Universality in the Three-Dimensional Random-Field Ising Model

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We solve a long-standing puzzle in statistical mechanics of disordered systems. By performing a high-statistics simulation of the $D = 3$ random-field Ising model at zero temperature for different shapes of the random-field distribution, we show that the model is ruled by a single universality class. We compute the complete set of critical exponents for this class, including the correction-to-scaling exponent, and we show, to high numerical accuracy, that scaling is described by two independent exponents. Discrepancies with previous works are explained in terms of strong scaling corrections.

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The random-field Ising model (RFIM) is one of the simplest and most investigated models for collective behavior in the presence of quenched disorder [1]. In spite of its simplicity, many problems in condensed matter physics can be studied through the RFIM: diluted antiferromagnets in a field [2], colloid-polymer mixtures [3,4], colossal magnetoresistance oxides [5,6] (more generally, phase coexistence in the presence of quenched disorder [7–9]), and nonequilibrium phenomena such as the Barkhausen noise in magnetic hysteresis [10,11] or the design of switchable magnetic domains [12], etc.

On the theoretical side, a scaling picture is available [13–16]. The paramagnetic-ferromagnetic phase transition is ruled by a fixed point [in the renormalization-group (RG) sense] at temperature $T = 0$ [1]. The spatial dimension $D$ is replaced by $D - \theta$ in hyperscaling relations ($\theta = D/2$). Nevertheless, many expect only two independent exponents [1,17,18], as in standard phase transitions (see, e.g., [19]). Unfortunately, establishing the scaling picture is far from trivial. Perturbation theory predicts that, in $D = 3$, the ferromagnetic phase disappears upon applying the tiniest random field [20]. Even if the statement holds at all orders in perturbation theory [21], the ferromagnetic phase is stable in $D = 3$ [22]. Nonperturbative phenomena are obviously at play [23,24]. Indeed, it has been suggested that the scaling picture breaks down because of spontaneous supersymmetry breaking, implying that more than two critical exponents are needed to describe the phase transition [25].

On the experimental side, a particularly well-researched realization of the RFIM is the diluted antiferromagnet in an applied magnetic field [2]. Yet, there are inconsistencies in the determination of critical exponents. In neutron scattering, different parametrizations of the scattering line shape yield mutually incompatible estimates of the thermal critical exponent, namely $\nu = 0.87(7)$ [26] and $\nu = 1.20(5)$ [27]. Moreover, the anomalous dimension $\eta = 0.16(6)$ [26] violates hyperscaling bounds (at least if one believes experimental claims of a divergent specific heat [28,29]). Clearly, a reliable parametrization of the line shape would be welcome. This program has been carried out for simpler, better understood problems [30]. Unfortunately, we do not have such a strong command over the RFIM universality class.

The RFIM also has been investigated by means of numerical simulations. However, typical Monte Carlo schemes get trapped into local minima with escape time exponential in $\xi^0$ ($\xi$ is the correlation length). Although sophisticated improvements have appeared [31–33], these simulations produced low-accuracy data because they were limited to linear sizes $L \leq 32$. Larger sizes can be achieved at $T = 0$, through the well-known mapping of the ground state to the maximum-flow optimization problem [34–44]. Yet, $T = 0$ simulations lack many tools, standard at $T > 0$. In fact, the numerical data at $T = 0$ and their finite-size scaling analysis mostly resulted in strong universality violations [33,36–39].

Here we show that the $D = 3$ RFIM is ruled by a single universality class, by considering explicitly four different models that belong to it. To this end, we perform high-statistics $T = 0$ simulations of the model, and we introduce a fluctuation-dissipation formalism in order to compute connected and disconnected correlation functions. Another asset of our implementation is the use of phenomenological renormalization [45,46], which allows us to extract effective size-dependent critical exponents, whose size evolution can be closely followed. Although the four models differ in their prediction for finite sizes, we show that, after a proper consideration of the scaling corrections, we can extrapolate to infinite-limit size, finding consistent results for all of them.

