POLYNOMIAL INEQUALITIES ON THE $\pi/4$–CIRCLE SECTOR


Abstract. A number of sharp inequalities are proved for the space $\mathcal{P}(\ell^2(D(\pi/4)))$ of 2-homogeneous polynomials on $\mathbb{R}^2$ endowed with the supremum norm on the sector $D(\pi/4) := \{e^{i\theta} : \theta \in [0, \pi/4]\}$. Among the main results we can find sharp Bernstein and Markov inequalities and the calculation of the polarization constant and the unconditional constant of the canonical basis of the space $\mathcal{P}(\ell^2(D(\pi/4)))$.

1. Preliminaries

The study of low dimensional spaces of polynomials can be an interesting source of examples and counterexamples related to more general questions. In this paper we mind 2-variable, real 2-homogeneous polynomials endowed with the supremum norm on the sector $D(\pi/4)$. In this paper we mind 2-variable, real 2-homogeneous polynomials endowed with the supremum norm on $D(\pi/4)$. Among the main results we can find sharp Bernstein and Markov inequalities and the calculation of the polarization constant and the unconditional constant of the canonical basis of the space $\mathcal{P}(\ell^2(D(\pi/4)))$.

Other publications in the same spirit can be found in [11, 12, 21, 22, 24, 25].

In Section 3, the notation $L^*(\ell^2(D(\pi/4)))$ will be useful to represent the symmetric bilinear forms on $\mathbb{R}^2$ endowed with the supremum norm on $D(\pi/4)$.

In order to obtain sharp polynomial inequalities in $\mathcal{P}(\ell^2(D(\pi/4)))$ we will use the so called Krein-Milman approach, which is based on the fact that norm attaining convex functions attain their norm at an extreme point of their domain. Hence, an explicit description of the norm $\|\cdot\|_{D(\pi/4)}$ and the extreme points of the unit ball $B_{D(\pi/4)}$, denoted by $\text{ext}\left(B_{D(\pi/4)}\right)$, will be required. Both are presented below:

Lemma 1.1. [20] Theorem 3.1 If $P(x,y) = ax^2 + by^2 + cxy$, then

$$
\|P\|_{D(\pi/4)} = \max \left\{ |a|, \frac{1}{2}|a + b + c|, \frac{1}{2}a + b + \text{sign}(c)\sqrt{(a-b)^2 + c^2} \right\}
$$

if $c(a-b) \geq 0$, and

$$
\max \left\{ |a|, \frac{1}{2}|a + b + c| \right\}
$$

if $c(a-b) \leq 0$.

Lemma 1.2. [20] Theorem 4.4 The extreme points of the unit ball of $\mathcal{P}(\ell^2(D(\pi/4)))$ are given by

$$
\text{ext}\left(B_{D(\pi/4)}\right) = \left\{ \pm P_t, \pm Q_s, \pm (1,1,0) : -1 \leq t \leq 1 \text{ and } 1 \leq s \leq 5 + 4\sqrt{2} \right\},
$$

where

$$
P_t : = \left(t, 4 + t + 4\sqrt{1 + t}, -2 - 2t - 4\sqrt{1 + t} \right),
$$

$$
Q_s : = \left(1, s, -2\sqrt{2(1 + s)} \right).
$$

2010 Mathematics Subject Classification. Primary 46G25; Secondary 46B28, 41A44.

Key words and phrases. Bernstein and Markov inequalities, unconditional constants, polarizations constants, polynomial inequalities, homogeneous polynomials, extreme points.

*Supported by PDSE/CAPES 8015/14-7.

**The second named author’s research was performed during his stay at the Mathematics Department of Kent State University, USA.

***Supported by the Spanish Ministry of Science and Innovation, grant MTM2012-34341.
Let us describe now the three inequalities that will be studied in this paper. Section 2 is devoted to obtain a Bernstein type inequality for polynomials in $\mathcal{P}(\mathbb{R}_+^2)$. Namely, for a fixed $(x, y) \in D\left(\frac{\pi}{4}\right)$, we find the best (smallest) constant $\Phi(x, y)$ in the inequality

$$\|\nabla P(x, y)\|_2 \leq \Phi(x, y) \|P\|_{D\left(\frac{\pi}{4}\right)},$$

for all $P \in \mathcal{P}(\mathbb{R}_+^2)$, where $\|\cdot\|_2$ denotes the euclidean norm in $\mathbb{R}^2$. Similarly, we also obtain a Markov type inequalities has a longstanding tradition. The interested reader can find further information on this classical topic in [2, 13, 14, 16, 18, 19, 23, 26, 28, 29, 30, 31].

In Section 3 we find the smallest constant $K > 0$ in the inequality

$$\|L\|_{D\left(\frac{\pi}{4}\right)} \leq K\|P\|_{D\left(\frac{\pi}{4}\right)},$$

where $P$ is an arbitrary polynomial in $\mathcal{P}(\mathbb{R}_+^2)$ and $L \in \mathcal{L}(\mathbb{R}_+^2)$ is the polar of $P$. Observe that here $\|L\|_{D\left(\frac{\pi}{4}\right)}$ stands for the sup norm of $L$ over $D\left(\frac{\pi}{4}\right)^2$. Hence, what we do is to provide the polarization constant of the space $\mathcal{P}(\mathbb{R}_+^2)$. The calculation of polarization constants in various polynomial spaces is largely motivated as the extensive, existing bibliography on the topic shows (see for instance [10, 13, 17, 27]).

Finally, in Section 4 we investigate the smallest constant $C > 0$ in the inequality

$$(1.1) \quad \|P\|_{D\left(\frac{\pi}{4}\right)} \leq C\|P\|_{D\left(\frac{\pi}{4}\right)},$$

for all $P \in \mathcal{P}(\mathbb{R}_+^2)$, where $|P|$ is the modulus of $P$, i.e., if $P(x, y) = ax^2 + by^2 + cxy$, then $|P|(x, y) = |a|x^2 + |b|y^2 + |c|xy$. The constant $C$ turns out to be the unconditional constant of the canonical basis of $\mathcal{P}(\mathbb{R}_+^2)$. It is interesting to note that already in 1914, H. Bohr [4] studied this type of inequalities for infinite complex power series. Actually, the study of Bohr radii is nowadays a fruitful field (see for instance [1, 8, 11, 15, 16, 17]). Observe that the relationship between unconditional constants in polynomial spaces and inequalities of the type (1.1) was already noticed in [7].

2. Bernstein and Markov-type inequalities for polynomials on sectors

In this section we provide sharp estimates on the Euclidean length of the gradient $\nabla P$ of a polynomial $P$ in $\mathcal{P}(\mathbb{R}_+^2)$.

**Theorem 2.1.** For every $(x, y) \in D\left(\frac{\pi}{4}\right)$ and $P \in \mathcal{P}(\mathbb{R}_+^2)$ we have

$$\|\nabla P\|_2 \leq \Phi(x, y) \|P\|_{D\left(\frac{\pi}{4}\right)},$$

where

$$\Phi(x, y) = \begin{cases} 4 \left(13 + 8\sqrt{2}\right) x^2 + \left(69 + 48\sqrt{2}\right) y^2 - 2 \left(28 + 20\sqrt{2}\right) xy & \text{if } 0 \leq y \leq \frac{\sqrt{2}-1}{2} x \text{ or } (4\sqrt{2} - 5) x \leq y \leq x, \\ \frac{x^2 + 4(x^2 + y^2)}{2(x^2 - y^2)} & \text{if } \frac{\sqrt{2}-1}{2} x \leq y \leq (\sqrt{2} - 1) x, \\ \frac{y^2}{x^2} & \text{if } (\sqrt{2} - 1) x \leq y \leq \left(4\sqrt{2} - 5\right) x. \end{cases}$$

**Proof.** In order to calculate $\Phi(x, y) := \sup\{\|\nabla P(x, y)\|_2 : \|P\|_{D\left(\frac{\pi}{4}\right)} \leq 1\}$, by the Krein-Milman approach, it is sufficient to calculate

$$\sup\{\|\nabla P(x, y)\|_2 : P \in \text{ext}(B_{D\left(\frac{\pi}{4}\right)})\}.$$

