Optimal Sequence of Landfills in Solid Waste Management*

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Abstract

Given that landfills are depletable and replaceable resources, the right approach, when dealing with landfill management, is that of designing an optimal sequence of landfills rather than designing every single landfill separately. In this paper we use Optimal Control models, with mixed elements of both continuous and discrete time problems, to determine an optimal sequence of landfills, as regarding their capacity and lifetime. The resulting optimization problems involve splitting a time horizon of planning into several subintervals, the length of which has to be decided. In each of the subintervals some costs, the amount of which depends on the value of the decision variables, have to be borne. The obtained results may be applied to other economic problems such as private and public investments, consumption decisions on durable goods, etc.

Keywords: Optimal Control, Optimal Capacity, Landfilling, Recycling.

1 Introduction

The aim of this paper is to analyse how the optimal capacity and the switching time of a sequence of landfills has to be decided, considering both the construction and management costs.

As noted in Ready and Ready (1995), landfills are depletable and replaceable resources. Unlike other natural resources, whose depletion is irreversible, once a landfill is full it can be replaced at some cost, by constructing a new one. The new landfill will also be depleted and so on. As a consequence, the capacity of a landfill should not be decided from a static point of view, just by considering the costs associated with the present landfill, but also the costs linked to the following ones. Therefore, instead of optimally designing a landfill, the appropriate approach is that of designing an optimal sequence of landfills. In Jacobs and Everett (1992), Ready and Ready (1995), Huhtala (1997) and Gaudet, Moreaux and Salant (1998) the sequential aspect of landfills is recognized. However, in these papers, landfill capacity is a given and therefore the problem of obtaining the optimal capacity is not explicitly considered.

The smaller is the capacity of the landfill to be constructed, the smaller is the construction cost, but also the shorter is the lifetime, so that the construction of a new landfill will have to be undertaken sooner. This conflict between present and future costs gives rise to an economic dynamic problem, so that present

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and future decisions are not independent, but have to be jointly taken. A planning time horizon has to be divided into several subintervals of length endogenously determined, and in each of the subintervals some costs, the amount of which depends on the decision variables, are realized. In this paper, the described problem is formalized and the solution is discussed under different assumptions. The models we state are Optimal Control problems, with mixed elements of both continuous and discrete time problems.

The rest of the paper has the following structure: In section 2 the basic problem, under the assumption of constant waste generation, is stated in continuous time and solved by expressing it as a discrete time Optimal Control problem, where the discrete time is given by the landfill index and the switching time plays a role of state variable. The solution is characterized by the so-called Optimal Capacity Condition.

In section 3 a generalization is studied, assuming that the instantaneous generation of waste follows a given evolution through time. We propose a solution method that gives rise to either a discrete time Calculus of Variations problem or a discrete time Optimal Control problem. Section 4 states the joint problem of optimal landfill capacity and optimal waste treatment when two methods (landfilling and recycling) exist. We obtain a multiple-stage Optimal Control problem, which is solved by applying the results of Tomiyama (1985) and Tomiyama and Rossana. Section 5 suggests some guidelines for empiral applications and further research. Section 6 summarizes the main results of the article and shows how they could be useful to describe other economic problems with similar dynamic structure, such as private and public investments, consumption decisions on durable goods, etc. The proof of the mathematical results are given in an appendix (section 7).

2 Problem with constant waste generation

A social planner has to take the following actions in order to manage, with the smallest possible cost, the waste produced in a time horizon of length τ:

1. At instant $t = 0$, to construct a landfill, with arbitrary capacity Y_0. The set up cost depends on Y_0, according to the increasing, convex and $C^{(2)}$ cost function $C(Y_0)$.

2. While the first landfill is being used, he has to pay the instantaneous waste management cost, given by the linear function $h_0(Q(t)) = \phi_0 Q(t)$, where ϕ_0 is the unit management cost, basically representing the collection, transportation and processing costs of one unit of waste, and $Q(t)$ is the amount of waste produced at instant t, which is assumed to be exogenous. In this section, we further assume that $Q(t) = Q$ is constant. This assumption is relaxed in section 3.

3. When the capacity of the first landfill is exhausted, which happens at time T_1, implicitly determined by the condition $\int_0^{T_1} Q(t)\,dt = Y_0$, the planner has to close it and to construct a new one, in another place, with capacity Y_1, which will last until time T_2 given by $\int_{T_1}^{T_2} Q(t)\,dt = Y_1$.

4. Between T_1 and T_2, he has also to pay the management costs of the waste produced in this period. These costs are given by the function $h_1(Q(t)) = \phi_1 Q(t)$, where the unit cost ϕ_1, in general, is different from ϕ_0, mainly due to the different transportation costs.

And so on, until the last landfill, denoted by $K-1$, being K a decision variable. In general, a landfill constructed at T_i with a capacity Y_i lasts until T_{i+1}, implicitly defined by the equation $\int_{T_i}^{T_{i+1}} Q(t)\,dt = Y_i$, and the instantaneous management costs associated with such a landfill are given by $h_i(Q(t)) = \phi_i Q$.

From a mathematical point of view, the described problem has a particular structure which incorporates some continuous time and some discrete time elements. On the one hand, the time variable t is continuous, waste is generated in continuous time and the management costs $h_i(Q(t))$ happen in continuous time. The variables T_i, which refer to time, can take any real value, as corresponds to a continuous time Optimal Control model. On the other hand, the construction costs happen at a finite number of times, as in discrete time Optimal Control problems.

Assuming that the capacities of all landfills are wholly depleted under the solution, the problem can be expressed as one in discrete time, that consists of finding a number of landfills K, and a sequence of capacities $\{Y_0, Y_1, ..., Y_{K-1}\}$, in order to minimize the function

$$\sum_{i=0}^{K-1} e^{-\delta T_i} C(Y_i) + \sum_{i=0}^{K-1} \left[\int_{T_i}^{T_{i+1}} e^{-\delta t} h_i(Q(t))\, dt \right] = \sum_{i=0}^{K-1} e^{-\delta T_i} \left[C(Y_i) + \int_{T_i}^{T_{i+1}} e^{-\delta(t-T_i)} \phi_i Q dt \right] \qquad \text{(P)}$$

subject to the following constraints:

$$\begin{aligned} T_0 &= 0,\, T_K = \tau, \\ T_{i+1} &= T_i + \frac{Y_i}{Q}, \qquad i = 0, 1, 2, ..., K-1, \\ \underline{Y} &\leq Y_i \leq \overline{Y}, \end{aligned} \qquad (1)$$

where δ is the discount rate and $\underline{Y}$, $\overline{Y}$ represent the minimum and maximum capacity constraints. The most interesting case, as for its economic interpretation, is the one where these constraints are not binding. For that reason, in what follows we focus mainly on interior solutions. Note that (P) can be regarded as a discrete time Optimal Control problem, where the "discrete time" is not given by the chronological time t, but by the different landfill index $i = 0, 1, \ldots, K-1$, and (1) is the state equation.

This problem is conceptually similar to that of exploiting a sequence of deposits of a natural resource, as studied in Herfindahl (1967), Hartwick (1978), Weitzman (1976) or Hartwick, Kemp and Long (1986), where the role of extraction cost is played by management costs in our problem. Anyway, there are two important differences: first, in our case, the capacity depletion rate, analogous to a resource extraction rate, can not be decided because it is given by the exogenous generation of waste. Second, the initial landfill capacity (analogous to the initial resource stock) is not given in our problem, as it is in natural resource extraction models, but it is a decision variable. The classic result by Herfindahl (1967) for various natural resource deposits, which states that the deposits has to be exploited in an increasing order of marginal extraction costs, applies here. If the only difference among the various places available for building landfills is the attached unit waste management cost, then it is optimal to make use of such places beginning from the lowest cost one and following in the order of increasing unit cost[1].

Because K is a decision variable, (P) is a free time horizon problem. The easiest way to solve it consists of finding the solution for all possible values of K, and choosing that which provides the minimum total cost. K can take any integer value from the set $\{K_{\min}, K_{\min}+1, ..., K_{\max}-1, K_{\max}\}$, where

$$K_{\min} = \begin{cases} \dfrac{\tau Q}{\overline{Y}} & \text{if } \dfrac{\tau Q}{\overline{Y}} \text{ is an integer,} \\ Int\left(\dfrac{\tau Q}{\overline{Y}} + 1\right) & \text{otherwise,} \end{cases} \Bigg| \; ; \quad K_{\max} = Int\left(\frac{\tau Q}{\underline{Y}}\right),$$

[1]Let us assume that the solution is given by the sequence $\{Y^*\} = \left\{Y_0^*, \ldots, Y_i^*, Y_{i+1}^*, \ldots\right\}$, where $\phi_i > \phi_{i+1}$. The discounted cost of $\{Y^*\}$ can be reduced just by changing the order of landfills i and $i+1$.