Our $S_x = \pm 1$ spins are on a cubic lattice with size $L$ and periodic boundary conditions. The Hamiltonian is

$$\mathcal{H} = -J \sum_{\langle x,y \rangle} S_x S_y - \sum_x h_x S_x. \quad (1)$$
$J = 1$ is the nearest-neighbors’ ferromagnetic interaction. Independent quenched random fields $h_i$ are extracted from one of the following double Gaussian (dG) or Poissonian (P) distributions (with parameters $h_R$, $\sigma$),

$$dG(\sigma)(h_i; h_R) = \frac{1}{\sqrt{8\pi\sigma}} \left[e^{-((h_i - h_R)^2)/2\sigma^2} + e^{-(h_i + h_R)^2}/2\sigma^2}\right].$$

The limiting cases $\sigma = 0$ and $h_R = 0$ of Eq. (2) correspond to the well-known bimodal (b) and Gaussian (G) distributions, respectively. In the P and G cases the strength of the random fields is parametrized by $\sigma$, while in the dG case we shall take $\sigma = 1$ and 2 and vary $h_R$. The phase diagram for the double Gaussian distribution is sketched in Fig. 1. Note the bimodal shape of Eq. (2) for $\sigma = 1$, with peaks near $\pm h_R$.

An instance of the random fields $\{h_i\}$ is named a sample. The only relevant spin configurations at $T = 0$ are ground states, which are nondegenerate for continuous random-field distributions [47]. Thermal mean values are denoted as $\langle \cdot \cdot \cdot \rangle$. The subsequent average over samples is indicated by an overline (e.g., for the magnetization density $m = \sum_s x_s/L^3$, we consider both $\langle m \rangle$ and $\langle \overline{m} \rangle$).

We considered four disorder distributions: P, G, and dG with $\sigma = 1, 2$. We obtained the ground states using the push-relabel algorithm [48]. We implemented in C the algorithm in [41,42], with periodic global updates. Our lattice sizes were $L = 12, 16, 24, 32, 48, 64, 96, 128$, and $192$ [16 $\leq L \leq 128$ for $dG^{(\sigma = 1)}$ and 12 $\leq L \leq 128$ for $dG^{(\sigma = 2)}$]. For each $L$, we averaged over $10^7$ samples [5 $\times 10^7$ samples for $dG^{(\sigma = 1)}$]. Previous studies were limited to $\sim 10^4$ samples [40,41].

We have generalized the fluctuation-dissipation formalism of [49] to compute connected $G_{xy} = \partial \langle S_x S_y \rangle/\partial h_y$ and disconnected $G_{xy}^{(dis)} = \langle S_x S_y \rangle$ correlation functions. We compute from them the second-moment connected ($\xi$) and disconnected ($\xi^{(dis)}$) correlation lengths [19,50].

We have also extended reweighting methods from percolation studies [51,52]. From a single simulation, we extrapolate the mean value of observables to nearby parameters of the disorder distribution [we varied $\sigma$ for the G and P distributions (see Fig. 2) and $h_R$ for the dG case]. Computing derivatives with respect to $\sigma$ or $h_R$ is straightforward. Consider, for instance, the P case (see [53] for other distributions). Let $D = \sum_i (|h_i| - \sigma)/\sigma^2$. The connected correlation function is $G_{xy} =\langle h_y S_x S_y \rangle/\langle h_y \rangle$, while the $\sigma$ derivative and the reweighting extrapolation to $\sigma + \delta \sigma$ of a generic observable $O$ are

$$D \langle O \rangle = \langle O \rangle + \delta \sigma \frac{\langle O \rangle}{\sigma} + \langle O \rangle^{(dis)} = \langle O \rangle^{(dis)}$$

with

$$R = \exp \left[ D \sigma \frac{\delta \sigma}{\sigma + \delta \sigma} + L \left( \log \frac{\sigma}{\sigma + \delta \sigma} + \frac{\sigma + \delta \sigma}{\sigma} \right) \right].$$

To extract the value of critical points, critical exponents, and dimensionless quantities, we employ the quotients method [19,45,46]. We compare observables computed in a pair of lattices ($L, 2L$). We start imposing scale invariance by seeking the $L$-dependent critical point: the value of $\sigma$ ($h_R$ for the dG), such that $\xi_{2L}/\xi_L = 2$ [i.e., the crossing point for $\xi_L/L$, (see Fig 2)]. Now, for dimensionful quantities $O$, scaling in the thermodynamic limit as $\xi_{\alpha/v}$, we

![Phase diagram of the double Gaussian RFIM, Eq. (2), at $T = 0$. The dotted critical line (a simple guide to the eye) separates the paramagnetic phase (large $\sigma, h_R$) from the ordered phase (low $\sigma, h_R$). Transition points are computed here (data from [54]), but for the limit $\sigma = 0$ which corresponds to binary random fields [39]. The arrow along the critical line indicates the RG flow. Inset: bimodal shape of the critical double Gaussian distribution (2) with $\sigma = 1$.](227201-2)

