}{2}V^2) + t_3(U^3 - \frac{3}{2}UV^2) + t_4(U^4 - 3U^2V^2 + \frac{3}{8}V^4) \\ & + t_5(U^5 - 5U^3V^2 + \frac{15}{8}UV^4) + \dots \end{aligned} \quad (29)$$

This formula explicitly shows the character of elliptic singularities exhibited for the function W for various values of parameters t_2, t_3, \dots .

Basic equations (25), (13) and also conditions (28) defining subspaces $\mathcal{M}_n^{\text{sing}}$ can be easily rewritten in terms of the original variables u and v . Since

$$\frac{\partial W}{\partial \beta} = -\frac{\partial W}{\partial u} + i\sqrt{-v}\frac{\partial W}{\partial v},$$

the hodograph equation (25) becomes (for $v \neq 0$)

$$\frac{\partial W}{\partial u} = 0, \quad \frac{\partial W}{\partial v} = 0, \quad (30)$$

while the Euler-Poisson-Darboux equation and equation (30) take the form

$$\frac{\partial^2 W}{\partial u^2} - v\frac{\partial^2 W}{\partial v^2} = 0. \quad (31)$$

For the subspace $\mathcal{M}_1^{\text{sing}}$ conditions (28) are

$$\frac{\partial W}{\partial \beta} = 0, \quad \frac{\partial^2 W}{\partial \beta^2} = 0, \quad \frac{\partial^3 W}{\partial \beta^3} \neq 0. \quad (32)$$

Since

$$\frac{\partial^2 W}{\partial \beta^2} = \frac{\partial^2 W}{\partial u^2} - 2i\sqrt{-v}\frac{\partial^2 W}{\partial u \partial v} + v\frac{\partial^2 W}{\partial v^2} + \frac{1}{2}\frac{\partial W}{\partial v},$$

one concludes taking into account equation (30) and (31) that the second condition (32) is satisfied if and only if

$$\frac{\partial^2 W}{\partial u^2} = 0, \quad \frac{\partial^2 W}{\partial v^2} = 0, \quad \frac{\partial^2 W}{\partial u \partial v} = 0. \quad (33)$$

Thus, the subspace $\mathcal{M}_1^{\text{sing}}$ is characterized by the conditions (30), (33) and by requirement of nonvanishing third order derivatives of W .

In order to compare these conditions with those of paper [4], we first observe that the $B(1)$ system (1) is converted into the system (1.8) by the substitution $u \rightarrow v, v \rightarrow -u$. Then, with the choice

$$W = f(u, v) + xv - uvt,$$

the hodograph equations (30) become equations (2.4) of [4] and equation (31) is reduced to their equation (2.5). Finally, with such a choice, the conditions (33) are converted to the condition (2.12) from the paper [4].

Finally, we note that according to the proposition 4 for the subspace $\mathcal{M}_1^{\text{sing}}$, the codimension of the corresponding subspace of (x, t_2, t_3, \dots) is equal to two and the function W with $t_n = 0, n \geq 4$ i.e.

$$W = xU + t_2(U^2 - \frac{1}{8}V^2) + t_3(U^3 - \frac{3}{2}UV^2),$$

exhibits the elliptic umbilic singularity according to Thom's classification [20] (see also [17]-[19]). These results reproduce those originally obtained in the paper [4] (formula (4.2))

5 5. dToda hierarchy.

Now let us consider the function

$$W_T(x, \beta_1, \beta_2) = \int \frac{d\lambda}{2\pi i} V_T(x, \lambda) \sqrt{(1 - \frac{\beta_1}{\lambda})(1 - \frac{\beta_2}{\lambda})} \quad (34)$$

where $V_T(x, \lambda) = \sum_{n \geq 0} \lambda^n x_n$. Critical points for this function are defined by the equations

$$\frac{\partial W_T}{\partial \beta_1} = 0, \quad \frac{\partial W_T}{\partial \beta_2} = 0. \quad (35)$$

It is a simple check to see that W_T obeys the Euler-Poisson-Darboux equation of the type $E(-1/2, -1/2)$

$$2(\beta_1 - \beta_2) \frac{\partial^2 W_T}{\partial \beta_1 \partial \beta_2} = -(\frac{\partial W_T}{\partial \beta_1} - \frac{\partial W_T}{\partial \beta_2}). \quad (36)$$