$Int\,(\xi)$ denoting the integer part of ξ. Henceforth, $K_{\max} - K_{\min} + 1$ discrete time Optimal Control problems have to be solved. Let $\hat{C}_K$ be the optimal discounted cost which can be obtained constructing K landfills. The optimal value of K is given by $K^* = \underset{\{K=K_{\min},\ldots,K_{\max}\}}{\arg\min} \hat{C}_K$. For each possible value of K, the (interior) solutions are characterized by the following proposition:

Proposition 1 *Given a value of K, in an interior solution to problem (P), for two consecutive landfills, k and $k+1$ $(k = 0, 1, \ldots, K-2)$, the following Optimal Capacity Condition holds:*

$$C'(Y_k) = e^{-\frac{\delta}{Q}Y_k}\left[C'(Y_{k+1}) + \frac{\delta}{Q}C(Y_{k+1}) + \Delta\phi_k\right] = e^{-\delta(T_{k+1}-T_k)}\left[C'(Y_{k+1}) + \frac{\delta}{Q}C(Y_{k+1}) + \Delta\phi_k\right] \quad (2)$$

where $\Delta\phi_k = \phi_{k+1} - \phi_k$ is the unit management cost increment from landfill k to landfill $k+1$■

Proof: see subsection 7.1■

Condition (2) is a nonlinear first order difference equation which represents the relation between the optimal capacity of two consecutive landfills. In order to economically interpret this condition, think of a situation in which $\Delta\phi_k = 0\ \forall k$ and $\delta = 0$, that is, the unit management cost is identical for all the landfills and there is no time discount. Then (2) takes the form

$$C'(Y_k) = C'(Y_{k+1}), \quad (3)$$

which can be taken as a non-arbitrage condition: if $C'(Y_k) < (>)\, C'(Y_{k+1})$, then total cost could be reduced by reducing Y_{k+1} (Y_k) and increasing Y_k (Y_{k+1}). Condition (3) establishes the impossibility of reducing total cost by transferring some capacity from one landfill to another one. With a strictly positive discount rate and different unit management costs, the relevant equation is (2), which is still a non-arbitrage condition, but now the marginal effect of transferring capacity from one landfill to another has two additional components: the delay of future construction costs (the larger is Y_k, the later landfill $k+1$ will be necessary) and the difference between the management costs borne on both landfills. The greater is the expected cost increment $\Delta\phi_k$, the greater is the value of the right hand side of (2). In order to maintain the equality, the left hand side has to be greater too. Given that C is assumed to be a convex function, and therefore $C'(Y_k)$ is nondecreasing with Y_k, it follows that the greater is $\Delta\phi_k$, the greater is the optimal capacity of landfill k. This conclusion is reasonable from an economic point of view: if future landfills are subject to large management cost increments, it is optimal to increase the capacity of the present landfill in order to extend its lifetime and to delay future management costs associated with the next landfills.

2.1 Example

Let us assume that the building cost function is quadratic $C(Y) = a + bY + \frac{c}{2}Y^2$, and the unit management cost increases at a rate ε, from landfill i to landfill $i+1$, according to the equation $\phi_{k+1} = (1+\varepsilon)\,\phi_k$.

The numerical solution is obtained for the following parameter values[2]:

$$\begin{array}{lllll} a = 100000, & c = 0.5, & \bar{Y} = 5000, & \tau = 50, & \varepsilon = 0.1 \\ b = 1, & Q = 100, & \underline{Y} = 333.3, & \phi_0 = 1 & \delta = 0.02. \end{array} \quad (4)$$

[2] As we have a discrete time, finite horizon, optimal control problem, it can be treated as a static one, taking the state equation as a constraint among the variables of the problem. The numerical solution is obtained using the Matlab optimization toolbox, that implements standard optimization algorithms. This note also applies for example 3.1.

from which, we know that $K_{\min} = 1$ and $K_{\max} = 15$. The solution is shown in figure 2.1.

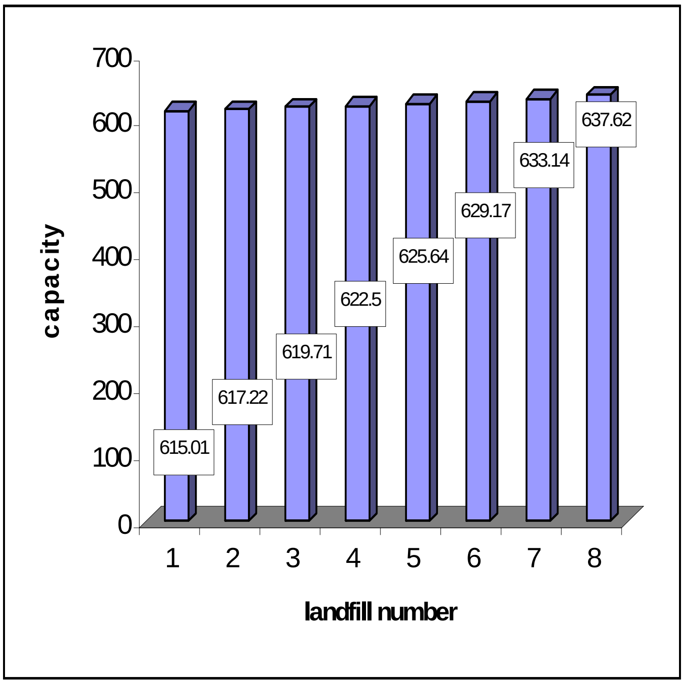

Figure 2.1. Solution for example 2.1

As shown in the figure, $K^* = 8$ and landfill capacities are slightly increasing. The increasing or decreasing character of the solution depends on the concrete form of the cost function and the parameter values, in such a way that no general statements can be made. To obtain a further insight about how the solution depends on the parameters, we now perform some sensitivity analysis. Figures 2.2, 2.3 and 2.4 show a summary of the effects on K^* and the average capacity of landfills in the solution, that is, $Y = \frac{1}{K}\sum_{i=0}^{K-1} Y_i$.

Increasing parameter a, which measures fixed construction costs, makes it optimal to build fewer landfills with a bigger capacity. The opposite occurs when increasing the variable building cost parameters b and c: it is optimal to build more landfills with a smaller capacity. Increasing the rate of discount δ leads to an increase in the weight given in the objective function to short term costs versus long term costs. As a consequence, it is better to build more landfills with smaller capacity in order to delay costs.

Changing parameters a, b, c or δ does not alter the overall quantity of waste produced throughout the planning period, given by τQ, so that, although the individual capacities Y_i change, the sum of capacities $\sum_i Y_i$ does not. Conversely, increasing Q or τ enlarges the overall waste generated and makes a bigger total capacity to be necessary. Note that "small" increments of τ or Q lead to increase the average individual capacity and keep K^* unchanged, up to a point that the increase of τQ is large enough to cause a new landfill to be profitable, allowing a reduction in average capacity. Henceforth, K^*, as a function of Q and τ, has a stair shape and $\tilde{Y}$, as a function of Q and τ, has a sawtooth shape.

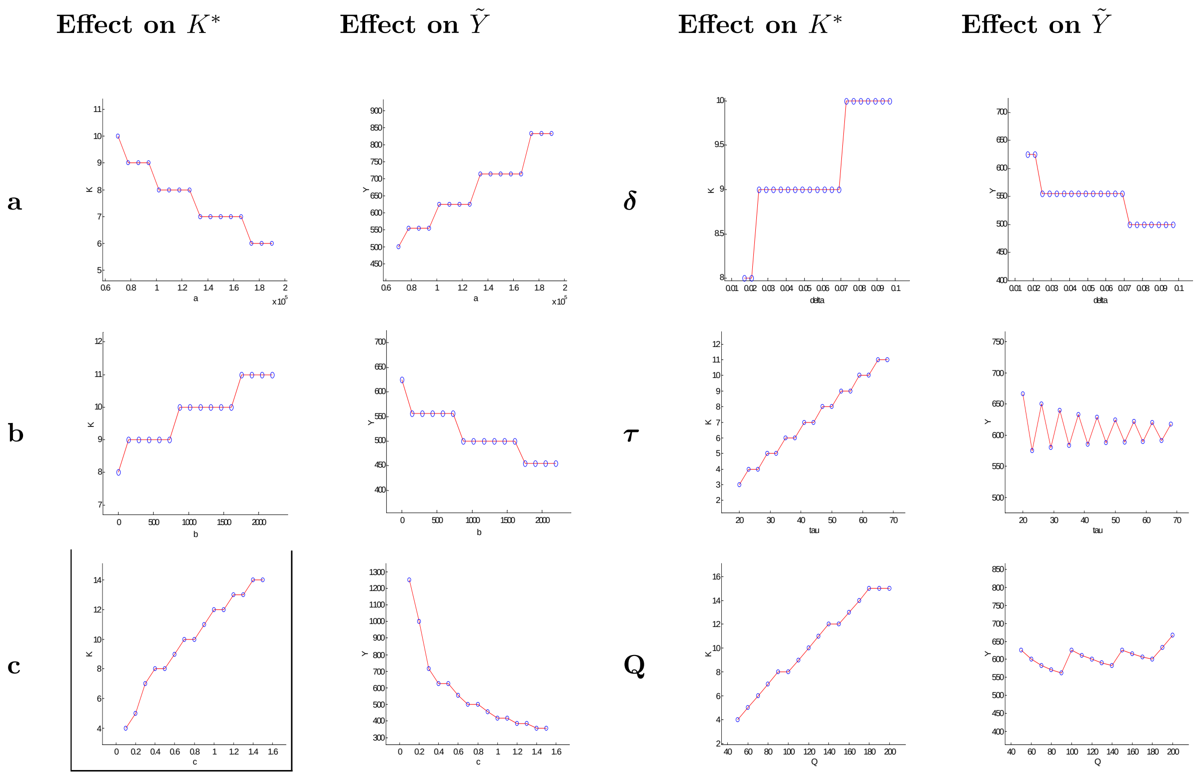

Figure 2.2. Effect of parameters on K^* and $\tilde{Y}$

Note that, in this example, $\Delta\phi_k = \phi_{k+1} - \phi_k = \varepsilon\phi_0 (1+\varepsilon)^k$; henceforth, increasing the value of ε or ϕ_0 makes the difference in unit management costs larger from landfill to landfill. The effect of ε is depicted in figures 2.3 and 2.4 and that of ϕ_0 is qualitatively analogous. As ε increases, the optimal solution implies a sequence of more sharply decreasing capacities, that is, the capacity of the initial landfills becomes larger and larger and the capacity of the latter becomes smaller and smaller (as shown in figure 2.3). When the increment in ε is large enough, it is optimal to decrease the number of landfills K^*.

