Summability of the perturbative expansion for a zero-dimensional disordered spin model

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Abstract. We show analytically that the perturbative expansion for the free energy of the zero dimensional (quenched) disordered Ising model is Borel-summable in a certain range of parameters, provided that the summation is carried out in two steps: first, in the strength of the original coupling of the Ising model and subsequently in the variance of the quenched disorder. This result is illustrated by some high-precision calculations of the free energy obtained by a straightforward numerical implementation of our sequential summation method.

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1. Introduction

One of the simplest cases to study the effects of impurities in magnetic systems occurs in some doped uniaxial antiferromagnets such as $Fe_{0.46}Zn_{0.54}F_2$ [1], where the Zn impurities do not induce competing interactions and Monte Carlo simulations with the three-dimensional random-site Ising model reproduce accurately the behavior close to the Curie temperature [2]. Nevertheless, the successful application of renormalization group techniques [3] to the calculation of critical exponents for the pure Ising model makes perturbative expansions a very appealing alternative to the Monte Carlo calculations mentioned above.

The appropriate field theory for this problem is the usual $u\phi^4$ theory in three dimensions modified with a random mass term, and the analytic difficulties raised by the quenched impurities are usually bypassed with the replica trick: the system is temporarily replaced by n non-interacting copies for which the disorder is now annealed, yielding an $O(n)$ theory with coupling constant proportional to minus the variance of the quenched disorder w, with an additional cubic anisotropy term with coupling constant u. This theory can be studied perturbatively and the results corresponding to the original system recovered (in principle) in the limit $n \to 0$.

Indeed, there are rather accurate analytic calculations of the critical exponents of the random-site Ising model [4], but the situation is less satisfactory than for pure systems [3]. For instance, the asymptotic parameter in disordered systems is $\sqrt{\epsilon}$ instead of ϵ (where $\epsilon = 4 - d$, d being the dimensionality of the space), and the β-function computed at three loops shows no stable fixed point [5]. In an attempt to understand disordered systems in a simpler setting, Bray *et al* [6] and McKane [7] studied the asymptotic expansion for the free energy in the zero-dimensional case, with discouraging results: the toy-model was found non Borel-summable, in sharp contrast with the (ordered) zero-dimensional $u\phi^4$ theory, where Borel-summability has been rigorously proved [3]. (Indeed, Borel summability in three dimensions has been also proved [8].)

We consider in this paper the same zero-dimensional problem of references [6, 7], and show that a slightly more elaborated procedure can recover the free energy from the perturbative expansion: the Borel-summation has to be carried out first in the strength of the original coupling of the Ising model u and subsequently in the variance of the quenched disorder w.

The layout of the rest of the paper is as follows. We first review the main results of Bray *et al* and McKane in section 2; the derivation of the Borel-summable double asymptotic expansion for the free energy is carried out in the third section; in section 4 we give the details of our numerical procedure and some illustrative examples, and the paper ends with a brief summary.

2. Previous Results

The free energy of the zero-dimensional Ising model with quenched dilution can be studied with a $u\phi^4$ theory with a random mass term

$$f(u,w) = -\int_{-\infty}^{\infty} \frac{\mathrm{d}\psi}{\sqrt{4\pi w}}\, \mathrm{e}^{-\psi^2/4w} \log Z(\psi,u) \tag{1}$$

where

$$Z(\psi,u) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\phi}{\sqrt{2\pi}}\, \mathrm{e}^{-\frac{1}{2}(1+\psi)\phi^2 - \frac{1}{4}u\phi^4} \tag{2}$$

is the partition function. Bray *et al* [6] fix

$$\lambda = w/u > 0 \tag{3}$$

and write the asymptotic expansion for the free energy as

$$f(u,w) \sim -\sum_{K=1}^{\infty} A_K(\lambda) u^K \tag{4}$$

where the $A_K(\lambda)$ are polynomials in λ with rational coefficients and degree K. Then they use the replica trick an a saddle point argument to infer that the asymptotic behavior of $A_K(\lambda)$ as $K \to \infty$ is given by two terms:

$$A_K(\lambda) \sim \begin{cases} A_K^{(1)}(\lambda) + A_K^{(\infty)}(\lambda), & 0 < \lambda < 1 \\ A_K^{(\infty)}(\lambda), & 1 \le \lambda \end{cases} \tag{5}$$

