Integrable Deformations of Algebraic Curves. *

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Abstract

A general scheme for determining and studying integrable deformations of algebraic curves, based on the use of Lenard relations, is presented. We emphasize the use of several types of dynamical variables : branches, power sums and potentials.

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1 Introduction

The theory of algebraic curves is a fundamental ingredient in the analysis of integrable nonlinear differential equations as it is shown, for example, by its relevance in the description of the finite-gap solutions or the formulation of the Whitham averaging method [1]-[7]. A particularly interesting problem is characterizing and classifying integrable deformations of algebraic curves. In [6]-[7] Krichever formulated a general theory of dispersionless hierarchies of integrable models associated with the deformations of algebraic curves arising in the Whitham averaging method. In [8]-[9] a different approach for determining integrable deformations of algebraic curves curves $\mathcal{C}$ defined by monic polynomial equations

$$F(p,k) := p^N - \sum_{n=1}^{N} u_n(k) p^{N-n} = 0, \quad u_n \in \mathbb{C}[k], \tag{1}$$

was proposed. It applies for finding the deformations $\mathcal{C}(x,t)$ consistent with the degrees of the polynomials u_n and characterized by the existence of an *action* function $\boldsymbol{S} = \boldsymbol{S}(k,x,t)$ verifying

1. The multiple-valued function $\boldsymbol{p} = \boldsymbol{p}(k)$ determined by (1) can be expressed as
$$\boldsymbol{p} = \boldsymbol{S}_x.$$
2. The function $\boldsymbol{S}_t$ represents, like $\boldsymbol{p} = \boldsymbol{S}_x$, a meromorphic function on $\mathcal{C}(x,t)$ with poles only at $k = \infty$.

As a consequence of these conditions $\boldsymbol{p}$ obeys the equation

$$\partial_t \boldsymbol{p} = \partial_x \boldsymbol{Q}, \tag{2}$$

where $\boldsymbol{Q} := \boldsymbol{S}_t$ is of the form [8, 10] (i.e. $\boldsymbol{Q} \in \mathbb{C}[k,p]/\rangle$),

$$\boldsymbol{Q} = \sum_{r=1}^{N} a_r(k,x,t) \boldsymbol{p}^{N-r}, \quad a_r \in \mathbb{C}[k].$$

The form of equation (2) motivates the use of several sets of dynamical variables to formulate these deformations conveniently. One should also note that the equation (2) provides an infinite number of conservation laws, when one expands $\boldsymbol{p}$ and $\boldsymbol{S}$ in the Laurent series in k. In this sense, we say that the equation (2) is integrable.

2 Algebraic curves and dynamical variables

In order to describe deformations of the curve $\mathcal{C}$ defined by (1) we will use not only the coefficients u_n (*potentials*) but also some other alternative sets of dynamical variables. Firstly, we have the N branches $p_i = p_i(k)$ $(i = 1, \ldots, N)$ of the multiple-valued function function $\boldsymbol{p} = \boldsymbol{p}(k)$ satisfying

$$F(p, k) = \prod_{i=1}^{N} (p - p_i(k)) = 0. \tag{3}$$

There is an important result concerning the branches p_i. Let $\mathbb{C}((\lambda))$ denote the field of Laurent series in λ with at most a finite number of terms with positive powers

$$\sum_{n=-\infty}^{N} c_n \lambda^n, \quad N \in \mathbb{Z}.$$

Then we have [11, 12] :

Theorem 1. ***(Newton Theorem)***
There exists a positive integer l such that the N branches

$$p_i(z) := \Big(p_i(k)\Big)\Big|_{k=z^l}, \tag{4}$$

are elements of $\mathbb{C}((z))$. Furthermore, if $F(p,k)$ is irreducible as a polynomial over the field $\mathbb{C}((k))$ then $l_0 = N$ is the least permissible l and the branches $p_i(z)$ can be labelled so that

$$p_i(z) = p_N(\epsilon^i z), \quad \epsilon := \exp \frac{2\pi i}{N}.$$

Notation convention *Henceforth, given an algebraic curve $\mathcal{C}$ we will denote by z the variable associated with the least positive integer l_0 for which the substitution $k = z^{l_0}$ implies $p_i \in \mathbb{C}((z))$, $\forall i$.*

