Interpolation of compact bilinear operators among quasi-Banach spaces and applications

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Dedicated to Professor Thomas Kühn on the occasion of his 65th birthday.

We study the interpolation properties of compact bilinear operators by the general real method among quasi-Banach couples. As an application we show that commutators of Calderón-Zygmund bilinear operators $S : L_p \times L_q \longrightarrow L_r$ are compact provided that $1/2 < r < 1$, $1 < p, q < \infty$ and $1/p + 1/q = 1/r$.

1 Introduction

This paper refers to interpolation theory, a consolidated branch of functional analysis which has found important applications in harmonic analysis, partial differential equations and operator theory, as one can see in the monographs by Butzer and Berens [6], Bergh and Löfström [4], Triebel [44, 45], König [32] or Bennett and Sharpley [1]. Inside this theory, interpolation of compact linear operators is a very active research area. It started with the pioneering results of Krasnosel'skiĭ [34], Lions and Peetre [37] and Persson [42] in the early 1960. Since then it has attracted the attention of many authors (see [9] and the references given there).

As for the real interpolation method $(A_0, A_1)_{\theta,q}$, it was a long standing problem to show that if any restriction of the operator is compact, then the interpolated operator is also compact. It was solved in 1992 by Cwikel [17] and Cobos, Kühn and Schonbek [13]. Later the result was extended to couples of quasi-Banach spaces by Cobos and Persson [15].

Interpolation properties of compact bilinear (or multilinear) operators were studied by Calderón [7] in his foundational paper on the complex interpolation method. The case of the real method has been investigated

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much more recently by Fernandez and Silva [22] and by Fernández-Cabrera and Martínez [23, 24] by using the techniques developed by Cobos and Peetre [14] and Cobos, Kühn and Schonbek [13] to deal with linear operators.

Results of [23, 24] refer to the general real method which is defined by replacing the weighted L_q norm used in $(A_0, A_1)_{\theta,q}$ by a more general lattice norm. The outcome is a very flexible method. For example, working with the couple (L_1, L_∞), the real method only produce Lebesgue spaces and Lorentz spaces. However, the general real method can generate any interpolation space with respect to (L_1, L_∞) (see [5] or [39]). In particular, Orlicz spaces and Lorentz-Zygmund spaces arise by using the general real method.

An important motivation for the investigations on interpolation properties of compact bilinear operators has been the fact that this kind of operators occurs rather naturally in harmonic analysis. This has been shown in the last few years in the papers by Bényi and Torres [3], Bényi and Oh [2], Hu [27] and other authors. In particular, Bényi and Torres [3] have established compactness of commutators S of Calderón-Zygmund bilinear operators (see Section 6 below) when acting from $L_p \times L_q$ into L_r provided that $1 < p, q < \infty, 1 \leq r < \infty$ and $1/p + 1/q = 1/r$. These operators are bounded in a more broad range for the parameter r. Namely, for $1/2 < r < \infty$ and $1/p + 1/q = 1/r$ (see the papers by Lerner et al [36] and Pérez et al [41]). So, it is natural to wonder for compactness of S in the range $1/2 < r < 1$, where the target space is no longer a Banach space but a quasi-Banach space and therefore duality arguments cannot be used. In this paper we solve this problem by means of interpolation techniques.

We start by reviewing in Section 2 the construction of the general real method for quasi-Banach couples. We also establish there some auxiliary results for our later considerations. In Section 3 we show the interpolation theorem for bounded bilinear operators, with a handy estimate for the norm of the interpolated operator. Then we review the properties of compact bilinear operators among quasi-Banach spaces and we prove other two auxiliary results. Section 4 contains the abstract results on interpolation of compact bilinear operators in the setting of the quasi-Banach spaces. The results extend those of Fernández-Cabrera and Martínez [23, 24] for the Banach case. We omit details when the arguments of [23, 24] need only minor modifications, but sometimes we must give separate proofs. Applications of these abstract results are given in the last two sections. In Section 5 we establish a reinforced version of an interpolation result of Calderón and Zygmund [8] on bounded bilinear operators among L_p spaces. Finally, in Section 6, we prove compactness of commutators of bilinear Calderón-Zygmund operators $S : L_p \times L_q \longrightarrow L_r$ for $1 < p, q < \infty, 1/2 < r < 1$ and $1/p + 1/q = 1/r$.