Written explicitly the function W_T is the series

$$W_T = -\frac{1}{2}x_0(\beta_1 + \beta_2) - \frac{1}{8}x_1(\beta_1 - \beta_2)^2 - \frac{1}{16}x_2(\beta_1 + \beta_2)(\beta_1 - \beta_2)^2 - \frac{1}{128}x_3(5\beta_1^2 + 6\beta_1\beta_2 + 5\beta_2^2)(\beta_1 - \beta_2)^2 + \dots \quad (37)$$

while the hodograph equations take the form

$$\begin{aligned} x_0 + \frac{1}{2}x_1(\beta_1 - \beta_2) + \frac{1}{8}x_2(3\beta_1^2 - 2\beta_1\beta_2 - \beta_2^2) + \dots &= 0, \\ x_0 - \frac{1}{2}x_1(\beta_1 - \beta_2) + \frac{1}{8}x_2(3\beta_2^2 - 2\beta_1\beta_2 - \beta_1^2) + \dots &= 0. \end{aligned}$$

These hodograph equations provide us with the solutions of the system

$$\frac{\partial\beta_1}{\partial x_1} = \frac{1}{2}(\beta_1 - \beta_2)\frac{\partial\beta_1}{\partial x_0}, \quad \frac{\partial\beta_2}{\partial x_1} = -\frac{1}{2}(\beta_1 - \beta_2)\frac{\partial\beta_2}{\partial x_0}. \quad (38)$$

In terms of the variables $u = -(\beta_1 + \beta_2), v = \frac{1}{4}(\beta_1 - \beta_2)^2$ one has the dToda system (2). Considering the higher times x_2, x_3, \dots one gets the whole dToda hierarchy.

Similar to the Benney case the function W_T is the generating function for classical singularities for functions of two variables. Indeed, in the variables $X = \frac{1}{2}(\beta_1 + \beta_2), Y = \frac{1}{2}(\beta_1 - \beta_2)$ it is of the form

$$W_T = -x_0X - \frac{1}{2}x_1Y^2 - \frac{1}{2}x_2XY^2 - \frac{1}{8}x_3(4X^2 + Y^2)Y^2 + \dots \quad (39)$$

The third term here represents the parabolic umbilic singularity both for hyperbolic and elliptic cases.

The formulas for the dToda hierarchy presented here coincide with those given in the paper [21] after the identification

$$V_T(x, \lambda) = -2T\lambda + \lambda V'_H(t, \lambda). \quad (40)$$

i.e. $x_0 = -2T, x_n = nt_n, n = 1, 2, 3, \dots$

It is obvious that the descriptions of the regular and singular sectors of the dToda hierarchy completely coincide with those of 1-layer Benney hierarchy.

6. Interrelations between the Euler-Poisson-Darboux equations with different indices and those for function W and densities of integrals of motion.

1-layer Benney hierarchy and dToda hierarchy are two examples of hydrodynamical type systems for which functions W obey the Euler-Poisson-Darboux equations

$$L_\varepsilon W_\varepsilon := \left[\frac{\partial^2}{\partial x \partial y} - \frac{\varepsilon}{x-y} \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right) \right] W_\varepsilon = 0, \quad (41)$$

with different indexes ε . Such linear equations are well studied (see e.g. [8]). The operators L_ε have a number of remarkable properties. One of them (probably missed before) is given by the identity

$$L_{\varepsilon+1} L_\mu = L_{\mu+1} L_\varepsilon \quad (42)$$

for arbitrary indices ε and μ . This identity implies, for instance, that for any solution W_ε the function $L_\mu W_\varepsilon$ with arbitrary μ obeys the Euler-Poisson-Darboux equation with index $\varepsilon + 1$, more precisely $L_\mu W_\varepsilon = \varepsilon(\varepsilon - \mu)W_{\varepsilon+1}$.

In particular, at $\varepsilon = -\frac{1}{2}$ and $\mu = 0$ one has $L_{\frac{1}{2}}L_0 = L_1L_{-\frac{1}{2}}$. In terms of the operators \tilde{L}_ε defined as $\tilde{L}_\varepsilon = (x - y)L_\varepsilon$ the last relation takes the form

$$\partial_x \partial_y \tilde{L}_{-\frac{1}{2}} = \tilde{L}_{\frac{1}{2}} \partial_x \partial_y. \quad (43)$$

This identity clearly demonstrates the duality between the Euler-Poisson-Darboux equations with indices $\frac{1}{2}$ and $-\frac{1}{2}$ and consequently between 1-layer Benney and dToda hierarchies.