The potentials can be expressed as elementary symmetric polynomials s_n [13]-[14] of the branches p_i

$$u_n = (-1)^{n-1} s_n(p_1, p_2, \ldots) = (-1)^{n-1} \sum_{1 \le i_1 < \ldots < i_n \le N} p_{i_1} \cdots p_{i_n}. \tag{5}$$

Notice that, according to the famous Abel theorem [13], for $N > 4$ the branches p_i of the generic equation (1) cannot be written in terms of the potentials u_n by means of rational operations and radicals.

Our second set of alternative dynamical variables is provided by the *power sums* [13]-[14]

$$\mathcal{P}_k = \frac{1}{k}(p_1^k + \cdots + p_N^k), \quad k \geq 1, \tag{6}$$

where we have added a factor $1/k$ to their standard definition. One can relate potentials and power sums through Newton recurrence formulas, the solution of which is given by Waring's formula [14]

$$\mathcal{P}_k = \sum_{1 \leq i \leq k}^{(k)} \frac{1}{i}(u_1 + \cdots + u_N)^i, \tag{7}$$

where the superscript (k) in the summation sign indicates that only the terms of weight k are retained in the expansion of $(u_1 + \cdots + u_N)^i$ (it is assumed that the weight of the potential u_n is given by n). The corresponding inverse formula is [12]

$$\begin{aligned} u_n &= -\mathcal{S}_n(-\mathcal{P}_1, -\mathcal{P}_2, \ldots) \\ &= -\sum_{i_1 + 2i_2 + \cdots + k i_k = n} \frac{1}{i_1! \cdots i_k!}(-\mathcal{P}_1)^{i_1} \cdots (-\mathcal{P}_k)^{i_k}, \end{aligned} \tag{8}$$

where $\mathcal{S}_n$ are the Schur's polynomials defined by the identity $\exp(\sum_{n \geq 1} \lambda^n x_n) = \sum_{\geq 0} \lambda^n \, \mathcal{S}_n(x))$. The formula (8) is the consequence of the identity [15]

$$\exp\Big(-\sum_{n \geq 1} \lambda^n \mathcal{P}_n\Big) = \sum_{n \geq 0} (-\lambda)^n \, s_n(p_1, p_2, \ldots).$$

Examples

For $N = 2$, the equation for the curve is

$$F := p^2 - u_1 \, p - u_2 = 0, \tag{9}$$

and the first power sums are

$$\mathcal{P}_1 = u_1, \quad \mathcal{P}_2 = \frac{1}{2} u_1^2 + u_2.$$

For $N=3$

$$F := p^3 - u_1\,p^2 - u_2\,p - u_3 = 0, \tag{10}$$

we have

$$\begin{aligned}
&\mathcal{P}_1 = u_1, \quad \mathcal{P}_2 = u_2 + \frac{1}{2}u_1^2, \quad \mathcal{P}_3 = u_3 + u_1\,u_2 + \frac{1}{3}u_1^3,\\
&\mathcal{P}_4 = u_1\,u_3 + \frac{1}{2}u_2^2 + \ u_2\,u_1^2 + \frac{1}{4}u_1^4,
\end{aligned}$$

Power sums have also the meaning of moments of the logarithmic derivative of the function $F(p,k)$ as shown by the formula (see chapter 7 of [16])

$$\mathcal{P}_k = \frac{1}{2\pi\,i\,k}\oint_{\partial D} p^k \frac{\partial}{\partial p}\log\,F(p,k)\,\mathrm{d}\,p,$$

where ∂D is the boundary of the domain of the p complex plane which contains all the zeros p_i of $F(p,k)$.