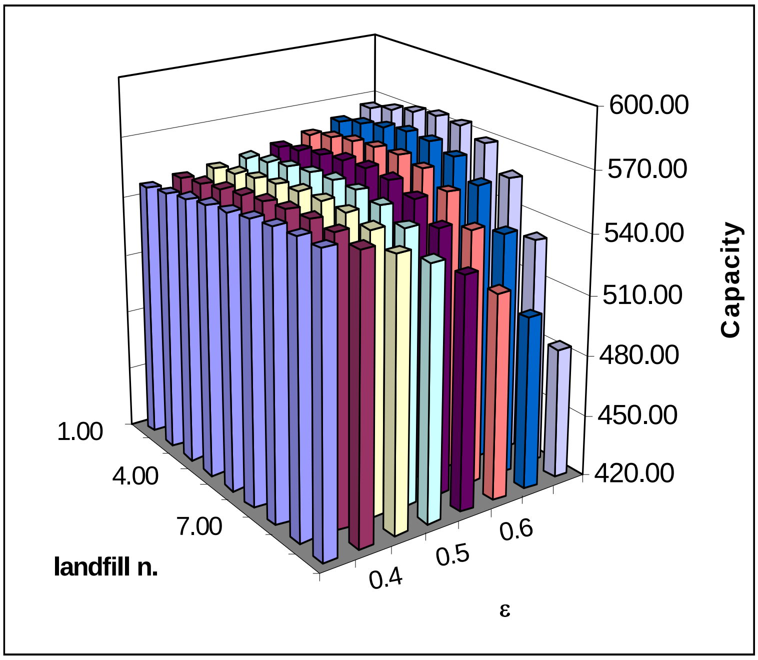

Figure 2.3. Solution for different values of ε.

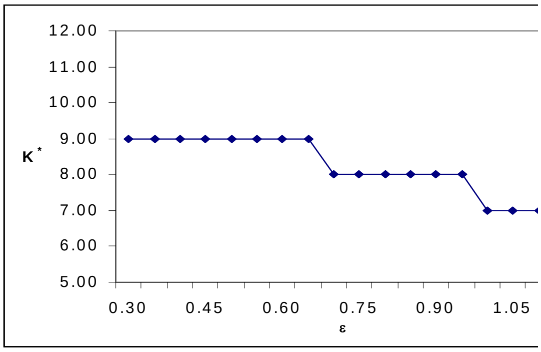

Figure 2.4. Optimal value of K^* for different values of ε.

3 A variable quantity of waste

Assume that the social planner does not expect the quantity of landfilled waste to be constant throughout the planning horizon. This belief may come from several circumstances, such as a foresight for economic growth or technological change, that will alter the production and consumption patterns, some forthcoming environmental regulation concerning packaging, recycling incentives, etc.

Let $Q(t)$ be the amount of waste generated by the population at instant t, and assume that such amount evolves according to the following differential equation:

$$\dot{Q}(t) = G(Q(t), t), \tag{5}$$

the concrete expression for function $G(t)$ depending on the expectations about the future evolution of waste. The problem is that to find a number of landfills K and a sequence of capacities $\{Y_0, Y_1, ..., Y_{K-1}\}$

to minimize

$$\sum_{i=0}^{K-1}\left[e^{-\delta T_i}C\left(Y_i\right)+\int_{T_i}^{T_{i+1}}e^{-\delta t}\phi_i Q\left(t\right)dt\right]$$

subject to the constraints

$$\begin{aligned}\dot{Q}\left(t\right) &= G\left(Q\left(t\right),t\right), \\ \int_{T_i}^{T_{i+1}}Q\left(t\right)dt &= Y_i, \quad i=0,1,2,\ldots,K-1 \\ T_0 &= 0,\ T_K=\tau,\ Q\left(0\right)=Q_0,\end{aligned} \tag{6}$$

where Q_0 is known and represents the instantaneous generation of waste at time $t=0$.

Problem (6), like the one studied in section 2, contains some continuous time and some discrete time elements. The evolution of $Q\left(t\right)$ and the isoperimetric constraint about landfill i capacity are formulated in continuous time, but the objective function does not have the typical form of an Optimal Control problem in continuous time, because it consists of a sum, as occurs in discrete time problems. Next, a way of approaching problem (6) employing usual dynamic optimization techniques is proposed. The method has the following steps:

1. Solve the differential equation $\dot{Q}\left(t\right)=G\left(Q\left(t\right),t\right)$, with initial condition $Q\left(0\right)=Q_0$, obtaining the expression for $Q\left(t\right)$ as a function of time.

2. Substitute the expression obtained in step **1** in the equation $Y_i=\int_{T_i}^{T_{i+1}}Q\left(t\right)dt$, solve this definite integral and obtain $Y_i=F\left(T_i,T_{i+1}\right)$, F being a function that measures the total amount of waste generated between two times T_i and T_{i+1}. The total discounted management cost of landfill i is given by $MC\left(T_i,T_{i+1}\right)=\int_{T_i}^{T_{i+1}}e^{-\delta t}\phi_i Q\left(t\right)dt$. Note that

$$\begin{array}{ll} F_1\left(T_i,T_{i+1}\right)=-Q\left(T_i\right)<0, & F_2\left(T_i,T_{i+1}\right)=Q\left(T_{i+1}\right)>0, \\ MC_1\left(T_i,T_{i+1}\right)=-e^{-\delta T_i}\phi_i Q\left(T_i\right)<0, & MC_2\left(T_i,T_{i+1}\right)=e^{-\delta T_{i+1}}\phi_i Q\left(T_{i+1}\right)>0. \end{array} \tag{7}$$

3. From here on, there are two possibilities:

3.1. Substitute $Y_i=F\left(T_i,T_{i+1}\right)$ in the objective function. The resulting problem is that of finding a sequence of construction times $\{T_1,T_2,...,T_{K-1}\}$ which minimizes

$$\sum_{i=0}^{K-1}H\left(T_i,T_{i+1}\right)=\sum_{i=0}^{K-1}\left\{e^{-\delta T_i}C\left(F\left(T_i,T_{i+1}\right)\right)+MC\left(T_i,T_{i+1}\right)\right\} \tag{8}$$

with the initial condition $T_0=0$ and the final condition $T_K=\tau$.

Taking $i=0,1,2,...,K-1$ as the time index, (8) is a discrete time Calculus of Variations problem, being T_i the state variable, and $\sum_{i=0}^{K-1}H\left(T_i,T_{i+1}\right)$ the objective function. Observe that, due to the different periods length, the term $e^{-\delta T_i}$ can not be interpreted as a discount, but as a part of the objective function. In order to solve this problem, the following Euler equation has to be applied[3]:

$$\begin{aligned} 0 &= H_2\left(T_i,T_{i+1}\right)+H_1\left(T_{i+1},T_{i+2}\right) \\ &= e^{-\delta T_i}C'\left(F\left(T_i,T_{i+1}\right)\right).F_2\left(T_i,T_{i+1}\right)+MC_2\left(T_i,T_{i+1}\right) \\ &\quad +e^{-\delta T_{i+1}}C'\left(F\left(T_{i+1},T_{i+2}\right)\right).F_1\left(T_{i+1},T_{i+2}\right)-\delta e^{-\delta T_{i+1}}C\left(F\left(T_{i+1},T_{i+2}\right)\right)+MC_1\left(T_{i+1},T_{i+2}\right) \\ &= \left\{e^{-\delta T_i}C'\left(F\left(T_i,T_{i+1}\right)\right)+e^{-\delta T_{i+1}}\left[\phi_i-\phi_{i+1}-C'\left(F\left(T_{i+1},T_{i+2}\right)\right)\right]\right\}.Q\left(T_{i+1}\right) \\ &\quad -\delta e^{-\delta T_{i+1}}C\left(F\left(T_{i+1},T_{i+2}\right)\right) \end{aligned}$$

[3] See, for example, Stockey and Lucas (1989).

where (7) has been used to substitute F_1, F_2, MC_1 and MC_2.

3.2. If it is possible to solve $Y_i = F(T_i, T_{i+1})$ for T_{i+1}, an expression like $T_{i+1} = \Phi(T_i, Y_i)$ is obtained and, using the chain rule, we know that

$$\Phi_1(T_i, Y_i) = \frac{-F_1}{F_2} = \frac{Q(T_i)}{Q(T_{i+1})} > 0 \qquad \Phi_2(T_i, Y_i) = \frac{1}{F_2} = \frac{1}{Q(T_{i+1})} > 0 \ .$$

Using the function Φ, we have a discrete time Optimal Control model, being T_i the state variable and Y_i the control variable. A solution is a sequence of capacities $\{Y_0, Y_1, ..., Y_{K-1}\}$ and the associated sequence of switching times $\{T_0, T_1, ..., T_{K-1}\}$ which minimize the objective functional

$$J = \sum_{i=0}^{K-1} \left\{ e^{-\delta T_i} C(Y_i) + MC(T_i, \Phi(T_i, Y_i)) \right\}$$

subject to the state equation $T_{i+1} = \Phi(T_i, Y_i)$, the initial condition $T_0 = 0$ and the final condition $T_K = \tau$. Let us define the Lagrangian function

$$\mathcal{L} = \sum_{i=0}^{K-1} e^{-\delta T_i} C(Y_i) + MC(T_i, \Phi(T_i, Y_i)) + \lambda_0 [\Phi(0, Y_0) - T_1] + ... + \lambda_{K-1} [\Phi(T_{K-1}, Y_{K-1}) - \tau],$$