By symmetry, we may just study the polynomials of Lemma 1.2 with positive sign. Let us start first with $P_t(x, y) = tx^2 + (1 + t + 4\sqrt{1 + t}) y^2 - 2 \left(1 + t + 2\sqrt{1 + t}\right) xy$, $t \in [-1, 1]$. Then,

$$\nabla P_t(x, y) = (2tx - 2 \left(1 + t + 2\sqrt{1 + t}\right) y, 2 \left(4 + t + 4\sqrt{1 + t}\right) y - 2 \left(1 + t + 2\sqrt{1 + t}\right)x),$$

for all $x, y \geq 0$. More generally, the following inequality holds:

$$\|\nabla P(x, y)\|_2 \leq \Phi(x, y) \|P\|_{D\left(\frac{\pi}{4}\right)}.$$
so that
\[
\| \nabla P_t(x, y) \|_2^2 = 4t^2x^2 + 4\left(1 + t + 2\sqrt{1+t}\right)^2 y^2 - 8t \left(1 + t + 2\sqrt{1+t}\right) xy
\]
\[
+ 4 \left(4 + t + 4\sqrt{1+t}\right)^2 y^2 + 4 \left(1 + t + 2\sqrt{1+t}\right)^2 x^2
\]
\[
- 8 \left(4 + t + 4\sqrt{1+t}\right) \left(1 + t + 2\sqrt{1+t}\right) xy
\]

Make now the change \( u = \sqrt{1+t} \in [0, \sqrt{2}] \), so that
\[
\| \nabla P_u(x, y) \|_2^2 = 8(x - y)^2u^4 + 16 \left(x^2 - 4xy + 3y^2\right) u^3
\]
\[
+ 8 \left(x^2 - 10xy + 13y^2\right) u^2 + 32 \left(3y^2 - xy\right) u + 4 \left(x^2 + 9y^2\right).
\]

Since
\[
\frac{\partial}{\partial u} \| \nabla P_u(x, y) \|_2^2 = 16 \left(2(x - y)^2 u^2 + \left(x^2 - 8xy + 7y^2\right) u + 2y(3y - x) \right) (u + 1),
\]
it follows that the critical points of \( \| DP_u(x, y) \|_2^2 \) are \( u = \frac{2y}{x-y}, u = \frac{3y-x}{2(x-y)} \) and \( u = -1 \) if \( x \neq y \) and \( u = 4 \) and \( u = -1 \) if \( x = y \). Since we need to consider \( 0 \leq u \leq \sqrt{2} \), we can directly omit the case \( x = y \).

Therefore, we can write
\[
\frac{\partial}{\partial u} \| \nabla P_u(x, y) \|_2^2 = 32(x - y)^2 \left( u - \frac{2y}{x-y} \right) \left( u - \frac{3y-x}{2(x-y)} \right) (u + 1).
\]

Let \( u_1 = \frac{2y}{x-y} \) and \( u_2 = \frac{3y-x}{2(x-y)} \). (Again, since we need to consider \( 0 \leq u \leq \sqrt{2} \), we can omit the solution \( u = -1 \)). Also, we have the extra conditions \( u_1 \in [0, \sqrt{2}] \) whenever \( 0 \leq y \leq (\sqrt{2} - 1) x \) and \( u_2 \in [0, \sqrt{2}] \) whenever \( \frac{1}{2}x \leq y \leq (4\sqrt{2} - 5) x \). Considering all these facts, we need to compare the quantities
\[
C_1(x, y) := \| \nabla P_{u_1}(x, y) \|_2^2 = \| \nabla P_{u_1} \|_2^2 = 4 \left(\frac{x^6 - 4x^3y + 7x^2y^2 - 8x^4y^3 + 7x^2y^4 + 4xy^5 + y^6}{(x-y)^4}\right)
\]
\[
= 4 \left(x^2 + y^2\right),
\]
for \( 0 \leq y \leq (\sqrt{2} - 1)x \) \( t_1 = \frac{3y+2xy-3y^2}{(y-x)^2} \), \( C_2(x, y) := \| \nabla P_{u_2}(x, y) \|_2^2 = \| \nabla P_{u_2} \|_2^2 = \frac{9x^6 - 30x^3y + 55x^4y^2 - 68x^3y^3 + 55x^2y^4 - 30xy^5 + 9y^6}{2(x-y)^4}
\]
\[
= \frac{(3x^2 - 2xy + 3y^2)^2}{2(x-y)^2},
\]
for \( \frac{1}{2}x \leq y \leq (4\sqrt{2} - 5)x \) \( t_2 = \frac{5x^2+2xy-3x^2}{4(x-y)^2} \),
\[
C_3(x, y) := \| \nabla P_{u_3}(x, y) \|_2^2 = 4 \left(2x + 9y^2\right),
\]
and
\[
C_4(x, y) := \| \nabla P_{u_4}(x, y) \|_2^2 \equiv 4 \left(13 + 8\sqrt{2}\right) x^2 + \left(69 + 48\sqrt{2}\right) y^2 - 2 \left(28 + 20\sqrt{2}\right) xy
\]
\[
\text{Let us focus now on } Q_s = \left(1, s, -2\sqrt{2(1+s)}\right), 1 \leq s \leq 5 + 4\sqrt{2}. \text{ Then, we have}
\]
\[
\| \nabla Q_s(x, y) \|_2^2 = \left(x^2 + 1 + s\right) (x^2 + 1 + s) y^2 + 8(1+s)(x^2 + y^2) - 8(1+s)\sqrt{2(1+s)} xy.
\]

Making the change \( v = \sqrt{2(1+s)} \in [2, 2 + 2\sqrt{2}] \), we need to study the function
\[
\| \nabla Q_v(x, y) \|_2^2 = v^2 (y^2 v^2 - 4xyv + 4x^2) + 4 \left(x^2 + y^2\right).
\]

If \( x = y = 0 \) we have \( \| \nabla Q_v(0, y) \|_2^2 = 0 \), so we will assume both \( x \neq 0 \) and \( y \neq 0 \). The critical points of \( \| \nabla Q_v(x, y) \|_2^2 \) are \( v = \frac{2}{y}, v = \frac{2x}{y} \) \( \text{and } v = 0 \) (but \( 0 \not\in [2, 2 + 2\sqrt{2}] \)). Observe that \( v_1 = \frac{2}{y} \in [2, 2 + 2\sqrt{2}] \) whenever \( \frac{\sqrt{2} - 1}{2} \leq y \), and \( v_2 = \frac{2x}{y} \in [2, 2 + 2\sqrt{2}] \) whenever \( y \geq (\sqrt{2} - 1)x \). Thus, we also need to compare the quantities
\[
C_5(x, y) := \| \nabla Q_{v_1}(x, y) \|_2^2 = \| \nabla Q_{v_2}(x, y) \|_2^2 = \frac{x^4}{y^2} + 4 \left(x^2 + y^2\right),
\]
\[
}\]
The graphs of the mappings $C_1(1, \lambda)$, $C_6(1, \lambda)$, $C_7(1, \lambda)$.

For $\frac{\sqrt{2}-1}{2} x \leq y \leq \frac{1}{2} x$ and $s_1 = \frac{x^2-2y^2}{2y^2}$, 
\[ C_6(x, y) := \|\nabla Q_{s_2}(x, y)\|_2^2 = \|\nabla Q_{s_2}(x, y)\|_2^2 = 4 \left( x^2 + y^2 \right), \]
for $(\sqrt{2} - 1) x \leq y \leq x$ and $s_2 = \frac{2x^2-y^2}{y^2}$, and also
\[ C_7(x, y) := \|\nabla Q_{s_3=1}\|_2^2 = 4 \left( x^2 + y^2 \right) + 16(x-y)^2, \]
and
\[ C_8(x, y) := \|\nabla Q_{s_4=5+4\sqrt{2}}\|_2^2 \]
\[ = \left( 12 + 8\sqrt{2} \right) \left[ 4x^2 + \left( 12 + 8\sqrt{2} \right) y^2 - \left( 8 + 8\sqrt{2} \right) xy \right] + 4 \left( x^2 + y^2 \right) \]
\[ = 4 \left[ \left( 13 + 8\sqrt{2} \right) x^2 + \left( 69 + 48\sqrt{2} \right) y^2 - 2 \left( 28 + 20\sqrt{2} \right) xy \right]. \]

Note that (the reader can take a look at Figures 1, 2 and 3)
\[ C_1(x, y), C_6(x, y) \leq C_7(x, y) \leq \begin{cases} C_4(x, y) & \text{if } 0 \leq y \leq \frac{2-\sqrt{2}}{2} x \text{ or } \frac{1}{2} x \leq y \leq x, \\ C_5(x, y) & \text{if } \frac{2\sqrt{2}-1}{2} x \leq y \leq \frac{1}{2} x, \end{cases} \]
\[ C_3(x, y) \leq \begin{cases} C_2(x, y) & \text{if } \frac{1}{2} x \leq y \leq (4\sqrt{2} - 5) x, \\ C_4(x, y) & \text{if } 0 \leq y \leq \frac{1}{2} x \text{ or } (4\sqrt{2} - 5) x \leq y \leq x, \\ C_8(x, y) = C_1(x, y). \end{cases} \]

Hence, for $(x, y) \in D \left( \frac{\pi}{4} \right)$,
\[ \Phi(x, y) = \sup \left\{ \|\nabla P(x, y)\|_2 : P \in \text{ext} \left( B_D \left( \frac{\pi}{4} \right) \right) \right\} \]
\[ = \begin{cases} C_4(x, y) & \text{if } 0 \leq y \leq \frac{\sqrt{2}-1}{2} x \text{ or } (4\sqrt{2} - 5) x \leq y \leq x, \\ C_5(x, y) & \text{if } \frac{\sqrt{2}-1}{2} x \leq y \leq (\sqrt{2} - 1) x, \\ C_2(x, y) & \text{if } (\sqrt{2} - 1) x \leq y \leq (4\sqrt{2} - 5) x. \end{cases} \]

In order to illustrate the previous step, the reader can take a look at Figure 4. 

