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# Canonical transformations for hyperkahler structures and hyperhamiltonian dynamics

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We discuss generalizations of the well known concept of canonical transformations for symplectic structures to the case of hyperkahler structures. Different characterizations, which are equivalent in the symplectic case, give rise to non-equivalent notions in the hyperkahler framework; we will thus distinguish between hyperkahler and canonical transformations. We also discuss the properties of hyperhamiltonian dynamics in this respect. © *2014 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4878121]

## I. INTRODUCTION

Following the pioneering work by Atiyah, Hitchin, and their co-workers,[3–5,24,25] hyperkahler manifolds and structures were recognized to be relevant not only to Geometry[9,10] but also to Physics, in particular in the context of Field Theory and in connection to instantons and their moduli spaces.[5,6,11–16,24,29,32]

More recently, an extension of standard Hamilton dynamics based on hyperkahler structures (and defined on hyperkahler manifolds) has been formulated,[18] and christened *hyperhamiltonian dynamics*; the physical motivation behind this was an attempt to have a classical framework for spin dynamics. It was recently shown, indeed, that several of the fundamental equations for the dynamics of spin (Pauli equation and the Dirac equation, the latter in both the Foldy-Wouthuysen and the Cini-Touschek frameworks) can be cast in the framework of hyperhamiltonian dynamics.[21] This parallel with Hamiltonian dynamics calls naturally for an extension (if possible) of the concepts and constructions which are at the roots of Hamiltonian dynamics; several of these have been obtained, in particular a variational formulation and a study of (quaternionic) integrable systems.[18–20,30]

A key ingredient which is obviously missing from this parallel is that of *canonical* transformations; it should be stressed that this is of independent interest: characterizing the group of transformations which leave a given hyperkahler structure invariant (in a sense to be detailed below) is of interest independently of the hyperhamiltonian dynamics motivation (indeed in this sense it has been studied in the literature;[33] thus, albeit our approach is from the point of view of hyperhamiltonian dynamics, it is not surprising that some of our findings will reproduce – with different approach and methods – results which are known from Differential Geometry[33]).

The purpose of this work is to start a detailed study of this problem taking into account previous results of the theory of hyperkahler manifolds and introducing new concepts and tools to fit them into the frame of hyperhamiltonian dynamics, that is, to properly define – and then study – canonical transformations for hyperhamiltonian dynamics and hyperkahler structures. As well known, in the symplectic (or Hamiltonian) case canonical transformations can be defined in several equivalent ways (see, e.g., Ref. 2); direct naive extensions of these to the hyperkahler framework are equally not viable, so suitable generalizations should be considered, and it turns out generalizations starting

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from notions which are equivalent in the symplectic case will produce non-equivalent notions in the hyperkahler case.

Then, after Sec. II, where we introduce some basic notions and set our notations, we will distinguish in Sec. III between hyperkahler (or equivalently hypersymplectic) transformations, preserving in a certain sense the hyperkahler structure (and the associated hypersymplectic one), see Definition 3.2, and canonical transformations, preserving a certain four form associated to the hyperkahler structure, see Definition 3.3 and more generally the discussion of Sec. III. In this sense, it is not entirely trivial that our discussion, based on a dynamical systems point of view, ends up on the one hand focusing on concepts already used in the differential geometric approach,[26,33] but also, on the other hand, showing that preservation (in a suitable sense) of the hyperkahler structure is not the natural requirement to be considered dynamically – we will in fact distinguish hyperkahler and canonical transformations, see below.

The present paper focuses to a large extent on the discussion of what are the suitable generalizations mentioned above, i.e., what are the appropriate definitions of hyperkahler and canonical transformations in general (see Sec. III); we will also characterize them by providing equations to be satisfied by the transformations. In a companion paper,[23] we will obtain a full characterization (that is, we solve the characterizing equations) of hyperkahler maps in the Euclidean case; this is related to the general case via the result presented here in Sec. IV. Albeit such a full characterization is obtained only in the Euclidean (flat) case, it should be emphasized that this covers a number of physically relevant cases:[21] not only the Dirac equation – which can be recast in terms of hyperhamiltonian dynamics – lives in flat Minkowski space, but many of the physically relevant non-flat hyperkahler manifolds and structures are obtained through a momentum map type construction[25] from Euclidean $\mathbf{R}^{4n}$ with standard hyperkahler structures (see, e.g., Ref. 22).

As for canonical transformations, these are characterized in Sec. V. The key requirement, based on a reformulation of the familiar area condition for canonical transformations in symplectic dynamics, will be the preservation of a certain four-form, invariantly attached to the quaternionic structure identified by the hyperkahler one. Application to the Dirac equation requires to consider dual hyperkahler structures, as discussed in Ref. 21; it is thus natural to consider and study hyperkahler and canonical maps for these as well, which is done in Sec. VI.

We will then finally consider our original motivation, namely, hyperhamiltonian dynamics (Sec. VII). It will turn out that this does not necessarily preserve the hyperkahler structure (in any of the senses discussed in Secs. III–V). Our main result in this context (holding in general, i.e., with no limitation to the Euclidean case) will be that, similar to what happens for Hamilton dynamics, *a hyperhamiltonian flow generates a one-parameter group of canonical transformations*.

*Notational convention.* We will consider smooth real manifolds $M$ of dimension $4n$, equipped with a Riemannian metric and three structures of several types (complex, Kahler, symplectic). We will be using Latin indices (running from 1 to $m = 4n$) for the local coordinates on the manifold $M$, and Greek letters for the label (running from 1 to 3) attached to the complex (Kahler, symplectic) structures on $M$; note that we should distinguish between covariant and contravariant Latin indices, as we deal with a generic Riemannian metric $g$, while the metric in the $\alpha$ space is Euclidean, i.e., Greek labels could be written equally as lower or upper indices (and we will sometimes move them for typographical convenience). The Einstein summation convention will be used unless otherwise stated; when confusion could arise we will indicate explicitly summation.

## II. HYPERKAHLER STRUCTURES

Let us start by recalling some basic definitions, mainly of geometrical nature, which we will use in the following (see, e.g., Ref. 1, 4, and 5 for further detail). All manifolds and related geometric objects to be considered will always be real and smooth; we will sometimes omit to indicate this for the sake of brevity.

Let $(M, g)$ be a smooth real Riemannian manifolds; as well known there is a unique torsion-free metric connection on it, the Levi-Civita connection $\nabla$.

### A. Kahler manifolds

Consider a smooth real Riemannian manifold $(M, g)$ of dimension $m = 2k$. An almost complex structure on this is a field of orthogonal transformations in T$M$, i.e., a (1,1) type tensor field $J$ such that $J^2 = -I$, with $I$ the identity map.

A *Kahler manifold* $(M, g, J)$ is a smooth orientable real Riemannian manifold $(M, g)$ of dimension $m = 2k$ equipped with an almost-complex structure $J$ which has vanishing covariant derivative under the Levi-Civita connection, $\nabla J = 0$.

Note the latter condition actually implies – due to the Newlander-Nirenberg theorem[31] – the integrability of $J$; so $(M, g, J)$ is a complex manifold.

The two-form $\omega \in \Lambda^2(M)$ associated to $J$ and $g$ via the Kahler relation

$$\omega(v, w) = g(v, Jw) \tag{1}$$

is closed and non-degenerate; hence, it defines a symplectic structure in $M$, and each Kahler manifold is also symplectic. (The converse is not true, and there are symplectic manifolds which do not admit any Kahler structure.)

### B. Hyperkahler manifolds

A *hyperkahler manifold* is a real smooth orientable Riemannian manifold $(M, g)$ of dimension $m = 4n$ equipped with three almost-complex structures $J_1, J_2, J_3$ which:

(i) are covariantly constant under the Levi-Civita connection, $\nabla J_\alpha = 0$ (hence they are actually complex structures on $(M, g)$, see above); and
(ii) satisfy the quaternionic relations, i.e.,

$$J_\alpha J_\beta = \epsilon_{\alpha\beta\gamma} J_\gamma - \delta_{\alpha\beta} I \tag{2}$$

with $\epsilon_{\alpha\beta\gamma}$ the completely antisymmetric (Levi-Civita) tensor.

Simple examples of hyperkahler manifolds are provided by quaternionic vector spaces $\mathbf{H}^k$ and by the cotangent bundle of complex manifolds.[26]

Note that the relations (2) imply that the $J_\alpha$ satisfy the $SU(2)$ commutation relations, but also involve the multiplication structure.

We denote the ordered triple $\mathbf{J} = (J_1, J_2, J_3)$ as a *hyperkahler structure* on $(M, g)$. We will denote a hyperkahler manifold as $(M, g; J_1, J_2, J_3)$, or simply as $(M, g; \mathbf{J})$.

Obviously, a hyperkahler manifold is also Kahler with respect to any linear combination $J = \sum_\alpha c_\alpha J_\alpha$ such that $|c|^2 := c_1^2 + c_2^2 + c_3^2 = 1$; thus, we have a $S^2$ sphere of Kahler structures on $M$. More precisely, we introduce the space

$$\mathbf{Q} := \Big\{ \sum_\alpha c_\alpha J_\alpha ,\ c_\alpha \in \mathbf{R} \Big\} \approx \mathbf{R}^3, \tag{3}$$

also called the **quaternionic structure** on $(M, g)$ spanned by $(J_1, J_2, J_3)$;[1] and denote by $\mathbf{S} \approx S^2$ the unit sphere in this space. Points in $\mathbf{S}$ are in one to one correspondence with those Kahler structures on $(M, g)$ which are in the linear span of the given basis structures $J_\alpha$, and opposite points correspond to complex conjugate structures. The sphere $\mathbf{S}$ will play a central role in our discussion and deserves a special name.

*Definition 2.1. The unit sphere in* $\mathbf{Q}$, *i.e., the set*

$$\mathbf{S} := \Big\{ \sum_\alpha c_\alpha J_\alpha ,\ c_\alpha \in \mathbf{R} \ : \ |c|^2 := \sum_\alpha c_\alpha^2 = 1 \Big\} \approx S^2 \subset \mathbf{R}^3, \tag{4}$$

*is the* **Kahler sphere** *corresponding to the hyperkahler structure* $\mathbf{J} = (J_1, J_2, J_3)$.

*Definition 2.2. Two hyperkahler structures on $(M, g)$ defining the same quaternionic structure* $\mathbf{Q}$, *and hence the same Kahler sphere* $\mathbf{S}$, *are said to be* **equivalent**. *An equivalence class of hyperkahler structures is identified with the corresponding quaternionic structure, and vice versa.*

*Remark 2.1.* Note that the quaternions $\mathbf{H}$ act (linearly) in a natural way on $\mathbf{Q}$; moreover, the group $\mathbf{H}_0$ of quaternions of unit norm acts preserving $\mathbf{S}$. More precisely, unit quaternions other than the real unit (which acts by the identity) will generate a rotation of the sphere $\mathbf{S} \simeq S^2$. The action mentioned here is of course given, for the quaternion $h = h_0 + ih_1 + jh_2 + kh_3$, by $J \mapsto H^{-1}JH$, with $H = h_0 I + h_1 J_1 + h_2 J_2 + h_3 J_3$ (where $h_i \in \mathbf{R}$). In the case $|h| = 1$, where of course $|h| := (h_0^2 + h_1^2 + h_2^2 + h_3^2)^{1/2}$, we have $H^{-1} = h_0 I - h_1 J_1 - h_2 J_2 - h_3 J_3$. ⊙

*Remark 2.2.* There are obvious symplectic counterparts to the notions defined above, the correspondence being through the Kahler relation (1) (note this implies that the metric will play a role, at difference with the standard symplectic case; in fact, in order to have a more usual analogue one should think of the Kahler case). Thus, the symplectic forms $\omega_\alpha$ correspond to the $J_\alpha$, and $(M, g; \omega_1, \omega_2, \omega_3)$ is a *hypersymplectic manifold*. Any nonzero linear combination of the $\omega_\alpha$, i.e., any $\mu \neq 0$ in

$$\mathcal{Q} \ := \ \left\{ \mu = \sum_\alpha c_\alpha \omega_\alpha, \ c_\alpha \in \mathbf{R} \right\} \ \approx \ \mathbf{R}^3 \tag{5}$$

is also a symplectic structure on $M$; in other words, we have a punctured three-dimensional space $\mathbf{R}^3 \backslash \{0\}$ of symplectic structures in $M$. Denote by $\mathcal{S}$ the unit sphere in $\mathcal{Q}$; the $\mu \in \mathcal{S}$ are *unimodular* symplectic structures in $M$. Obviously, the sphere $\mathcal{S}$ corresponds to $\mathbf{S}$ via the Kahler relation; hence, $\mathcal{S}$ is the **symplectic Kahler sphere** for the hyperkahler structure $(J_1, J_2, J_3)$, and two hypersymplectic structures defining the same $\mathcal{S}$ are *equivalent*. ⊙

### C. Relations between equivalent structures

The notion of equivalent structures will play a key role in a large part of the following; it is thus worth presenting some remarks to further characterize them.

Let us consider two equivalent structures $\mathbf{J}$ and $\widetilde{\mathbf{J}}$; by definition these generate the same three-dimensional linear space $\mathbf{Q}$, hence each of them can be written in term of the other. In particular, we can write

$$\widetilde{J}_\alpha \ = \ R_{\alpha\beta} \, J_\beta \tag{6}$$

(the metric in $\mathbf{Q}$ is Euclidean, so we will write both indices as lower ones for typographical convenience). Now the requirement that $\widetilde{J}_\alpha^2 = J_\alpha^2 = -I$ forces the (real, three-dimensional) matrix $R$ to be orthogonal, $R \in O(3)$. Moreover, the quaternionic relations (2) require $\mathtt{Det}(\widetilde{J}_\alpha) = \mathtt{Det}(J_\alpha) = 1$, hence also $\mathtt{Det}(R) = 1$; in other words, we must actually have $R \in SO(3)$.