The term $A_K^{(1)}(\lambda)$ is dominant for $0 < \lambda \le \frac{1}{2}$ and Borel-summable, while

$$A_K^{\infty}(\lambda) = -\frac{K!(4\lambda)^k}{\sqrt{\pi}K^{3/2}} \exp(-\gamma\sqrt{K} + \sigma)\cos(\mu\sqrt{K} + \delta) \tag{6}$$

with coefficients γ, σ, μ and δ that they ultimately adjust by comparison between this asymptotic formula and the numerical values of the $A_K(\lambda)$ for K up to 200 and several values of λ greater than $\frac{1}{2}$. The increasing-period cosine oscillations suggest them that there is an essential singularity of the Borel-transform in the positive real u axis, and therefore the series is not Borel-summable for any finite disorder (although the term $A_K^{(\infty)}(\lambda)$ is subdominant in the region $0 < \lambda \le \frac{1}{2}$). Despite the non-summability of the series (4), i.e. acknowledging that there would be some error even if they could compute with all the terms, Bray *et al* set $u = 1$ and try to extract an accurate answer from (4) using a conformal transformation suitable for oscillatory series, but they find that as soon as $\lambda > \frac{1}{2}$ the method fails.

McKane [7] avoids the replica trick in favor of a more direct analysis that clearly identifies the two sources of non-analytic behavior of the free energy leading to the two contributions is equation (5): the branch cut along the negative u axis and the zeros of the partition function (2). Moreover, he obtains the exact values of γ, σ, μ and δ in terms of the zeros of the partition function and guesses the form of the nonperturbative contributions to the free energy that yield the asymptotic behavior (5).

The possibility of alternative summation methods, however, remained unaddressed in these works.

3. Summable asymptotic expansion for the free energy

For analyticity considerations, we remark that in terms of the new variable

$$z = \frac{1+\psi}{2u^{1/2}} \tag{7}$$

the partition function (2) can be written as

$$Z(\psi, u) = \frac{2^{1/2}}{\pi^{1/2}u^{1/4}} \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-zt^2-\frac{1}{4}t^4} \tag{8}$$

which in turn [7] can be expressed in terms of a parabolic cylinder function or a modified Bessel function:

$$Z(\psi, u) = u^{-1/4}(2\pi)^{-1/2}\mathrm{e}^{\frac{1}{2}z^2} z^{1/2} K_{1/4}(\tfrac{1}{2}z^2) \tag{9}$$

$$= (2u)^{-1/4}\mathrm{e}^{\frac{1}{2}z^2} D_{-1/2}(\sqrt{2}z). \tag{10}$$

There is, however, another expression particularly suitable for later summability considerations: via a Mellin-Barnes integral [9], the partition function can be written as a Laplace transform in z^2,

$$Z(\psi, u) = \frac{1}{(4u)^{1/4}\Gamma(\frac{1}{4})} \int_0^\infty \frac{\mathrm{d}t\, \mathrm{e}^{-z^2t}}{[t(t+1)]^{3/4}} \qquad (\mathrm{Re}(z^2) > 0) \tag{11}$$

$$= \frac{z}{(4u)^{1/4}\Gamma(\frac{1}{4})} \int_0^\infty \frac{\mathrm{d}t\, \mathrm{e}^{-t}}{[t(t+z^2)]^{3/4}} \qquad (\mathrm{Re}\, z > 0). \tag{12}$$

Equations (8) or (10) show that

$$F(z) = u^{1/4} Z(\psi, u) \tag{13}$$

is an entire function of z, and from equation (9) and reference [10] it follows that the only zeros of $F(z)$ appear in complex-conjugate pairs $z_k, \bar{z}_k$ in the $\mathrm{Re}\, z < 0$ half-plane, and are given asymptotically by

$$z_k^2 \sim -\tfrac{1}{2}\log 2 - \mathrm{i}\tfrac{\pi}{2}(4k+3) \qquad (\mathrm{Im}\, z_k > 0, k = 0, 1, 2, \ldots). \tag{14}$$