**Corollary 2.2.** If $P \in \mathcal{P} \left( D \left( \frac{\pi}{4} \right) \right)$, then
\[ \sup \left\{ \|\nabla P(x, y)\|_2 : (x, y) \in D \left( \frac{\pi}{4} \right) \right\} \leq 4(13 + 8\sqrt{2})\|P\|_{D \left( \frac{\pi}{4} \right)}, \]
with equality for the polynomials $P_1(x, y) = \pm \left( x^2 + (5 + 4\sqrt{2})y^2 - 2(2 + 2\sqrt{2})xy \right)$. 

\[ \square \]
In this section we find the exact value of the polarization constant of the space $\mathcal{P}(2D\left(\pi/4\right))$. In order to do that, we prove a Bernstein type inequality for polynomials in $\mathcal{P}(2D\left(\pi/4\right))$. Observe that if $P \in \mathcal{P}(2D\left(\pi/4\right))$ and $(x, y) \in D\left(\pi/4\right)$ then the differential $DP(x, y)$ of $P$ at $(x, y)$ can be viewed as a linear
Let us define
\[ \lambda \]
where
\[ \text{By symmetry, we may just study the polynomials of Lemma 1.2 with positive sign. Let us start first with} \]

**Proof.** In order to calculate \( \Psi \)
\[ (3.1) \]
For every \( (x, y) \in D(\frac{\pi}{4}) \) and \( P \in P(2D(\frac{\pi}{4})) \) we have that
\[ ||DP(x, y)||_{D(\frac{\pi}{4})} \leq \Psi(x, y)||P||_{D(\frac{\pi}{4})}, \]
where
\[ \Psi(x, y) = \begin{cases} \sqrt{2} \left[ (1 + 2\sqrt{2}) x - (3 + 2\sqrt{2}) y \right] & \text{if } 0 \leq y < \frac{2\sqrt{2} - 1}{2} x, \\ \frac{\sqrt{2}}{2y} \left( x^2 + b^2 \right) & \text{if } \frac{2\sqrt{2} - 1}{2} x \leq y < (\sqrt{2} - 1) x, \\ 2 \left( x + \frac{y^2}{4} \right) & \text{if } (\sqrt{2} - 1) x \leq y < (2 - \sqrt{2}) x, \\ 4 \left( 1 + \sqrt{2} \right) y - 2x & \text{if } (2 - \sqrt{2}) x \leq y \leq x \end{cases} \]

Moreover, inequality \( (3.1) \) is optimal for each \( (x, y) \in D(\frac{\pi}{4}) \).

**Theorem 3.2.**

**Proof.** In order to calculate \( \Psi(x, y) := \sup\{ ||DP(x, y)||_{D(\frac{\pi}{4})} : ||P||_{D(\frac{\pi}{4})} \leq 1 \} \), by the Krein-Milman approach, it suffices to calculate
\[ \sup\{ ||DP(x, y)||_{D(\frac{\pi}{4})} : P \in \text{ext}(B_{D(\frac{\pi}{4})}) \}. \]

By symmetry, we may just study the polynomials of Lemma 1.2 with positive sign. Let us start first with
\[ P_t(x, y) = tx^2 + (4 + t + 4\sqrt{1 + t}) y^2 - (2 + 2t + 4\sqrt{1 + t}) xy. \]

So we may write
\[ \nabla P_t(x, y) = (2tx - (2 + 2t + 4\sqrt{1 + t}) y, 2(4 + t + 4\sqrt{1 + t}) y - (2 + 2t + 4\sqrt{1 + t}) x), \]
from which
\[ ||DP_t(x, y)||_{D(\frac{\pi}{4})} = \sup_{0 \leq \theta \leq \frac{\pi}{4}} \left[ 2 \left[ tx - (1 + t + 2\sqrt{1 + t}) y \right] \cos \theta \right. \]
\[ + 2 \left[ (4 + t + 4\sqrt{1 + t}) y - (1 + t + 2\sqrt{1 + t}) x \right] \sin \theta \left. \right] \]
\[ = 2x \sup_{0 \leq \theta \leq \frac{\pi}{4}} |f_\lambda(t, \theta)|, \]
for \( f_\lambda(t, \theta) = \left[ t - (1 + t + 2\sqrt{1 + t}) \lambda \right] \cos \theta \]
\[ + \left[ (4 + t + 4\sqrt{1 + t}) \lambda - (1 + t + 2\sqrt{1 + t}) \right] \sin \theta, \]
where \( \lambda = \frac{x}{y}, x \neq 0 \) (the case \( x = 0 \) is trivial, since the only point in \( D(\frac{\pi}{4}) \) where \( x = 0 \) is \((0, 0), \) in which case \( P_t(0, 0) = ||DP_t(0, 0)||_{D(\frac{\pi}{4})} = 0). \)

We need to calculate
\[ \sup_{-1 \leq t \leq 1} ||DP_t(x, y)||_{D(\frac{\pi}{4})} = 2x \sup_{0 \leq \lambda \leq \frac{\pi}{4}} |f_\lambda(t, \theta)|, \]
Let us define \( C_1 = [-1, 1] \times [0, \frac{\pi}{4}] \). We will analyze 5 cases.

(1) \((t, \theta) \in (-1, 1) \times (0, \frac{\pi}{4}). \)

We are interested just in critical points. Hence,
\[
\begin{align*}
\frac{\partial f_\lambda}{\partial t}(t, \theta) &= \left[ \left( 1 + \frac{2}{\sqrt{1+t}} \right) \lambda - \left( 1 + \frac{1}{\sqrt{1+t}} \right) \right] \sin \theta \\
&\quad \quad + \left[ 1 - \left( 1 + \frac{1}{\sqrt{1+t}} \right) \right] \cos \theta = 0,
\end{align*}
\]

(3.2)

\[
\begin{align*}
\frac{\partial f_\lambda}{\partial \theta}(t, \theta) &= \left[ (1 + t + 2\sqrt{1+t}) \lambda - t \right] \sin \theta \\
&\quad \quad + \left[ (4 + t + 4\sqrt{1+t}) \lambda - (1 + t + 2\sqrt{1+t}) \right] \cos \theta = 0,
\end{align*}
\]

(3.3)

Equation (3.3) tells us that

\[
\sin \theta = \frac{(4 + t + 4\sqrt{1+t}) \lambda - (1 + t + 2\sqrt{1+t})}{t - (1 + t + 2\sqrt{1+t}) \lambda} \cos \theta.
\]

(3.4)

If we now plug (3.4) in equation (3.2), we obtain

\[
0 = \left\{ \left[ 1 - \left( 1 + \frac{1}{\sqrt{1+t}} \right) \right] + \left[ \left( 1 + \frac{2}{\sqrt{1+t}} \right) \lambda - \left( 1 + \frac{1}{\sqrt{1+t}} \right) \right] \right\} \frac{(4 + t + 4\sqrt{1+t}) \lambda - (1 + t + 2\sqrt{1+t})}{t - (1 + t + 2\sqrt{1+t}) \lambda} \cos \theta.
\]