The same argument also applies to equivalent hypersymplectic structures: in this case, we also conclude that if $\{\omega_1, \omega_2, \omega_3\}$ and $\{\widetilde{\omega}_1, \widetilde{\omega}_2, \widetilde{\omega}_3\}$ are equivalent hypersymplectic structures, then necessarily $\widetilde{\omega}_\alpha = R_{\alpha\beta}\omega_\beta$, with $R \in SO(3)$.

*Remark 2.4.* A simple example of $R \in O(3)$ but not in $SO(3)$, for which it is immediate to check that the Kahler sphere is mapped into itself but the quaternionic relations are not preserved, is given by $R = \mathrm{diag}(-1, 1, 1)$. ⊙

*Remark 2.5.* Here, we are considering generic maps in $\mathbf{Q}$ or $\mathbf{S}$ (or more generally in the space of tensors defined on $M$). If we consider only maps induced by maps in $M$, then the situation is different. In particular, due to their tensorial nature, the quaternionic relations are automatically preserved under any (non-singular) map $\varphi: M \to M$. ⊙

### D. Standard structures in $\mathbf{R}^4$

In the following, we will make reference to "standard" hyperkahler and hypersymplectic structures in $\mathbf{R}^{4n}$; these are obtained from standard structures in $\mathbf{R}^4$ (with Euclidean metric).[18,21] We will consider the standard volume form $\Omega = \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^4$ in $\mathbf{R}^4$.

There are two such standard structures, differing for their orientation. The positively oriented standard hyperkahler structure is given by

$$\mathcal{Y}_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad \mathcal{Y}_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{Y}_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{7}$$

To these complex structures correspond the symplectic structures, satisfying $(1/2)(\omega_\alpha \wedge \omega_\alpha) = \Omega$ (no sum on $\alpha$),

$$\begin{aligned} \omega_1 &= \mathrm{d}x^1 \wedge \mathrm{d}x^2 + \mathrm{d}x^3 \wedge \mathrm{d}x^4, \ \omega_2 = \mathrm{d}x^1 \wedge \mathrm{d}x^4 + \mathrm{d}x^2 \wedge \mathrm{d}x^3, \\ \omega_3 &= \mathrm{d}x^1 \wedge \mathrm{d}x^3 + \mathrm{d}x^4 \wedge \mathrm{d}x^2. \end{aligned} \tag{8}$$

The negatively oriented standard hyperkahler structure is given by

$$\hat{\mathcal{Y}}_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \hat{\mathcal{Y}}_2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \hat{\mathcal{Y}}_3 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{9}$$

In this case, to these complex structure correspond the symplectic structures

$$\begin{aligned} \widehat{\omega}_1 &= \mathrm{d}x^1 \wedge \mathrm{d}x^3 + \mathrm{d}x^2 \wedge \mathrm{d}x^4, \ \widehat{\omega}_2 = \mathrm{d}x^4 \wedge \mathrm{d}x^1 + \mathrm{d}x^2 \wedge \mathrm{d}x^3, \\ \widehat{\omega}_3 &= \mathrm{d}x^2 \wedge \mathrm{d}x^1 + \mathrm{d}x^3 \wedge \mathrm{d}x^4 \,; \end{aligned} \tag{10}$$

these satisfy $(1/2)(\omega_\alpha \wedge \omega_\alpha) = -\Omega$ (again with no sum on $\alpha$).

*Remark 2.6.* Note that $[Y_\alpha, \widehat{Y}_\beta] = 0$ for all $\alpha$, $\beta$. The existence of these two equivalent (and oppositely oriented) mutually commuting real representations of $su(2)$ (and hence of the group $SU(2)$ as well) is of course related to the quaternionic nature of $SU(2)$ in the classification given by the real version of Schur Lemma (see, e.g., Chap. 8 of Ref. 27, in particular Theorem 3 there). ⊙

*Remark 2.7.* Note also that while the $SU(2)$ commutation relations are satisfied by any representation, the condition $J_\alpha^2 = -I$ imply that the tensors $J_\alpha$ are represented, at any given point, by a sum of copies of the two (oppositely oriented) fundamental representations, i.e., the standard ones defined above. ⊙

*Remark 2.8.* The orientation of hyperkahler structures is detected by an algebraic invariant (of matrices representing the complex structures $J_\alpha$), defined on generic matrices $A$ of order $2m$ as

$$\mathcal{P}_m(A) := (1/p_m) \sum_{i_s, j_s = 1}^{2m} \epsilon_{i_1 j_1 \ldots i_m j_m} \, A_{i_1 j_1} \ldots A_{i_m j_m}, \tag{11}$$

with $p_m = 2^m(m!)$ a combinatorial coefficient. This will appear in Sec. V and is discussed in Appendix A. It is immediate to check that $\mathcal{P}_2(\mathcal{Y}_\alpha) = 1$, $\mathcal{P}_2(\hat{\mathcal{Y}}_\alpha) = -1$. ⊙

### E. Hyperkahler structures in coordinates

The results we want to prove are of local nature, so we can work on a single chart of the hyperkahler manifold $(M, g; \mathbf{J})$. In the following, we will use local coordinates $x^i$ ($i = 1, \dots, 4n$); it will be useful to have a standard notation for expressing the objects introduced above in coordinates.

The metric $g$ is defined in coordinates by $g_{ij}\, \mathrm{d}x^i\, \mathrm{d}x^j$ (we will use the same letter for its corresponding matrix); when using shorthand notation (with no indices) we will denote the contravariant metric tensor $g^{ij}$ by $g^{-1}$.

The complex structures $J_\alpha$ and the associated Kahler symplectic forms $\omega_\alpha$ will be written as

$$\begin{aligned} J_\alpha &= (Y_\alpha)^i{}_j\, \partial_i \otimes \mathrm{d}x^j, \\ \omega_\alpha &= (K_\alpha)_{ij}\, \mathrm{d}x^i \wedge \mathrm{d}x^j; \end{aligned} \tag{12}$$

where the wedge product is defined as $\mathrm{d}x^i \wedge \mathrm{d}x^j = (1/2)(\mathrm{d}x^i \otimes \mathrm{d}x^j - \mathrm{d}x^j \otimes \mathrm{d}x^i)$. We will also consider tensors of type (2, 0) associated to these, i.e.,

$$M_\alpha^{ij} = g^{i\ell}\, K^\alpha_{\ell m}\, g^{mj}. \tag{13}$$

Note that here $M_\alpha$, $Y_\alpha$, $K_\alpha$ are in general functions of the point $x$, and are of course not independent (we prefer to have distinct notations for the tensor fields $Y_\alpha$, $K_\alpha = gY_\alpha$, $M_\alpha = Y_\alpha g^{-1}$ as these will be useful in writing subsequent equations in compact form without the need to write down all the indexes; note $K_\alpha^{-1} = -M_\alpha$, and of course $Y_\alpha^{-1} = -Y_\alpha$).

The quaternionic relations (2) are reflected into the same relations being satisfied by the matrices $Y_\alpha$, and similar ones – involving also $g$ – by the $K_\alpha$ and $M_\alpha$, i.e.,

$$\begin{aligned} Y_\alpha\, Y_\beta &= \epsilon_{\alpha\beta\gamma}\, Y_\gamma - \delta_{\alpha\beta}\, I, \\ K_\alpha\, g^{-1}\, K_\beta &= \epsilon_{\alpha\beta\gamma}\, K_\gamma - \delta_{\alpha\beta}\, g, \\ M_\alpha\, g\, M_\beta &= \epsilon_{\alpha\beta\gamma}\, M_\gamma - \delta_{\alpha\beta}\, g^{-1}. \end{aligned} \tag{14}$$

Similarly, the fact that the $J_\alpha$ are covariantly constant implies that $\nabla Y_\alpha = 0$ as well; as $g$ is by definition also covariantly constant under its associated Levi-Civita connection, we also have $\nabla K_\alpha = 0$, $\nabla M_\alpha = 0$.

*Remark 2.9.* Note that the $M_\alpha$ defined in (13) are three Poisson tensors on the manifold $M$ (we hope no confusion is possible between $M$ and the $M_\alpha$), with $M_\alpha$ the Poisson tensor corresponding in the usual way to the symplectic structure $\omega_\alpha$. The $M_\alpha$ are in involution in the sense they combine (with the action of the metric tensor $g$) as detailed in (14); they are not compatible in different senses, and in particular in the sense familiar from the theory of bi-hamiltonian systems.[28]

## III. HYPERKAHLER AND CANONICAL TRANSFORMATIONS

In this section, we will set our definitions of hyperkahler and of canonical transformations. These will be built by (non-trivial) analogy with the standard case of canonical transformations in Hamiltonian mechanics. We will thus start by briefly recalling, in Subsection III A, this standard case, referring, e.g., to Ref. 2 for details.

It should be stressed that while the definitions of hyperkahler (or hypersymplectic) and quaternionic transformations, considered in Subsection III B, is rather straightforward, the task (which is the subject of Subsection III C) of identifying the correct notion of canonical transformation in this context, i.e., Definition 3.3 below, is less immediate. One could of course just provide Definition 3.3 axiomatically, but we prefer to describe the heuristics which led to this definition – which is then justified *a posteriori* by the results of Sec. VII, see in particular Theorem 7.1 and its reformulation given in Remark 7.4. Thus, we will first discuss the situation in the Euclidean case (Sec. III C 1); here, the complex structures making the HK structure are not just covariantly constant but – as the Levi Civita connection is trivial – actually constant. As a consequence of this, they provide a splitting of T$M$ into four-dimensional subspaces not only at a reference point, but globally on $M$. This splitting allows for a direct generalization of the situation – and of the characterization of canonical maps based on Darboux submanifolds, see the discussion in Subsections III A and III C 1 – met on symplectic manifolds. This splitting does *not* apply in the general case, i.e., to manifolds with nontrivial Levi-Civita connection. On the other hand, still working in the Euclidean case, we are able to reformulate the notion of canonical maps identified through this analogy in terms of four-forms, and more precisely of the four-form $\omega \wedge \omega$ (for $\omega \in \mathcal{S}$). Once we have done this, all the discussion through the parallel with the symplectic case, and having just an heuristic value, can be thrown away, and we are left simply with Definition 3.3.

### A. Symplectic maps

Let $(M, \omega)$ be a symplectic manifold (of dimension $2n$); we say that a map $\varphi: M \to M$ is *symplectic* if it preserves the symplectic form $\omega$, i.e., if

$$\varphi^*(\omega) = \omega. \tag{15}$$

An equivalent characterization is also quite common (we refer, e.g., to Sec. 44 of Ref. 2 for detail). As well known, by Darboux theorem[2] one can introduce local coordinates $(p_a, q^a)$, $a = 1, \ldots, n$, in a neighborhood $U \subset M$ such that $\omega = \mathrm{d}p_a \wedge \mathrm{d}q^a$. Then, one considers local manifolds of minimal dimension on which $\omega$ is non-degenerate; these are two-dimensional and are spanned by $q^a$ and $p_a$ (with same $a$). They are known as Darboux submanifolds and denoted as $U_a$; these also correspond to leaves of the Abelian distribution generated by the Hamiltonian vector fields associated with canonical coordinates.

Let us consider a given point in $U$ and the manifolds $U_a$ through this. Denote by $\iota_a$ the embedding $\iota_a: U_a \hookrightarrow U \subseteq M$; then the restriction $\iota_a^*\omega$ of the symplectic form to $U_a$ provides a volume form $\Omega_a = \mathrm{d}p_a \wedge \mathrm{d}q^a$ (no sum on $a$) on $U_a$. Then, for any two-chain $A$ in $U$ and with $\pi_a A$ the projection of $A$ to $U_a$,

$$\int_A \omega = \int_A \sum_{a=1}^n \mathrm{d}p_a \wedge \mathrm{d}q^a = \sum_{a=1}^n \int_A \Omega_a = \sum_{a=1}^n \mathrm{area}[\pi_a A]\ ;$$

thus preservation of $\omega$ is equivalent to preservation of the sum of oriented areas of projection of any $A$ to Darboux submanifolds. That is, the map $\varphi$ is canonical if

$$\sum_{a=1}^n \mathrm{area}[\pi_a A] = \sum_{a=1}^n \mathrm{area}[\pi_a(\varphi A)].$$

It should be noted that if we start from a manifold equipped with a Riemannian metric, passing to Darboux coordinates will in general not preserve the representation of the metric tensor in coordinates, i.e., not preserve the (matrix $g_{ij}$ representing the) metric. Thus, this construction is in general not viable if one requires preservation of the metric.

In the case of a Kahler manifold, the symplectic form $\omega$ corresponds to a complex structure $J$ through the Kahler relation (1). This satisfies $J^2 = -I$, and provides a splitting of $\mathrm{T}_0M$ (at any point $m_0 \in M$) into two-dimensional invariant subspaces; the volume form $\Omega$ defined in $M$ induces volume forms $\Omega_a$ in each of these, and $\omega = \sum \Omega_a$. Thus, again canonical transformations can be characterized as those satisfying

$$\sum_{a=1}^n \Omega_a = \sum_{a=1}^n \varphi^*(\Omega_a). \tag{16}$$

Note this construction does not make use of Darboux coordinates or manifolds, but only of the splitting of $\mathrm{T}M$ induced by the action of the complex structure; moreover, we only consider volume forms.

### B. Hyperkahler transformations

Let us now pass to consider hyperkahler structures. As already noted, the tensorial nature of the quaternionic relation (2) guarantees they will be preserved under any map $\varphi: M \to M$. Note also that here the Riemannian metric is an essential part of the structure, so if we look at maps which preserve the hyperkahler (or the quaternionic) structure it is natural to only consider maps $\varphi$: $M \to M$ which are orthogonal with respect to $g$.

#### *1. Strongly hyperkahler maps*

It may seem natural to generalize (15) by requiring that the three symplectic forms $\omega_\alpha$ (and hence all symplectic form in $\mathcal{Q}$) are preserved; from the point of view of the complex structures, this means considering *tri-holomorphic* maps. However, this criterion would be exceedingly restrictive, and we will deal with a weaker form of it. We will reserve a different name for this case.