To stay close to the notation of Bray *et al* and McKane and yet show that we are dealing with a two-variable problem, we set

$$\lambda = w/u \tag{15}$$

$$\mu = 1/(2u^{1/2}) \tag{16}$$

$$\Phi(\mu, \lambda) = f(u, w). \tag{17}$$

In terms of these variables, the free energy can be written as

$$-\Phi(\mu, \lambda) = \tfrac{1}{2}\log(2\mu) + \frac{1}{(\lambda\pi)^{1/2}} \int_{-\infty}^\infty \mathrm{d}z\, \mathrm{e}^{-(z-\mu)^2/\lambda} \log[F(z)] \tag{18}$$

were the integral in the right-hand side is taken along the real axis, and is well defined for any complex μ and $\mathrm{Re}\, \lambda > 0$. Note that as z varies along the real axis in the increasing sense, $F(z)$ traces the positive axis in the decreasing sense, but as we shift (say) upwards the integration path and z picks up a constant (positive) imaginary part

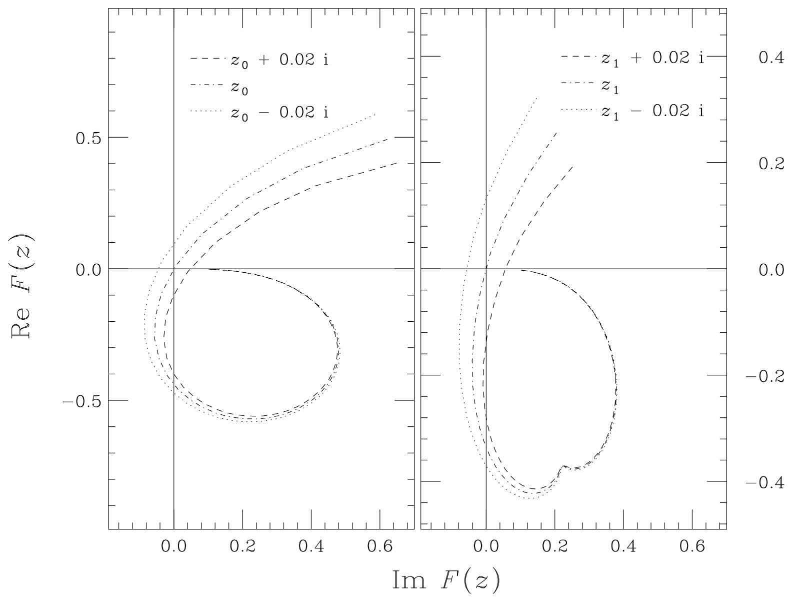

Figure 1. Paths in the complex plane followed by the function $F(z)$ as z varies along straight lines parallel to the real axis slightly below, on and slightly above the first two zeros of $F(z)$ in the upper half plane ($z = z_k + t$, $z = z_k \pm \mathrm{i}\epsilon + t$, $-\infty < t < \infty$). The dip in the right plot is a trace of the first loop of the spiral (shown in the left plot) which unfolds via a cusp (when $F'(z) = 0$) as the integration path shifts upwards.

[i.e. $z = t + i\epsilon, t \in (-\infty, \infty)$], equation (10) shows that $F(z)$ traces a spiral ending at the origin for $t = \infty$. In particular, if $z = t + z_k$ the spiral passes again through the origin (see figure 1). Therefore, if we shift vertically the integration path to $z = t + \mu$ (with $0 < \mathrm{Im}\,\mu \neq \mathrm{Im}\,z_k$), we pick up a finite number of contributions from the zeros of $F(z)$ whose evaluation in terms of the complementary error function is given by the following self-explanatory chain of equations:

$$
\begin{aligned}
-\Phi(\mu,\lambda) &= \tfrac{1}{2}\log(2\mu) + \frac{1}{(\lambda\pi)^{1/2}} \int_{-\infty+\mu}^{\infty+\mu} \mathrm{d}z\, \mathrm{e}^{-(z-\mu)^2/\lambda} \log[F(z)] \\
&\quad + \sum_{0<\mathrm{Im}\,z_k<\mathrm{Im}\,\mu} \frac{1}{(\lambda\pi)^{1/2}} \int_{-\infty, z_k^{(+)}} \mathrm{d}z\, \mathrm{e}^{-(z-\mu)^2/\lambda} \log[F(z)] \qquad (19) \\
&= \tfrac{1}{2}\log(2\mu) + \frac{1}{(\lambda\pi)^{1/2}} \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{-t^2/\lambda} \log[F(t+\mu)] \\
&\quad + \sum_{0<\mathrm{Im}\,z_k<\mathrm{Im}\,\mu} \frac{2\pi\mathrm{i}}{(\lambda\pi)^{1/2}} \int_{0}^{\infty} \mathrm{d}t\, \mathrm{e}^{-(z_k-t-\mu)^2/\lambda} \qquad (20) \\
&= \tfrac{1}{2}\log(2\mu) + \frac{1}{(\lambda\pi)^{1/2}} \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{-t^2/\lambda} \log[F(t+\mu)] \\
&\quad + \mathrm{i}\pi \sum_{0<\mathrm{Im}\,z_k<\mathrm{Im}\,\mu} \mathrm{erfc}\left(\frac{\mu - z_k}{\lambda^{1/2}}\right). \qquad (21)
\end{aligned}
$$

There are three key points to note in these expressions. First, the determination of the logarithms: they are real for real argument (as $t \to \infty$) and must be followed by continuity according to figure 1. Second, for $|\,\mathrm{Im}\,\mu| < \mathrm{Im}\,z_0$ there are no contributions

Table 1. Comparison among the $K = 3$ partial sum of equation (4), the shifted integral (22), the shifted integral plus corrections (21), and the unshifted integral (18) for $\lambda = 1$ and three values of μ with $0 \le \operatorname{Im}\mu < \operatorname{Im} z_0$, $\operatorname{Im} z_0 < \operatorname{Im}\mu < \operatorname{Im} z_1$ and $\operatorname{Im} z_1 < \operatorname{Im}\mu < \operatorname{Im} z_2$ respectively.

	$\mu = 2$	$\mu = 1 + 2\mathrm{i}$	$\mu = 1 + 5\mathrm{i}/2$
Partial sum	0.0155640	−0.00752925 − i0.0099890	−0.00624927 − i0.00593754
Equation (22)	0.0155379	−0.00752689 − i0.0099516	−0.00624739 − i0.00593804
Equation (21)	0.0155379	−0.00620929 − i0.0107989	−0.00908002 − i0.00573683
Equation (18)	0.0155379	−0.00620929 − i0.0107989	−0.00908002 − i0.00573683

from the zeros and we do have

$$-\Phi(\mu,\lambda) = \tfrac{1}{2}\log(2\mu) + \frac{1}{(\lambda\pi)^{1/2}}\int_{-\infty}^{\infty} \mathrm{d}z\, \mathrm{e}^{-z^2/\lambda} \log[F(z+\mu)]. \qquad (22)$$

We stress that the right-hand side of equation (22) is well defined whenever $\operatorname{Im}\mu \neq \operatorname{Im} z_k$, but it is equal to the free energy only for $|\operatorname{Im}\mu| < \operatorname{Im} z_0$. As a numerical illustration of this point, in the last three rows of table 1 we show numerical calculations of the shifted integral (22), the shifted integral plus corrections (21), and the unshifted integral (18) for $\lambda = 1$ and three values of μ with $0 \le \operatorname{Im}\mu < \operatorname{Im} z_0$, $\operatorname{Im} z_0 < \operatorname{Im}\mu < \operatorname{Im} z_1$ and $\operatorname{Im} z_1 < \operatorname{Im}\mu < \operatorname{Im} z_2$ respectively. Third, equation (22) can be rewritten as a Laplace transform in λ^{-1}

$$-\Phi(\mu,\lambda) = \tfrac{1}{2}\log(2\mu) + \frac{1}{\lambda^{1/2}}\int_0^{\infty} \mathrm{d}t\, \mathrm{e}^{-t/\lambda} B(\mu,t) \qquad (23)$$

where

$$B(\mu,t) = \frac{1}{(4\pi t)^{1/2}} \log\left[F(\mu + t^{1/2})F(\mu - t^{1/2})\right]. \qquad (24)$$

The singularities of $B(\mu,t)$ as a function of t are readily calculated in terms of the z_k and (apart from the trivial singularity at the origin) stay at a finite distance of the positive real axis whenever $\operatorname{Im}\mu \neq \operatorname{Im} z_k$.