3 Deformations of algebraic curves

We are looking for evolution equation of the potentials leading to equations for $\boldsymbol{p}$ of the form (2) or, equivalently, in terms of branches, p_i for $i=1,\ldots,N$,

$$\partial_t p_i = \partial_x Q_i, \quad \text{with} \quad Q_i := \sum_{r=1}^{N} a_r(k,x,t) p_i^{N-r}. \tag{11}$$

By using the notations

$$\mathbf{p} := (p_1,\ldots,p_N)^\top, \quad \mathbf{a} := (a_N,\ldots,a_1)^\top,$$

the system (11) can be written as

$$\partial_t\,\mathbf{p} = \partial_x(V\,\mathbf{a}), \tag{12}$$

where V is the Vandermonde matrix

$$V := \begin{pmatrix} 1 & p_1 & \cdots & p_1^{N-1} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 1 & p_N & \cdots & p_N^{N-1} \end{pmatrix}, \tag{13}$$

which in turn is the Jacobian of the transformation from branches to power sum variables (6). Hence, for power sums we have

$$\partial_t \mathbf{P} = J\,\mathbf{a}, \quad \mathbf{P} := (\mathcal{P}_1, \dots, \mathcal{P}_N)^\top, \tag{14}$$

where J is the Hamiltonian operator

$$J := V^\top \partial_x V. \tag{15}$$

Notice that

$$\begin{aligned} &J_{11} = N\,\partial_x, \\ &J_{ij} = (i+j-2)\mathcal{P}_{i+j-2}\,\partial_x + (j-1)\,\mathcal{P}_{i+j-2,x}, \quad (i,j) \neq (1,1). \end{aligned} \tag{16}$$

Finally let us consider the evolution law for the potentials. In virtue of the well-known formula $\frac{\partial \mathcal{S}_n(x)}{\partial x_k} = \mathcal{S}_{n-k}(x)$, it is straightforward to deduce from (8) that the Jacobian of the transformation from power sums to potentials is given by

$$T := \begin{pmatrix} 1 & -u_1 & \cdots & -u_{N-1} \\ 0 & 1 & \cdots & -u_{N-2} \\ \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 1 \end{pmatrix}. \tag{17}$$

Therefore one has

$$\partial_t \mathbf{u} = J_0 \mathbf{a}, \quad \mathbf{u} := (u_1, \dots, u_N)^\top \tag{18}$$

where J_0 is the matrix differential operator

$$J_0 := T^\top J = T^\top V^\top \partial_x V. \tag{19}$$

The problem now is to determine expressions for $\mathbf{a}$ depending on k and $\mathbf{u}$ such that (18) is consistent with the polynomial dependence of $\mathbf{u}$ on the variable k. That is to say, if $d_n := \text{degree}(u_n)$ are the degrees of the coefficients u_n as polynomials in k , then (18) must satisfy

$$\text{degree}(J_0 \mathbf{a})_n \leq d_n, \quad n = 1, \dots N,$$

where we are taking into account our notation (18) for $\mathbf{u}$. We observe that if (18) is consistent then, as a consequence of (12), the coefficients of the expansions in z of the branches p_i are conserved densities.

Our strategy for finding consistent deformations is to use Lenard type relations

$$J_0\boldsymbol{r} = 0, \quad \boldsymbol{r} := (r_1, \ldots, r_N)^\top, \quad r_i \in \mathbb{C}((k)), \tag{20}$$

to generate systems of the form

$$\mathbf{u}_t = J_0\mathbf{a}, \quad \mathbf{a} := \boldsymbol{r}_+. \tag{21}$$

Here $(\,\cdot\,)_+$ and $(\,\cdot\,)_-$ indicate the parts of non-negative and negative powers in k, respectively. The point is that from the identity

$$J_0\mathbf{a} = J_0\boldsymbol{r}_+ = -J_0\boldsymbol{r}_-,$$

it is clear that a sufficient condition for the consistency of (21) is that

$$\operatorname{degree}\Big(J_0\Big)_{nm} \le d_n + 1, \tag{22}$$

for all n and all m such that $a_{N-m+1} = (\boldsymbol{r}_+)_m \neq 0$.

If we impose (22) for all $1 \le n, m \le N$, we get a sufficient condition for consistency, which only depends on the curve (1) and does not refer to the particular solution of the Lenard relation used.