*Definition 3.1. Let* $(M, g; J_1, J_2, J_3)$ *be a hyperkahler manifold. We say that the orthogonal map* $\varphi$: $M \to M$ *is* **strongly hyperkahler** *if it leaves the three complex structures* $J_\alpha$ *invariant.*

*Remark 3.1.* We have stated that this class of maps is exceedingly restrictive. To show this is the case, let us consider the map generated by a Hamiltonian flow, say under the symplectic structure $\omega_1$. It is easy to check that in this case (even in the simplest integrable case, with hamiltonian $|x|^2/2$), the forms $\omega_2$, $\omega_3$ are not preserved. In fact, for the transformed forms $\widetilde{\omega}_\alpha$ we have

$$\widetilde{\omega}_1 = \omega_1,\ \widetilde{\omega}_2 = \cos(\theta)\omega_2 - \sin(\theta)\omega_3,\ \widetilde{\omega}_3 = \sin(\theta)\omega_2 + \cos(\theta)\omega_3\ ;$$

here $\theta$ is an angle, depending on time. Thus, the forms $\omega_2$, $\omega_3$ are rotated in the plane they span in $\mathcal{Q}$. In other words, the hypersymplectic structure is in this case mapped into an equivalent – but different – one. ⊙

#### *2. Hyperkahler maps*

The above remark suggest that (as discussed also in Refs. 18 and 19) the appropriate generalization of symplectic transformations in the hyperkahler case should *not* require the preservation of the three symplectic (Kahler) forms; we should rather require – besides the preservation of the metric – the milder condition that the hyperkahler structure is mapped into an equivalent one.

*Definition 3.2. Let* $(M, g; \mathbf{J})$ *be a hyperkahler manifold. We say that the orthogonal map* $\varphi$: $M \to M$ *is* **hyperkahler** *if it maps the hyperkahler structure into an equivalent one, i.e., if* $\varphi^*$: $\mathbf{S} \to \mathbf{S}$.

*Remark 3.2.* With this definition, the Hamiltonian flow considered in Remark 3.1 will generate a one-parameter group of hyperkahler maps. Note that a generic Hamiltonian flow will not preserve the metric and hence will not qualify as generating (a family of) hyperkahler maps. ⊙

*Remark 3.3.* Hyperkahler maps will preserve the quaternionic structure; we will thus also refer to them as **quaternionic** maps. ⊙

Finally, we note that, as obvious, the concepts considered in this section can also be expressed referring to symplectic (rather than complex) structures; we will in this framework have the corresponding

*Definition 3.1′. Let* $(M, g; \omega_1, \omega_2, \omega_3)$ *be a hypersymplectic manifold. We say that the orthogonal map* $\varphi$: $M \to M$ *is* **strongly hypersymplectic** *if it leaves the hypersymplectic structures invariant, i.e., if* $\varphi^*(\omega_\alpha) = \omega_\alpha$ *for* $\alpha = 1, 2, 3$.

*Definition 3.2′. If* $(M, g; \omega_1, \omega_2, \omega_3)$ *is a hypersymplectic manifold, we say that the orthogonal map* $\varphi$: $M \to M$ *is* **hypersymplectic** *if it maps the hypersymplectic structure into an equivalent one, i.e., if* $\varphi^* : \mathcal{S} \to \mathcal{S}$.

### C. Canonical transformations

We will reserve the name "canonical transformations" (or maps) for those which satisfy the (generalization of) the criterion based on conservation of projected areas, see (16).

In the hyperkahler case, the three complex structures induce a splitting of $\mathrm{T}_0 M$ (at any given point $m_0 \in M$ of the $4n$-dimensional manifold $M$) into four-dimensional invariant subspaces $U_a$ (it may be worth remarking again, in this respect, that the quaternionic relations (2) imply the $J_\alpha$ satisfy the $su(2)$ Lie algebra commutation relations, but also involve the multiplicative structure. In particular, they imply that the $J_\alpha$ (at a given point) provide a representation of $su(2)$ as the sum of $n$ four-dimensional real irreducible representations).

#### *1. The Euclidean case*

In the Euclidean case (thus $M = \mathbf{R}^{4n}$ and $g = I_{4n}$), the Levi-Civita connection is flat and the $J_\alpha$ are actually constant (it is easy to see that in this case the complex structures are given by a sum of structures in standard form). Thus, the splitting actually applies to the full $M$ (the decomposition is of course in terms of $\mathbf{R}^4$ subspaces). More generally, if $M$ is locally Euclidean, the invariant four-dimensional subspaces of $\mathrm{T}_x M$ (for $x \in U \subset M$) form a distribution which has invariant four-dimensional integral manifolds $U_a$. Considering the embedding $\iota_a \colon U_a \hookrightarrow U$, the volume form on $U_a$ is obtained as

$$\Omega_a \ = \ \iota_a^* \left( \frac{1}{2} \, \omega \wedge \omega \right), \tag{17}$$

for $\omega$ any symplectic form in $\mathcal{S}$ (that this is independent of $\omega \in \mathcal{S}$ is easily checked via the explicit form of the $K_\alpha$ in standard form, and the remark made above that in the Euclidean case the structures can be written in standard form of either orientation. Actually, this remark amounts, in Lie theoretic terms, to the fact that there are only two real irreducible representations of the $su(2)$ Lie algebra of dimension four).

It follows easily that the maps which preserve the sum of oriented volumes, thus the sum of the $\Omega_a$, are precisely those which preserve the four-forms $\omega \wedge \omega$ for any $\omega \in \mathcal{S}$, and in particular for $\omega = \omega_\alpha$ (with $\alpha = 1, 2, 3$).

#### *2. The general case*

Motivated by the above discussion for the Euclidean case, we will extend the characterization of canonical transformations found in that case to the general situation.

*Definition 3.3. Let* $(M, g; \mathbf{J})$ *be a hyperkahler manifold, and* $\mathcal{Q}$ *the corresponding symplectic Kahler sphere. We say that the map* $\varphi \colon M \to M$ *is* **canonical** *if, for any* $\omega \in \mathcal{S}$*, it preserves the form* $\omega \wedge \omega$*.*

*Remark 3.4.* It is clear that the two notions of canonical and hyperkahler (or quaternionic) maps proposed here are *not* equivalent (at difference with the notion holding in the symplectic or Kahler case which they generalize). In a way, quaternionic maps preserve the quaternionic structure, while canonical ones only preserve the square of forms associated to it; moreover, note that we are *not* requiring canonical maps to be orthogonal. Consider, e.g., $\omega_1$ (see Sec. II): under the map $x^1 \to \lambda x^1$, $x^2 \to \lambda x^2$, $x^3 \to \lambda^{-1} x^3$, $x^4 \to \lambda^{-1} x^4$, the form $\omega_1$ is not preserved (note $g$ is not preserved as well) nor mapped to a different form in $\mathcal{S}$, but $\omega_1 \wedge \omega_1$ is invariant. More generally, a canonical map could even mix the positively and negatively oriented structures. ⊙

The criterion for a transformation to be canonical can also be stated in terms of a basis for $\mathcal{S}$, i.e., of the $\omega_\alpha$ associated to the $J_\alpha$. In terms of these, we have the equivalent definition:

*Definition 3.4. The map* $\varphi \colon M \to M$ *is* **canonical** *for the hyperkahler structure* $(g; \mathbf{J})$ *if and only if* (*with no sum on* $\alpha$)

$$\varphi^*(\omega_\alpha \wedge \omega_\alpha) \ = \ \omega_\alpha \wedge \omega_\alpha \quad \alpha = 1, 2, 3.$$

To see this is equivalent to the previous one, just note that any $\omega \in \mathcal{S}$ can be written as $\omega = c_\alpha \omega_\alpha$ with $c_\alpha c_\alpha = 1$; hence, – recalling $\omega_\alpha \wedge \omega_\beta = 0$ for $\alpha \neq \beta$ – we have

$$\omega \wedge \omega \ = \ \sum_\alpha c_\alpha^2 \, (\omega_\alpha \wedge \omega_\alpha).$$

It is obvious that preservation of $\omega \wedge \omega$ for all $\omega \in \mathcal{S}$ implies in particular it is preserved for $\omega = \omega_\alpha$; and conversely it follows from the above, and linearity of the pullback operation, that preserving $\omega_\alpha \wedge \omega_\alpha$ (no sum on $\alpha$) for each $\alpha$ implies preservation of $\omega \wedge \omega$ for any $\omega \in \mathcal{S}$.

*Remark 3.5.* In the Euclidean case, we can introduce the $U_a$ local submanifolds and the embedding $i_a$, as seen above. In this case, our Definition 3.3 can be recast in a different way making use of these: *The map* $\varphi$: $M \to M$ *is* **canonical** *for the hyperkahler structure* $(g; \mathbf{J})$ *if and only if (with no sum on* $\alpha$*)*

$$\iota_a^*(\omega_\alpha \wedge \omega_\alpha) \ = \ \iota_a^*[\varphi^*(\omega_\alpha \wedge \omega_\alpha)] \quad \alpha = 1, 2, 3, \quad a = 1, \dots, n.$$

In order to see that this is equivalent to the previous ones, it suffices to note that any $\mu \in \mathcal{S}$ is written as $\mu = c_\alpha \omega_\alpha$, and that independence of the $\omega_\alpha$ (required by the quaternionic relations) imply that $\iota_a^*(\omega_\alpha \wedge \omega_\beta) = 0$ when $\alpha \neq \beta$. Thus,

$$\iota_a^*(\mu \wedge \mu) \ = \ \sum_{\alpha,\beta=1}^{3} c_\alpha c_\beta \, \iota_a^*(\omega_\alpha \wedge \omega_\beta) \ = \ \sum_{\alpha=1}^{3} c_\alpha^2 \iota_a^*(\omega_\alpha \wedge \omega_\alpha).$$

Given the arbitrariness of $\mu$, i.e., of the $c_\alpha$, we conclude that indeed this definition is equivalent to Definition 3.4 and hence to Definition 3.3 in the Euclidean case. ⊙

## IV. CHARACTERIZATION OF HYPERKAHLER MAPS

We can now discuss hyperkahler (quaternionic) transformations, i.e., applications $\Phi$: $M \to M$ which map the hyperkahler structure into an equivalent one. It is clear that these form a group, which will be denoted as `hSp`, or more precisely `hSp`$(M, g, \mathbf{J})$. It is obvious that (see Definition 3.2′) this is equivalently the group of hypersymplectic transformations. (In fact, the notation `hSp` stands for "hypersymplectic.")

It should be stressed that there is an essential difference between this and the symplectic group which is familiar from standard Hamiltonian dynamics or from symplectic geometry. In fact, in the symplectic case the Darboux theorem allows to reduce (locally) any symplectic structure to the standard form $\omega = \mathrm{d}p_i \wedge \mathrm{d}q^i$; with this $\omega$ is (locally) constant, and the maps which preserve $\omega$ at a given point $x_0$ will also – when extended as constant ones – preserve it in a full neighborhood of $x_0$. Thus, as well known, one effectively reduces to a problem in linear algebra. On the other hand, there is no Darboux theorem for hyperkahler structures, and the latter are in general *not* constant (even locally), but instead covariantly constant. Thus, the analysis made at a single reference point $x_0$ will not immediately provide "hypersymplectic" maps in a neighborhood of it, the extension requiring to have covariantly constant maps.

### A. Hyperkahler maps for Euclidean versus general manifolds

Characterization of hyperkahler maps is much easier in the Euclidean case – where we can in practice reduce to consider the standard structures introduced in Sec. II – than in the general one, even at the local level. In both cases, one would like first to reduce the structure to standard form at least in a reference point (for the Euclidean case this will hold on the whole manifold). Note that in the following we will sometime say, for ease of writing, "new metric" (and so on) to mean "expression of the metric in the new coordinates" (and so on); we hope the reader will forgive this little abuse of language.

Let us consider a neighborhood $U \subset M$ and local coordinates $x^i$ in it; we denote the covariant derivative with respect to $x^i$ defined by the Levi-Civita connection as $\nabla_i$. This acts on (1, 1) tensor fields $J$ as $\nabla_i J = \partial_i J + [A_i, J]$; hence, the $J_\alpha$ satisfy

$$\nabla_i J_\alpha = \partial_i J_\alpha + [A_i, J_\alpha] = 0. \tag{18}$$

Let us now consider a change of variables; we denote its Jacobian by $\Lambda$, i.e., $\Lambda^i{}_j = (\partial x^i / \partial \widetilde{x}^j)$. Under this change of coordinates the metric, represented in the old coordinates by the matrix $g$ is represented by the matrix $\widetilde{g}$ with

$$\widetilde{g} = \Lambda^T g \Lambda ; \tag{19}$$

correspondingly the coordinate expression of the Levi-Civita connection changes (we write $\widetilde{\nabla}$ for the new expression), and the covariant derivatives under this (acting on (1,1) tensor fields) are written in coordinates as

$$\widetilde{\nabla}_i = \widetilde{\partial}_i + [\widetilde{A}_i, .], \tag{20}$$

where

$$\widetilde{A}_i = \Lambda^{-1} A_i \Lambda - (\partial_i \Lambda^{-1}) \Lambda \equiv \Lambda^{-1} A_i \Lambda + \Lambda^{-1} (\partial_i \Lambda). \tag{21}$$

The (1,1) tensor fields $J_\alpha$ are changed into new (1,1) tensor fields $\widetilde{J}_\alpha$ with

$$\widetilde{J}_\alpha = \Lambda^{-1} J_\alpha \Lambda. \tag{22}$$

Obviously (as the considered relations do not depend on coordinates), the $\widetilde{J}_\alpha$ are still orthogonal and covariantly constant, and satisfy the quaternionic relations; in other words, they are again a hyperkahler structure.

Let us now fix a reference point $x_0 \in U$, and choose a first change of variables (with Jacobian $\Lambda^{(0)}$) so that at this point the new metric is just given by $\widetilde{g}(x_0) = \delta$, which is always possible choosing a suitable $\Lambda^{(0)}$.