Equations (12) and (13) imply that we can obtain a Borel-summable asymptotic expansion for $F(z)$ and its derivatives, and equations (23) and (24) that, as far as we stay away from the zeros of F, we can also obtain a Borel-summable series in λ. Our final result will be, therefore, a Borel-summable series (in λ) with each of its coefficients given by a Borel-summable series (in $\mu^{-2} = 4u$).

Now we proceed to the details of the derivation. We first expand $\log[F(z+\mu)]$ in equation (22) in (convergent) Taylor series around $z = 0$—or equivalently $B(\mu,t)$ in equation (23) around $t = 0$—and integrate term by term to get

$$-\Phi(\mu,\lambda) \sim \tfrac{1}{2}\log(2\mu) + \sum_{n=0}^{\infty} (\log\circ F)^{(2n)}(\mu)\frac{\lambda^n}{4^n n!}. \qquad (25)$$

Then, we use the Borel-summable asymptotic expansion for $F(z)$ that follows directly from equation (12)

$$F(z) \sim (2z)^{-1/2}{}_2F_0(\tfrac{1}{4}, \tfrac{3}{4};; -z^{-2}) \qquad (-\tfrac{\pi}{2} < \arg z < \tfrac{\pi}{2}) \qquad (26)$$

to obtain asymptotic expansions for the derivatives $(\log\circ F)^{(2n)}(\mu)$. Incidentally, we mention that the summability sector of $F(z)$ in terms of u is just $-\pi < \arg u < \pi$, i.e. we are rederiving the well-known summability of the (ordered) zero-dimensional $u\phi^4$ theory as explained, for example, in reference [3].

Note that for the zero-th derivative we have

$$\log[F(\mu)] \sim -\tfrac{1}{2}\log(2\mu) + \log\left[{}_2F_0(\tfrac{1}{4},\tfrac{3}{4};;-\mu^{-2})\right] \tag{27}$$

$$= -\tfrac{1}{2}\log(2\mu) + \sum_{k=1}^{\infty}\frac{(-1)^k b_k}{\mu^{2k}} \tag{28}$$

and the logarithm cancels the first term in the right-hand side of equation (22). (Equation (28) is in fact the definition of the coefficients b_k). Higher derivatives have pure asymptotic power series that can be computed easily in terms of the zero-th derivative series

$$(\log\circ F)^{(2n)}(\mu) \sim \frac{1}{\mu^{2n}}\left[\frac{\Gamma(2n)}{4} + \sum_{k=1}^{\infty}\frac{\Gamma(2n+2k)}{\Gamma(2n)}\frac{(-1)^k b_k}{\mu^{2k}}\right]. \tag{29}$$

The final result is the double series

$$-\Phi(\mu,\lambda) \sim \sum_{k=1}^{\infty}\frac{(-1)^k b_k}{\mu^{2k}} + \sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{\lambda}{4\mu^2}\right)^n\left[\frac{\Gamma(2n)}{4} + \sum_{k=1}^{\infty}\frac{\Gamma(2n+2k)}{\Gamma(2n)}\frac{(-1)^k b_k}{\mu^{2k}}\right] \tag{30}$$

whose structure in terms of the original variables is ($c_{00} = 0$)

$$-f(u,w) \sim \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty} c_{n,k}u^k\right) w^n\,. \tag{31}$$

This double series is formally equivalent to equations (B10), (B11) and (B13) in reference [6] (the only difference is that we have transposed the indices in $c_{n,k}$), but with a definite ordering that cannot be altered without further considerations: to take advantage of the summability we should first Borel-sum the series in u (see the precise definition in the next section)

$$c_n(u) = \text{Borel}\sum_{k=0}^{\infty} c_{n,k}u^k \tag{32}$$

and then in w

$$-f(u,w) = \text{Borel}\sum_{n=0}^{\infty} c_n(u)w^n \tag{33}$$

instead of performing first the finite sum

$$A_K(\lambda) = \sum_{n=0}^{K} c_{n,K-n}\lambda^n \tag{34}$$

followed by some kind of summation in u. Although this last reordering preserves the asymptotic nature of the series to the function defined by equation (22) (as we also illustrate in the first row of table 1), the numerical and analytic evidence presented by Bray *et al* shows that it spoils the summability of equation (30).