λ_i measuring the total discounted cost reduction that happens when landfill i lifetime is marginally prolonged, so that it can be called the (opposite of) "shadow price of time", referring to the lifetime of landfill i. The first order conditions are

$$\frac{\partial \mathcal{L}}{\partial T_i} = -\delta e^{-\delta T_i} C(Y_i) + MC_1 + MC_2 \Phi_1 - \lambda_{i-1} + \lambda_i \Phi_1(T_i, Y_i) = 0, \quad i = 1, 2, \ldots, K-1 \tag{9}$$

$$\frac{\partial \mathcal{L}}{\partial Y_i} = e^{-\delta T_i} C'(Y_i) + MC_2 \Phi_2 + \lambda_i \Phi_2(T_i, Y_i) = 0, \quad i = 0, 1, \ldots K-1, \tag{10}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_i} = 0 \rightarrow T_{i+1} = \Phi(T_i, Y_i), \quad i = 0, 1, \ldots, K-1, \tag{11}$$

with $T_0 = 0$ and $T_K = \tau$. From (10) we obtain that the costate variable λ_i is negative

$$\lambda_i = -\frac{e^{-\delta T_i} C'(Y_i) + MC_2 \Phi_2}{\Phi_2(T_i, Y_i)} < 0 \qquad \forall i.$$

From (9) we obtain the following difference equation that rules the evolution of λ_i:

$$\lambda_i = \frac{\lambda_{i-1} + \delta e^{-\delta T_i} C(Y_i) - MC_1 - MC_2 \phi_1}{\Phi_1(T_i, Y_i)}.$$

Condition (10) is the Optimal Capacity Condition for landfill i and it states the equality between the marginal cost and the marginal profit of increasing the capacity of landfill i. The marginal cost is given by $e^{-\delta T_i} C'(Y_i)$, that is, the (discounted) derivative of the building cost. The marginal profit is the "value of time gained", that is, the discounted saving produced by using landfill i for a longer time. This saving is obtained by multiplying Φ_2, the marginal increase of landfill i's lifetime due to an increment in Y_i, times λ_i (the shadow price of that increment).

3.1 Example

Assume that the construction cost function is quadratic, $C(Y) = a + bY + \frac{c}{2} Y^2$, and the instantaneous generation of waste follows the differential equation $\dot{Q}(t) = \alpha Q(t)$, implying that waste generation increases or decreases at a constant rate equal to α. The concrete value of α depends on the expectations about the future evolution of waste. If the sole reason for the expected increment of waste generation is economic growth, a proxy variable for α may be the GNP or the Industrial Production growth rate. Solving the equation for $Q(t)$ with the initial condition $Q(0) = Q_0$ we obtain $Q(t) = Q_0 e^{\alpha \cdot t}$, and the relation among Y_i, T_i and T_{i+1} is given by

$$Y_i = \int_{T_i}^{T_{i+1}} \left(Q_0 e^{\alpha \cdot t}\right) dt = \frac{Q_o}{\alpha} \left[e^{\alpha T_{i+1}} - e^{\alpha T_i}\right]. \tag{12}$$

Assume, moreover, that waste management cost is identical for all the landfills, in such a way that this component can be taken as a constant, and the problem can be solved taking into account just the construction costs. Substituting (12) in the objective function, the problem can be formulated as the following Calculus of Variations problem, with initial condition $T_0 = 0$ and final condition $T_K = \tau$:

$$\min_{\{T_1, T_2 \ldots, T_{K-1}\}} \sum_{i=0}^{K-1} \left\{ e^{-\delta T_i} \left[a + b \frac{Q_o}{\alpha} \left[e^{\alpha T_{i+1}} - e^{\alpha T_i}\right] + \frac{c}{2} \frac{Q_o^2}{\alpha^2} \left[e^{\alpha T_{i+1}} - e^{\alpha T_i}\right]^2 \right] \right\}$$

or, solving (12) for T_{i+1}, as the following Optimal Control problem in discrete time:

$$\min_{\{Y_0, Y_1, \ldots, Y_{K-1}\}} \sum_{i=0}^{K-1} \left\{ e^{-\delta T_i} \left[a + b Y_i + \frac{c}{2} Y_i^2 \right] \right\}$$

subject to the state equation $T_{i+1} = \frac{1}{\alpha} \log \left(\frac{\alpha}{Q_0} Y_i + e^{\alpha T_i} \right)$, the initial condition $T_0 = 0$ and the final condition $T_K = \tau$. Figure 3.1 shows the solution to the problem with the following parameter values[4]:

$$\begin{array}{ccc} a = 50000, & \delta = 0.04, & K_{\min} = 1, \\ b = 1, & Q_0 = 30, & K_{\max} = 15, \\ c = 0.4, & \tau = 60, & \alpha = 0.021. \end{array}$$

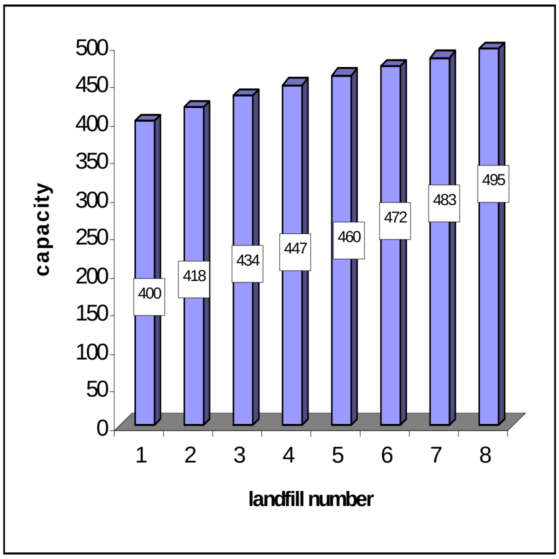

Figure 3.1. Solution to the problem with variable waste.

[4]See footnote 2.

Figures 3.2.a. and 3.2.b. show the effect of parameter α on the optimal number of landfills, K^*, and the average capacity of the landfills, $Y = \frac{1}{K}\sum_{i=0}^{K-1} Y_i$. Note that the total volume of waste generated in period $[0, \tau]$ increases with α. To keep feasibility, either the number of landfills or their average capacity should increase. As can be seen in the graphics, for small increments of α, it is optimal to increase the average capacity (leaving K^* unchanged), whereas, for large increments of α it is optimal to increase K^* (perhaps decreasing $\tilde{Y}$). The comparative dynamics results concerning the rest of parameters are not substantially different from those shown in example 2.1.

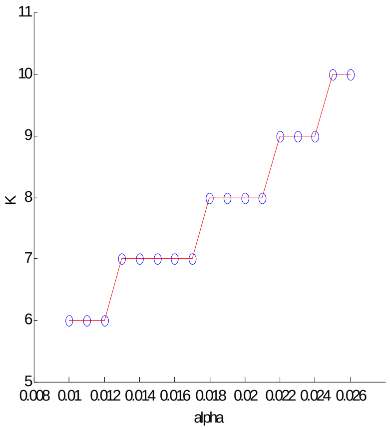

Figure 3.2.a. Effect of α on K^*

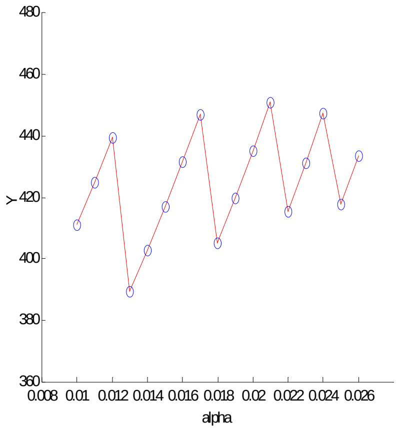

Figure 3.2.b. Effect of α on Y

4 Landfilling and Recycling

A relevant matter, concerning waste management, is that of deciding which method, or combinations of methods, among the available ones (landfilling, incineration, recycling, composting, etc.) to use for the treatment of a given amount of waste. In this section the optimal capacity of landfills is decided taking into account the existence of a different technology aside from landfilling. Recycling is selected as the alternative technology because it is being the object of a great and increasing interest nowadays for its economic and environmental advantages (see, for example, Weinstein and Zeckhauser (1974), Lund (1990), Highfill and McAsey (1997) or Huhtala (1995, 1997, 1999)).

Assume a constant instantaneous waste quantity $Q(t) = Q$ is generated. From the total amount, a portion $R(t)$ is recycled and the rest $V(t)$ is landfilled. The following *mass balance* condition must hold:

$$V(t) + R(t) = Q, \qquad \forall\, t. \tag{13}$$

Disposing of any quantity of waste in landfill i has a unit cost ϕ_i, and the cost of recycling an amount $R(t)$ is given by $r_i(R(t))$, where r_i is a $C^{(2)}$ function holding $r_i' > 0$, $r_i'' > 0$[5]. The recycling cost functions are assumed to be increasing and convex to represent the different technical recycling complexity attached to different materials. For example, glass is more easily recyclable than paper, and paper more than plastic. In practice, it is reasonable to recycle first those materials which are technically easier (and hence, cheaper) to recycle. As a bigger amount of waste is to be recycled, more complex materials are affected and the recycling marginal cost increases faster and faster.

[5] Income obtained from recycled products trading is not explicitely taken into account in the model. This shortcoming may be overcome by interpreting r_i as recycling cost minus recycling income.

A landfill of capacity Y_i built at time T_i is depleted at T_{i+1}, given by $\int_{T_i}^{T_{i+1}} V(t)\,dt = Y_i$. Let us define the variable $Y_i(t)$, denoting the available capacity of landfill i at instant t. $Y_i(t)$ evolves according to

$$\dot{Y}_i(t) = \begin{cases} 0 & t < T_i, \\ -V(t) = -Q + R(t) & T_i \le t \le T_{i+1}, \qquad i = 0, 1, \dots K-1 \\ 0 & t > T_{i+1}, \end{cases} \tag{14}$$

with the boundary conditions $Y_i(T_i) = Y_i$ and $Y_i(T_{i+1}) = 0$, being Y_i a decision variable.