![FIG. 2 (color online). For several system sizes, we show $\xi/L$ vs the strength of the Poissonian random field $\sigma$ [see Eq. (3)] ($\xi$: correlation length from the connected correlator, $L$: system size). Lines join data obtained from reweighting extrapolation, Eq. (4) (discontinuous lines of the same color come from independent simulations). In the large-$L$ limit, $\xi/L$ is $L$ independent at the critical point $\sigma^c$. In the quotients method, we consider the $\xi/L$ curves for a pair of lattices ($L, 2L$) and seek the $\sigma$ where they cross. This crossing is employed for computing effective, $L$-dependent critical exponents with Eq. (5).](227201-3)
TABLE I. For our four field distributions, size-dependent critical exponents of the $D=3$ RFIM as computed from the quotients method.

<table>
<thead>
<tr>
<th>Distribution</th>
<th>$(L_1, L_2)$</th>
<th>$\nu$</th>
<th>$\eta$</th>
<th>$2\eta - \tilde{\eta}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>G</td>
<td>(16,32)</td>
<td>1.48(3)</td>
<td>0.5168(6)</td>
<td>0.0038(11)</td>
</tr>
<tr>
<td></td>
<td>(24,48)</td>
<td>1.45(3)</td>
<td>0.5155(5)</td>
<td>0.0022(11)</td>
</tr>
<tr>
<td></td>
<td>(32,64)</td>
<td>1.36(4)</td>
<td>0.5150(5)</td>
<td>0.0019(10)</td>
</tr>
<tr>
<td></td>
<td>(48,96)</td>
<td>1.43(6)</td>
<td>0.5154(5)</td>
<td>0.0033(9)</td>
</tr>
<tr>
<td></td>
<td>(64,128)</td>
<td>1.38(9)</td>
<td>0.5142(5)</td>
<td>0.0014(10)</td>
</tr>
<tr>
<td></td>
<td>(96,192)</td>
<td>1.38(11)</td>
<td>0.5144(5)</td>
<td>0.0021(11)</td>
</tr>
<tr>
<td>$dG^{(\sigma=1)}$</td>
<td>(16,32)</td>
<td>3.04(14)</td>
<td>0.5035(7)</td>
<td>0.0016(15)</td>
</tr>
<tr>
<td></td>
<td>(24,48)</td>
<td>2.26(9)</td>
<td>0.5083(7)</td>
<td>0.0034(14)</td>
</tr>
<tr>
<td></td>
<td>(32,64)</td>
<td>1.87(8)</td>
<td>0.5093(7)</td>
<td>0.0010(13)</td>
</tr>
<tr>
<td></td>
<td>(48,96)</td>
<td>1.56(9)</td>
<td>0.5121(7)</td>
<td>0.0026(14)</td>
</tr>
<tr>
<td></td>
<td>(64,128)</td>
<td>1.67(12)</td>
<td>0.5125(8)</td>
<td>0.0015(17)</td>
</tr>
<tr>
<td></td>
<td>(96,192)</td>
<td>1.38(11)</td>
<td>0.5144(5)</td>
<td>0.0021(11)</td>
</tr>
<tr>
<td>$dG^{(\sigma=2)}$</td>
<td>(16,32)</td>
<td>1.48(5)</td>
<td>0.5154(6)</td>
<td>0.0020(12)</td>
</tr>
<tr>
<td></td>
<td>(24,48)</td>
<td>1.50(6)</td>
<td>0.5151(7)</td>
<td>0.0020(13)</td>
</tr>
<tr>
<td></td>
<td>(32,64)</td>
<td>1.41(8)</td>
<td>0.5142(7)</td>
<td>0.0004(13)</td>
</tr>
<tr>
<td></td>
<td>(48,96)</td>
<td>1.36(10)</td>
<td>0.5148(7)</td>
<td>0.0024(14)</td>
</tr>
<tr>
<td></td>
<td>(64,128)</td>
<td>1.31(11)</td>
<td>0.5154(6)</td>
<td>0.0041(13)</td>
</tr>
<tr>
<td>P</td>
<td>(16,32)</td>
<td>1.20(2)</td>
<td>0.5183(9)</td>
<td>-0.0006(19)</td>
</tr>
<tr>
<td></td>
<td>(24,48)</td>
<td>1.26(3)</td>
<td>0.5168(8)</td>
<td>0.0011(17)</td>
</tr>
<tr>
<td></td>
<td>(32,64)</td>
<td>1.30(4)</td>
<td>0.5153(8)</td>
<td>0.0005(17)</td>
</tr>
<tr>
<td></td>
<td>(48,96)</td>
<td>1.37(7)</td>
<td>0.5143(9)</td>
<td>0.0004(18)</td>
</tr>
<tr>
<td></td>
<td>(64,128)</td>
<td>1.33(7)</td>
<td>0.5148(8)</td>
<td>0.0024(16)</td>
</tr>
<tr>
<td></td>
<td>(96,192)</td>
<td>1.43(13)</td>
<td>0.5146(8)</td>
<td>0.0026(17)</td>
</tr>
</tbody>
</table>