4. Numerical algorithm and results

Of the two standard methods to achieve the analytic continuation implicit in the Borel-summation (conformal mapping or Padé approximants) we have chosen the second because it does not require a precise knowledge of the singularities of the function, although some Padé approximants may introduce spurious singularities close to the integration path. Equation (29) suggests that an appropriate numerical Borel-sum of the u series is

$$c_n(u) \approx \frac{1}{u^{2n+1/2}} \int_0^\infty \mathrm{e}^{-t/u} t^{2n-1/2} P^{[p,q]}(t)\,\mathrm{d}t \tag{35}$$

where $P^{[p,q]}(t)$ is the $[p,q]$-Padé approximant for

$$\hat{c}_n(t) = \sum_{k=0}^{p+q} \frac{c_{n,k} t^k}{\Gamma(2n+k+\frac{1}{2})}. \tag{36}$$

Equation (35) has the additional advantage over other equivalent forms that the Padé approximants and their zeros (which we use later) need to be calculated only once. Furthermore, to avoid problems with the numerical integration we expand the Padé approximant as a polynomial plus partial fractions, i.e. assuming that all the poles are simple

$$P^{[p,q]}(t) = \sum_{k=0}^{p-q} p_k t^k + \frac{R(t)}{S(t)} = \sum_{k=0}^{p-q} p_k t^k + \sum_{k=1}^{q} \frac{R(t_k)}{S'(t_k)(t-t_k)} \tag{37}$$

and the integration in equation (35) can be carried out in terms of complete and incomplete gamma functions evaluated at the q poles t_k of $P^{[p,q]}(t)$:

$$c_n(u) \approx \sum_{k=0}^{p-q} p_k u^k \Gamma(2n+k+\tfrac{1}{2}) + \frac{\Gamma(2n+\frac{1}{2})}{u^{2n+1/2}} \sum_{k=1}^{q} \frac{R(t_k)}{S'(t_k)} \mathrm{e}^{-t_k/u} (-t_k)^{2n-1/2} \Gamma(-2n+\tfrac{1}{2}, -t_k/u). \tag{38}$$

We have performed our calculations with the same 200 terms of the expansion (28) used in reference [6], using the arbitrary precision numerical capabilities of *Mathematica*. Figure 2 shows some typical results of this first stage of our double-summation method. For four values of u and the lowest eleven coefficients $c_0(u), \ldots, c_{10}(u)$ we plot a measure Δ of the number of correct digits to the right of the decimal point,

$$\Delta = -\log_{10} |c_n(u)^{\text{exact}} - c_n(u)^{\text{summed}}| \tag{39}$$

as a function of the order p of the diagonal Padé approximants $P^{[p,p]}(t)$ in equation (35). As a general trend, the precision of the summed coefficients increases with the order of the approximant, although with smaller slope for higher values of u (note that we have used the same scales in the four plots). The irregularities in figure 2 are often due to ill-conditioned linear systems of equations in the calculation of the Padé approximants, and are usually corrected using higher precision or higher order approximants. We have also checked that neighboring off-diagonal Padé approximants give similar results.