2 Real interpolation of quasi-Banach spaces

Important spaces as the Lebesgue spaces L_p or the Schatten-von Neumann operator spaces $S_p(H)$ are defined for $0 < p < 1$. Then they are not Banach spaces but quasi-Banach spaces, that is to say, the triangle inequality needs an additional constant $c \geq 1$.

Let $(A, \|\cdot\|_A)$ be a quasi-Banach space with constant $c = c_A \geq 1$ in the quasi-triangle inequality and let $0 < p \leq 1$ such that $c = 2^{1/p-1}$. Then there is another quasi-norm $|||\cdot|||$ on A which is equivalent to $\|\cdot\|_A$ and such that $|||\cdot|||^p$ satisfies the triangle inequality (that is to say, $|||\cdot|||$ is a p-norm). See [33, §5.10] or [32, Proposition 1.c.5]. We say that A is a p-normed quasi-Banach space. Clearly, if $0 < r < p$, then A is also an r-normed quasi-Banach space.

By a (p*-normed*) *quasi-Banach couple* $\bar{A} = (A_0, A_1)$ we mean two (p-normed) quasi-Banach spaces A_j which are continuously embedded in the same Hausdorff topological vector space. Given $t > 0$, *Peetre's* K*- and* J*-functionals* are defined by

$$K(t, a) = K(t, a; A_0, A_1) = \inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} : a = a_0 + a_1, a_j \in A_j\}$$

where $a \in A_0 + A_1 = \Sigma(\bar{A})$, and

$$J(t,a) = J(t,a;A_0,A_1) = \max\{\|a\|_{A_0}, t\|a\|_{A_1}\}, \quad a \in A_0 \cap A_1 = \Delta(\bar{A})$$

(see [4, 44, 5]). Functionals $K(t,\cdot)$ and $J(t,\cdot)$ are quasi-norms in $\Sigma(\bar{A})$ and $\Delta(\bar{A})$, respectively. Note that we can take the same constant $c \geq 1$ in the quasi-triangle inequality for any $t > 0$. The functional $K(1,\cdot)$ coincides with the quasi-norm $\|\cdot\|_{\Sigma(\bar{A})}$ of $\Sigma(\bar{A})$ and $J(1,\cdot)$ is $\|\cdot\|_{\Delta(\bar{A})}$.

Observe that if $\|\cdot\|_{A_0}$ and $\|\cdot\|_{A_1}$ are p-norms then $J(t,\cdot)$ is a p-norm on $\Delta(\bar{A})$, and the functional

$$K_p(t,a) = \inf\left\{\left(\|a_0\|_{A_0}^p + t^p\|a_1\|_{A_1}^p\right)^{1/p} : a = a_0 + a_1, a_j \in A_j\right\}$$

is a p-norm on $\Sigma(\bar{A})$, which is equivalent to $K(t,\cdot)$. Namely,

$$K(t,a) \leq K_p(t,a) \leq 2^{1/p}K(t,a), \quad a \in \Sigma(\bar{A}).$$

The general real interpolation method has been studied in the monographs by Peetre [40] and by Brudnyĭ and Krugljak [5], and the articles by Cwikel and Peetre [18], Nilsson [38, 39], Cobos, Fernández-Cabrera, Manzano and Martínez [10] and Cobos, Fernández-Cabrera and Martínez [11, 12] among other papers. Following [38], here we consider this method realized in discrete way. Subsequently, by a *quasi-Banach sequence lattice* Γ we mean a quasi-Banach space of real valued sequences with $\mathbb{Z}$ as index set which satisfies the following properties:

(i) Γ contains all sequences with only finitely many non-zero co-ordinates.