consider the quotient $Q_\Omega = O_{2L}/O_L$ at the crossing. For dimensionless magnitudes $g$, we focus on $g_{2L}$. In either case, one has

$$Q_{\Omega}^{\text{cross}} = 2x_\sigma + O(L^{-\omega}), \quad g_{2L}^{\text{cross}} = g^* + O(L^{-\omega}),$$

(5)

where $x_\sigma/\nu$, $g^*$ and the scaling-corrections exponent $\omega$ are universal. Examples of dimensionless quantities are $\xi/L$, $\xi^{(\text{dis})}/L$, and $U_4 = (m^4)/(m_\sigma^2)$. Instances of dimensionful quantities are the derivatives of $\xi$, $\xi^{(\text{dis})}$ ($x_\xi = 1 + \nu$), the connected and disconnected susceptibilities $\chi$ and $\chi^{(\text{dis})}$ [\(x_\chi = \nu(2 - \eta), x_{\chi^{(\text{dis})}} = \nu(4 - \tilde{\eta})\)], and the ratio $U_{22} = \chi^{(\text{dis})}/\chi^2 [x_{U_{22}} = \nu(2\eta - \tilde{\eta})]$.

The application of Eq. (5) to our four random-field distributions is summarized in Table I and Figs. 3 and 4 (the numerical values are tabulated in [54]). We start inspecting $\xi/L$ in Fig. 3. At fixed $L$, the dependence on the distribution is substantial. However, the strong size evolution suggests a common $L \to \infty$ limit. The behavior of the critical exponents, $\xi^{(\text{dis})}/L$ and $U_4$, is similar.

In order to extrapolate to $L \to \infty$, one fits the data of Table I to polynomials in $L^{-\omega}$. Although the procedure is standard [52], it has not been attempted before for the RFIM. Our extrapolations are documented in Fig. 4 and Table II. The following comments are in order: (a) For dimensionless quantities we needed a third-order polynomial in $L^{-\omega}$ (only a subclass of the subleading corrections-to-scaling [19]).

FIG. 3 (color online). Inspection of the size dependence of universal quantities. We show the universal ratio $\xi/L$ vs $1/L$, as computed at the corresponding crossing points (see Fig. 2) for the four disorder distributions considered in this work. The inset is an enlargement for $L \geq 48$.

Leading-order corrections sufficed for critical exponents. (b) We are not aware of any other computation of $\omega$, $\xi/L$, $\xi^{(\text{dis})}/L$, and $U_4$. All of them are universal. (c) The full critical line belongs to a single universality class (justifying

FIG. 4 (color online). Computation of the corrections-to-scaling exponent $\omega$ [see Eq. (5)] by means of a joint fit for $\xi/L$ (a), $\xi^{(\text{dis})}/L$ (b), and $U_4 = (m^4)/(m_\sigma^2)$ (c); see Table II and [63]. Black circles depicted at $L^{-\omega} = 0$ are our extrapolations to $L = \infty$. Stars denote extrapolations obtained using only the diagonal terms of the covariance matrix.
We perform joint fits for sizes $L \approx L_{\text{min}}$ to polynomials in $L^{-\omega}$, imposing common extrapolations for all four random-field distributions. The $\chi^2$ figure of merit was computed with the full covariance matrix (DOF: number of degrees of freedom in the fit). For the exponent $\nu$, we considered derivatives of both $\xi$ and $\xi^{(\text{dis})}$. The error induced by the uncertainty in $\omega$ is given as a second error estimate. The extrapolation of the critical points slightly differs [55].