If we want to consider further transformations which do not alter the metric at this reference point, we have to consider only changes such that their Jacobian, denoted by $\Lambda^{(1)}$, satisfies $\Lambda^{(1)}(x_0) = B \in O(4n)$. It is quite clear that by a suitable choice of this $B$, hence of the overall change of coordinates with Jacobian $\Lambda = \Lambda^{(1)}\Lambda^{(0)}$, we can obtain that the new complex structures $\widetilde{J}_\alpha$ satisfy $\widetilde{J}_\alpha(x_0) = \mathcal{Y}_\alpha$ with $\mathcal{Y}_\alpha$ the "standard" complex structures considered in Refs. 18 and 21 and given in Sec. II D.

We summarize our discussion in the form of a Lemma.

*Lemma 4.1. Given a hyperkahler manifold* $(M, g; \mathbf{J})$ *and a point* $x_0 \in M$, *it is always possible to change local coordinates around* $x_0$ *so that the metric and the complex structures are written as* $\widetilde{g}$, $\widetilde{J}_\alpha$, *with* $\widetilde{g}(x_0) = \delta$, $\widetilde{J}_\alpha(x_0) = \mathcal{Y}_\alpha$.

*Remark 4.1.* The Lemma deals with a single point $x_0$; but, we are of course interested not only in what happens at $x_0$, but at least in an open neighborhood $U$ of it. The form of the $\widetilde{J}_\alpha$ at other points of $U$ is rather general, and only subject to the condition of being covariantly constant, $\widetilde{\nabla}\widetilde{J}_\alpha = 0$. Note that if $\widetilde{\nabla}$ has a nontrivial holonomy, this does not uniquely define the $\widetilde{J}_\alpha$. ⊙

*Remark 4.2.* The holonomy group $\mathcal{H}$ must be a subgroup of the invariance group for the (integrable) quaternionic structure, i.e., $\mathcal{H} \subseteq \mathtt{hSp}$. We can expect that, unless the hyperkahler structure has some special (invariance) property, the two will just coincide (in Ref. 23 we will find this is the case in Euclidean spaces; this fact should be seen as a check that our notion of hyperkahler maps is an appropriate one). ⊙

Let us now discuss the relation between the groups $\mathtt{hSp}(M, g, \mathbf{J})$ and $\mathtt{hSp}_0(4n) := \mathtt{hSp}(\mathbf{R}^{4n}, g_0, \mathbf{J}_0)$; here and in the following we denote by $\mathtt{hSp}_0(4n)$ the group of hypersymplectic transformations for metric in Euclidean form $g_0 = \delta$ and standard hyperkahler structures $\mathbf{J}_0$ at a reference point $x_0$. In fact, as it is known, the group coincides with the holonomy group of the structure, $Sp(n)$ if we consider hyperkahler manifolds, that is, strong transformations in our notation, or $Sp(n) \times Sp(1)$ (in the case of quaternionic manifolds).

*Lemma 4.2. Let $R(x)$ be a map taking $(g, \mathbf{J})$ into standard form $(g_0, \mathbf{J}_0)$ at the point $x_0$; then*

$$\mathtt{hSp}(4n, g, \mathbf{J}) = R^{-1}(x_0)\, \mathtt{hSp}(4n, g_0, \mathbf{J}_0)\, R(x_0) = R^{-1}(x_0)\, \mathtt{hSp}_0(4n)\, R(x_0). \tag{23}$$

*Proof.* This just follows from $R$: $g \to g_0$ and $R$: $\mathbf{J} \to \mathbf{J}_0$. Note that $R$ is not uniquely defined, as any map $\widehat{R} = S \cdot R$ with $S = S(x)$ such that $S(x_0) \in \mathtt{hSp}_0(4n)$ will have the same effect, but this lack of uniqueness will not affect (23). △

### B. Characterization for structures in standard form

Thanks to Lemma 4.2, we can just focus on $\mathtt{hSp}_0(4n)$, i.e., deal with metric and hyperkahler structures which are in standard form at an arbitrary reference point $x_0$. We will from now on write $g, \nabla, J_\alpha$, to denote the metric, the associated connection, and the complex structures in this case.

In view of Definition 3.1, we have to look for changes of coordinates $\Phi$ with Jacobian $\Lambda$ which preserve $g$ (and hence $\nabla$ and its coefficients $A_i$) and which map $\mathbf{J}$ into an equivalent $\widetilde{\mathbf{J}}$. In other words, we have to require that

$$\widetilde{J}_\alpha = R_{\alpha\beta}\, J_\beta \quad \text{with} \quad R \in SO(3). \tag{24}$$

*Remark 4.3.* One could think of a generalization of (24) with $R$ a matrix field with values in $SO(3)$ rather than a constant one; this is actually forbidden by the condition $\nabla \widetilde{J}_\alpha = 0$, which implies $R$ is constant. In fact, we have immediately $\nabla_i \widetilde{J}_\alpha = \partial_i \widetilde{J}_\alpha + [A_i, \widetilde{J}_\alpha] = \partial_i (R_{\alpha\beta} J_\beta) + R_{\alpha\beta}[A_i, J_\beta] = (\partial_i R_{\alpha\beta}) J_\beta + R_{\alpha\beta}(\nabla_i J_\beta) = (\partial_i R_{\alpha\beta})\, J_\beta$. Hence, $\nabla \widetilde{J}_\alpha = 0$ if and only if $(\partial_i R_{\alpha\beta}) = 0$, i.e., if and only if the $R_{\alpha\beta}$ are constant. ⊙

In order to discuss (24), we will suppose that $J_\alpha(x_0)$ is represented by $\mathcal{Y}_\alpha$ in blocks $1, \ldots, m$ and by $\hat{\mathcal{Y}}_\alpha$ in blocks $m + 1, \ldots, n$ (a reordering of blocks is needed for the general case, but inessential).

It will be convenient to write the matrix $\Lambda$ in terms of four-dimensional blocks; we set a standard notation for this, and write (no confusion should be possible between the sub-matrices $A_{ij}$ and the connection coefficients $A_i$)

$$\Lambda = \begin{pmatrix} A_{11} & A_{12} & \ldots & A_{1n} \\ A_{21} & A_{22} & \ldots & A_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ A_{n1} & A_{n2} & \ldots & A_{nn} \end{pmatrix}; \quad \Lambda^T = \begin{pmatrix} A^T_{11} & A^T_{21} & \ldots & A^T_{n1} \\ A^T_{12} & A^T_{22} & \ldots & A^T_{n2} \\ \ldots & \ldots & \ldots & \ldots \\ A^T_{1n} & A^T_{2n} & \ldots & A^T_{nn} \end{pmatrix}. \tag{25}$$

It will also be convenient to deal with $K_\alpha$ (rather than $\mathcal{Y}_\alpha$), so to avoid inversion of the matrix $\Lambda$; the condition $\widetilde{J}_\alpha = R_{\alpha\beta} J_\beta$ is equivalent to $\widetilde{K}_\alpha = R_{\alpha\beta} K_\beta$.

We will write the (block-diagonal, once we pass to standard form) matrices $K_\alpha$ and the (in general, not block-diagonal) $\widetilde{K}_\alpha = \Lambda^T K_\alpha \Lambda$ as

$$K^\alpha = \begin{pmatrix} K^\alpha_{11} & 0 & \ldots & 0 \\ 0 & K^\alpha_{22} & \ldots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & \ldots & 0 & K^\alpha_{nn} \end{pmatrix}, \quad \widetilde{K}_\alpha = \Lambda^T K_\alpha \Lambda = \begin{pmatrix} \widetilde{K}^\alpha_{11} & \widetilde{K}^\alpha_{12} & \ldots & \widetilde{K}^\alpha_{1n} \\ \widetilde{K}^\alpha_{21} & \widetilde{K}^\alpha_{22} & \ldots & \widetilde{K}^\alpha_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ \widetilde{K}^\alpha_{n1} & \ldots & \ldots & \widetilde{K}^\alpha_{nn} \end{pmatrix}.$$

It is easily checked that $\widetilde{K}^\alpha_{ij} = A^T_{\ell i} K^\alpha_{\ell m} A_{mj}$, and in particular, using $K^\alpha_{ij} = 0$ for $i \neq j$ we get

$$\widetilde{K}^\alpha_{ii} = \sum_m A^T_{mi}\, K^\alpha_{mm}\, A_{mi} \quad \text{(no sum on } i\text{)}.$$

The admitted $\Lambda$ are thus identified as those built with the $A_{ij}$ satisfying the conditions

$$\sum_m A^T_{mi} K^\alpha_{mm} A_{mj} = 0 \quad \text{for } i \neq j, \tag{26}$$

$$\sum_m A^T_{mi} \, K^\alpha_{mm} \, A_{mi} = R_{\alpha\beta} \, K^\beta_{ii}. \tag{27}$$

A discussion of solutions to (26) and (27) is more conveniently conducted in terms of infinitesimal generators; the computations, yielding the known holonomy groups of this manifold, will be presented in a companion work.[23]

## V. CHARACTERIZATION OF CANONICAL MAPS

We will now discuss canonical transformations (see Definitions 3.3 and 3.4 above); again it is clear that these form a group, which will be denoted as $\mathtt{Can}(M)$, or more precisely $\mathtt{Can}(M, g, \mathbf{J})$. We will proceed as for hyperkahler maps, i.e., first discuss the relation between the general case and the case where the structure is in standard form at least at a given point, and then discuss the characterization of canonical maps for structures in standard form.

### A. Canonical maps for Euclidean versus general manifolds

Let us again fix a reference point $x_0 \in M$ and perform the change of coordinates $R$ which takes the Riemannian metric into standard form at $x_0$, and subsequently the change of coordinates $S$ which, leaving $g$ in standard form at $x_0$, takes the complex structures $J_\alpha$ (and hence the symplectic forms $\omega_\alpha$) into standard form at $x_0$. Proceeding as in Sec. IV, we obtain the following.

*Lemma 5.1. Let* $(M, g; \mathbf{J})$ *be a hyperkahler manifold; let* $R(x)$ *be the transformation taking* $g$ *and* $J_\alpha$ *into standard form* $(g_0 = \delta, \mathbf{J}_0)$ *at the point* $x \in M$. *The group of canonical transformations* at the point $x$ *is given by*

$$\mathtt{Can}(g, \mathbf{J}) \;=\; R^{-1}(x) \, \mathtt{Can}(\delta, \mathbf{J}_0) \, R(x),$$

*where* $\mathtt{Can}(\delta, \mathbf{J}_0)$ *is the group of canonical transformations for* $g$ *and* $\mathbf{J}$ *in standard form.*

### B. Characterization for structures in standard form

We have then to characterize the group $\mathtt{Can}_0 := \mathtt{Can}(\delta, \mathbf{J}_0)$ of maps which preserve $\omega \wedge \omega$ for standard hyperkahler structures; actually most of the discussion will be the same for standard or generic form of these.

The key observation is that for any symplectic form $\omega$ we can write the volume form on any of the local four-dimensional manifolds $U_a$ built in Sec. III as

$$\Omega_{(a)} \;=\; \pm \, \iota_a^*[(1/2)(\omega \wedge \omega)], \tag{28}$$

the sign depending on the orientation of $\iota_a^* \omega$. (In the same way, the volume form $\Omega$ on the $4n$-dimensional manifold $M$ can be written as $\Omega = \pm \, [(1/(2n!))(\omega \wedge \ldots \wedge \omega)]$.)

In local coordinates, we have $\omega = K_{ij} \mathrm{d}x^i \mathrm{d}x^j$ and hence

$$\frac{1}{2} \, (\omega \wedge \omega) \;=\; \frac{1}{2} \, K_{ij} \, K_{\ell m} \, \mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^\ell \wedge \mathrm{d}x^m.$$

Under a map with Jacobian $\Lambda$, $K$ is transformed into $\widetilde{K} = \Lambda^T K \Lambda$; correspondingly, the form $(1/2)(\omega \wedge \omega)$ is rewritten as

$$\frac{1}{2}(\widetilde{\omega} \wedge \widetilde{\omega}) \;=\; \frac{1}{2} \, (\widetilde{K}_{ij} \, \widetilde{K}_{\ell m}) \, \mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^\ell \wedge \mathrm{d}x^m.$$

When we look at the volume form on $U_a$, only coordinates $i, j, \ell, m$ in the range $\mathcal{R}_a := [4(a-1)+1, \dots, 4a]$ should appear; in other words, the operation $\iota_a^*$ sets to zero all four-forms $dx^i \wedge dx^j \wedge dx^\ell \wedge dx^m$ except those with exactly (any permutation of) the four suitable coordinates.

Thus we have, with $i, j, \ell, m \in \mathcal{R}_a$,

$$V_{(a)} = \frac{1}{2} (\varepsilon_{ij\ell m} K_{ij} K_{\ell m})\, \Omega_{(a)}\,; \quad \widetilde{V}_{(a)} = \frac{1}{2} (\varepsilon_{ij\ell m} \widetilde{K}_{ij} \widetilde{K}_{\ell m})\, \Omega_{(a)}.$$

The central object is thus the quantity

$$\mathcal{P}_2(K) := (1/8)\,(\varepsilon_{ij\ell m} K_{ij} K_{\ell m}) \tag{29}$$

(see Sec. II and Appendix A); and a map is canonical if and only if

$$\sum_a \iota_a^*[\mathcal{P}_2(\tilde{K})] \equiv \sum_a \iota_a^*[\mathcal{P}_2(\Lambda^T K \Lambda)] = \sum_a \iota_a^*[\mathcal{P}_2(K)] \tag{30}$$

for any $K$ corresponding to a symplectic form $\omega \in \mathcal{S}$.