Given that the total number of landfills K is a decision variable, $K_{\max} - K_{\min} + 1$ Optimal Control problems have to be solved, where $K_{\min} = 1$ and $K_{\max} = Int\left(\frac{\tau Q}{\underline{Y}}\right)$. The maximum number of landfills $K_{\max}$ (which is the relevant number when no waste is recycled, $V(t) = Q\ \forall t$, and all landfills are minimum capacity, $Y_0 = Y_1 = \cdots = Y_{K_{\max}-1} = \underline{Y}$) has the same expression as in section 2. Nevertheless, $K_{\min} = 1$. Assuming (rather realistically) that the constraint $R(t) \le Q$ is never binding, because of the high marginal cost of recycling the whole amount of waste, a positive amount of waste is landfilled at every instant t, and henceforth, at least one landfill is necessary. But, in principle, the landfilled amount could be so small that one single landfill would be enough. For each possible value of K, the social planner faces the following dynamic optimization problem:

$$\min_{\{Y_0, Y_1, \dots, Y_{K-1}\}, [R(t)]_{t=0}^{\tau}} \sum_{i=0}^{K-1} \left[e^{-\delta T_i} C(Y_i) + \int_{T_i}^{T_{i+1}} e^{-\delta t} \{\phi_i [Q - R(t)] + r_i(R(t))\}\, dt \right]$$

subject to

$$\begin{aligned} &\dot{Y}_i(t) = -[Q - R(t)] \qquad\qquad \text{for } T_i \le t \le T_{i+1}, \\ &\underline{Y} \le Y_i \le \overline{Y}, \qquad 0 \le R(t) \le Q, \\ &T_0 = 0, \quad T_K = \tau, \quad Y_i(T_i) = Y_i, \quad Y_i(T_{i+1}) = 0, \end{aligned}$$

where (13) has been used to substitute the variable $V(t)$.

This problem fits in the category of multiple-stage Optimal Control problems, whose solution can be found by applying the results of Tomiyama (1985) and Tomiyama and Rossana (1989)[6]. The main idea implies managing the whole problem as made of a sequence of K Optimal Control problems, each related to a time interval $[T_i, T_{i+1})$, for $i = 0, 1, \dots$, and solving them backwards, as shown below:

1. First, solve the sub-problem related to landfill $K-1$, deciding the capacity Y_{K-1} and the recycling path $[R(t)]_{T_{K-1}}^{\tau}$, taking T_{K-1} and τ as given, to minimize

$$J(T_{K-1}) = e^{-\delta T_{K-1}} C(Y_{K-1}) + \int_{T_{K-1}}^{\tau} e^{-\delta t} \left\{\phi_{K-1} [Q - R(t)] + r_{K-1}(R(t))\right\} dt$$

subject to

$$\begin{aligned} &\dot{Y}_{K-1}(t) = -[Q - R(t)] \qquad\qquad \text{for } T_{K-1} \le t \le \tau, \\ &\underline{Y} \le Y_{K-1} \le \overline{Y}, \qquad 0 \le R(t) \le Q, \\ &Y_{K-1}(T_{K-1}) = Y_{K-1}, \qquad Y_{K-1}(\tau) = 0. \end{aligned}$$

Once the solution is obtained, given by Y_{K-1}^* and $[R^*(t)]_{T_{K-1}}^{\tau}$, it is substituted in the objective function, and we define the value function as

$$J^*(T_{K-1}) = \min_{Y_{K-1}, [R(t)]_{T_{K-1}}^{\tau}} J(T_{K-1}).$$

[6] Both papers deal with two-stage problems, but the extension of their results to problems with more than two stages is straightforward.

2. The next step is to solve the sub-problem related to landfill $K-2$, which implies deciding Y_{K-2}, T_{K-1} and $[R(t)]_{T_{K-2}}^{T_{K-1}}$ taking T_{K-2} as given.

3. The value function $J^*(T_{K-2})$, obtained in step 2, is used to solve the problem related to landfill $K-3$, and so on, up to landfill $k=0$, delimited by $t \in [0, T_1)$.

For each $k = 0, 1, 2, \ldots, K-1$, we have a continuous time Optimal Control problem with a state variable, $Y_k(t)$, and a control variable, $R(t)$, taking T_k as given and T_{k+1} as a decision variable, except for the case $k = K-1$, in which $T_K = \tau$ is also given. For the $k-th$ interval, $[T_k, T_{k+1})$, the current-value Hamiltonian is defined as

$$\mathcal{H}_k = \phi_k [Q - R(t)] + r_k(R(t)) + \Psi_k(t)[Q - R(t)] \qquad t \in [T_k, T_{k+1})$$

and the current-value Lagrangian is given by

$$\mathcal{L}_k = \phi_k [Q - R(t)] + r_k(R(t)) + \Psi_k(t)[Q - R(t)] + \xi_k R(t) \qquad t \in [T_k, T_{k+1}),$$

$-\Psi_k(t)$ (with $-\Psi_k(t) \leq 0$) being the costate variable related to the available capacity of landfill k at instant t, representing the effect of a marginal increase in $Y_k(t)$ on the objective function. ξ_k is the Kuhn-Tucker multiplier associated with the non-negativity constraint for $R(t)$.

The first order conditions for each control problem, $k = 0, 1, \ldots, K-1$ are

$$\left.\begin{array}{l}
1.\ r_k'(R(t)) - \Psi_k(t) - \phi_k \geq 0 \quad (\text{with }" = "\text{ if } R(t) > 0) \\
2.\ \dot{\Psi}_k(t) = \delta \Psi_k(t) \\
3.\ C'(Y_k) - \Psi_k(T_k) + \lambda_k + \mu_k = 0 \\
\quad 3'.\ \lambda_k \left[\bar{Y} - Y_k\right] = 0; \ \ \mu_k \left[Y_k - \underline{Y}\right] = 0 \\
\quad 3''.\ \lambda_k \geq 0; \quad \mu_k \leq 0 \\
4.\ \mathcal{H}_k\left(T_{k+1}^-\right) = \delta C\left(Y_{k+1}^*\right) - C'\left(Y_{k+1}^*\right) \frac{\partial Y_{k+1}^*}{\partial T_{k+1}} + \mathcal{H}_{k+1}\left(T_{k+1}^+\right)
\end{array}\right\} t \in [T_k, T_{k+1}),$$

λ_k and μ_k being the multipliers attached to maximum and minimum admissible capacity constraints for landfill k, and $\mathcal{H}_k\left(T_{k+1}^-\right)$ denoting $\lim\limits_{t \to T_{k+1}^-} \mathcal{H}_k(t)$. These conditions can be interpreted as follows:

Equation 1 is the first order maximization condition of $\mathcal{H}_k$ subject to $R(t) \geq 0$, that is, $\dfrac{\partial \mathcal{L}_k}{\partial R} = 0$, which insures that total cost cannot be reduced by increasing or decreasing the amount recycled. If the marginal cost of recycling is greater than that of landfilling, that is, if condition 1 holds with strict inequality, then, under the optimal solution, no waste is recycled, $R(t) = 0$. In the case of an interior solution with $R(t) > 0$, the optimal quantity of recycled waste is determined according to

$$r_k'(R(t)) = \phi_k + \Psi_k, \tag{15}$$

so that, at every time, the marginal cost of recycling, given by $r_k'(R(t))$, must equal the marginal cost of landfilling, given by the unit cost ϕ_k plus the shadow price of available landfill capacity.

Condition 2, $\dot{\Psi}_k(t) = \delta \Psi_k(t) + \dfrac{\partial \mathcal{H}_k(\cdot)}{\partial Y_k(t)}$, rules the optimal time evolution of the costate variable Ψ_k and takes the form of the classical Hotelling rule, stating that the shadow price of the landfill capacity $Y_k(t)$ grows at a rate equal to δ, because it is a nonrenewable resource. Given that ϕ_k is constant for each landfill, the right side of (15) is increasing throughout $[T_k, T_{k+1}]$. To maintain the equality, the left side must also be increasing. Given the assumption $r_k'' > 0$, we conclude that, during the useful life of a landfill, the recycled amount increases with time.

Conditions 3, $3'$ and $3''$ are the transversality conditions[7] for the initial state Y_k, which is a decision variable with maximum and minimum threshold values. If threshold conditions are not binding, the optimal capacity of landfill k is determined by condition 3 alone, which takes the form $C'(Y_k) = \Psi_k T_k$, being the Optimal Capacity Condition for landfill k, and stating the equality between marginal cost of Y_k, given by the increase in building cost, and its marginal gain, given by the shadow price $\Psi_k(T_k)$ of the available landfill capacity at time T_k, that measures the effect of increasing Y_k on the total discounted costs from T_k on, coupling the saving in the management costs attached to landfill k and the discounted cost saving that may be obtained by delaying the need for future landfills[8].

Equation 4 is the transversality condition for T_{k+1}. In the case $k = K-1$, equation 4 is replaced by the final condition $T_K = \tau$. The left side of equation 4 represents the marginal cost of enlarging the lifetime of landfill k[9]. The right side of 4 represents the marginal gain obtained from enlarging such lifetime, which is the effect of increasing T_{k+1} on $J^*(T_{k+1})$. According to condition 4, at T_{k+1} a jump happens from the value of the $k-th$ Hamiltonian to the $(k+1)-th$ one. This conclusion is also obtained in Hartwick, Kemp and Long (1986), in the context of the exploitation of many deposits of an exhaustible resource, with the peculiarity that, in Hartwick et. al.'s paper, the jump is always the same size because all the deposits have the same initial capacity, while in this paper the jump size, given by $\frac{\partial}{\partial T_{k+1}}\left[e^{-\delta T_{k+1}} C\left(Y_{k+1}^*\right)\right]$, depends on Y_{k+1}^*, that is a decision variable.