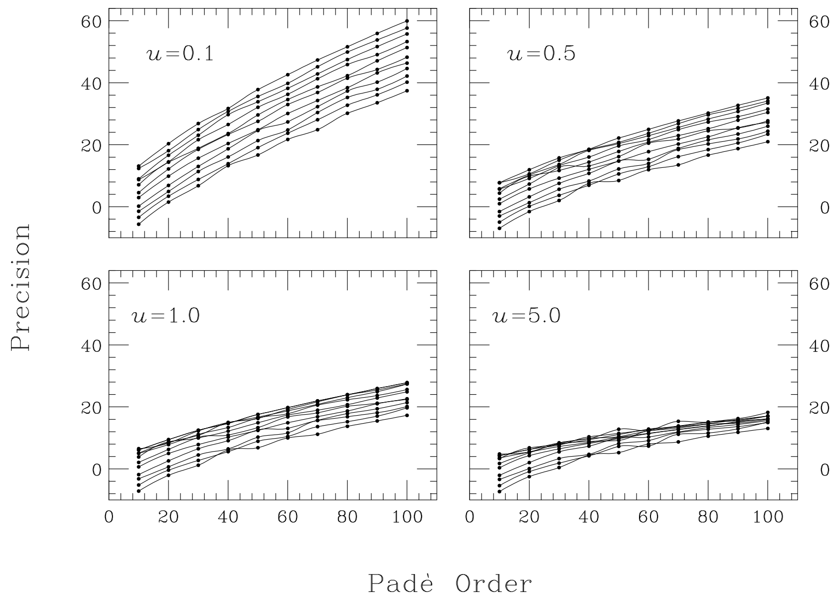

Figure 2. Number of correct digits to the right of the decimal point in the Borel-summed coefficients $c_0(u),\ldots,c_{10}(u)$ (from top to bottom at $p=10$) as a function of the order of the diagonal Padé approximant in equation (35).

The second stage, the summation in w, is equivalent to the $n=0$ summation of the first stage. Now the Borel-transformed coefficients $c_n(u)/\Gamma(n+\frac{1}{2})$ are precisely the coefficients of the convergent Taylor expansion of $t^{1/2}B(\mu,t)$, and to illustrate how well the Padé approximants continue the Taylor series beyond the radius of convergence, we plot in figure 3 the exact integrand $B(\frac{1}{2},t)$ corresponding to $u=1$ in equation (23), and its approximations $t^{-1/2}P^{[p,p]}(t)$ for $p=10$, 15 and 20. The radius of convergence of the Taylor series is marked in the plot by the vertical line at $t=|z_0-\frac{1}{2}|^2\approx 6.54$.

In figure 4 (again with $u=1$ and for several values of λ) we plot the number of correct digits to the right of the decimal point in the Borel-summed free energy as a function of the order of the diagonal Padé approximant. We have performed the calculations with all the $c_n(u)$ of the same precision, so that the numerical error is due only to this second-stage summation. The first feature worth mentioning is the steady increase in the precision of the summed free energy (with a few exceptions that are related again to the Padé approximants), although the convergence is slower for larger values of λ. Note that the plateau in the summation of Bray *et al* for $\lambda=0.2$ ends approximately at fifty terms of the series, while at $p=40$ (i.e. with 81 terms of the series) we have steady numerical convergence; $\lambda=0.5$ corresponds to the switch between asymptotic behaviors in equation (5), while in our reordering it is not a distinguished point; finally, for $\lambda=0.8$ Bray *et al* are already unable to get a good result, while our summation still gives four correct digits for $\lambda=5$.