It should be noted that – as easy to check, e.g., by direct computation (see also Appendix A) – for a generic antisymmetric matrix $K$ it results

$$\mathcal{P}_2(\Lambda^T K \Lambda) = \mathcal{P}_2(K)\, \mathtt{Det}(\Lambda). \tag{31}$$

It will again be convenient to write the matrix $\Lambda$, as well as the $K = K_\alpha$, in terms of four-dimensional blocks; we will use the notation set up in Sec. IV. With this, it turned out that $\widetilde{K}_{ij} = A^T_{\ell i} K^0_{\ell m} A_{mj}$, and in particular, using $K^0_{ij} = 0$ for $i \neq j$ (no sum on $i$)

$$\widetilde{K}_{ii} = \sum_m A^T_{mi}\, K^0_{mm}\, A_{mi}.$$

Thus, the condition to have a canonical transformation (30) reads now

$$\sum_i \mathcal{P}_2 \left[ \sum_m A^T_{mi}\, K^0_{mm}\, A_{mi} \right] = \sum_i \mathcal{P}_2 \left[ K^0_{mm} \right]; \tag{32}$$

using now $\mathcal{P}_2[K^0_{mm}] = 1$, this is also written as

$$\frac{1}{n} \sum_i \mathcal{P}_2 \left[ \sum_m A^T_{mi}\, K^0_{mm}\, A_{mi} \right] = 1. \tag{33}$$

This can be written in a slightly different form by introducing the notation

$$\mathcal{P}_1(A, B) = (1/8)\, \epsilon_{ijkm}\, A_{ij}\, B_{km}\,; \tag{34}$$

it is easily checked that $\mathcal{P}_1(B, A) = \mathcal{P}_1(A, B)$ and $\mathcal{P}_2(A) = \mathcal{P}_1(A, A)$. Moreover,

$$\mathcal{P}_2(A + B) = \mathcal{P}_1(A, A) + \mathcal{P}_1(A, B) + \mathcal{P}_1(B, A) + \mathcal{P}_1(B, B)$$

$$= \mathcal{P}_2(A) + \mathcal{P}_2(B) + 2\,\mathcal{P}_1(A, B).$$

Using this, condition (33) can be rewritten as

$$\frac{1}{n} \sum_{i=1}^{n} \left[ \sum_m \mathcal{P}_2(A^T_{mi}\, K^0_{mm}\, A_{mi}) + 2 \sum_{m,\ell} \mathcal{P}_1 \left( A^T_{mi} K^0_{mm} A_{mi},\, A^T_{\ell i} K^0_{\ell\ell} A_{\ell i} \right) \right] = 1. \tag{35}$$

We can summarize our discussion as follows.

*Lemma 5.2. The group* $\mathtt{Can}_0$ *is the group of all the matrices* $\Lambda$ *of the form* (25) *which satisfy* (35).

## VI. DUAL HYPERKAHLER STRUCTURES

In Euclidean space, we have two kinds of standard hyperkahler and hypersymplectic structures, characterized by their different orientation, as recalled in Sec. II D. Both of these are needed when we want to describe Dirac mechanics in hyperhamiltonian terms (in this frame they are associated to opposite helicity states).[21]

### A. Construction of dual structures and standard forms

We want to discuss briefly the relation between the map taking a hyperkahler structure into standard form and its action on the associated hyperkahler structure of opposite orientation. The construction of Sec. III allows to essentially reduce the discussion to the four-dimensional case.

The orientation-reversing map on T$M$ at the reference point $x_0$ can be described in terms of a block diagonal matrix $R_0$ satisfying $R_0^T = R_0^{-1} = R_0$ and $\mathtt{Det}(R_0) = -1$. In coordinates, the simplest such map can be either the reversing of a coordinate axis (say the first one), or the exchange of two coordinate axes (say the first two); we will refer to these as reversing and parity-reversing maps, respectively. In the first case, we write it as $R_0 = \rho_0$, while in the second one we write $R_0 = \eta_0$ (in each block); that is,

$$\rho_0 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad \eta_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Needless to say, if we want to change orientation in several (or all) of the four-dimensional blocks, the operation should be applied to each of these.

The $R$ defined on $M$ is identified by $R(x_0) = R_0$ and by the requirement that $\nabla R = 0$. The metric $g$ and hence the Levi-Civita connection $\nabla$ and its coefficients $A_i$ are invariant under $R$.

As $R$ is covariantly constant (and thus are $J_\alpha$ and $\omega_\alpha$), it follows immediately that the transformed complex structures $\widetilde{J}_\alpha$ (respectively, symplectic structures $\widetilde{\omega}_\alpha$) are also covariantly constant.

Let us now consider a hyperkahler structure $(g^0, \mathbf{J}^0)$ on $M$, and take it into standard form $(g, \mathbf{J})$ – with positive orientation – via a map $\Phi$ with Jacobian $\Lambda$, as discussed in Sec. IV.

*Lemma 6.2. The map $R_0$ takes $(g, \mathbf{J})$ into a hyperkahler structure $(\overline{g}, \overline{\mathbf{J}})$ which is also in standard form but with negative orientation.*

*Proof.* In order to check that $(\overline{g}, \overline{\mathbf{J}})$ still provides a hyperkahler structure on $M$, it suffices to check that the $\overline{J}_\alpha$ are covariantly constant under the connection $\overline{\nabla}$ corresponding to $\overline{g}$ and satisfy the quaternionic relations. The first fact follows from the previous observation, and the second from the tensorial nature of the $J_\alpha$. Finally, it is obvious that $R$ changes orientation. We should still check that the $\overline{J}_\alpha$ are in standard form (with reversed orientation); this follows easily from an explicit computation at the reference point $x_0$. Note also that $\overline{g} = g$ (and hence $\overline{\nabla} = \nabla$), as $R$ is orthogonal. △

Inverting the map $\Phi$ with Jacobian $\Lambda$ (i.e., considering the map $\Phi^{-1}$ with Jacobian $\Lambda^{-1}$) we now get $(\widetilde{g}^0, \widetilde{\mathbf{J}}^0)$, given explicitly by

$$\widetilde{g}^0 = (\Lambda^T)^{-1}\, \widetilde{g}\, (\Lambda)^{-1} = g^0 \ ; \quad \widetilde{J}^0_\alpha = \Lambda\, \widetilde{J}_\alpha\, \Lambda^{-1}. \tag{36}$$

We say that $(\widetilde{g}^0, \widetilde{\mathbf{J}}^0)$ is the hyperkahler structure on $M$ **dual** to $(g^0, \mathbf{J}^0)$. Note that dual hyperkahler structures share the same Riemannian metric.

### B. Dirac structures

On physical grounds – e.g., in providing a hyperhamiltonian description of the Dirac equation – it is sometimes needed to consider a pair of dual hyperkahler structures.

*Definition 6.1. A pair of mutually dual hyperkahler structures* $\mathbf{J}$ and $\widehat{\mathbf{J}}$ *on* $(M, g)$ *is said to be a* Dirac structure *on* $(M, g)$, *and denoted as* $(\mathbf{J}, \widehat{\mathbf{J}})$.

*Remark 6.1.* A Dirac structure is characterized not by a single unit sphere $\mathbf{S}$ in the space $\mathbf{Q}$ of Kahler structures, but by a pair of dual unit spheres; referring to their orientation we will denote these by $\mathbf{S}_+$ and $\mathbf{S}_-$. ⊙

*Remark 6.2.* The discussion given here in terms of hyperkahler structures could have been performed in terms of hypersymplectic structures; in this framework, we could speak of Dirac-symplectic structures, and denote the unit spheres in $\mathcal{Q}$ characterizing such a structure as $\mathcal{S}_+$ and $\mathcal{S}_-$. ⊙

*Remark 6.3.* In this sense, and in view of a discussion of canonical maps, it is essential to note that – as apparent from the construction of dual hyperkahler structures – the splitting of T$M$ into four-dimensional invariant subspaces is the same for both members of a pair of dual hyperkahler structures. In other words, these are also invariant subspaces for the Dirac structure, and no distinction between the dual structures can be made on the basis of the induced splitting.

### C. Hypersymplectic transformations for Dirac structures

Consideration of Dirac structures calls for a discussion of their canonical and (the equivalent of their) hyperkahler transformations. While in the former case the definition can be extended unaltered from the hyperkahler case, in the latter we will need a slight generalization in order to consider both structures at the same time and allow some mixing.

*Definition 6.2. Let* $(M, g)$ *be a real Riemannian manifold of dimension* $4n$, *and let* $(\mathbf{J}^{(1)}, \widehat{\mathbf{J}}^{(1)})$, $(\mathbf{J}^{(2)}, \widehat{\mathbf{J}}^{(2)})$, *be two Dirac structures in it. We say that these are equivalent if* $\mathbf{J}^{(1)}$ *is equivalent to* $\mathbf{J}^{(2)}$ *and* $\widehat{\mathbf{J}}^{(1)}$ *is equivalent to* $\widehat{\mathbf{J}}^{(2)}$.

*Definition 6.3. Let* $(M, g)$ *be a real Riemannian manifold of dimension* $4n$, *equipped with a Dirac structure* $(\mathbf{J}, \widehat{\mathbf{J}})$. *We say that the orthogonal map* $\varphi: M \to M$ *is* Dirac-hyperkahler *(or* Dirac-quaternionic *) if it maps the Dirac structure into an equivalent one. Equivalently, if its pullback* $\varphi^*$ *satisfies* $\varphi^* : \mathcal{S}_\pm \to \mathcal{S}_\pm$.

*Remark 6.4.* We can also define *strongly Dirac-symplectic* maps as those leaving the Dirac structure invariant; that is, those for which $\varphi^*(\omega_\alpha) = \omega_\alpha$, $\varphi^*(\widehat{\omega}_\alpha) = \widehat{\omega}_\alpha$ for $\alpha = 1, 2, 3$. The requirement for a map to be strongly Dirac-symplectic is very restrictive, and in general one should expect these maps, apart from trivial ones, to be quite exceptional. ⊙

*Remark 6.5.* As for Dirac-hyperkahler maps, a large class of them is provided by standard Hamiltonian flows under any of the involved symplectic structures. It should be stressed that if we consider the flow related to say $\omega_1$, this will be strongly hypersymplectic for the hyperkahler structure with reverse orientation (in that the $\widehat{\omega}_\alpha$ are left invariant, as follows from $[Y_\beta, \widehat{Y}_\gamma] = 0$), and hypersymplectic for the hyperkahler structure to which $\omega_1$ belongs.

⊙

### D. Canonical transformations for Dirac structures

Let us now look at canonical transformations. As noted in Remark 6.3 above, the invariant subspaces of T$M$ are just the same for two dual hypersymplectic structures, and are hence attached to the full Dirac structure. Moreover, we have $\omega_\alpha \wedge \omega_\alpha = \widehat{\omega}_\beta \wedge \widehat{\omega}_\beta$ (no sum on $\alpha$ and $\beta$). In other words, canonical transformations will be the same for dual hyperkahler (or hypersymplectic) structures, and these will also be the canonical transformations for the corresponding Dirac structure.

We can then just rephrase our definition of canonical transformation in the present framework, and give a formalization of the above remark.

*Definition 6.4. Let* $(M, g)$ *be a real Riemannian manifold of dimension* $4n$*, equipped with a Dirac structure* $\mathcal{D} = (\mathbf{J}, \widehat{\mathbf{J}})$*; let* $\mathcal{S}_\pm$ *be the corresponding symplectic Kahler spheres. The map* $\varphi: M \to M$ *is said to be canonical for* $\mathcal{D}$ *if, for any* $\omega \in \mathcal{S}_+$ *and* $\widehat{\omega} \in \mathcal{S}_-$*, it preserves the four-forms* $\omega \wedge \omega$ *and* $\widehat{\omega} \wedge \widehat{\omega}$*.*

*Lemma 6.3. Let* $(M, g)$ *be a real Riemannian manifold of dimension* $4n$*, equipped with a Dirac structure* $\mathcal{D} = (\mathbf{J}, \widehat{\mathbf{J}})$*. The set* $\mathtt{Can}(g, \mathbf{J}, \widehat{\mathbf{J}})$ *of canonical transformations for it coincides with the sets of canonical transformations for each of the associated hyperkahler structures,*

$$\mathtt{Can}(g, \mathbf{J}, \widehat{\mathbf{J}}) \ = \ \mathtt{Can}(g, \mathbf{J}) \ = \ \mathtt{Can}(g, \widehat{\mathbf{J}}). \tag{37}$$

*Proof.* The forms $\Omega_+ = (1/2)(\omega \wedge \omega)$ and $\Omega_- = (1/2)(\widehat{\omega} \wedge \widehat{\omega})$ (where $\omega \in \mathcal{S}_+$, $\widehat{\omega} \in \mathcal{S}_-$) built from dual hyperkahler structures are equal up to a sign, $\Omega_- = -\Omega_+$; thus, preservation of one of them implies preservation of the other one as well. ⊙

## VII. CANONICAL PROPERTY OF THE HYPERHAMILTONIAN FLOW

We want now to show that the hyperhamiltonian vector fields, first introduced in Ref. 18 (see also Refs. 20 and 30) provide an unfolding of canonical transformations, pretty much in the same way as Hamiltonian vector fields in the case of symplectic structures. We will first recall the basic definitions of hyperhamiltonian vector fields, and then show they enjoy the canonicity property.

### A. Hyperhamiltonian vector fields

Let $(M, g; \mathbf{J})$ be a hyperkahler manifold. Given a triple of functions on $M$, $\overrightarrow{\mathcal{H}} = \{\mathcal{H}^1, \mathcal{H}^2, \mathcal{H}^3\}$, these identify three hamiltonian vector fields via the standard hamiltonian relation (no sum on $\alpha$)

$$X_\alpha \lrcorner\, \omega_\alpha \ = \ \mathrm{d}\,\mathcal{H}^\alpha. \tag{38}$$

The hyperhamiltonian vector field $X$ associated to the triple $(\mathcal{H}^1, \mathcal{H}^2, \mathcal{H}^3)$ is defined as the sum of the $X_\alpha$'s, i.e.,

$$X \ := \ \sum_\alpha X_\alpha. \tag{39}$$

This was introduced – and several of its properties discussed – in Ref. 18; see also Ref. 19 for a discussion of the integrable case,[21] for some physical applications (to systems with spin), and Ref. 30 for a complex analysis approach.