Using that \(0 < \theta < \frac{\pi}{4}\), we can conclude

\[
0 = \left[ 1 - \left( 1 + \frac{1}{\sqrt{1+t}} \right) \right] + \left[ \left( 1 + \frac{2}{\sqrt{1+t}} \right) \lambda - \left( 1 + \frac{1}{\sqrt{1+t}} \right) \right] \frac{(4 + t + 4\sqrt{1+t}) \lambda - (1 + t + 2\sqrt{1+t})}{t - (1 + t + 2\sqrt{1+t}) \lambda}
\]

and thus

\[
0 = \left[ 1 - \left( 1 + \frac{1}{\sqrt{1+t}} \right) \right] \cdot [t - (1 + t + 2\sqrt{1+t}) \lambda]
\]

\[
= \left[ 1 - \left( 1 + \frac{2}{\sqrt{1+t}} \right) \lambda - \left( 1 + \frac{1}{\sqrt{1+t}} \right) \right] \cdot [(4 + t + 4\sqrt{1+t}) \lambda - (1 + t + 2\sqrt{1+t})]
\]

\[
= \frac{\lambda^2}{\sqrt{1+t}} (1 + t + 2\sqrt{1+t}) + \left( 1 + \frac{2}{\sqrt{1+t}} \right) (4 + t + 4\sqrt{1+t}) \lambda^2
\]

\[
- \left( 1 + \frac{2}{\sqrt{1+t}} \right) (1 + t + 2\sqrt{1+t}) \lambda - \left( 1 + \frac{1}{\sqrt{1+t}} \right) (4 + t + 4\sqrt{1+t}) \lambda
\]

\[
+ \left( 1 + \frac{1}{\sqrt{1+t}} \right) (1 + t + 2\sqrt{1+t})
\]

\[
= t (1 - 2\lambda + 2\lambda^2 - 2\lambda + 1) + (-2\lambda + 2\lambda^2 + 4\lambda^2 - 2\lambda - 4\lambda + 2) \sqrt{1+t}
\]

\[
+ \frac{t}{\sqrt{1+t}} (-\lambda + \lambda^2 + 2\lambda^2 - 2\lambda - \lambda + 1) + \frac{1}{\sqrt{1+t}} (\lambda^2 + 8\lambda^2 - 2\lambda - 4\lambda + 1)
\]

\[
+ (-\lambda + \lambda^2 + 2\lambda^2 + 4\lambda^2 - 4\lambda + 1 + 2 + 8\lambda^2 - 8\lambda)
\]

\[
= 2t(\lambda - 1)^2 + 6\sqrt{1+t}(\lambda - 1) \left( \lambda - \frac{1}{3} \right) + 3 \frac{t}{\sqrt{1+t}} (\lambda - 1) \left( \lambda - \frac{1}{3} \right)
\]

\[
+ \frac{1}{\sqrt{1+t}} (3\lambda - 1)^2 + 15 \left( \lambda - \frac{1}{3} \right) \left( \lambda - \frac{3}{5} \right).
\]
Working with this last expression, we get
\[
0 = 2t\sqrt{1+t}(\lambda - 1)^2 + 6(1+t)(\lambda - 1) \left( \lambda - \frac{1}{3} \right) + 3t(\lambda - 1) \left( \lambda - \frac{1}{3} \right) \\
+ (3\lambda - 1)^2 + 15\sqrt{1+t} \left( \lambda - \frac{1}{3} \right) \left( \lambda - \frac{3}{5} \right)
\]
and hence, rearranging terms,
\[
(3.5) \quad \sqrt{1+t} \left[ 15 \left( \lambda - \frac{1}{3} \right) \left( \lambda - \frac{3}{5} \right) + 2t(\lambda - 1)^2 \right] = -9t(\lambda - 1) \left( \lambda - \frac{1}{3} \right) - 15 \left( \lambda - \frac{1}{3} \right) \left( \lambda - \frac{3}{5} \right).
\]
If \( \lambda = 1 \), we obtain
\[
\sqrt{1+t} + 1 = 0
\]
and so, in particular, we have \( \lambda \neq 1 \). Equation (3.5) has two solutions,
\[
t_1(\lambda) = \frac{-1 + 2\lambda + 3\lambda^2}{(\lambda - 1)^2} \quad \text{and} \quad t_2(\lambda) = \frac{5\lambda^2 + 2\lambda - 3}{4(\lambda - 1)^2}.
\]
Using equation (3.2), we may see
\[
\tan \theta = \frac{\left( 1 + \frac{1}{\sqrt{1+t}} \right) \lambda - 1}{\left( 1 + \frac{2}{\sqrt{1+t}} \right) \lambda - \left( 1 + \frac{1}{\sqrt{1+t}} \right)}.
\]
In particular, evaluating in \( t_1(\lambda) \) we obtain
\[
\tan \theta_1 = \frac{(1 + \frac{1-\lambda}{2\lambda}) \lambda - 1}{(1 + \frac{1}{\lambda^2}) \lambda - (1 + \frac{1-\lambda}{2\lambda})} = \lambda,
\]
in which case we have
\[
D_{1,1}(\lambda) := |f_\lambda(t_1, \theta_1)| = \left| -\sqrt{1+\lambda^2} \right| = \sqrt{1+\lambda^2}.
\]
Regarding \( t_2(\lambda) \), we obtain
\[
\tan \theta_2 = \frac{\left( 1 + \frac{\sqrt{4(\lambda-1)^2}}{4(\lambda-1)} \right) \lambda - 1}{\left( 1 + \frac{2}{\sqrt{4(\lambda-1)^2}} \right) \lambda - \left( 1 + \frac{\sqrt{4(\lambda-1)^2}}{4(\lambda-1)} \right)},
\]
Since \( \theta_2 \in (0, \frac{\pi}{4}) \), we need to guarantee \( 0 < \tan \theta_2 < 1 \), and for this we need \( 0 < \lambda < \frac{1}{3} \). Therefore
\[
\tan \theta_2 = \frac{5\lambda - 1}{7\lambda - 3}
\]
and in this case,
\[
D_{1,2}(\lambda) := |f_\lambda(t_2, \theta_2)|
\]
\[
= \left| \frac{5\lambda^2 + 2\lambda - 3}{4(\lambda - 1)^2} - \left( \frac{9\lambda^2 - 6\lambda + 1}{4(\lambda - 1)^2} + \frac{3\lambda - 1}{\lambda - 1} \right) \lambda \right| \frac{3 - 7\lambda}{\sqrt{74\lambda^2 - 52\lambda + 10}} \\
+ \left| \left( 3 + \frac{9\lambda^2 - 6\lambda + 1}{4(\lambda - 1)^2} + \frac{6\lambda - 2}{\lambda - 1} \right) \lambda - \left( \frac{9\lambda^2 - 6\lambda + 1}{4(\lambda - 1)^2} + \frac{3\lambda - 1}{\lambda - 1} \right) \right| \frac{1 - 5\lambda}{\sqrt{74\lambda^2 - 52\lambda + 10}} \\
= -\frac{78\lambda^4 - 208\lambda^3 + 196\lambda^2 - 80\lambda + 14}{4(\lambda - 1)^2 \sqrt{74\lambda^2 - 52\lambda + 10}} \\
= -\frac{39\lambda^2 - 26\lambda + 7}{2\sqrt{74\lambda^2 - 52\lambda + 10}}.
\]
(2) \( \theta = 0, -1 \leq t \leq 1. \)
We have 
\[ f_\lambda(t,0) = t - \left(1 + t + 2\sqrt{1+t}\right)\lambda. \]
Then,
\[ f_\lambda(-1,0) = -1, \quad f_\lambda(1,0) = 1 - 2\left(1 + \sqrt{2}\right)\lambda, \]
and hence
\[ |f_\lambda(1,0)| = \left\{ \begin{array}{ll}
1 - 2\left(1 + \sqrt{2}\right)\lambda & \text{if } 0 \leq \lambda < \frac{\sqrt{2} - 1}{2}, \\
2\left(1 + \sqrt{2}\right)\lambda - 1 & \text{if } \frac{\sqrt{2} - 1}{2} \leq \lambda \leq 1.
\end{array} \right. \]
Working now on \((-1,1),\) since
\[ f'_\lambda(t,0) = 1 - \left(1 + \frac{1}{\sqrt{1+t}}\right)\lambda, \]
the critical point of \(f_\lambda(t,0)\) is
\[ t = \frac{\lambda^2}{(1 - \lambda)^2} - 1. \]
Recall that we need to make sure that \(-1 < t < 1.\) Therefore, in this case we also need to ask
\[ \lambda < \frac{\sqrt{2}}{1 + \sqrt{2}} = 2 - \sqrt{2}. \]
Plugging the critical point of \(f_\lambda(t,0)\) into \(f_\lambda(t,0),\) we obtain
\[ f_\lambda \left(\frac{\lambda^2}{(1 - \lambda)^2} - 1, 0\right) = \frac{\lambda^2}{(\lambda - 1)^2} - 1 - \left[\frac{\lambda^2}{(\lambda - 1)^2} + 2\lambda \frac{2}{1 - \lambda}\right] \lambda = \frac{\lambda^2}{\lambda - 1} - 1, \]
and hence
\[ \left| f_\lambda \left(\frac{\lambda^2}{(1 - \lambda)^2} - 1, 0\right) \right| = 1 + \frac{\lambda^2}{1 - \lambda}. \]
- Assume first \(0 \leq \lambda < \frac{\sqrt{2} - 1}{2}.\) Then,
  \[ \sup_{-1 \leq t \leq 1} |f_\lambda(t,0)| = \max \left\{ 1, 1 - 2\left(1 + \sqrt{2}\right)\lambda, 1 + \frac{\lambda^2}{1 - \lambda}\right\} = 1 + \frac{\lambda^2}{1 - \lambda}. \]
- Assume now \(\frac{\sqrt{2} - 1}{2} \leq \lambda < 2 - \sqrt{2}.\) Then,
  \[ \sup_{-1 \leq t \leq 1} |f_\lambda(t,0)| = \max \left\{ 1, 2\left(1 + \sqrt{2}\right)\lambda - 1, 1 + \frac{\lambda^2}{1 - \lambda}\right\} = 1 + \frac{\lambda^2}{1 - \lambda}. \]
- Assume finally \(2 - \sqrt{2} \leq \lambda \leq 1.\) Then,
  \[ \sup_{-1 \leq t \leq 1} |f_\lambda(t,0)| = \max \left\{ 1, 2\left(1 + \sqrt{2}\right)\lambda - 1\right\} = 2\left(1 + \sqrt{2}\right)\lambda - 1. \]
So, in conclusion,
\[ \sup_{-1 \leq t \leq 1} |f_\lambda(t,0)| = \left\{ \begin{array}{ll}
1 + \frac{\lambda^2}{1 - \lambda} & \text{if } 0 \leq \lambda < 2 - \sqrt{2}, \\
(2 + 2\sqrt{2})\lambda - 1 & \text{if } 2 - \sqrt{2} \leq \lambda \leq 1,
\end{array} \right. \]
\[ =: \begin{cases} D_{2,1}(\lambda) & \text{if } 0 \leq \lambda < 2 - \sqrt{2}, \\
D_{2,2}(\lambda) & \text{if } 2 - \sqrt{2} \leq \lambda \leq 1.
\end{cases} \]