4.1 Example

Assume that building costs are given by[10] $C(Y) = a + bY$ and recycling costs, which are identical for all the landfills, have the form $r(R_t) = d \cdot [R(t)]^2$, d being a parameter, and no maximum or minimum capacity constraints exist. We need to obtain the solution for each possible value of K. To illustrate, we show the $K = 2$ case, that gives rise to the following optimization problem:

$$\begin{aligned}\min_{\{Y_0, Y_1, R(t), T_1\}} & [a + bY_0] + \int_0^{T_1} e^{-\delta t}\left\{\phi_0 [Q - R(t)] + d [R(t)]^2\right\} dt \\ & + e^{-\delta T_1}[a + bY_1] + \int_{T_1}^{\tau} e^{-\delta t}\left\{\phi_1 [Q - R(t)] + d [R(t)]^2\right\} dt\end{aligned}$$

subject to

$$\begin{aligned}&\dot{Y}_0 = -Q + R(t) && 0 \le t \le T_1, \\ &\dot{Y}_1 = -Q + R(t) && T_1 < t \le \tau, \\ &Y_0(0) = Y_0, \quad Y_1(T_1) = Y_1, && \\ &Y_0(T_1) = Y_1(\tau) = 0,\ R(t) \le Q. &&\end{aligned}$$

[7] See Hestenes (1966) for a general treatment of transversality conditions.

[8] We say "the saving that may be obtained" and not "the saving that is obtained" because the amount of waste dumped is a decision variable and it is not sure, *a priori*, that a higher value of Y_k implies a delay of future landfils.

[9] See Caputo and Wilen (1995).

[10] Given that this model contains some additional complexity compared to the previous ones, a linear function, instead of a quadratic one, is selected in order to somewhat simplify the calculus. No qualitative contents is lost because of this choice.

As shown in subsection 7.2, the solution to this problem is

$$\left.\begin{aligned} Y_0^* &= \left[Q - \frac{\phi_0}{2d}\right] T_1 + \frac{b}{2d\delta}\left[1 - e^{\delta T_1}\right] \\ Y_0^*(t) &= \left[Q - \frac{\phi_0}{2d}\right](T_1 - t) + \frac{b}{2d\delta}\left[e^{\delta t} - e^{\delta T_1}\right] \\ \Psi_0^*(t) &= be^{\delta t} \\ R^*(t) &= \frac{\phi_0}{2d} + \frac{b}{2d}e^{\delta t}. \end{aligned}\right\} 0 \leq t \leq T_1 \tag{16}$$

$$\left.\begin{aligned} Y_1^* &= \left[Q - \frac{\phi_1}{2d}\right](\tau - T_1) + \frac{b}{2d\delta}\left[1 - e^{\delta(\tau - T_1)}\right] \\ Y_1^*(t) &= \left[Q - \frac{\phi_1}{2d}\right](\tau - t) + \frac{b}{2d\delta}\left[e^{\delta(t-T_1)} - e^{\delta(\tau - T_1)}\right] \\ \Psi_1^*(t) &= be^{\delta(t-T_1)} \\ R^*(t) &= \frac{\phi_1}{2d} + \frac{b}{2d}e^{\delta(t-T_1)} \end{aligned}\right\} T_1 \leq t \leq \tau, \tag{17}$$

The optimal value of T_1 is implicitly determined by the following condition:

$$\begin{aligned} & 4dc_0Q - \phi_0^2 - 2c_0be^{\delta T_1} - b^2e^{2\delta T_1} + 4dQbe^{\delta T_1} \\ = \; & 4da\delta + 4db\delta Q(\tau - T_1) - 2c_1b\delta(\tau - T_1) - b^2 + 2b \\ & -2b^2e^{\delta(\tau - T_1)} - 2be^{\delta(\tau - T_1)} + 8dbQ - 4c_1b + 4dc_1Q - \phi_1^2. \end{aligned} \tag{18}$$

To illustrate the results, let us show the solution for the following parameter values[11]:

$$\begin{array}{llll} a = 10, & \phi_0 = 2, & d = 4, & Q = 20, \\ b = 0.8, & \phi_1 = 3, & \delta = 0.04, & \tau = 30, \end{array} \tag{19}$$

which is given by

$$\begin{array}{ll} R^*(t) = \frac{1}{4} + \frac{1}{10}e^{0.04t} & 0 \leq t < 15, \\ R^*(t) = \frac{3}{8} + \frac{1}{10}e^{\delta(t-15)} & 15 \leq t \leq 30. \end{array} \tag{20}$$

$$T_1^* = 15, \quad Y_0^* = 294.2, \quad Y_1^* = 292.3$$

In figure number 4.1 the optimal shape of $R(t)$ is shown. Figure 4.2 shows sensibility analysis results by changing one parameter and holding the rest at the benchmark values (19). In figures 4.2. the effect of different parameters on the optimal values of Y_0 and Y_1 are shown.

Figure 4.1 Solution for $R(t)$

[11] By the bisection method, we obtain the numerical value T_1^* that solves (18), and from T_1^* and the parameter values, the optimal value of Y_0 and Y_1 is obtained using (16) and (17).

For "low" values of a it is optimal to build two landfills with capacities Y_0^* and Y_1^*, while from a certain threshold value of a, the fixed cost attached to the building of a landfill is so high that it is not optimal to build two, but only one with enough capacity to dispose of all the waste generated throughout the period $[0, \tau]$. From that threshold we have $Y_1^* = 0$.

The higher the value of parameter b, which represents the marginal cost of each landfill built capacity, the lower the optimal value for Y_0 and Y_1, and hence, the total landfilled amount of waste. For the solution to be still feasible, total recycled waste throughout $[0, \tau]$ must increase as b increases.

The higher the parameter ϕ_0, measuring the unit disposal cost of the first landfill, the lower the optimal value of Y_0 and the higher the value of Y_1. As for parameter ϕ_1, when it is below a certain threshold value, small increments do not affect the optimal value Y_0 and produce a slight decrease in Y_1 (the scale of the plot do not allow the latter effect to be perceived visually). So that, the solution does not change in the interval $[0, T_1)$, while recycling is more and landfilling less intensively used in the interval $[T_1, \tau]$. When ϕ_1 exceeds a certain threshold value, the second landfill ceases to be profitable, and it becomes optimal to build a single landfill.

Increasing parameter d makes recycling more expensive as compared with landfilling, and it leads to an increase in both landfills capacity in order to allow more waste landfilling and less recycling.

Because of the linearity of building costs, and given $\phi_1 > \phi_0$, for very low values of δ there is no reason to use two landfills, and bear twice the fixed cost a, but it is better to build a single landfill. So, for low values of δ, we find $Y_1^* = 0$. For "medium" values of δ, Y_0^* and Y_1^* approximately have the values given in (20), with Y_0 slightly decreasing and Y_1 slightly increasing (in a range that can not be visually perceived with the plot scale). Finally, from a certain threshold value of δ, a negative (positive) leap happens for Y_0 (Y_1).

The instantaneous waste generation, represented by Q affects the optimal values of Y_0 and Y_1 in a linear and positive way.

The time horizon variable τ has a two-piece effect: for low values of τ, it is optimal to build a single landfill, and hence, $Y_1 = 0$. For "small" τ increments, the optimal value of Y_0 increases and that of Y_1 stays at zero. For a high enough τ increment, a leap happens in the solution: building two landfills becomes optimal, so that Y_0 sharply decreases and Y_1 switches from zero to a strictly positive value. From that point, further τ increments leads to increase both landfills' capacity.