Examples

For $N = 2$ the conditions (22) reduce to

$$d_1 \le d_2 + 1. \tag{23}$$

This indicates that one can take $u_2(k)$ to be a polynomial of arbitrary degree, which implies that there exist integrable deformations for hyperelliptic curves with arbitrary genus.

For $N = 3$ the consistency conditions (22) become

$$\begin{gathered} d_1 \le 1, \quad d_2 \le d_1 + 1, \\ d_3 \le d_2 + 1, \quad d_2 \le d_3 + 1, \end{gathered} \tag{24}$$

which lead to the following twelve nontrivial choices for (d_1, d_2, d_3)

$$\begin{aligned}&(0,0,1),\ (0,1,0),\ (0,1,1),\ (0,1,2),\\ &(1,0,0),\ (1,0,1),\ (1,1,0),\ (1,1,1)\\ &(1,1,2),\ (1,2,1),\ (1,2,2),\ (1,2,3).\end{aligned}$$

This implies that our deformations allows one to have only trigonal curves with genus less than or equal to one. This subject will be discussed elsewhere.

There is a natural class of solutions of the Lenard relations (15). Indeed, from the expression (19) of J_0 it is obvious that for any constant vector $\boldsymbol{c} \in \mathbb{C}^N$

$$\boldsymbol{r} := V^{-1}\, \boldsymbol{c},$$

is a solution of (15). Hence by taking into account that V and T are the Jacobians for the transformation from branches to power sums and power sums to potentials, respectively, we have that

$$\begin{gathered}\boldsymbol{r}_i := \nabla_{\mathbf{P}} p_i = T\, \nabla_{\mathbf{u}} p_i, \quad i = 1, \ldots, N,\\ \nabla_{\mathbf{P}} p_i := \Big(\frac{\partial p_i}{\partial \mathcal{P}_1} \cdots \frac{\partial p_i}{\partial \mathcal{P}_N}\Big)^{\top}, \quad \nabla_{\mathbf{u}} p_i := \Big(\frac{\partial p_i}{\partial u_1} \cdots \frac{\partial p_i}{\partial u_N}\Big)^{\top},\end{gathered} \tag{25}$$

are solutions of the Lenard relations.

In summary, given an algebraic curve (1) satisfying (22) then evolution equations of the form

$$\partial_t \mathbf{u} = J_0 \Big(T\, \nabla_{\mathbf{u}} R\Big)_+, \quad R(z, \boldsymbol{p}) = \sum_i f_i(z)\, p_i, \tag{26}$$

where $f_i \in \mathbb{C}[z]$ define a deformation of the curve provided $R \in \mathbb{C}[k]$.

We conclude with the observation that the nonlinear equations discussed above have a very simple form in terms of branches. Indeed by applying Cramer rule to find the columns of V^{-1} it follows that

$$\frac{\partial p_i}{\partial \mathcal{P}_j} = \frac{|\mathbf{1}\, \mathbf{p} \ldots \mathbf{e}_i \ldots \mathbf{p}^{N-1}|}{|\mathbf{1}\, \mathbf{p} \ldots \mathbf{p}^j \ldots \mathbf{p}^{N-1}|}, \tag{27}$$

where $|\mathbf{1}\, \mathbf{p} \ldots \mathbf{p}^j \ldots \mathbf{p}^{N-1}|$ denotes the determinant of Vandermonde matrix V and $|\mathbf{1}\, \mathbf{p}\ \ldots \mathbf{e}_i \ldots \mathbf{p}^{N-1}|$ stands for the determinant of the matrix resulting from substituting in V the j-th column by the vector $(\mathbf{e}_i)_j = \delta_{ij}$. Thus from (12) we find

$$\partial_t \mathbf{p} = \partial_x \Big(V \, (\nabla_{\mathbf{P}} R)_+ \Big),$$

$$(\nabla_{\mathbf{P}} R)_i = \frac{|\mathbf{1}\,\mathbf{p} \ldots \mathbf{R}_i \ldots \mathbf{p}^{N-1}|}{|\mathbf{1}\,\mathbf{p} \ldots \mathbf{p}^{i-1} \ldots \mathbf{p}^{N-1}|}, \quad i = 1, \ldots, N, \tag{28}$$

where $|\mathbf{1}\,\mathbf{p} \ldots \mathbf{R}_i \ldots \mathbf{p}^{N-1}|$ is the determinant of the matrix resulting from substituting in V the i-th column by the vector $(\mathbf{R}_i)_j = f_j(z)$.