(ii) Whenever $|\xi_m| \leq |\eta_m|$ for each $m \in \mathbb{Z}$ and $(\eta_m) \in \Gamma$, then $(\xi_m) \in \Gamma$ and $\|(\xi_m)\|_\Gamma \leq \|(\eta_m)\|_\Gamma$.

We say that Γ is *K-non-trivial* if $\left(\min(1,2^m)\right) \in \Gamma$.

If $\bar{A} = (A_0,A_1)$ is a quasi-Banach couple and Γ is K-non-trivial, the *K-space* $\bar{A}_{\Gamma;K} = (A_0,A_1)_{\Gamma;K}$ is formed of all $a \in \Sigma(\bar{A})$ such that $(K(2^m,a)) \in \Gamma$. We put

$$\|a\|_{\bar{A}_{\Gamma;K}} = \|(K(2^m,a))\|_\Gamma.$$

Since

$$K(2^m,a) \leq \min(1,2^m)J(1,a),\ a \in \Delta(\bar{A}),\ m \in \mathbb{Z},$$

and

$$\min(1,2^m)K(1,a) \leq K(2^m,a),\ a \in \Sigma(\bar{A}),\ m \in \mathbb{Z},$$

one can check that $\bar{A}_{\Gamma;K}$ is an *intermediate space* with respect to $\bar{A}$, that is to say,

$$A_0 \cap A_1 \hookrightarrow (A_0,A_1)_{\Gamma;K} \hookrightarrow A_0 + A_1\,.$$

Here $\hookrightarrow$ means continuous embeddings.

Let $\bar{A} = (A_0,A_1), \bar{B} = (B_0,B_1)$ be quasi-Banach couples. By $T \in \mathcal{L}(\bar{A},\bar{B})$ we mean that T is a linear operator from $\Sigma(\bar{A})$ into $\Sigma(\bar{B})$ whose restriction to each A_j defines a bounded operator from A_j into B_j with quasi-norm $\|T\|_{A_j,B_j}$, $j = 0,1$.

If $T \in \mathcal{L}(\bar{A},\bar{B})$, it is not hard to check that the restriction

$$T : (A_0,A_1)_{\Gamma;K} \longrightarrow (B_0,B_1)_{\Gamma;K}$$

is bounded with quasi-norm

$$\|T\|_{(A_0,A_1)_{\Gamma;K},(B_0,B_1)_{\Gamma;K}} \leq \max\{\|T\|_{A_0,B_0}, \|T\|_{A_1,B_1}\}.$$

A better estimate can be obtained if we know the behaviour of the norms of the shift operators on Γ (see [11, 12]). Given $k \in \mathbb{Z}$, the shift operator τ_k is defined by $\tau_k \xi = (\xi_{m+k})_{m\in\mathbb{Z}}$ for $\xi = (\xi_m)_{m\in\mathbb{Z}}$. In view of [11, Lemma 2.6], it will be useful for our aims to assume in what follows that τ_k is bounded in Γ for all $k \in \mathbb{Z}$ and

$$\lim_{n\to\infty} 2^{-n}\|\tau_n\|_{\Gamma,\Gamma} = 0 \quad \text{and} \quad \lim_{n\to\infty} \|\tau_{-n}\|_{\Gamma,\Gamma} = 0. \tag{2.1}$$

Following [12] we put

$$f(t) = f_\Gamma(t) = \|\tau_{[\log_2 t]}\|_{\Gamma,\Gamma} \quad , \quad t > 0\,,$$

where the logarithm is taken in base 2 and $[\,\cdot\,]$ is the greatest integer function.