(3) \(\theta = \frac{\pi}{4}\) and \(-1 \leq t \leq 1.\)
We have
\[ f_\lambda \left(\frac{\pi}{4}, t\right) = \frac{\sqrt{2}}{2} \left[ t - (1 + t + 2\sqrt{1+t})\lambda + (4 + t + 4\sqrt{1+t})\lambda - (1 + t + 2\sqrt{1+t}) \right] \]
\[ = \frac{\sqrt{2}}{2} \left[ (3 + 2\sqrt{1+t})\lambda - (1 + 2\sqrt{1+t}) \right]. \]
Again, we have

\[ f_\lambda \left(-1, \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} (3\lambda - 1), \]
\[ f_\lambda \left(1, \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \left[ \left(3 + 2\sqrt{2}\right) \lambda - \left(1 + 2\sqrt{2}\right) \right], \]
\[ f_\lambda' \left(t, \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \left[ \frac{\lambda}{\sqrt{1+t}} - \frac{1}{\sqrt{1+t}} \right]. \]

and \( f_\lambda'(t, \frac{\pi}{4}) = 0 \) implies \( \lambda = 1 \) (in which case \( f_\lambda(t, \frac{\pi}{4}) = \sqrt{2} \) for every \( t \)).

- Assume first \( 0 \leq \lambda < \frac{1}{2} \). Then,
  \[
  \sup_{-1 \leq t \leq 1} |f_\lambda \left(t, \frac{\pi}{4}\right)| = \frac{\sqrt{2}}{2} \max \left\{ \left(1 + 2\sqrt{2}\right) - \left(3 + 2\sqrt{2}\right) \lambda, 1 - 3\lambda \right\}
  = \frac{\sqrt{2}}{2} \left[ \left(1 + 2\sqrt{2}\right) - \left(3 + 2\sqrt{2}\right) \lambda \right]
  \]

- Assume now \( \frac{1}{3} \leq \lambda < 4\sqrt{2} - 5 \). Then,
  \[
  \sup_{-1 \leq t \leq 1} |f_\lambda \left(t, \frac{\pi}{4}\right)| = \frac{\sqrt{2}}{2} \max \left\{ \left(1 + 2\sqrt{2}\right) - \left(3 + 2\sqrt{2}\right) \lambda, 3\lambda - 1 \right\}
  = \left\{ \begin{array}{ll}
  \frac{\sqrt{2}}{2} \left[ \left(1 + 2\sqrt{2}\right) - \left(3 + 2\sqrt{2}\right) \lambda \right] & \text{if } \frac{1}{3} \leq \lambda < \frac{2\sqrt{2}+1}{4} \\
  \frac{\sqrt{2}}{2} (3\lambda - 1) & \text{if } \frac{2\sqrt{2}+1}{4} \leq \lambda < 4\sqrt{2} - 5.
  \end{array} \right.
  \]

- Assume finally \( 4\sqrt{2} - 5 \leq \lambda \leq 1 \). Then,
  \[
  \sup_{-1 \leq t \leq 1} |f_\lambda \left(t, \frac{\pi}{4}\right)| = \frac{\sqrt{2}}{2} \max \left\{ 3\lambda - 1, \left(3 + 2\sqrt{2}\right) \lambda - \left(1 + 2\sqrt{2}\right) \right\}
  = \frac{\sqrt{2}}{2} (3\lambda - 1).
  \]

Hence, we can say that

\[
\sup_{-1 \leq t \leq 1} |f_\lambda \left(t, \frac{\pi}{4}\right)| = \left\{ \begin{array}{ll}
  \frac{\sqrt{2}}{2} \left[ \left(1 + 2\sqrt{2}\right) - \left(3 + 2\sqrt{2}\right) \lambda \right] & \text{if } 0 \leq \lambda < \frac{2\sqrt{2}+1}{4} \\
  \frac{\sqrt{2}}{2} (3\lambda - 1) & \text{if } \frac{2\sqrt{2}+1}{4} \leq \lambda \leq 1.
  \end{array} \right.
\]

(4) \( t = -1, 0 \leq \theta \leq \frac{\pi}{4} \).

Applying lemma 3.1 we obtain

\[
\sup_{0 \leq \theta \leq \frac{\pi}{4}} f_\lambda(-1, \theta) = \left\{ \begin{array}{ll}
  1 & \text{if } 0 \leq \lambda < \frac{1+\sqrt{2}}{4} \\
  \frac{\sqrt{2}}{2} (3\lambda - 1) & \text{if } \frac{1+\sqrt{2}}{4} \leq \lambda \leq 1.
  \end{array} \right.
\]

(5) \( t = 1, 0 \leq \theta \leq \frac{\pi}{4} \).

We use again lemma 3.1 with \( a = 1 - \left(2 + 2\sqrt{2}\right) \lambda \) and \( b = \left(5 + 4\sqrt{2}\right) \lambda - \left(2 + 2\sqrt{2}\right) \). Through standard calculations, we see that \( \frac{b}{a} \leq 0 \) if and only if \( \lambda \in \left[0, \frac{\sqrt{2}-1}{2}\right] \cup \left(\frac{9-4\sqrt{2}}{2}, 1\right] \) and \( \frac{b}{a} > 1 \) if and only
if $\frac{\sqrt{2} - 1}{2} < \lambda < \frac{3 + 4\sqrt{2}}{23}$. Therefore,

\[
\sup_{0 \leq \theta \leq \frac{\pi}{4}} |f_{\lambda}(1, \theta)|
\]

\[
= \left\{ \begin{array}{ll}
\lambda \vphantom{\frac{\sqrt{2}}{2}} & \text{if } 0 \leq \lambda < \frac{3 + 4\sqrt{2}}{23}, \\
\lambda \frac{\sqrt{2}}{2} & \text{if } \frac{3 + 4\sqrt{2}}{23} \leq \lambda < \frac{5 - 2\sqrt{2}}{9}, \\
\lambda \frac{3 + 2\sqrt{2}}{2} & \text{if } \frac{5 - 2\sqrt{2}}{9} \leq \lambda \leq 1.
\end{array} \right.
\]

Since $0 \leq \lambda < \sqrt{2} - 1$ implies $|1 - (2 + 2\sqrt{2})| < \frac{\sqrt{2}}{2} \frac{3 + 2\sqrt{2}}{2}$, it follows that

\[
|1 - (2 + 2\sqrt{2})| \leq \frac{\sqrt{2}}{2} (3 + 2\sqrt{2}) \frac{3 + 2\sqrt{2}}{2} \frac{3 + 4\sqrt{2}}{23} \frac{5 - 2\sqrt{2}}{9} \leq 1.
\]