Duality between the functions W and densities of integrals of motions is the another type of duality typical for the so-called ε integrable hydrodynamical type systems. Indeed, due to the Tsarev's result [22], a symmetry w_i of a semi-Hamiltonian hydrodynamical system

$$\frac{\partial \beta_i}{\partial t} = \lambda_i(\beta) \frac{\partial \beta_i}{\partial x}, \quad i = 1, \dots, n, \quad (44)$$

i.e. a solution of the system

$$\frac{\partial \beta_i}{\partial \tau} = w_i(\beta) \frac{\partial \beta_i}{\partial x}, \quad i = 1, \dots, n \quad (45)$$

which commutes with the system (44), are defined by the system

$$\frac{\frac{\partial w_k}{\partial \beta_i}}{w_i - w_k} = \frac{\frac{\partial \lambda_k}{\partial \beta_i}}{\lambda_i - \lambda_k}, \quad i \neq k. \quad (46)$$

Such w_i provide us with the solutions of the systems (44) via the hodograph equations

$$\Omega_i := -x + \lambda_i(\beta)t + w_i = 0, \quad i = 1, \dots, n. \quad (47)$$

For such system (44) densities P of integrals of motion obey the equations [22]

$$\frac{\partial^2 P}{\partial \beta_i \partial \beta_k} = \frac{\frac{\partial \lambda_i}{\partial \beta_k}}{\lambda_i - \lambda_k} \frac{\partial P}{\partial \beta_i} - \frac{\frac{\partial \lambda_k}{\partial \beta_i}}{\lambda_i - \lambda_k} \frac{\partial P}{\partial \beta_k}, \quad i \neq k. \quad (48)$$

Let us define ε -systems as those (for particular class of such systems see e.g. [23]) for which

$$\frac{\frac{\partial \lambda_i}{\partial \beta_k}}{\lambda_i - \lambda_k} = \frac{\frac{\partial \lambda_k}{\partial \beta_i}}{\lambda_i - \lambda_k} = \frac{\varepsilon}{\beta_i - \beta_k} \quad (49)$$

For such systems densities of integrals obey Euler-Poisson-Darboux equations

$$\frac{\partial^2 P}{\partial \beta_i \partial \beta_k} = \frac{\varepsilon}{\beta_i - \beta_k} \frac{\partial P}{\partial \beta_i} - \frac{\varepsilon}{\beta_i - \beta_k} \frac{\partial P}{\partial \beta_k}, \quad i \neq k. \quad (50)$$

At the same time the equations for w_i become

$$\frac{\partial w_k}{\partial \beta_i} = -\varepsilon \frac{w_i - w_k}{\beta_i - \beta_k}, \quad i \neq k \quad (51)$$

Symmetry of these equations with respect to the transposition of indices i and k implies that $\frac{\partial w_k}{\partial \beta_i} = \frac{\partial w_i}{\partial \beta_k}$. Hence

$$w_i = \frac{\partial \widetilde{W}}{\partial \beta_i}, \quad i = 1, \dots, n,$$

for a certain function \widetilde{W} . Thus, equations (51) are the Euler-Poisson-Darboux equations of the type $E(-\varepsilon, -\varepsilon)$ for the function \widetilde{W}

$$\frac{\partial^2 \widetilde{W}}{\partial \beta_i \partial \beta_k} = - \left(\frac{\varepsilon}{\beta_i - \beta_k} \frac{\partial \widetilde{W}}{\partial \beta_i} - \frac{\varepsilon}{\beta_i - \beta_k} \frac{\partial \widetilde{W}}{\partial \beta_k} \right), \quad i \neq k. \quad (52)$$

The fact that the generating function for symmetries of the Whitham equations and some other integrable hydrodynamical systems obey the Euler-Poisson-Darboux equations has been observed earlier in the papers [9, 13, 23]. Also the duality between the Euler-Poisson-Darboux equations for the densities of integrals of motions and generating functions of symmetries has been noted before too. However the demonstration presented above seems to be different from those discussed earlier.