*Remark 7.1.* Equivalent hypersymplectic structures will not generate the same hyperhamiltonian dynamics for a given triple of Hamiltonians; needless to say, if we operate corresponding rotations in the space $\mathbf{Q}$ *and* in the space of Hamiltonians as well – i.e., consider complex structures $\widetilde{J}_\alpha = R_{\alpha\beta} J_\beta$ and hence symplectic forms $\widetilde{\omega}_\alpha = R_{\alpha\beta}\omega_\beta$, and Hamiltonians $\widetilde{\mathcal{H}}_\alpha = R_{\alpha\beta}\mathcal{H}_\beta$ with the same $R \in SO(3)$ – then we obtain the same dynamics. ⊙

The results we want to prove are of local nature, i.e., we can work on a single chart of the hyperkahler manifold $(M, g; \mathbf{J})$. In the following, we will use local coordinates $x^i$, $i = 1, \ldots, 4n$. With these (and recalling the notation introduced in Sec. II), the hyperhamiltonian vector field $X$

will be written as

$$X \;=\; f^i\,\partial_i\;;\quad \text{where}\;\; f^i \;=\; \sum_\alpha (M_\alpha)^{ij}\,\partial_j \mathcal{H}^\alpha. \tag{40}$$

### B. Hyperhamiltonian flows and canonical transformations

Let us look at the transformations undergone by an arbitrary symplectic form $\omega \in \mathcal{S}$, and by the associated volume form $V_a(\omega) := (1/2)\iota_a^*(\omega\wedge\omega)$ on $U_a$, under a hyperhamiltonian flow.

We will work in local coordinates around the reference point $x_0$ at which the metric and the hypersymplectic structure are in standard form. We freely move the indices $\alpha, \beta, \ldots$ (referring to the hyperkahler triple) up and down for typographical convenience.

#### *1. Lie derivative of the symplectic forms*

In order to know how the $\omega_\alpha$ change under the hyperhamiltonian flow we have to compute the Lie derivative $\mathcal{L}_X(\omega)$.

In our case, $\omega$ is by definition closed (being a symplectic form), hence we have $\mathcal{L}_X\omega = \mathrm{d}(X \lrcorner\, \omega)$.

We will use the shorthand notations (note $D_\alpha = D_\alpha^T$)

$$P^\beta_{\ k} \;:=\; (\partial H^\beta/\partial x^k)\;;\quad D^\alpha_{ij} \;:=\; \frac{\partial^2 H^\alpha}{\partial x^i \partial x^j}. \tag{41}$$

*Lemma 7.1. For X the hyperhamiltonian vector field, it results*

$$\mathcal{L}_X(\omega_\alpha) \;=\; \epsilon_{\alpha\beta\gamma}\,[\partial_i (P^\beta_{\ k} (Y_\gamma)^k_{\ j})]\mathrm{d}x^i\wedge \mathrm{d}x^j. \tag{42}$$

*Proof.* We have

$$X \lrcorner\, \omega_\alpha = (1/2)\,\left(K^\alpha_{ij} f^i \mathrm{d}x^j - K^\alpha_{ij} f^j \mathrm{d}x^i\right).$$

By a rearrangement of indices, and using $K_\alpha^T = -K_\alpha$, this is also rewritten as

$$X \lrcorner\, \omega_\alpha \;=\; f^i\,K^\alpha_{ij}\,\mathrm{d}x^j \;:=\; \lambda^\alpha_{\ j}\,dx^j\;; \tag{43}$$

here, we defined the quantities $\lambda^\alpha_{\ j} = f^i (K^\alpha)_{ij}$ on the r.h.s. as they will appear repeatedly in the following. With this, we get

$$\mathcal{L}_X(\omega_\alpha) \;=\; \mathrm{d}\lambda^\alpha_j \wedge \mathrm{d}x^j \;=\; (\partial_i \lambda^\alpha_j)\,\mathrm{d}x^i\wedge\mathrm{d}x^j. \tag{44}$$

These hold for a generic vector field; now we specialize to the hyperhamiltonian case. With the notation (41), the hyperhamiltonian vector field is given by

$$f^m \;=\; M^{mk}_\beta\,\partial_k H^\beta \;=\; P^\beta_{\ k} (M^T_\beta)^{km} \;=\; -\,P^\beta_{\ k} M^{km}_\beta. \tag{45}$$

A simple computation shows that

$$\begin{aligned}\lambda^\alpha_j &= -P^\beta_{\ k} M^{km}_\beta K^\alpha_{mj} \;=\; -\,P^\beta_{\ k}\,(Y_\beta)^k_{\ m}\,(Y_\alpha)^m_{\ j}\\ &= -\,P^\beta_{\ k}\,\left[\epsilon_{\beta\alpha\gamma}(Y_\gamma)^k_{\ j} \;-\; \delta_{\beta\alpha}\delta^k_{\ j}\right] \;=\; P^\alpha_{\ j} \;+\; \epsilon_{\alpha\beta\gamma}\,P^\beta_{\ k}(Y_\gamma)^k_{\ j}.\end{aligned} \tag{46}$$

Then (42) follows at once recalling that $\mathcal{L}_X(\omega_\alpha) \;=\; \mathrm{d}\lambda^\alpha_j\wedge\mathrm{d}x^j$ and $\partial_i P^\alpha_{\ j} = D^\alpha_{ij}$. Indeed, differentiating (46) we get

$$d\lambda^\alpha_j \;=\; D^\alpha_{ij}\,\mathrm{d}x^i \;+\; \epsilon_{\alpha\beta\gamma}\,[\partial_i (P^\beta_{\ k} (Y_\gamma)^k_{\ j})]\mathrm{d}x^i\;; \tag{47}$$

the first term does not contribute to the final result since $D^\alpha_{ij}\mathrm{d}x^i\wedge\mathrm{d}x^j = 0$ due to $D_\alpha^T = D_\alpha$. △

*Remark 7.2.* For a generic form $\omega = c_\alpha \omega_\alpha \in \mathcal{S}$ (hence with $|c| = \sum c_\alpha^2 = 1$), we obtain easily that

$$\mathcal{L}_X(\omega \wedge \omega) \ = \ c_\eta \cdot c_\eta \, \epsilon_{\alpha\beta\gamma} \, \left[ \partial_i \left( P^\beta_q (Y_\gamma)^q{}_j \right) \right] K^\alpha_{\ell m} \, \mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^\ell \wedge \mathrm{d}x^m .$$

If $\partial_i Y^\gamma = 0$ (which is verified in the Euclidean case) we get simply (no sum on $\alpha$)

$$\mathcal{L}_X(\omega_\alpha \wedge \omega_\alpha) \ = \ \epsilon_{\alpha\beta\gamma} \, K^\alpha_{\ell m} D^\beta_{iq} (Y_\gamma)^q{}_j \, \mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^\ell \wedge \mathrm{d}x^m . \qquad \odot$$

### 2. Lie derivative of the volume forms $\Omega_a$

Now we consider volume forms $\Omega_a$ in the $U_a$. The key observation here is that $\Omega_a = \mathrm{d}x^{4a-3} \wedge \mathrm{d}x^{4a-2} \wedge \mathrm{d}x^{4a-1} \wedge \mathrm{d}x^{4a}$ (no sum on $a$), can be written as

$$\Omega_a \ = \ \sigma_a(\omega) \, \iota_a^* [(1/2) \, (\omega \wedge \omega)] \tag{48}$$

for any unimodular $\omega$, where $\sigma_a(\omega) = 1$ for $\iota_a^* \omega \in \mathcal{S}$ and $\sigma_a(\omega) = -1$ for $\iota_a^* \omega \in \widehat{\mathcal{S}}$ (that is, $\sigma_a(\omega) = \mathcal{P}[\iota_a^*(\omega)]$).

Any symplectic form in four dimensions is written at the reference point $x_0$ as the sum of the standard positively and negatively oriented ones (this just follows from $Y_\alpha$ and $\widehat{Y}_\alpha$ being a basis for the set of all the possible antisymmetric matrices in dimension four); thus, we may set

$$\iota_a^*(\omega) = \sum_\alpha c_\alpha \omega_\alpha + \sum_\alpha \widehat{c}_\alpha \widehat{\omega}_\alpha . \tag{49}$$

It follows from a standard explicit computation that for such $\omega$,

$$\iota_a^*(\omega \wedge \omega) \ = \ \iota_a^* \left[ \sum_\alpha \left[ c_\alpha^2 \, (\omega_\alpha \wedge \omega_\alpha) \ + \ \widehat{c}_\alpha^2 \, (\widehat{\omega}_\alpha \wedge \widehat{\omega}_\alpha) \right] \right],$$

with exactly the same $c_\alpha$ and $\widehat{c}_\alpha$ as above; in fact, it is easy to check that $\omega_\alpha \wedge \omega_\beta = 0$ for $\alpha \neq \beta$, and that $\omega_\alpha \wedge \widehat{\omega}_\beta = 0$ for all $\alpha$ and $\beta$. (Needless to say, for $\omega \in \mathcal{S}$ only the $c_\alpha$ are nonzero, and conversely for $\widehat{\omega} \in \widehat{\mathcal{S}}$.)

We also recall that for symplectic forms (or complex structures) in standard form, all the matrix elements $K_{ij}$ (or $Y^i{}_j$) with $i$ and $j$ not belonging to the same four-dimensional block are zero.

Equation (42) yields (no sum on $\alpha$)

$$\mathcal{L}_X(\omega_\alpha \wedge \omega_\alpha) = \epsilon_{\alpha\beta\gamma} \, K^\alpha_{\ell m} \left( D^\beta_{iq} (Y_\gamma)^q{}_j + P^\beta_q \partial_i (Y_\gamma)^q{}_j \right) \mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^\ell \wedge \mathrm{d}x^m . \tag{50}$$

For a general $\omega = c_\eta \omega_\eta$, this provides

$$\mathcal{L}_X(\omega \wedge \omega) = (c_\eta \cdot c_\eta) \epsilon_{\alpha\beta\gamma} \, K^\alpha_{\ell m} \left( D^\beta_{iq} (Y_\gamma)^q{}_j + P^\beta_q \partial_i (Y_\gamma)^q{}_j \right) \times \tag{51}$$

$$\times \, \mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^\ell \wedge \mathrm{d}x^m .$$

Here, everything can be computed by evaluating matrices at the single reference point $x_0$, except the derivative $\partial_i (Y_\gamma)^q{}_j$. However, this can also be transformed into an algebraic quantity by recalling that $\nabla J_\gamma = 0$. In coordinates, this reads

$$\partial_i (Y_\gamma) \ + \ [A_i , Y_\gamma] \ = \ 0 \, ; \tag{52}$$

here, $A_i$ is the connection matrix, defined by $(A_i)^j{}_k = \Gamma^j_{ik}$, with $\Gamma^j_{ik} = \Gamma^j_{ki}$ the Christoffel symbols for the metric $g$.

Using (52) allows to rewrite (51) as

$$\mathcal{L}_X(\omega \wedge \omega) = (c_\eta \cdot c_\eta) \epsilon_{\alpha\beta\gamma} \left[ K^\alpha_{\ell m} \left( D^\beta_{iq} (Y_\gamma)^q{}_j \ + \ P^\beta_q ((A_i)^q{}_m (Y_\gamma)^m{}_j - (Y_\gamma)^q{}_m (A_i)^m{}_j) \right) \right] \times$$

$$\times \, \mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^\ell \wedge \mathrm{d}x^m .$$

*Remark 7.3.* In all these formulas, the action of $\iota_a^*$ amounts to setting to zero all variables (and its differential) not belonging to the $a$th block. ⊙

We are now ready to complete our computations; we will set their results (respectively, for the case of constant $Y$ and the general case) in the form of a Lemma (for the special case of constant $Y$) and a theorem for the general case. (We also discuss, in Appendix B, an alternative – combinatorial – approach to the proof of our main result in Theorem 7.1.)

*Lemma 7.2. In the case where, for all $\gamma$ and all $i$, $\partial_i Y_\gamma = 0$, any hyperhamiltonian flow preserves $\iota_a^*(\omega \wedge \omega)$ and hence the volume forms $\Omega_a$ on $U_a$.*

*Proof.* In the case $\partial_i Y^\gamma = 0$, formula (51) reduces to

$$\mathcal{L}_X(\omega \wedge \omega) \;=\; (c_\eta \cdot c_\eta)\, \epsilon_{\alpha\beta\gamma}\, K^\alpha_{\ell m} D^\beta_{iq} (Y_\gamma)^q{}_j \, \mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^\ell \wedge \mathrm{d}x^m. \tag{53}$$

It suffices to write down this, or more precisely its pullback under $\iota_a^*$, in explicit terms; for ease of notation we will consider the case $a = 1$, so that only variables $\{x^1, \ldots, x^4\}$ are nonzero (any function should be considered as evaluated with $x^k = 0$ for $k > 4$). We get

$$\begin{aligned}
&(1/2)\, \mathcal{L}_X\, (\omega \wedge \omega) \;= \\
&= \{c_1^2 \, \left[ \left( D^2_{14} + D^2_{23} - D^2_{32} - D^2_{41} \right) + \left( D^3_{13} - D^3_{24} - D^3_{31} + D^3_{42} \right) \right] \\
&+ c_2^2 \, \left[ \left( D^1_{12} - D^1_{21} + D^1_{34} - D^1_{43} \right) + \left( D^3_{13} - D^3_{24} - D^3_{31} + D^3_{42} \right) \right] \\
&+ c_3^2 \, \left[ \left( D^1_{12} - D^1_{21} + D^1_{34} - D^1_{43} \right) + \left( D^2_{14} + D^2_{23} - D^2_{32} - D^2_{41} \right) \right] \} \times \\
&\times \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^4.
\end{aligned}$$

Recalling that $D^\alpha_{ij} = D^\alpha_{ji}$ we conclude that each of the coefficients of the $c_\alpha^2$ vanish separately, hence $\mathcal{L}_X(\omega \wedge \omega) = 0$ as stated. △

**Theorem 7.1.** *Any hyperhamiltonian flow preserves $\iota_a^*(\omega \wedge \omega)$ and hence the volume forms $\Omega_a$ on the $U_a$.*

*Proof.* The variation of $\iota_a^*(\omega \wedge \omega)$ under $X$ is given by $\iota_a^*[\mathcal{L}_X(\omega \wedge \omega)]$. To evaluate this, we make use of (53) and of Remark VII B.3; moreover, for ease of notation, we will focus on $a = 1$. That is, we should compute (53) with all $i, k, \ell, m$ indices restricted to the range $1, \ldots, 4$.