Since (see Figures 5 and 6)

\[
D_{1,1}(\lambda) \leq \begin{cases} 
D_{2,1}(\lambda) & \text{if } 0 \leq \lambda < 2 - \sqrt{2}, \\
D_{2,2}(\lambda) & \text{if } 2 - \sqrt{2} \leq \lambda \leq 1, \\
D_{3,1}(\lambda) & \text{for } 0 < \lambda < \frac{1}{4},
\end{cases}
\]

we can rule out case (1). Since

\[
D_{3,1}(\lambda) = D_{5,1}(\lambda) \quad \text{for } 0 \leq \lambda \leq \frac{3 + 4\sqrt{2}}{23},
\]

\[
D_{3,2}(\lambda) = D_{4,2}(\lambda) \quad \text{for } \frac{3 + \sqrt{2}}{3} \leq \lambda \leq 1,
\]

we can directly rule out case (3). Since (see Figures 5 and 7)

\[
D_{4,1}(\lambda) = 1 \leq \begin{cases} 
D_{2,1}(\lambda) & \text{if } 0 \leq \lambda < 2 - \sqrt{2}, \\
D_{2,2}(\lambda) & \text{if } 2 - \sqrt{2} \leq \lambda < \frac{1 + \sqrt{2}}{3},
\end{cases}
\]

\[
D_{4,2}(\lambda) \leq D_{2,2} \text{ for } \frac{1 + \sqrt{2}}{3} \leq \lambda \leq 1,
\]

we can rule out case (4). Finally, since (see Figure 8)

\[
D_{5,2}(\lambda) \leq D_{2,1}(\lambda) \quad \text{for } \frac{3 + 4\sqrt{2}}{23} \leq \lambda < \frac{5 - 2\sqrt{2}}{9},
\]

\[
D_{5,3}(\lambda) = D_{2,2}(\lambda) \quad \text{for } 2 - \sqrt{2} \leq \lambda \leq 1,
\]

we can rule out the expressions $D_{5,2}(\lambda)$ and $D_{5,3}(\lambda)$ of case (5).
Thus, putting all the above cases together, we may reach the conclusion

\[
\sup_{(t,\theta)\in\mathcal{C}_1} |f_\lambda(t,\theta)| = \begin{cases} 
D_{5,1}(\lambda) & \text{if } 0 \leq \lambda < \frac{(2-3\sqrt{2})\sqrt{4\sqrt{2}+7+5\sqrt{2}+6}}{14}, \\
D_{2,1}(\lambda) & \text{if } \frac{(2-3\sqrt{2})\sqrt{4\sqrt{2}+7+5\sqrt{2}+6}}{14} \leq \lambda < 2 - \sqrt{2}, \\
D_{2,2}(\lambda) & \text{if } 2 - \sqrt{2} \leq \lambda \leq 1,
\end{cases}
\]

and hence

\[
\sup_{-1\leq \xi \leq 1} \|DP_1(x, y)\|_{D(\mathcal{C}_1)} = 2x \sup_{(t,\theta)\in\mathcal{C}_1} |f_\lambda(t,\theta)| = \begin{cases} 
\sqrt{2} \left( (1 + 2\sqrt{2}) x - (3 + 2\sqrt{2}) y \right) & \text{if } 0 \leq y < \frac{(2-3\sqrt{2})\sqrt{4\sqrt{2}+7+5\sqrt{2}+6}}{14} x, \\
2 \left( x + \frac{y^2}{x+y} \right) & \text{if } \frac{(2-3\sqrt{2})\sqrt{4\sqrt{2}+7+5\sqrt{2}+6}}{14} x \leq y < (2 - \sqrt{2}) x, \\
4 \left( 1 + \sqrt{2} \right) y - 2x & \text{if } (2 - \sqrt{2}) x \leq y \leq x,
\end{cases}
\]

assuming in every moment \( x \neq 0 \) (in order to illustrate the previous step, the reader can take a look at Figure 9).
Let us deal now with the polynomials

\[ Q_s(x, y) = x^2 + sy^2 - 2\sqrt{2(1 + s)}xy, \quad 1 \leq s \leq 5 + 4\sqrt{2}. \]
Then,

$$\nabla Q_s(x, y) = \left(2x - 2\sqrt{2(1+s)}y, \ 2sy - 2\sqrt{2(1+s)}x\right),$$

and thus

$$\|DQ_s(x, y)\|_{D(\xi)} = \sup_{0 \leq \theta \leq \frac{\pi}{4}} \left|2x \left[1 - \sqrt{2(1+s)}\lambda\right] \cos \theta + \left(s\lambda - \sqrt{2(1+s)}\right) \sin \theta\right|,$$

and thus

$$\sup_{1 \leq s \leq 5 + 4\sqrt{2}} \|DQ_s(x, y)\|_{D(\xi)} = 2x \sup_{(s, \theta) \in C_2} |g_\lambda(s, \theta)|,$$

with

$$g_\lambda(s, \theta) = \left(1 - \sqrt{2(1+s)}\lambda\right) \cos \theta + \left(s\lambda - \sqrt{2(1+s)}\right) \sin \theta$$

and $C_2 = [1, 5 + 4\sqrt{2}] \times [0, \frac{\pi}{4}]$. Again, we have several cases:

(6) $(s, \theta) \in (1, 5 + 4\sqrt{2}) \times (0, \frac{\pi}{4})$.

Let us first calculate the critical points of $g_\lambda$ over $C_2$.

$$\frac{\partial g_\lambda}{\partial s}(s_0, \theta_0) = \frac{-\lambda}{\sqrt{2(1+s_0)}} \cos \theta_0 + \left(1 - \frac{1}{\sqrt{2(1+s_0)}}\right) \sin \theta_0,$$

$$\frac{\partial g_\lambda}{\partial \theta}(s_0, \theta_0) = \left(s_0\lambda - \sqrt{2(1+s_0)}\right) \cos \theta_0 - \left(1 - \sqrt{2(1+s_0)}\lambda\right) \sin \theta_0,$$

so, if $Dg_\lambda(s_0, \theta_0) = 0$, using the first expression, we obtain $\tan \theta_0 = \frac{\lambda}{\sqrt{2(1+s_0)}\lambda-1}$, and, using the second one, we obtain $\tan \theta_0 = \frac{s_0\lambda - \sqrt{2(1+s_0)}}{1 - \sqrt{2(1+s_0)}\lambda}$.

Hence, we may say

$$\frac{s_0\lambda - \sqrt{2(1+s_0)}}{1 - \sqrt{2(1+s_0)}\lambda} = \frac{\lambda}{\sqrt{2(1+s_0)}\lambda-1},$$

and thus

$$s_0 = \frac{2 - \lambda^2}{\lambda^2}. $$

Then, $\tan \theta_0 = \lambda$ and also, if we want to guarantee that $1 < s_0 < 5 + 4\sqrt{2}$, we need $\sqrt{2} - 1 < \lambda < 1$.

In that case, $\sin \theta_0 = \frac{\lambda}{\sqrt{1+\lambda^2}}$ and $\cos \theta_0 = \frac{1}{\sqrt{1+\lambda^2}}$, and then

$$g_\lambda(s_0, \theta_0) = \frac{-1}{\sqrt{1+\lambda^2}} + \frac{-\lambda^2}{\sqrt{1+\lambda^2}} = -\sqrt{1+\lambda^2},$$

so

$$D_0(\lambda) := |g_\lambda(s_0, \theta_0)| = \sqrt{1+\lambda^2}.$$

(7) $s = 1, \ 0 \leq \theta \leq \frac{\pi}{4}.$

Apply lemma 3.1 with $a = 1 - 2\lambda$ and $b = \lambda - 2$. Using $0 \leq \lambda \leq 1$, observe that we always have $b < 0$ and $b \leq a$. Also, $a < (1 - \sqrt{2})b$ if and only if $\lambda > \frac{5 - 3\sqrt{2}}{2}$.