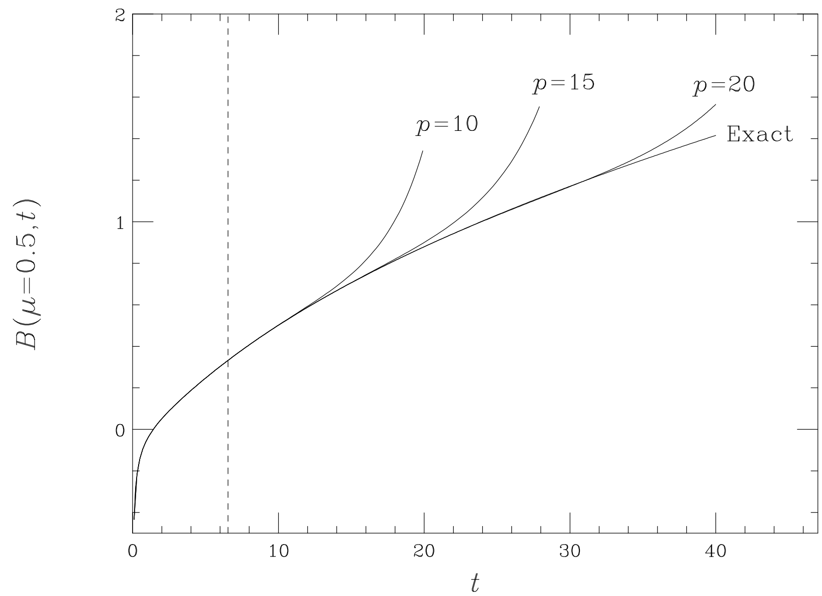

Figure 3. Exact integrand $B(\frac{1}{2},t)$ corresponding to $u=1$ in equation (23) and its approximations $t^{-1/2}P^{[p,p]}(t)$ for $p=10,15$ and 20. The vertical line marks the radius of convergence of the Taylor series for $B(\frac{1}{2},t)$.

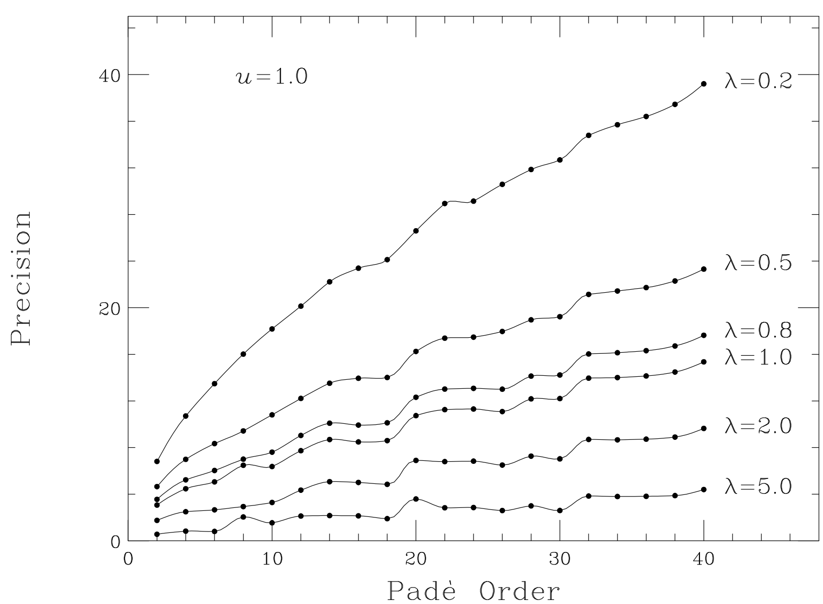

Figure 4. Number of correct digits to the right of the decimal point in the Borel-summed free energy $f(1,\lambda)$ as a function of the order of the diagonal Padé approximant used in the second stage of the double summation.

5. Summary

We have analyzed the Borel-summability of the perturbative expansion for the free energy in one of the simplest disordered systems, the zero dimensional quenched diluted

Ising model. This expansion is a double asymptotic series in the original coupling u of the Ising model and in the variance w of the quenched disorder, and we have shown that the expansion is in fact Borel-summable by a sequential method: first we sum in the coupling u and subsequently in the variance w. This summability results are valid at least for $-\pi < \arg u < \pi$ and $\mathrm{Re}\, w > 0$, with the understanding that if we want to recover the free energy and not the integral (22), additional erfc terms are needed in the domains specified in section 3.

A straightforward numerical implementation of our double summation method has permitted us to obtain (with the same number of terms of the original series) accurate free energies for values of the parameters well beyond the region where Bray *et al* were already unable to get an estimate.

We expect that the insight provided by these zero dimensional results will be helpful in the understanding of the more complicated higher-dimensional case, despite the significant qualitative differences motivated by Griffiths singularities [11] in the generalized sense of Dotsenko [12]. In fact we feel that a clear understanding of the Griffiths phase in the field-theoretical setting will be needed before the problem will be finally solved.

Finally, we would like to mention as open problems the study of the negative λ region, and a more thorough understanding of the solution of this problem using replicas.

Acknowledgments

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