In addition one can note that equations (49) imply that for ε -systems also $\lambda_i = \frac{\partial g}{\partial \beta_i}$ with some function g . As the result the hodograph equations (47) for the ε -systems take the form

$$\Omega_i = -x + t \frac{\partial g}{\partial \beta_i} + \frac{\partial \widetilde{W}}{\partial \beta_i} = \frac{\partial W}{\partial \beta_i} = 0, \quad i, \dots, n \quad (53)$$

where $W = -x(\beta_1 + \beta_2) + gt + \widetilde{W}$. Thus, hodograph equations for the integrable hydrodynamical type systems are nothing but the equations defining the critical points of the function W . It seems that this fact has been missing in the previous publications. Moreover, due to the equations (49) the function g also obeys the $E(-\varepsilon, -\varepsilon)$ Euler-Poisson-Darboux equation and, hence, the function W does the same. Note that particular class of ε -systems for which λ_i are linear functions of β_i has been discussed in [23].

So, for integrable hydrodynamical systems of the ε type, the densities of integrals and the functions W (as well as the functions \widetilde{W} generating symmetries) play a dual role obeying the Euler-Poisson-Darboux equations with opposite sign of the index ε . This property resembles a lot the well-known duality between the generating functions of integrals of motion and symmetries for the dispersionful integrable equations.

Acknowledgements

The authors wish to thank the Spanish Ministerio de Educación y Ciencia (research project FIS2008-00200/FIS).

References

- [1] B. A. Dubrovin and S. P. Novikov, Russian Math. Surveys **44**, 35 (1989)
- [2] D. Y. Benney, Stud. Appl. Math. **52** 45-50 (1973).
- [3] V. E. Zakharov, Func. Anal. Appl. **14**, 89 (1980).
- [4] B. Dubrovin, T. Grava and C. Klein, J. Nonlinear Science **19** 57 (2009).
- [5] C.P. Boyer and J.D. Finley, J. Math. Phys. **23** 1126 (1982).
- [6] M.Mineev-Weinstein, P. Wiegman and A. Zabrodin, Phys. Rev. Lett. **84** 5106 (2000).
- [7] L. Martínez Alonso and E. Medina, J. Phys. A: Math. Gen. **41**, 335202 (2008)
- [8] G. Darboux, *Lecons sur la theorie general des surfaces II* , Gauthier Villars (1915).
- [9] V. R. Kudashev and S. E. Sharapov, Phys. Lett. A **154**,445 (1991); Theor. Math. Phys. **87**, 40 (1991).
- [10] F. R. Tian, Commun. Pure. Appl. Math. **46** 1093 (1993).
- [11] F. R. Tian, Duke Math. J. **74** 203 (1994).
- [12] B.G. Konopelchenko, L. Martínez Alonso and E. Medina, J.Phys.A:Math.Theor. **43** 434020 2010; arXiv:1003.2892.
- [13] M. V. Pavlov, *Hamiltonian formulation of electroforesis equations. Integrable hydrodynamic equations* Preprint, Landau Inst. Theor. Phys., Chernogolovka (1987).
- [14] Y. Kodama and B.G. Konopelchenko, J. Phys. A: Math. Gen. **35**, L489-L500 (2002).
- [15] B.G. Konopelchenko and L. Martínez Alonso, J. Phys. A: Math. Gen. **37**, 7859 (2004)
- [16] I. M. Krichever, Commun. Pure. Appl. Math. **47** 437 (1994)
- [17] L. Alphors, *Lectures on quasiconformal mappings*, D. van Nostrand Co, Princeton, 1996.
- [18] V. I. Arnold, Func. Anal. Appl. **6** no.4, 3 (1972) ; Russian Math. Surveys **29** no. 2, 10 (1974); Russian Math. Surveys **30** no. 5, 3 (1975).
- [19] V. I. Arnold, S. M. Gusein-Zade and A. N. Varchenko, *Singularities of differentiable maps*, Birkhäuser Boston, Inc. (1985).

- [20] R. Thom, *Structural stability and Morphogenesis: An outline of a General theory of models*, Addison-Wesley Reading (1989).
- [21] B.Konopelchenko, L.Martínez Alonso and E.Medina, *On the singular sector of the Hermitian random matrix model in the large N limit*, Phys. Lett.A (to be published), arXiv:1005.4773 (2010).
- [22] S. P. Tsarev, Sov. Math. Dokl. **31** 488 (1985)
- [23] M.Pavlov, J.Math. Phys. **44** 4134 (2003).