It follows from (2.1) that

$$f(t) = o(\max(1,t)). \tag{2.2}$$

Let $M_1 = \max(1, \|\tau_1\|_{\Gamma,\Gamma})$, $M_2 = \sup\{f(t) : 0 < t \leq 1\} = \sup\{\|\tau_{-n}\|_{\Gamma,\Gamma} : n \geq 0\}$ and $M_3 = \sup\{f(t)/t : 1 \leq t < \infty\} = \sup\{2^{-n}\|\tau_n\|_{\Gamma,\Gamma} : n \geq 0\}$. Using that $\|\tau_{m+k}\|_{\Gamma,\Gamma} \leq \|\tau_m\|_{\Gamma,\Gamma}\|\tau_k\|_{\Gamma,\Gamma}$, $m,k \in \mathbb{Z}$, one can easily derive that:

For any $s,t > 0$, we have $f(st) \leq M_1 f(s) f(t)$. Hence, if (2.3)

$s < t$ we get that $f(s) \leq M_1 M_2 f(t)$ and $f(t)/t \leq M_1 M_3 f(s)/s$.

The argument used in [12, Lemma 4.3] in the Banach case also work in the more general quasi-Banach case considered here with the effect that if $T \in \mathcal{L}(\bar{A}, \bar{B})$ then

$$\|T\|_{\bar{A}_{\Gamma;K},\bar{B}_{\Gamma;K}} \leq \begin{cases} 0 \quad \text{if} \quad \|T\|_{A_j,B_j} = 0 \quad \text{for } j = 0 \text{ or } 1, \\ f(2)\|T\|_{A_0,B_0} f(\|T\|_{A_1,B_1}/\|T\|_{A_0,B_0}) \text{ otherwise.} \end{cases} \tag{2.4}$$

The following result is proved in [23, (5.2)] for Banach couples but the argument uses the Hahn-Banach theorem, so we give a new proof which is valid in the quasi-Banach case.

Subsequently, we write δ_m^k for the Kronecker delta. We also put $e_0 = (\delta_m^0)_{m\in\mathbb{Z}}$.

Lemma 2.1 *Let $\bar{A} = (A_0, A_1)$ be a quasi-Banach couple and let Γ be a K-non-trivial quasi-Banach sequence lattice satisfying (2.1). Then there is a constant $C > 0$ such that*

$$\sup_{0<t<\infty} \frac{K(t,a)}{f(t)} \leq C\|a\|_{\bar{A}_{\Gamma;K}} \quad , \quad a \in \bar{A}_{\Gamma;K}.$$

P r o o f. Given any $t > 0$, we can choose $k \in \mathbb{Z}$ such that $2^k \leq t < 2^{k+1}$. We have

$$\begin{aligned} K(t,a) \leq 2K(2^k,a) &= \frac{2}{\|e_0\|_\Gamma}\|K(2^k,a)e_0\|_\Gamma \\ &\leq \frac{2}{\|e_0\|_\Gamma}\|\tau_k\|_{\Gamma,\Gamma}\|\tau_{-k}(K(2^k,a)e_0)\|_\Gamma \\ &\leq \frac{2}{\|e_0\|_\Gamma}\|\tau_k\|_{\Gamma,\Gamma}\|(K(2^m,a))\|_\Gamma \\ &\leq \frac{2}{\|e_0\|_\Gamma} f(t)\|a\|_{\bar{A}_{\Gamma;K}}. \end{aligned}$$

□

For $0 < p \leq 1$, the quasi-Banach sequence lattice Γ is said to be (p, J)-*non-trivial* if

$$\sup\Big\{\Big(\sum_{m=-\infty}^{\infty}\big(\min(1,2^{-m})|\xi_m|\big)^p\Big)^{1/p} : \|(\xi_m)\|_\Gamma \leq 1\Big\} < \infty.$$

Clearly, if Γ is (p, J)-non-trivial then Γ is also (r, J)-non-trivial for any $p \leq r \leq 1$.