Putting everything together, we can say

$$\sup_{0 \leq \theta \leq \frac{\pi}{4}} |g_\lambda(1, \theta)| = \begin{cases} 1 - 2\lambda & \text{if } 0 \leq \lambda < \frac{5 - 3\sqrt{2}}{2}, \\ \frac{\lambda^2}{2}(1 + \lambda) & \text{if } \frac{5 - 3\sqrt{2}}{2} \leq \lambda \leq 1, \end{cases}$$

$$=: \begin{cases} D_{\tau,1}(\lambda) & \text{if } 0 \leq \lambda < \frac{5 - 3\sqrt{2}}{2}, \\ D_{\tau,2}(\lambda) & \text{if } \frac{5 - 3\sqrt{2}}{2} \leq \lambda \leq 1. \end{cases}$$

(8) $s = 5 + 4\sqrt{2}, \ 0 \leq \theta \leq \frac{\pi}{4}.$
Apply again lemma [3.1] this time to \( a = 1 - 2 \left( 1 + \sqrt{2} \right) \lambda \) and \( b = (5 + 4\sqrt{2}) \lambda - 2 \left( 1 + \sqrt{2} \right) \). As usual, we notice that \( a < 0 \) if and only if \( \lambda > \frac{\sqrt{2} - 1}{2} \), \( b < 0 \) if and only if \( \lambda < \frac{2 - \sqrt{2}}{2} \) and \( a < b \) if and only if \( \lambda > \frac{3 + 4\sqrt{2}}{23} \). All together, we can say that, for \( \frac{3 + 4\sqrt{2}}{23} < \lambda < \frac{6 - 2\sqrt{2}}{7} \), we have

\[
\sup_{0 \leq \theta \leq \frac{\pi}{4}} |g_{\lambda}(5 + 4\sqrt{2}, \theta)| = \sqrt{a^2 + b^2} = \sqrt{13 + 8\sqrt{2} - (56 + 40\sqrt{2}) \lambda + (69 + 48\sqrt{2}) \lambda^2}.
\]

Also, notice that, for any \( \lambda \in [0, 1] \), we are going to have \( b < -\left( 1 + \sqrt{2} \right) a \) and \( a < \left( 1 - \sqrt{2} \right) b \). Hence,

\[
\sup_{0 \leq \theta \leq \frac{\pi}{4}} |g_{\lambda}(5 + 4\sqrt{2}, \theta)| = \begin{cases} \frac{\sqrt{2}}{2} \left[ (1 + 2\sqrt{2}) - (3 + 2\sqrt{2}) \lambda \right] & \text{if } 0 \leq \lambda < \frac{3 + 4\sqrt{2}}{23}, \\ \sqrt{13 + 8\sqrt{2} - (56 + 40\sqrt{2}) \lambda + (69 + 48\sqrt{2}) \lambda^2} & \text{if } \frac{3 + 4\sqrt{2}}{23} \leq \lambda < \frac{6 - 2\sqrt{2}}{7}, \\ 2 \left( 1 + \sqrt{2} \right) - \lambda & \text{if } \frac{6 - 2\sqrt{2}}{7} \leq \lambda \leq 1, \end{cases}
\]

(9) \( \theta = 0, 1 \leq s \leq 5 + 4\sqrt{2} \).

We have

\[
g_{\lambda}(s, 0) = 1 - \sqrt{2(1 + s)}\lambda,
\]

\[
g_{\lambda}(1, 0) = 1 - 2\lambda,
\]

\[
g_{\lambda}(5 + 4\sqrt{2}, 0) = 1 - 2 \left( 1 + \sqrt{2} \right) \lambda,
\]

\[
g_{\lambda}'(s, 0) = -\frac{\lambda}{\sqrt{2(1 + s)}} \neq 0 \text{ for } \lambda \neq 0.
\]

Then,

\[
\sup_{1 \leq s \leq 5 + 4\sqrt{2}} |g_{\lambda}(s, 0)| = \max \left\{ |1 - 2\lambda|, |1 - 2(1 + \sqrt{2})\lambda| \right\} = \begin{cases} 1 - 2\lambda & \text{if } 0 \leq \lambda < \frac{2 - \sqrt{2}}{2}, \\ 2 \left( 1 + \sqrt{2} \right) - \lambda - 1 & \text{if } \frac{2 - \sqrt{2}}{2} \leq \lambda \leq 1, \end{cases}
\]

(10) \( \theta = \frac{\pi}{4}, 1 \leq s \leq 5 + 4\sqrt{2} \).

We have

\[
g_{\lambda} \left( s, \frac{\pi}{4} \right) = \frac{\sqrt{2}}{2} \left[ 1 + s\lambda - \sqrt{2(1 + s)}(1 + \lambda) \right].
\]

Then

\[
g_{\lambda} \left( 1, \frac{\pi}{4} \right) = -\frac{\sqrt{2}}{2}(1 + \lambda),
\]

\[
g_{\lambda} \left( 5 + 4\sqrt{2}, \frac{\pi}{4} \right) = \frac{\sqrt{2}}{2} \left[ \left( 3 + 2\sqrt{2} \right) \lambda - \left( 1 + 2\sqrt{2} \right) \right],
\]

\[
g_{\lambda}' \left( s_0, \frac{\pi}{4} \right) = 0 \text{ if and only if } s_0 = \frac{(1 + \lambda)^2}{2\lambda^2} - 1
\]

and since we need to ensure that \( 1 < s_0 < 5 + 4\sqrt{2} \), we need \( \frac{2\sqrt{2} - 1}{7} < \lambda < 1 \). In that case,

\[
g_{\lambda} \left( s_0, \frac{\pi}{4} \right) = -\frac{\sqrt{2}}{4}(1 + 3\lambda^2).
\]
we can rule out the expression $D$ we can rule out case (7). Since

we can conclude that

Hence,

$$
\sup_{1 \leq s \leq 5+4\sqrt{2}} |g_{\lambda}(s, \frac{\pi}{4})| = \begin{cases} 
\frac{\sqrt{2}}{\lambda} \left[ (1 + 2\sqrt{2}) - (3 + 2\sqrt{2}) \lambda \right] & \text{if } 0 \leq \lambda < \frac{2\sqrt{2}-1}{7}, \\
\frac{\sqrt{2}}{\lambda^2 + 3\lambda^2} & \text{if } \frac{2\sqrt{2}-1}{7} \leq \lambda \leq 1,
\end{cases}
$$

we can rule out case (6). Since (see Figures 11 and 12)

$$
D_{7,1}(\lambda) \leq D_{10,1}(\lambda) \text{ for } 0 \leq \lambda < \frac{5-3\sqrt{2}}{7},
$$

we can rule out case (7). Since

$$
D_{8,1}(\lambda) = D_{10,1}(\lambda) \text{ for } 0 \leq \lambda < \frac{2\sqrt{2}-1}{7},
$$

we can rule out the expression $D_{8,1}(\lambda)$ of case (8). Since

$$
D_{9,1}(\lambda) = D_{7,1}(\lambda) \text{ for } 0 \leq \lambda < \frac{5-3\sqrt{2}}{7},
$$

we can directly rule out case (9). Furthermore, since (see Figure 13)

$$
D_{8,2}(\lambda) \leq D_{10,2}(\lambda) \text{ for } \frac{4+4\sqrt{2}}{7} \leq \lambda < \frac{6-2\sqrt{2}}{7},
$$

we can conclude that

$$
\sup_{(s, \theta) \in C_{2}} |g_{\lambda}(s, \theta)| = \begin{cases} 
D_{10,1}(\lambda) & \text{if } 0 \leq \lambda < \frac{2\sqrt{2}-1}{7}, \\
D_{10,2}(\lambda) & \text{if } \frac{2\sqrt{2}-1}{7} \leq \lambda < \frac{(4\sqrt{2}-5)\sqrt{4\sqrt{2}+7+8-5\sqrt{2}}}{7}, \\
D_{8,3}(\lambda) & \text{if } \frac{(4\sqrt{2}-5)\sqrt{4\sqrt{2}+7+8-5\sqrt{2}}}{7} \leq \lambda \leq 1,
\end{cases}
$$

and hence

$$
\sup_{1 \leq x \leq 5+4\sqrt{2}} \|DQ_{4}(x, y)\|_{D(\frac{\pi}{4})} = \begin{cases} 
\sqrt{2} \left[ (1 + 2\sqrt{2}) x - (3 + 2\sqrt{2}) y \right] & \text{if } 0 \leq y < \frac{2\sqrt{2}-1}{7} x, \\
\frac{\sqrt{2}}{2y} \left[ \frac{5+3\sqrt{2}}{2y} \right] & \text{if } \frac{2\sqrt{2}-1}{7} x \leq y < \frac{(4\sqrt{2}-5)\sqrt{4\sqrt{2}+7+8-5\sqrt{2}}}{7} x, \\
4 (1 + \sqrt{2}) y - 2x & \text{if } \frac{(4\sqrt{2}-5)\sqrt{4\sqrt{2}+7+8-5\sqrt{2}}}{7} x \leq y \leq x.
\end{cases}
$$
Finally, if we compare the results obtained with $P_t$ and $Q_s$, since $\sqrt{2(1+\lambda^2)} \geq 1 + \frac{x^2}{1-x}$ whenever $\lambda \leq \sqrt{2} - 1$, we obtain

$$\Phi(x, y) = \begin{cases} 
\sqrt{2} \left[ \frac{(1 + 2\sqrt{2}) x - (3 + 2\sqrt{2}) y}{\sqrt{2}(x^2 + 3y^2)} \right] & \text{if } 0 \leq y < \frac{2\sqrt{2} - 1}{x}, \\
\frac{2y}{x+y} & \text{if } \frac{2\sqrt{2} - 1}{x} \leq y < (\sqrt{2} - 1) x, \\
4 \left( 1 + \sqrt{2} \right) y - 2x & \text{if } (\sqrt{2} - 1) x \leq y < (2 - \sqrt{2}) x, \\
4 \left( 1 + \sqrt{2} \right) y - 2x & \text{if } (2 - \sqrt{2}) x \leq y \leq x.
\end{cases}$$
We can see that $\Phi(x,y) \leq 4 + \sqrt{2}$, for all $(x, y) \in D \left( \frac{x}{2} \right)$. Furthermore, the maximum is attained by the polynomials

$$P_t(x, y) = x^2 + \left( 5 + 4\sqrt{2} \right) y^2 - \left( 4 + 4\sqrt{2} \right) xy = Q_{5+4\sqrt{2}}(x, y).$$