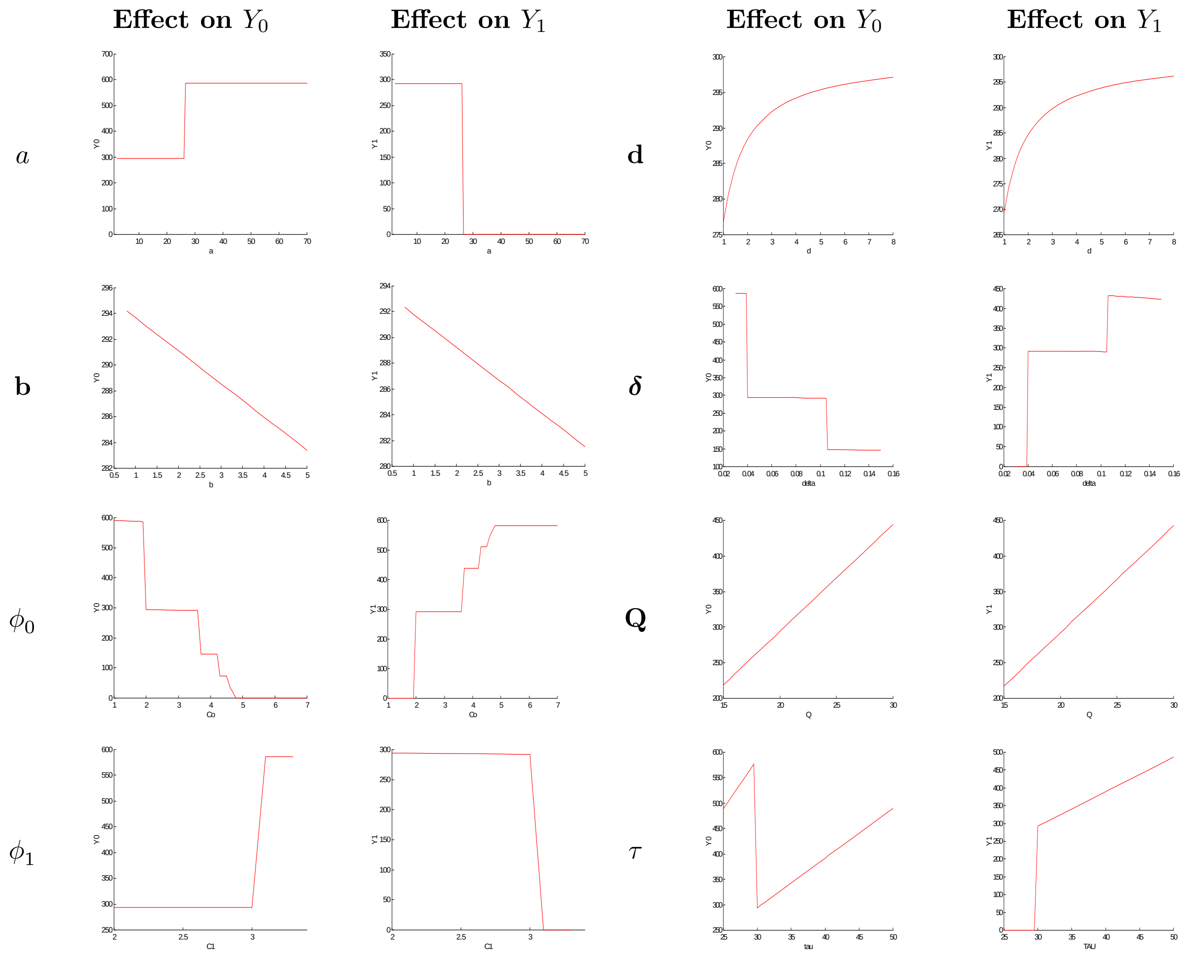

Figure 4.2 Effect of parameters on Y_0 and Y_1

5 An outline for policy making and further research

This paper deals with a set of theoretical decision models arised from a real world problem: the optimal management of solid waste. As such, an adapted model can be developed as a basis for empirical work and policy making. Aside from the optimal capacity and use of landfills, the agencies concerned with waste management may be interested in some further decision variables such as the use of other treatment technologies (composting, incineration , reuse,...).

Accordingly, assume there are L waste treatment options indexed by $l = 1, \ldots, L$. Let $q_l(t)$ denote que amount of waste treated by the $l-th$ procedure a time t. Then, the relevant *mass balance* condition al each instant is $\sum_{l=1}^{L} q_l(t) = Q(t)$.

Most waste treatments are not able to eliminate the whole existing waste, but they rather generate some residuals as a by-product (e.g. incineration generates ash) which often need to be landfilled. Assume that a given proportion θ_l of the waste treated with the method l remains as a residual[12]. Then, the state equation regarding the available capacity of landfill i is given by $\dot{Y}_i(t) = -\sum_{l=1}^{L} \theta_l q_l(t)$ when $t \in [T_i, T_{i+1}]$ and $\dot{Y}_i(t) = 0$ otherwise. Assume that managing the quantity q_l by means of the $l-th$ procedure, when landfill i is open, has a cost given by the function $r_{li}(q_l)$. Then, for each possible value

[12] As a particular case, if the $l-th$ method is precisely landfilling, then $\theta_l = 1$.

of K, the social planner faces the following dynamic optimization problem:

$$\min_{\{Y_0,Y_1,\ldots,Y_{K-1}\},[q_1(t),\ldots,q_L(t)]_{t=0}^{\tau}} \sum_{i=0}^{K-1} \left[e^{-\delta T_i} C(Y_i) + \int_{T_i}^{T_{i+1}} e^{-\delta t} \left(\sum_{l=1}^{L} r_{li}(q_l) \right) dt \right]$$

subject to the mass balance condition, the state equation, the boundary constraints $T_0 = 0$, $T_K = \tau$, $Y_i(T_i) = Y_i$ and $Y_i(T_{i+1}) = 0$, the feasibility constraints on Y_i and the relevant set of legal constraints on q_l, depending on the environmental legislation of each country[13]. A similar approach for determining optimal solid waste management (excluding the optimal capacity decision) is performed in Lund (1990) and Jacobs and Everett (1992).

A deeper study of waste management can be performed by regarding, not just the whole amount of waste, but also its composition. So, it makes sense a broader analysis considering the interaction between production decisions and waste management decisions[14].

Some additional relevant issues for further research include the stochastic generation of waste or the joint study of optimal capacity and optimal location of landfills.

6 Summary and relevance for economics

The optimal capacity of a sequence of landfills, which is usually taken as given in most economic articles, has been studied in the present paper within a dynamic framework. The basic dynamic nature of the problem has been pointed out and several specific cases have been explored. To deal with this matter, a class of Optimal Control problems, sharing some continuous time and some discrete time features, have been stated and solved.

In an interior solution, the optimal capacity of a certain landfill k is determined according to the so-called *Optimal Capacity Condition*, which states the equality between such capacity marginal cost and marginal gain. The marginal cost is given by the building cost plus the management cost attached to landfill k, while the marginal gain comes from all the discounted cost saving attached to future landfills that can be achieved by increasing the capacity of landfill k. Optimal capacity depends positively on the expected future waste management cost increment from the present landfill to the following one, in such a way that the more management costs increase, the more sharply decreasing is the sequence of capacities.

If instant waste generation is not constant, but follows a certain time evolution, a solution method is suggested, based on discretizing the continuous time problem by summing up the generated amount of waste between two consecutive (endogenously determined) landfill switching times. This strategy allows us to avoid the temporal nature of the switching time variable, that becomes the state variable of the problem. The time-variable role is played by the landfill index ($k = 0, 1, \ldots, K-1$).

Selecting management technologies and building landfills are related decisions. When both decisions are jointly considered, a multiple-stage Optimal Control results, whose solution requires the use of dynamic continuous time techniques (Pontryagin Maximum Principle) for every landfill sub-problem and a discrete time procedure (Dynamic Programming) to manage the whole problem. The recycled amount of waste is time increasing within every landfill's useful life.

[13] In Europe, for example, the European Directive for Packaging and Packaging Waste (European Parliament and Council Directive 94/62/EC) sets some recovery and recycling targets.

[14] From this point of view, André and Cerdá (2001) study the optimal combination of input resources in production taking into account the interaction between the production technology and the recycling technology.

The decision problems discussed in this article share a common structure that involves splitting a time horizon of planning into subintervals the length of which has to be decided. In each of the subintervals some costs, the amount of which depends on the value of the decision variables, have to be borne. This dynamic structure arised from the optimal capacity decision resembles other economic dynamic problems that, up to our knowledge, have not been addressed from this perspective. Take, as an example, a consumer's decision about the purchase of a durable good (for example, a computer): purchasing a last-generation computer implies a larger cost but is likely to have la longer lifetime, while a cheaper computer will become obsolete sooner. An Optimal Capacity Condition (similar to the one proposed in this article for landfill management) seems to fit quite well with the computer purchasing dynamic policy of a consumer, as well as the infrastructure policy of a firm or a public agency.

7 Appendix: Mathematical Conditions

7.1 Proof of Proposition 1

Applying recurrently the formula $T_{i+1} = T_i + \frac{Y_i}{Q}$, and assuming that all the landfills' capacity get exhausted under the optimal solution, we obtain

$$T_i = \frac{1}{Q}\sum_{j=0}^{i-1} Y_j, \qquad i = 1, 2, \ldots, K-1. \tag{21}$$

Solving the integral in the objective function of (P) and using the equation (21), we have

$$\int_{T_i}^{T_{i+1}} e^{-\delta(t-T_i)}\phi_i Q dt = \frac{Q\phi_i}{\delta}\left[1 - e^{-\delta(T_{i+1}-T_i)}\right] = \frac{Q\phi_i}{\delta}\left[1 - e^{-\delta\frac{Y_i}{Q}}\right],$$

and therefore, assuming interior solution, problem (P) consists of finding a sequence of capacities $\{Y_0, Y_1, ..., Y_{K-1}\}$ which minimize

$$C(Y_0) + \frac{Q\phi_0}{\delta}\left[1 - e^{-\delta\frac{Y_0}{Q}}\right] + \sum_{i=1}^{K-1} e^{-\frac{\delta}{Q}\sum_{j=0}^{i-1} Y_j}\left[C(Y_i) + \frac{Q\phi_i}{\delta}\left(1 - e^{-\delta\frac{Y_i}{Q}}\right)\right]$$

subject to the overall capacity constraint

$$Y_0 + Y_1 + \cdots + Y_{K-1} = \tau Q. \tag{22}$$

The Lagrangean of this problem is

$$C(Y_0) + \frac{Q\phi_0}{\delta}\left[1 - e^{-\delta\frac{Y_0}{Q}}\right] + \sum_{i=1}^{K-1} e^{-\frac{\delta}{Q}\sum_{j=0}^{i-1} Y_j}\left[C(Y_i) + \frac{Q\phi_i}{\delta}\left(1 - e^{-\delta\frac{Y_i}{Q}}\right)\right] - \lambda\left[\tau Q - \sum_{i=1}^{K-1} Y_{K-1}\right]$$

being λ the Lagrange multiplier attached to the constraint (22).

The first order conditions for Y_0, Y_1,..., Y_{K-1} are

$$C'(Y_0) + \phi_0 e^{-\delta\frac{Y_0}{Q}} - \frac{\delta}{Q}\sum_{i=1}^{K-1} e^{-\frac{\delta}{Q}\sum_{j=0}^{i-1} Y_j}\left[C(Y_i) + \frac{Q\phi_i}{\delta}\left(1 - e^{-\delta\frac{Y_i}{Q}}\right)\right] = \lambda,$$

$$\begin{aligned} &e^{-\frac{\delta}{Q}\sum_{j=0}^{k-1} Y_j}\left[C'(Y_k) + \phi_k e^{-\delta\frac{Y_k}{Q}}\right] - \\ &-\tfrac{\delta}{Q}\textstyle\sum_{i=k+1}^{K-1} e^{-\frac{\delta}{Q}\sum_{j=0}^{i-1} Y_j}\left[C(Y_i) + \frac{Q\phi_i}{\delta}\left(1 - e^{-\delta\frac{Y_i}{Q}}\right)\right] = \lambda, \qquad k = 1, 2 \ldots, K-2 \end{aligned}$$

and

$$e^{-\frac{\delta}{Q}\sum_{j=0}^{K-1}Y_j}\left[C'\left(Y_K\right)+\phi_{K-1}e^{-\delta\frac{Y_{K-1}}{Q}}\right]=\lambda$$

jointly with (22).