If $\bar{A} = (A_0, A_1)$ is a p-normed quasi-Banach couple and Γ is (p, J)-non-trivial, the J-*space* $\bar{A}_{\Gamma;J} = (A_0, A_1)_{\Gamma;J}$ consists of all sums $\sum_{m=-\infty}^{\infty} u_m$ (convergence in $\Sigma(\bar{A})$), where $(u_m) \subseteq A_0 \cap A_1$ and $(J(2^m, u_m)) \in \Gamma$. The quasi-norm on $\bar{A}_{\Gamma;J}$ is given by

$$\|a\|_{\bar{A}_{\Gamma;J}} = \inf\Big\{\|(J(2^m, u_m))\|_\Gamma : a = \sum_{m=-\infty}^{\infty} u_m\Big\}.$$

Since $\Sigma(\bar{A})$ is a p-normed quasi-Banach space, if $(J(2^m, u_m)) \in \Gamma$ then the series $\sum_{m=-\infty}^{\infty} u_m$ is convergent in $\Sigma(\bar{A})$ because

$$\sum_{m=-\infty}^{\infty}\|u_m\|^p_{\Sigma(\bar{A})} \leq \sum_{m=-\infty}^{\infty}\min(1,2^{-m})^p J(2^m, u_m)^p < \infty.$$

The following estimate is useful.

Lemma 2.2 *Let $\bar{A} = (A_0, A_1)$ be a p-normed quasi-Banach couple and let Γ be a (p, J)-non-trivial quasi-Banach sequence lattice satisfying (2.1). Then there is a constant $C > 0$ such that*

$$\|a\|_{\bar{A}_{\Gamma;J}} \leq C \inf_{t>0} f(t) J(t^{-1}, a) \quad , \quad a \in \Delta(\bar{A}).$$

P r o o f. Given $t > 0$, take $k \in \mathbb{Z}$ such that $2^{-k} \leq t < 2^{-k+1}$. If $a \in \Delta(\bar{A})$, using the representation $a = \sum_{m=-\infty}^{\infty} \delta_m^k a$ we get

$$\begin{aligned}\|a\|_{\bar{A}_{\Gamma;J}} &\leq \|(J(2^m, \delta_m^k a))\|_\Gamma = J(2^k, a)\|(\delta_m^k)\|_\Gamma \\ &= J(2^k, a)\|\tau_{-k}e_0\|_\Gamma \leq 2J(t^{-1}, a)\|\tau_{-k}\|_{\Gamma,\Gamma}\|e_0\|_\Gamma \\ &= 2J(t^{-1}, a)f(t)\|e_0\|_\Gamma.\end{aligned}$$

□

Corollary 2.3 *Let $\bar{A} = (A_0, A_1)$ be a p-normed quasi-Banach couple and let Γ be a (p, J)-non-trivial quasi-Banach sequence lattice satisfying (2.1). Then there is a constant $C > 0$ such that*

$$\|a\|_{\bar{A}_{\Gamma;J}} \leq C\|a\|_{A_0} f_\Gamma(\|a\|_{A_1}/\|a\|_{A_0}) \quad , \quad a \in \Delta(\bar{A}) \tag{2.5}$$

P r o o f. Take $t = \|a\|_{A_1}/\|a\|_{A_0}$ in Lemma 2.2. □

It turns out that $(A_0, A_1)_{\Gamma;K} \hookrightarrow (A_0, A_1)_{\Gamma;J}$. The converse embedding depends on the boundedness of the Calderón transform

$$\Lambda_p(\xi_m) = \Big(\big(\sum_{k=-\infty}^{\infty}(\min(1, 2^{m-k})|\xi_k|)^p\big)^{1/p}\Big)_{m\in\mathbb{Z}}.$$

Namely, if Λ_p is bounded in Γ then $(A_0, A_1)_{\Gamma;J} \hookrightarrow (A_0, A_1)_{\Gamma;K}$ (see [38, Lemma 2.5]).