We note that according to (53), $\mathcal{L}_X(\omega \wedge \omega)$, and therefore $\iota_a^*[\mathcal{L}_X(\omega \wedge \omega)]$ as well, is the sum of two terms; these correspond, respectively, to $K^\alpha_{\ell m} D^\beta_{iq} (Y_\gamma)^q{}_j$ and to $K^\alpha_{\ell m} P^\beta_q [A_i, Y_\gamma]^q{}_j$. The first term is exactly the one which was already evaluated in the case $\partial_i Y_\gamma = 0$; it vanishes as stated by Lemma 7.2 (and shown in its proof).

We therefore have only to show that

$$\begin{aligned}
&(c_\alpha/2) \epsilon_{\alpha\beta\gamma} \iota_a^* [K^\alpha_{\ell m} P^\beta_q \left( (A_i)^q{}_p (Y_\gamma)^p{}_j - (Y_\gamma)^q{}_p (A_i)^p{}_j \right) \; \times \\
&\mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^\ell \wedge \mathrm{d}x^m] \qquad\qquad (54)\\
&:= (c_\alpha/2)\, \epsilon_{\alpha\beta\gamma}\, \Theta^{(a)}_{\alpha\beta\gamma}
\end{aligned}$$

vanishes; here, we have of course defined

$$\Theta^{(a)}_{\alpha\beta\gamma} \;=\; \iota_a^* \left[ K^\alpha_{\ell m} P^\beta_q \left( (A_i)^q{}_p (Y_\gamma)^p{}_j - (Y_\gamma)^q{}_p (A_i)^p{}_j \right) \mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^\ell \wedge \mathrm{d}x^m \right].$$

Note that here the coefficients $c_\alpha$ (satisfying $|c|^2 = 1$) and the vectors $P^\beta$ are completely arbitrary; thus, the r.h.s. of (54) should vanish for any choice of these. In other words, we should have $\Theta^{(a)}_{\alpha\beta\gamma} = 0$ for all choices of the indices $\alpha, \beta, \gamma$, provided these are all different, $\alpha \neq \beta \neq \gamma \neq \alpha$.

The expression for $\Theta$ only involves quantities computed at the reference point $x_0$, and we can hence make use of the explicit expressions for the standard form of the $Y_\alpha$ and the $K_\alpha$.

Using these, choosing $a = 1$ (and omitting the index $a$), and the case of positive orientation in the first block, we get, e.g.,

$$\begin{aligned}\Theta_{123} = -2\,\{&[(\Gamma^1_{14} + \Gamma^1_{23} - \Gamma^1_{32} - \Gamma^1_{41}) - (\Gamma^3_{12} - \Gamma^3_{21} + \Gamma^3_{34} - \Gamma^3_{43})]\; P^2_1 \\ &+[(\Gamma^2_{14} + \Gamma^2_{23} - \Gamma^2_{32} - \Gamma^2_{41}) + (\Gamma^4_{12} - \Gamma^4_{21} + \Gamma^4_{34} - \Gamma^4_{43})]\; P^2_2 \\ &+[(\Gamma^1_{12} - \Gamma^1_{21} + \Gamma^1_{34} - \Gamma^1_{43}) + (\Gamma^3_{14} + \Gamma^3_{23} - \Gamma^3_{32} - \Gamma^3_{41})]\; P^2_3 \\ &+[(\Gamma^2_{21} - \Gamma^2_{12} + \Gamma^2_{43} - \Gamma^2_{34}) + (\Gamma^4_{14} + \Gamma^4_{23} - \Gamma^4_{32} - \Gamma^4_{41})]\; P^2_4\} \;\times \\ &\times\; \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^4\;;\end{aligned}$$

recalling the property $\Gamma^i_{jk} = \Gamma^i_{kj}$, valid for any torsion-free Riemannian metric $g$, it is immediately seen that $\Theta_{123}$ vanishes. The same holds for all forms $\Theta_{\alpha\beta\gamma}$ with $\alpha \neq \beta \neq \gamma \neq \alpha$ (explicit formulas are omitted for the sake of brevity). This concludes the proof for positive orientation. As usual, computations are the same up to certain signs for negative orientation as well, and lead to the same result. △

*Remark 7.4.* The result of the theorem above can be restated as follows: *Any hyperhamiltonian flow corresponds to a one-parameter family of canonical transformations for the underlying hyperkahler structure.* ⊙

*Remark 7.5.* We have shown that the hyperhamiltonian flow is canonical for the underlying hyperkahler structure (i.e., the one defining it through (39)); it turns out it is also canonical for the dual one. Indeed, the structures $\omega_\alpha$ and $\widehat{\omega}_\alpha$ define the same invariant subspaces in $\mathrm{T}_xM$ and define on these volume forms which only differ by a sign, $\Omega_a = -\widehat{\Omega}_a$; it is thus a triviality that preservation of the volume forms $\Omega_a$ for the defining structure entails preservation of the $\Omega_a$ for the dual one. In other words, $\mathcal{L}_X(\omega \wedge \omega) = 0$ implies $\mathcal{L}_X(\widehat{\omega} \wedge \widehat{\omega}) = 0$; this follows at once from $(\widehat{\omega} \wedge \widehat{\omega}) = -(\omega \wedge \omega)$. ⊙

## VIII. EXAMPLES: FOUR-DIMENSIONAL EUCLIDEAN SPACE

We will discuss in detail hyperkahler and canonical maps for flat hyperkahler structures in a companion paper;[23] in this section we will just discuss the simplest case of Euclidean space, $M = \mathbf{R}^4$ with Euclidean metric $g(x) = \delta$.

### A. Hyperkahler maps

We have $(M, g) = (\mathbf{R}^4, \delta)$ and either one of the standard hyperkahler structures (see Sec. II), to which we can always reduce; we will for short just focus on the $Y_\alpha$, the situation being completely analogous for the $\widehat{Y}_\alpha$.

To preserve the metric, we are bound to consider orthogonal transformations, i.e., $O(4)$. Moreover, we have to preserve orientation, which ensures $\mathtt{hSp}(4) \subseteq SO(4)$.

The six generators of the Lie algebra $so(4) \simeq su(2) \oplus su(2)$ can be chosen to be exactly $\{Y_\alpha; \widehat{Y}_\alpha\}$. It is immediate to check that the $\widehat{Y}_\alpha$ (each of them commutes with all of the $Y_\alpha$) generate strongly hyperkahler transformations, while the $Y_\alpha$ themselves generate (non-strongly) hyperkahler ones.

In a somewhat more detailed way, let us write a generic complex structure $J \in \mathbf{S}$ as $J = \sum_\alpha k_\alpha Y_\alpha$, where $k_\alpha$ are real constants and $|k|^2 := \sum_\alpha k_\alpha^2 = 1$. A generic element $\lambda$ of the algebra $so(4)$ will be written as $\lambda = p_\alpha Y_\alpha + q_\alpha \widehat{Y}_\alpha$, where $p_\alpha, q_\alpha \in \mathbf{R}$. The infinitesimal action of $\lambda$ on $J$ is given by

$$\begin{aligned} J \;\to\; J' &= J + \varepsilon\,[\lambda, J] = J + \varepsilon\left(p_\alpha [Y_\alpha, J] + q_\alpha[\widehat{Y}_\alpha, J]\right) \\ &= J + \varepsilon\,(p_\alpha[Y_\alpha, J]) = k_\beta\, Y_\beta + \varepsilon\; p_\alpha k_\beta\,[Y_\alpha, Y_\beta] \\ &= k_\beta\, Y_\beta + 2\varepsilon\; \epsilon_{\alpha\beta\gamma}\, p_\alpha k_\beta Y_\gamma = (k_\gamma + 2\varepsilon\epsilon_{\alpha\beta\gamma}\, p_\alpha k_\beta)\; Y_\gamma \\ &:= z_\gamma\; Y_\gamma . \end{aligned}$$

It is obvious that $J'$ is in the linear span of $(Y_1, Y_2, Y_3)$; to check we are indeed on the unit sphere, it suffices to recall we have to consider orthogonal transformations. We can also compute explicitly (at first order in $\varepsilon$)

$$|z|^2 \ = \ z_\alpha \, z_\alpha \ = \ k_\alpha k_\alpha + 2\varepsilon k_\alpha \, \epsilon_{\beta\gamma\alpha} \, p_\beta k_\gamma + O(\varepsilon^2) \ = \ k_\alpha k_\alpha \ + \ O(\varepsilon^2).$$

In conclusion, as stated above, $\mathtt{hSp}(4) \simeq SO(4)$; more precisely, all maps in the group $SO(4) \simeq SU(2) \times SU(2)$ generated by the $\{Y_\alpha, \widehat{Y}_\alpha\}$ are hyperkahler, and those in the $SU(2)$ factor generated by the $\widehat{Y}_\alpha$ are strongly hyperkahler.

In arbitrary $4n$ dimension, the invariance group will be still the direct product of two groups corresponding to hyperkahler and strong hyperkahler transformations. In accordance with general results on manifolds with special holonomy,[8] the invariance group will be $Sp(1) \times Sp(n)$, which reduces for $n = 1$ (four-dimensional case) to $Sp(1) \times Sp(1)$ which is isomorphic to the group we have obtained here.

### B. Hyperhamiltonian flows and canonical transformations

Let us now consider $(\mathbf{R}^4, \delta)$ with standard hyperkahler structure (with positive orientation) from the point of view of canonical maps. The volume form is just

$$\Omega \ = \ \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^4.$$

We consider an arbitrary $\omega \in \mathcal{S}$, i.e., $\omega = c_\alpha \omega_\alpha$ with $|c|^2 = c_1^2 + c_2^2 + c_3^2 = 1$; for this we have $(1/2)(\omega \wedge \omega) = \Omega$. For a vector field $X = f^i \partial_i$ it follows from standard computations (using also $|c|^2 = 1$) that

$$\begin{aligned}(X \lrcorner \, \omega) \wedge \omega = {} & f^1 \, \mathrm{d}x^2 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^4 \ - \ f^2 \mathrm{d}x^1 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^4 \\ & + f^3 \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^4 \ - \ f^4 \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3. \end{aligned} \tag{55}$$

Specifying now that $X$ is the hyperhamiltonian vector field corresponding to hamiltonians $\{\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3\}$, see Sec. VII A, we get

$$\begin{aligned}(X \lrcorner \, \omega) \wedge \omega = {} & (\partial_2 \mathcal{H}_1 + \partial_4 \mathcal{H}_2 + \partial_3 \mathcal{H}_3) \, \mathrm{d}x^2 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^4 \\ & + (\partial_1 \mathcal{H}_1 - \partial_3 \mathcal{H}_2 + \partial_4 \mathcal{H}_3) \, \mathrm{d}x^1 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^4 \\ & + (\partial_4 \mathcal{H}_1 - \partial_2 \mathcal{H}_2 - \partial_1 \mathcal{H}_3) \, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^4 \\ & + (\partial_3 \mathcal{H}_1 + \partial_1 \mathcal{H}_2 - \partial_2 \mathcal{H}_3) \, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3. \end{aligned} \tag{56}$$

It follows from this that

$$\mathcal{L}_X(\omega \wedge \omega) \ = \ \mathrm{d}[(X \lrcorner \, \omega) \wedge \omega] \ = \ 0. \tag{57}$$

In other words, we have shown by explicit computation that $\Omega = (1/2)(\omega \wedge \omega)$ is preserved under *any* hyperhamiltonian flow. (Actually our computation showed this only for positively oriented hypersymplectic structures; the computation goes the same way for negatively oriented ones.)