<table>
<thead>
<tr>
<th>Extrapolation</th>
<th>$\chi^2$/DOF</th>
<th>$L_{\text{min}}$</th>
<th>Order in $L^{-\omega}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(\xi/L)_{L=\infty} = 2.08(13)$</td>
<td>18.8/14</td>
<td>24</td>
<td>third</td>
</tr>
<tr>
<td>$(\xi^{(\text{dis})}/L)_{L=\infty} = 8.4(8)$</td>
<td>10.0/9</td>
<td>32</td>
<td>first</td>
</tr>
<tr>
<td>$U_{d=1} = 1.0011(18)$</td>
<td>10.5/17</td>
<td>16</td>
<td>first</td>
</tr>
<tr>
<td>$\eta = 0.5153(9)(2)$</td>
<td>3.1/3</td>
<td>16</td>
<td>second</td>
</tr>
<tr>
<td>$(2\eta - \bar{\eta})_{L=\infty} = 0$ (fixed)</td>
<td>2.5/1</td>
<td>24</td>
<td>second</td>
</tr>
<tr>
<td>$(2\eta - \bar{\eta})_{L=\infty} = 0.0026(9)(1)$</td>
<td>0.7/2</td>
<td>16</td>
<td>second</td>
</tr>
<tr>
<td>$\sigma^{[G]} = 2.27205(18)(4)$</td>
<td>3.0/3</td>
<td>16</td>
<td>second</td>
</tr>
<tr>
<td>$\sigma^{[\mathcal{D}(g=1)]} = 1.9955(6)(24)$</td>
<td>0.7/2</td>
<td>16</td>
<td>second</td>
</tr>
<tr>
<td>$\sigma^{[\mathcal{D}(g=2)]} = 1.0631(7)(10)$</td>
<td>0.7/2</td>
<td>16</td>
<td>second</td>
</tr>
</tbody>
</table>

The computation has been attempted within the ambitious field-theoretic study of the RFIM. The task is highly demanding, and some critical points were expected to be very close to the critical point. The computational effort required for such a study is enormous, and we have been able to perform the calculations only for a small number of system sizes. The results obtained are in good agreement with previous studies [30,31,32].

In conclusion, we have shown that the universality class of the RFIM is independent of the implementation of the random-field distribution, in disagreement with the current opinion in the literature [32,33,36–39] and with the early predictions of mean-field theory [61]. To reach this conclusion, we had to identify and control the role of scaling corrections, the Achilles heel in the study of the RFIM (this problem was emphasized in the pioneering work of [34], but it was overlooked in subsequent investigations). On technical parts, we have developed a fluctuation-dissipation formalism that allows us to compute correlation functions and to apply phenomenological renormalization. We have also adapted the approach of [62] to study the scaling of the energy, checking that our data are compatible with modified hyperscaling [53,54] (a rather slippery problem [40]). Hence, several contradictions of previous works have been resolved in a consistent picture, paving the way to more sophisticated, experimentally relevant computations.

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1. See T. Nattermann in [64].
2. See D. P. Belanger in [64].
[47] Severe ground-state degeneracy arises for discrete distributions, which needs to be taken into account; see S. Bastea and P. M. Duxbury, Phys. Rev. E 58, 4261 (1998).
[49] For instance, in the Gaussian RFIM $G_{xy} = \frac{1}{\sqrt{2\pi\Delta}} e^{-x^2/2\Delta}$, see M. Schwartz and A. Soffer, Phys. Rev. Lett. 55, 2499 (1985).
[53] N. G. Fytas and V. Martín-Mayor (to be published).
[55] Scaling corrections for the critical point are of order $L^{-D/(1+\nu)}$, $L^{-D/(2\nu)}$, etc. [19, 53].
[56] The two-exponent scaling relation $\eta = (4-D+2\beta/\nu)/2$ yields $\eta = 0.5111(8)$; see [31]. However, this error is underestimated, as it does not consider scaling corrections nor the uncertainty in the critical value of $\mu$.
[60] G. Tarjus (private communication).
[63] Joint fits share the value of some fitting parameters such as the $L \to \infty$ extrapolation (which is the same for all random-field distributions) or the corrections-to-scaling exponent $\omega$ (which is common to all magnitudes).