Equating the first order equations for two consecutive arbitrary landfills, k and $k+1$, ($k=1,2,\ldots,K-3$[15]) we obtain

$$\begin{aligned}
& e^{-\frac{\delta}{Q}\sum_{j=0}^{k-1}Y_j}\left[C'\left(Y_k\right)+\phi_k e^{-\delta\frac{Y_k}{Q}}\right]-\frac{\delta}{Q}\sum_{i=k+1}^{K-1}e^{-\frac{\delta}{Q}\sum_{j=0}^{i-1}Y_j}\left[C\left(Y_i\right)+\frac{Q\phi_i}{\delta}\left(1-e^{-\delta\frac{Y_i}{Q}}\right)\right] \\
= \; & e^{-\frac{\delta}{Q}\sum_{j=0}^{k}Y_j}\left[C'\left(Y_{k+1}\right)+\phi_{k+1}e^{-\delta\frac{Y_{k+1}}{Q}}\right]-\frac{\delta}{Q}\sum_{i=k+2}^{K-1}e^{-\frac{\delta}{Q}\sum_{j=0}^{i-1}Y_j}\left[C\left(Y_i\right)+\frac{Q\phi_i}{\delta}\left(1-e^{-\delta\frac{Y_i}{Q}}\right)\right].
\end{aligned}$$

Multiplying both sides by $e^{\frac{\delta}{Q}\sum_{j=0}^{k-1}Y_j}$, adding $\frac{\delta}{Q}\sum_{i=k+2}^{K-1}e^{-\frac{\delta}{Q}\sum_{j=0}^{i-1}Y_j}\left[C\left(Y_i\right)+\frac{Q\phi_i}{\delta}\left(1-e^{-\delta\frac{Y_i}{Q}}\right)\right]$ to both sides and rearranging, we obtain (2)■

7.2 Solution to Example 4.1

The first step is to solve the sub-problem attached to landfill 1, that is,

$$\min_{Y_1,R(t)} e^{-\delta T_1}\left[a+bY_1\right]+\int_{T_1}^{\tau}e^{-\delta t}\left\{\phi_1\left[Q-R(t)\right]+d\left[R(t)\right]^2\right\}dt \tag{23}$$

subject to

$$\begin{aligned}
&\dot{Y}_1=-Q+R(t) \qquad T_1\leq t\leq\tau, \\
&Y_1\left(T_1\right)=Y_1, \qquad Y_1(\tau)=0,
\end{aligned}$$

taking T_1 and τ as given. Being $e^{-\delta T_1}$ constant, minimizing (23) is the same as minimizing

$$\left[a+bY_1\right]+\int_{T_1}^{\tau}e^{-\delta(t-T_1)}\left\{\phi_1\left[Q-R(t)\right]+d\left[R(t)\right]^2\right\}dt$$

which, making the variable change $\omega=t-T_1$, may be expressed as

$$\left[a+bY_1\right]+\int_{0}^{\tau-T_1}e^{-\delta\omega}\left\{\phi_1\left[Q-R(\omega)\right]+d\left[R(\omega)\right]^2\right\}d\omega$$

and the problem constraints become

$$\begin{aligned}
&\dot{Y}_1(\omega)=-Q+R(\omega), \\
&Y_1(0)=Y_1, \qquad Y_1\left(\tau-T_1\right)=0, \\
&R(t)\leq Q,
\end{aligned}$$

with T_1 and τ given. The current value Hamiltonian is $\mathcal{H}_1=\phi_1\left[Q-R(\omega)\right]+d\left[R(\omega)\right]^2+\Psi_1(\omega)\left[Q-R(\omega)\right]$. The Pontryagin Maximum Principle conditions are

$$\frac{\partial\mathcal{H}_1}{\partial R(\omega)} = -\phi_1+2dR(\omega)-\Psi_1(\omega)=0, \tag{24}$$

$$\dot{\Psi}_1(\omega) = \delta\Psi_1(\omega), \tag{25}$$

$$\dot{Y}_1(\omega) = -Q+R(\omega). \tag{26}$$

[15] The intermediate expressions for landfills $k=0$ and $k=K-2$ are slightly different, but it is easy to show that equation (2) also holds for these two cases.

Solving equation (25), we have $\Psi_1(\omega) = \Psi_1(0)\, e^{\delta\omega}$, substituting in (24) and rearranging, we have

$$R(\omega) = \frac{\phi_1}{2d} + \frac{\Psi_1(0)}{2d} e^{\delta\omega}. \tag{27}$$

Substituting (27) in (26) and solving the resulting differential equation, whose general solution for Y_1, using the initial condition $Y_1(0) = Y_1$, is

$$Y_1(\omega) = Y_1 + \left[\frac{\phi_1}{2d} - Q\right]\omega + \frac{\Psi_1(0)}{2d\delta}\left[e^{\delta\omega} - 1\right] \tag{28}$$

and, using the final condition $Y_1(\tau - T_1) = 0$ and rearranging, provides the following expression for Y_1:

$$Y_1 = \left[Q - \frac{\phi_1}{2d}\right](\tau - T_1) + \frac{\Psi_1(0)}{2d\delta}\left[1 - e^{\delta(\tau - T_1)}\right]. \tag{29}$$

From the Y_1 optimality condition, the following value of $\Psi_1(0)$ is obtained:

$$C'(Y_1) = b = \Psi_1(0) \tag{30}$$

and substituting (29) and (30) in (28) the following final expression for $Y_1(\omega)$ is obtained:

$$Y_1(\omega) = \left[Q - \frac{\phi_1}{2d}\right](\tau - T_1) + \frac{b}{2d\delta}\left[1 - e^{\delta(\tau - T_1)}\right] + \left[\frac{\phi_1}{2d} - Q\right]\omega + \frac{b}{2d\delta}\left[e^{\delta\omega} - 1\right]$$

and the solution for $\Psi_1(\omega)$ and $R(\omega)$ is given by

$$\begin{aligned} \Psi_1(\omega) &= be^{\delta\omega}, \\ R(\omega) &= \frac{\phi_1}{2d} + \frac{b}{2d} e^{\delta\omega}. \end{aligned}$$

Undoing the variable change $\omega = t - T_1$, we have the expressions in (17) and the value function

$$J_1^*(T_1) = e^{-\delta T_1}\left[a + bY_1^*\right] + \int_{T_1}^{\tau} e^{-\delta t}\left\{\phi_1\left[Q - R^*(t)\right] + d\left[R^*(t)\right]^2\right\} dt,$$

which only depends on T_1 and parameters. The problem corresponding to $k = 0$ is

$$\min_{Y_0, R(t), T_1} \left[a + bY_0\right] + \int_0^{T_1} e^{-\delta t}\left\{\phi_0\left[Q - R(t)\right] + d\left[R(t)\right]^2\right\} dt + J_1^*(T_1)$$

subject to

$$\begin{aligned} &\dot{Y}_0 = -Q + R(t) && 0 \le t \le T_1, \\ &Y_0(0) = Y_0, \qquad Y_0(T_1) = 0, \\ &R(t) \le Q, \end{aligned}$$

the current value Hamiltonian being $\mathcal{H}_0 = \phi_0\left[Q - R(t)\right] + d\left[R(t)\right]^2 + \Psi_0(t)\left[Q - R(t)\right]$ and the Pontryagin Maximum Principle conditions

$$\frac{\partial \mathcal{H}_0}{\partial R(t)} = -\phi_0 + 2dR(t) - \Psi_0(t) = 0, \tag{31}$$

$$\dot{\Psi}_0(t) = \delta\Psi_0(t), \tag{32}$$

$$\dot{Y}_0(t) = -Q + R(t). \tag{33}$$

By a procedure similar to the one used for landfill $k = 1$, we obtain (16).

The optimal value of T_1 is obtained from the transversality condition $\mathcal{H}_0\left(T_1^-\right) = \delta C\left(Y_1^*\right) - C'\left(Y_1^*\right)\frac{\partial Y_1^*}{\partial T_1} + \mathcal{H}_1\left(T_1^+\right)$, that, using (17) and (16), becomes

$$\underbrace{\phi_0\left[Q - \frac{\phi_0}{2d} - \frac{b}{2d}e^{\delta T_1}\right] + d\left[\frac{\phi_0}{2d} + \frac{b}{2d}e^{\delta T_1}\right]^2 + be^{\delta T_1}\left[Q - \frac{\phi_0}{2d} - \frac{b}{2d}e^{\delta T_1}\right]}_{\mathcal{H}_0\left(T_1^-\right)}$$

$$= \delta\underbrace{\left\{a + b\left(\left[Q - \frac{\phi_1}{2d}\right](\tau - T_1) + \frac{b}{2d\delta}\left[1 - e^{\delta(\tau - T_1)}\right]\right)\right\}}_{C\left(Y_1^*\right)} + b\underbrace{\left\{\left[Q - \frac{\phi_1}{2d}\right] - \frac{b}{2d}e^{\delta(\tau - T_1)}\right\}}_{-\frac{\partial Y_1^*}{\partial T_1}}$$

$$\underbrace{+\phi_1\left[Q - \frac{(\phi_1 + b)}{2d}\right] + d\left[\frac{(\phi_1 + b)}{2d}\right]^2 + b\left[Q - \frac{(\phi_1 + b)}{2d}\right]}_{\mathcal{H}_1\left(T_1^+\right)}$$

that, after simplification, reduces to (18).

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