Let us go back to considering $\mathcal{L}_X(\omega)$; using the explicit expression for the hyperhamiltonian vector field, it turns out by a direct computation that this can be written as

$$\mathcal{L}_X(\omega) \ = \ \frac{1}{2} \ (p_\alpha \, \omega_\alpha \ + \ q_\alpha \, \widehat{\omega}_\alpha) \tag{58}$$

with coefficients $p_\alpha$, $q_\alpha$ given by (here $\triangle$ is the Laplacian)

$$p_1 = c_2 \triangle \mathcal{H}_3 - c_3 \triangle \mathcal{H}_2, \;\; p_2 = c_3 \triangle \mathcal{H}_1 - c_1 \triangle \mathcal{H}_3, \;\; p_3 = c_1 \triangle \mathcal{H}_2 - c_2 \triangle \mathcal{H}_1;$$

$$\begin{aligned} q_1 = c_1 & \left[ \left( \partial_1^2 \mathcal{H}_2 - \partial_2^2 \mathcal{H}_2 + \partial_3^2 \mathcal{H}_2 - \partial_4^2 \mathcal{H}_2 \right) - 2 \left( \partial_1 \partial_2 \mathcal{H}_3 + \partial_3 \partial_4 \mathcal{H}_3 \right) \right] \\ & - c_2 \left[ \left( \partial_1^2 \mathcal{H}_1 - \partial_2^2 \mathcal{H}_1 + \partial_3^2 \mathcal{H}_1 - \partial_4^2 \mathcal{H}_1 \right) - 2 \left( \partial_1 \partial_4 \mathcal{H}_3 - \partial_2 \partial_3 \mathcal{H}_3 \right) \right] \\ & + 2\, c_3 \left[ \left( \partial_1 \partial_2 \mathcal{H}_1 + \partial_3 \partial_4 \mathcal{H}_1 \right) + \left( \partial_1 \partial_4 \mathcal{H}_2 - \partial_2 \partial_3 \mathcal{H}_2 \right) \right], \\ q_2 = c_1 & \left[ \left( \partial_1^2 \mathcal{H}_3 - \partial_2^2 \mathcal{H}_3 - \partial_3^2 \mathcal{H}_3 + \partial_4^2 \mathcal{H}_3 \right) + 2 \left( \partial_1 \partial_2 \mathcal{H}_2 - \partial_3 \partial_4 \mathcal{H}_2 \right) \right] \\ & - c_3 \left[ \left( \partial_1^2 \mathcal{H}_1 - \partial_2^2 \mathcal{H}_1 - \partial_3^2 \mathcal{H}_1 + \partial_4^2 \mathcal{H}_1 \right) - 2 \left( \partial_1 \partial_3 \mathcal{H}_2 + \partial_2 \partial_4 \mathcal{H}_2 \right) \right] \\ & - 2\, c_2 \left[ \left( \partial_1 \partial_2 \mathcal{H}_1 - \partial_3 \partial_4 \mathcal{H}_1 \right) + \left( \partial_1 \partial_3 \mathcal{H}_3 + \partial_2 \partial_4 \mathcal{H}_3 \right) \right], \\ q_3 = c_2 & \left[ \left( -\partial_1^2 \mathcal{H}_3 - \partial_2^2 \mathcal{H}_3 + \partial_3^2 \mathcal{H}_3 + \partial_4^2 \mathcal{H}_3 \right) + 2 \left( \partial_1 \partial_4 \mathcal{H}_1 + \partial_2 \partial_3 \mathcal{H}_1 \right) \right] \\ & + c_3 \left[ \left( \partial_1^2 \mathcal{H}_2 + \partial_2^2 \mathcal{H}_2 - \partial_3^2 \mathcal{H}_2 - \partial_4^2 \mathcal{H}_2 \right) + 2 \left( \partial_1 \partial_3 \mathcal{H}_1 - \partial_2 \partial_4 \mathcal{H}_1 \right) \right] \\ & - 2\, c_1 \left[ \left( \partial_1 \partial_4 \mathcal{H}_2 + \partial_2 \partial_3 \mathcal{H}_2 \right) - \left( \partial_1 \partial_3 \mathcal{H}_3 - \partial_2 \partial_4 \mathcal{H}_3 \right) \right]. \end{aligned}$$

The essential point here is that – as these explicit formulas show – the Lie derivative $\mathcal{L}_X(\omega)$ of a symplectic form $\omega \in \mathcal{S} \subset \mathcal{Q}$ has components along $\hat{\mathcal{Q}}$, i.e., the negatively oriented forms.

This shows that in general the hyperhamiltonian flow, even in this simple case, is canonical but *not* hyperkahler; see also Remark 5.2.

An exception is provided by the choice $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}_3 = (1/2)(x_1^2 + x_2^2 + x_3^2 + x_4^2)$, corresponding to the "quaternionic oscillator" (which is an integrable case[17,19]). With this, we get $p_1 = 4(c_2 - c_3)$, $p_2 = 4(c_3 - c_1)$, $p_3 = 4(c_1 - c_2)$; $q_1 = q_2 = q_3 = 0$.

It is maybe worth pointing out also what happens when only one of the Hamiltonians, say $\mathcal{H}_1$, is nonzero; this corresponds to a standard Hamiltonian flow. Setting $\mathcal{H}_1 = H$, $\mathcal{H}_2 = \mathcal{H}_3 = 0$ in the general formulas above, we get

$$\begin{aligned} & p_1 = 0, \;\; p_2 = c_3 \triangle H, \;\; p_3 = -c_2 \triangle H \,; \\ & q_1 = -\, c_2 \left( \partial_1^2 H - \partial_2^2 H + \partial_3^2 H - \partial_4^2 H \right) + 2\, c_3 \left( \partial_1 \partial_2 H + \partial_3 \partial_4 H \right), \\ & q_2 = -\, c_3 \left( \partial_1^2 H - \partial_2^2 H - \partial_3^2 H + \partial_4^2 H \right) - 2\, c_2 \left( \partial_1 \partial_2 H \partial_3 \partial_4 H \right), \\ & q_3 = 2\, c_2 \left( \partial_1 \partial_4 H + \partial_2 \partial_3 H \right) + 2\, c_3 \left( \partial_1 \partial_3 H - \partial_2 \partial_4 H \right). \end{aligned}$$

Then, this Hamiltonian flow is in general *not* hyperkahler; the special choice $H = H(x_1^2 + x_2^2 + x_3^2 + x_4^2)$ will of course produce $q_\alpha = 0$ and hence gives an hyperkahler flow.

**Final comments.** As pointed out by the referee, who we thank for this remark, the topic discussed in this paper could also be approached in terms of su(2)-valued tensors, and, then, related to the work of Avramidi and Collopy[7] in the context of problems related to the stability of non-Abelian chromomagnetic vacuum of Yang-Mills theory in an Euclidean Einstein universe $S^1 \times S^3$. Similar remarks were also made by P. Morando after the completion of this work.

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## APPENDIX A: THE OPERATOR $\mathcal{P}_k(\omega)$

Let $\omega$ be a non-degenerate two-form on the $4n$-dimensional orientable manifold $M$; and let $\Omega$ be the volume form on $M$. We associate to $\omega$ its $2n$th external power, which we denote by (from now on $m = 2n$) $\Delta_m(\omega) \in \Lambda^{2m}(M)$

$$\Delta_m(\omega) \ = \ \omega \wedge \ldots \wedge \omega \, ; \tag{A1}$$

being a form of maximal rank on $M$, this is necessarily proportional to the volume form,

$$\Delta_m(\omega) \ = \ p_m(\omega)\, \Omega . \tag{A2}$$

Obviously, the scalar function $p_m(\omega)$: $M \to \mathbf{R}$ is homogeneous of degree $m$, i.e., $p_m(k\omega) = k^m p_m(\omega)$. Thus, it suffices to study $\Delta_m(\omega)$ on the unit sphere $\mathcal{Q} \in \mathbf{Q}$.

We also notice that $\Delta_m(\omega)$ is defined point-wise on $M$; as discussed in Sec. II D, we can always transform any hypersymplectic structure to a standard one at any given point: it is enough to consider $p_m(\omega)$ for a standard quaternionic symplectic structure, i.e., a block reducible one, spanned by the $\{\omega_\alpha\}$, or the $\{\widehat{\omega}_\alpha\}$, on each fundamental block.

We can write $\omega$ in coordinates as

$$\omega \ = \ K_{ij}(x)\, \mathrm{d}x^i \wedge \mathrm{d}x^j \tag{A3}$$

(we will just write $K$ for $K(x)$ in the following); the matrix $K$ is antisymmetric and of maximal rank. We can then write the $m$th external power of $\omega$ as

$$\Delta_m(\omega) = (1/m!)\epsilon_{i_1 j_1 \ldots i_m j_m} K_{i_1 j_1} \ \ldots \ K_{i_m j_m} \ \Omega \ := \ p_m(\omega)\, \Omega . \tag{A4}$$

We will focus on the scalar function $p_m(\omega)$, and look at it in terms of a function defined on the (antisymmetric) matrices $K$ corresponding to $\omega$,

$$\mathcal{P}_m(K) \ := \ \sum_{i_s, j_s = 1}^{4n} \epsilon_{i_1 j_1 \ldots i_m j_m} \ K_{i_1 j_1} \ \ldots \ K_{i_m j_m} . \tag{A5}$$

This is the function considered in Secs. II and V. The square of $\mathcal{P}_m(K)$ is given by

$$[\mathcal{P}_m(K)]^2 \ = \ \epsilon_{i_1 j_1 \ldots i_m j_m}\, \epsilon_{a_1 b_1 \ldots a_m b_m} \ K_{i_1 j_1} \ \ldots \ K_{i_m j_m} \ K_{a_1 b_1} \ \ldots \ K_{a_m b_m} . \tag{A6}$$

We can rewrite $\epsilon_{i_1 j_1 \ldots i_m j_m} = (-1)^{m/2} \epsilon_{i_1 \ldots i_m j_1 \ldots j_m}$, and the like for $\epsilon_{a_1 b_1 \ldots a_m b_m}$

$$[\mathcal{P}_m(K)]^2 \ = \ \epsilon_{i_1 \ldots i_m j_1 \ldots j_m}\, \epsilon_{a_1 \ldots a_m b_1 \ldots b_m} \ K_{i_1 j_1} \ \ldots \ K_{i_m j_m} \ K_{a_1 b_1} \ \ldots \ K_{a_m b_m} . \tag{A7}$$

Note that each of the $\epsilon$ symbols depends on $2m$ indices; hence, all the $4n$ coordinates must appear in it. We can then always operate a permutation in one of them, say the first one, so that the coordinate indices appear in consecutive order; this will give a $\pm 1$ sign for the permutation. If we operate the same permutation also on the indices of the second $\epsilon$ tensor (thus getting an equal sign which in any case cancels the one obtained from the previous permutation) we are reduced to an expression of the type

$$[\mathcal{P}_m(K)]^2 \ = \ \epsilon_{c_1 \ldots c_{2m}} \ K_{1 c_1} \ \ldots \ K_{2m, c_{2m}} . \tag{A8}$$

This is immediately recognized as the determinant of $K$. We have thus shown that

$$[\mathcal{P}_m(K)]^2 \ = \ \mathtt{Det}(K) \, ; \quad \mathcal{P}_m(K) \ = \ \pm \sqrt{\mathtt{Det}(K)} . \tag{A9}$$

It follows at once from this that for the product of two matrices we have

$$\mathcal{P}_m(AB) \ = \ [\pm \sqrt{\mathtt{Det}(A)}]\, [\pm \sqrt{\mathtt{Det}(B)}] \ = \ \pm \sqrt{\mathtt{Det}(AB)} . \tag{A10}$$

When we consider $\widehat{K} = A^T K A$ we thus have

$$\mathcal{P}_m(A^T K A) \ = \ \mathcal{P}_m(K)\, \mathtt{Det}(A) . \tag{A11}$$

Similar considerations, up to combinatorial factors, also hold for $\Delta_k(\omega)$ with $k < m$, and for projections of these to $2k$-dimensional submanifolds; in particular, to the invariant four-dimensional subspaces $U_a$.

## APPENDIX B: ALTERNATIVE PROOF OF THEOREM 7.1

The key step to our proof of Theorem 7.1 was to show that $\Theta_{\alpha\beta\gamma} = 0$. The vanishing of the $\Theta_{\alpha\beta\gamma}$ depends of course not only on the symmetry of Christoffel symbols but also on the combinatorial properties of the $K_\alpha$ and $Y_\alpha$. In this appendix, we discuss briefly how these lead to the vanishing of the $\Theta_{\alpha\beta\gamma}$.

Let us look separately at the two kinds of terms in (54). As for those of the form $K^\alpha_{\ell m}(Y_\gamma)^q{}_p(A_i)^p{}_j(\mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^\ell \wedge \mathrm{d}x^m)$, it follows immediately from the symmetry of $(A_i)^p{}_j$, and the antisymmetry of $\mathrm{d}x^i \wedge \mathrm{d}x^j$, that under the exchange of $i$ and $j$ these change sign, and hence their sum vanishes.

The other type of terms, i.e., those of the type $K^\alpha_{\ell m}(A_i)^q{}_p(Y_\gamma)^p{}_j(\mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^\ell \wedge \mathrm{d}x^m)$, require a slightly more careful discussion. Only two pair of indices $(\ell, m)$ produce nonzero results for the corresponding element $K^\alpha_{\ell m}$.

Once we have fixed $\alpha$ and $\gamma$, e.g., $\alpha = 1$ and $\gamma = 3$, for each element $K^1_{\ell m}$ only elements ($q$, $j$) of $Y_3$ with $j$ different from both $\ell$ and $m$ will contribute to (54). For example, consider (for $\alpha = 1, \gamma = 3$) the choice $\ell = 1, m = 2$; now only the elements $(Y_3)^1{}_3$ and $(Y_3)^2{}_4$ satisfy the requirement $(Y_3)^q{}_j \neq 0$ for $j \neq 1, 2$. Thus, when we remember that now it should also be $i \neq \ell, m, j$, the only terms actually contributing to products of the form $K_\alpha A_i Y_\gamma$ will be

$$K^1_{12}[(A_3)^q{}_2(Y_3)^2{}_4 - (A_4)^q{}_1(Y_3)^1{}_3]\ \Omega_{(1)}. \tag{B1}$$

Exchanging the indices $\ell$ and $m$ will give just the same result. On the other hand, also terms with $\ell = 3$ and $m = 4$ will give a nonzero $K^1_{\ell m}$; proceeding as above, this will give terms of the type

$$K^1_{34}[(A_1)^q{}_4(Y_3)^4{}_2 - (A_2)^q{}_3(Y_3)^3{}_1]\ \Omega_{(1)}. \tag{B2}$$

Here, again exchanging $\ell$ and $m$ will give the same result.

If now we sum (B1) and (B2), use $Y_\alpha = K_\alpha$ at the reference point, and collect terms using $(A_i)^j{}_k = \Gamma^j_{ik} = \Gamma^j_{ki}$, we obtain

$$\Gamma^q_{14}\ \left(K^1_{34}K^3_{42} - K^1_{12}K^3_{13}\right)\ +\ \Gamma^q_{23}\ \left(K^1_{12}K^3_{24} - K^1_{34}K^3_{31}\right)\ \Omega_{(1)}. \tag{B3}$$

Now we observe that, as the $\omega_\alpha$ have the same orientation, necessarily $K^1_{12}/K^1_{34} = K^3_{24}/K^3_{31}$, hence the form (B3) vanishes. The same discussion can be repeated for other choices of $\alpha$ and $\gamma$, and for negative orientation.


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