**Corollary 3.3.** Let $P \in \mathcal{P} \left( 2D \left( \frac{x}{2} \right) \right)$ and assume $L \in \mathcal{L}^s \left( 2D \left( \frac{x}{2} \right) \right)$ is the polar of $P$. Then

$$\|L\|_{D(z)} \leq \left( 2 + 2\sqrt{2} \right) \|P\|_{D(z)}.$$

Moreover, equality is achieved for $P_t(x, y) = Q_{5+4\sqrt{2}}(x, y) = x^2 + \left( 5 + 4\sqrt{2} \right) y^2 - \left( 4 + 4\sqrt{2} \right) xy$. Hence, the polarization constant of the polynomial space $\mathcal{P} \left( 2D \left( \frac{x}{2} \right) \right)$ is $2 + \sqrt{2}$.

4. Unconditional constants for polynomials on sectors

Here, we obtain a sharp estimate on the norm of the modulus of a polynomial in $\mathcal{P} \left( 2D \left( \frac{x}{2} \right) \right)$ in terms of its norm. That sharp estimate turns out to be the unconditional constant of the canonical basis of $\mathcal{P} \left( 2D \left( \frac{x}{2} \right) \right)$.

**Theorem 4.1.** The unconditional constant of the canonical basis of $\mathcal{P} \left( 2D \left( \frac{x}{2} \right) \right)$ is $5 + 4\sqrt{2}$. In other words, the inequality

$$\|P\|_{D(z)} \leq (5 + 4\sqrt{2}) \|P\|_{D(z)},$$

for all $P \in \mathcal{P} \left( 2D \left( \frac{x}{2} \right) \right)$. Furthermore, the previous inequality is sharp and equality is attained for the polynomials $\pm P_t(x, y) = \pm Q_{5+4\sqrt{2}}(x, y) = \pm \left[ x^2 + (5 + 4\sqrt{2})y^2 - (4 + 4\sqrt{2})xy \right]$.

**Proof.** We just need to calculate

$$\sup \left\{ \|P\|_{D(z)} : P \in \text{ext} \left( B_D(z) \right) \right\}.$$ 

In order to calculate the above supremum we use the extreme polynomials described in Lemma 1.2 If we consider first the polynomials $P_t$, then $|P_t| = (|t|, 4 + t + 4\sqrt{1 + t}, 2 + t + 4\sqrt{1 + t})$. Now, using Lemma 1.4 we have

$$\sup_{-1 \leq t \leq 1} \|P_t\|_{D(z)} = \sup_{-1 \leq t \leq 1} \max \left\{ |t|, \frac{1}{2} \left( |t| + 4 + t + 4\sqrt{1 + t} + 2 + 2t + 4\sqrt{1 + t} \right) \right\} = \sup_{-1 \leq t \leq 1} \frac{1}{2} \left( |t| + 6 + 3t + 8\sqrt{1 + t} \right) = 5 + 4\sqrt{2}.$$
Notice that the above supremum is attained at \( t = 1 \). On the other hand, if we consider the polynomials \( Q_s \), we have \( |Q_s| = (1, s, 2\sqrt{2(1 + s)}) \). Now, using Lemma 1.1 we have

\[
\sup_{1 \leq s \leq 5 + 4\sqrt{2}} \|Q_s\|_{D\left(\frac{\pi}{4}\right)} = \sup_{1 \leq s \leq 5 + 4\sqrt{2}} \max \left\{ 1, \frac{1}{2} \left( 1 + s + 2\sqrt{2(1 + s)} \right) \right\} = \sup_{1 \leq s \leq 5 + 4\sqrt{2}} \frac{1}{2} \left( 1 + s + 2\sqrt{2(1 + s)} \right) = 5 + 4\sqrt{2}.
\]

Observe that the last supremum is now attained at \( s = 5 + 4\sqrt{2} \).

\( \Box \)

5. Conclusions

Comparing the results obtained in [11] and [25] for polynomials on the simplex \( \Delta \), in [12] for polynomials on the unit square \( \square \), in [15] for polynomials on the sector \( D\left(\frac{\pi}{2}\right) \) and the results obtained in the previous sections, we have the following:

<table>
<thead>
<tr>
<th></th>
<th>( P(2\Delta) )</th>
<th>( P(2D\left(\frac{\pi}{2}\right)) )</th>
<th>( P(2D\left(\frac{\pi}{4}\right)) )</th>
<th>( P(2\square) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Markov constants</td>
<td>( 2\sqrt{10} )</td>
<td>( 2\sqrt{5} )</td>
<td>( 4(13 + 8\sqrt{2}) )</td>
<td>( \sqrt{13} )</td>
</tr>
<tr>
<td>Polarization constants</td>
<td>3</td>
<td>2</td>
<td>( 2 + \frac{\sqrt{2}}{2} )</td>
<td>( \frac{3}{2} )</td>
</tr>
<tr>
<td>Unconditional Constants</td>
<td>2</td>
<td>3</td>
<td>( 5 + 4\sqrt{2} )</td>
<td>5</td>
</tr>
</tbody>
</table>

Furthermore, all the constants appearing in the previous table are sharp. Actually, the extreme polynomials where the constants are attained are the following:

1. \( \pm (x^2 + y^2 - 6xy) \) for the simplex.
2. \( \pm (x^2 + y^2 - 4xy) \) for the sector \( D\left(\frac{\pi}{2}\right) \).
3. \( \pm (x^2 + (5 + 4\sqrt{2})y^2 - (4 + 4\sqrt{2})xy) \) for the sector \( D\left(\frac{\pi}{4}\right) \).
4. \( \pm (x^2 + y^2 - 3xy) \) for the unit square.

Compare the previous table with similar results that hold for 2-homogeneous polynomials on the Banach spaces \( l_1^2 \), \( l_2^2 \) and \( l_\infty^2 \):

<table>
<thead>
<tr>
<th></th>
<th>( P(2l_1^2) )</th>
<th>( P(2l_2^2) )</th>
<th>( P(2l_\infty^2) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Markov constants</td>
<td>4</td>
<td>2</td>
<td>( 2\sqrt{2} )</td>
</tr>
<tr>
<td>Polarization constants</td>
<td>2</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>Unconditional Constants</td>
<td>( \frac{1 + \sqrt{2}}{2} )</td>
<td>( \sqrt{2} )</td>
<td>( 1 + \sqrt{2} )</td>
</tr>
</tbody>
</table>

Observe that the Markov constants of the spaces \( P(2l_1^2) \) and \( P(2l_\infty^2) \) can be calculated taking into consideration the description of the geometry of those spaces given in [5]. Also, the Markov constant of \( P(2l_2^2) \) is twice its polarization constant, or in other words, 2.

On the other hand, the constants appearing in the second line of the previous table are well-known results (see for instance [27]).

Finally, the unconditional constants corresponding to the third line of the previous table were calculated in Theorem 3.5, Theorem 3.19 and Theorem 3.6 of [11].
References


5. Y. S. Choi and S. G. Kim, Exposed points of the unit balls of the spaces \( P(\mathbb{C}^2) \) \((p = 1, 2, \infty)\), Indian J. Pure Appl. Math. 35 (2004), 37–41.


DEPARTAMENTO DE ÁNALISIS MATEMÁTICO,
FACULTAD DE CIENCIAS MATEMÁTICAS,
PLAZA DE CIENCIAS 3,
UNIVERSIDAD COMPLUTENSE DE MADRID,
MADRID, 28040 (SPAIN).
E-mail address:
gdasara篓@gmail.com
pablo.jimenez.rod@gmail.com
gustavo.fernandez@mat.ucm.es
jseoane@mat.ucm.es