Finally, several remarks are in order:

1. If (1) is an irreducible curve the condition $R \in \mathbb{C}[k]$ is verified by $z^i \, \mathcal{L}_i$ where $\mathcal{L}_i$ are the so-called *Lagrange resolvents* [13]

$$\mathcal{L}_i = \sum_j \epsilon^{j\,i} \, p_j(z), \quad \epsilon := \exp \frac{2\pi i}{N}, \tag{29}$$

   where $i = 1, \ldots, N-1$.

2. The equations (26) admit a simpler expression when they are formulated in terms of power sums. Indeed, they reduce to

$$\partial_t \mathbf{P} = J \left(\nabla_{\mathbf{P}} R \right)_+ . \tag{30}$$

   This form should be particularly convenient for analyzing their Hamiltonian content as J defined in (15) is a Hamiltonian operator.

3. The main problem which remains in our analysis is the classification of deformations of algebraic curves (1) for general potentials of fixed degrees in k. In [9] a complete description of the deformations of hyperelliptic curves ($N = 2$) was given. Since for $N = 3$ only twelve cases of consistent sets of general potential degrees arise, one is lead to think that for higher N admissible cases different from the Gelfand-Dikii choice ($d_1 = \ldots = d_{N-1} = 0$, $d_N = 1$) should be very special. This of course must be expected for $N \geq 5$ because of Abel theorem.

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References

[1] S. P. Novikov, S. V. Manakov, L. P. Pitaevski and V. E. Zakharov *Theory of solitons*, Plenum, New York (1984).

[2] E. D. Belokolos, A. I. Bobenko , V. Z. Enol´ski, A. R. Its and V. B. Matveev, *Algebro-Geometric approach to nonlinear integrable equations*, Springer-Verlag, Berlin (1994).

[3] B. Dubrovin and S. Novikov, Russ. Math. Surv. **44**(6), 35 (1989).

[4] H. Flaschka, M.G. Forest and D.W. Mclauglin , Commun. Pure Appl. Math **33**, 739 (1980).

[5] B.A. Dubrovin, Commun. Math. Phys. **145**, 415 (1992)

[6] I. M. Krichever, Funct. Anal. Appl. **22**, 206 (1988).

[7] I. M. Krichever, Commun. Pure. Appl. Math. **47**, 437 (1994).

[8] Y. Kodama and B.G. Konopelchenko, J. Phys. A: Math. Gen. **35**, L489-L500 (2002); *Deformations of plane algebraic curves and integrable systems of hydrodynamic type* in *Nonlinear Physics: Theory and Experiment II*,edited by M.J. Ablowitz et al, World Scientific, Singapore (2003).

[9] B.G. Konopelchenko and L. Martínez Alonso, J. Phys. A: Math. Gen. **37**, 7859 (2004).

[10] C. L. Siegel, *Topics in complex function theory* Vol. I, Wiley (1969)

[11] R. Y. Walker, *Algebraic Curves* Springer-Verlag, Berlin (1978).

[12] S. S. Abhyankar, *Algebraic Geometry for Scientists and Engineers*, Mathematical Surveys and Monograps vol. 35, AMS (1990).

[13] B. L. van der Waerden,*Algebra*, Vol. I, Springer-Verlag, Berlin (1991).

[14] L. Redei,*Introduction to algebra*, Vol. I, Pergamon Press, Oxford (1967).

[15] I. G. Macdonald, *Symmetric functions and Hall polynomials*, Clarendom Press, Oxford (1979).

[16] L. Schwartz, *Analyse Mathematique*, Vol. II, Hermann, Paris (1967).