Sometimes in our later computations it is useful that $\bar{A}_{\Gamma;K} = \bar{A}_{\Gamma;J}$ with equivalence of quasi-norms. To get it, working with couples of p-normed spaces, we shall assume that

$$\Gamma \text{ is } K\text{-non-trivial , } (p,J)\text{-non-trivial and the operator } \Lambda_p \text{ is bounded in } \Gamma. \tag{2.6}$$

In that case we write $\bar{A}_\Gamma$ for any of the spaces $\bar{A}_{\Gamma;K}$ or $\bar{A}_{\Gamma;J}$ and we denote by $\|\cdot\|_{\bar{A}_\Gamma}$ any of the two quasi-norms. This however will not cause any confusion.

For $0 < q \leq \infty$ we let ℓ_q be the usual space of q-summable scalar sequences with $\mathbb{Z}$ as index set. Let (λ_m) be a sequence of positive numbers and let (W_m) be a sequence of quasi-Banach spaces with the same constant $c \geq 1$ in the quasi-triangle inequality for any W_m. We put

$$\ell_q(\lambda_m W_m) = \Big\{ w = (w_m) : w_m \in W_m \quad \text{and} \quad (\lambda_m \|w_m\|_{W_m}) \in \ell_q \Big\}.$$

The quasi-norm in $\ell_q(\lambda_m W_m)$ is given by $\|w\|_{\ell_q(\lambda_m W_m)} = \| \, (\lambda_m \|w_m\|_{W_m}) \, \|_{\ell_q}$. Note that in $\ell_q(\lambda_m W_m)$ the quasi-triangle inequality holds with constant $2^{1/q}c$. We define the space $\Gamma(\lambda_m W_m)$ similarly. If W_m is equal to the scalar field $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), then we simply write $\ell_q(\lambda_m)$.

Lemma 2.4 *Let $0 < q_0, q_1 \leq \infty$ and let (W_m) be a sequence of quasi-Banach spaces with the same constant in the quasi-triangle inequality, so $(l_{q_0}(W_m), l_{q_1}(2^{-m}W_m))$ is a p-normed quasi-Banach couple for some $0 < p \leq 1$. If Γ is a quasi-Banach sequence lattice satisfying (2.1) and (2.6), then we have with equivalence of quasi-norms*

$$(l_{q_0}(W_m), l_{q_1}(2^{-m}W_m))_\Gamma = \Gamma(W_m).$$

Proof. Since $p \leq \min(q_0, q_1)$, we have that

$$(l_p(W_m), l_p(2^{-m}W_m))_\Gamma \hookrightarrow (l_{q_0}(W_m), l_{q_1}(2^{-m}W_m))_\Gamma \hookrightarrow (l_\infty(W_m), l_\infty(2^{-m}W_m))_\Gamma.$$

Hence, it suffices to show that

$$\Gamma(W_m) \hookrightarrow (l_p(W_m), l_p(2^{-m}W_m))_\Gamma \quad \text{and} \quad (l_\infty(W_m), l_\infty(2^{-m}W_m))_\Gamma \hookrightarrow \Gamma(W_m). \tag{2.7}$$

Let $w = (w_m) \in \Gamma(W_m)$ and write $u_k = (\delta_m^k w_k)_{m \in \mathbb{Z}}$ for the vector valued sequence having all co-ordinates equal to 0 except for the k-th one which is w_k. We have that $w = \sum_{k=-\infty}^{\infty} u_k$ and

$$J(2^k, u_k) = \max\left(\|u_k\|_{\ell_p(W_m)}, 2^k \|u_k\|_{\ell_p(2^{-m}W_m)}\right) = \|w_k\|_{W_k}.$$

Hence

$$\|w\|_{(\ell_p(W_m), \ell_p(2^{-m}W_m))_\Gamma} \leq \|(J(2^m, u_m))\|_\Gamma = \|(\|w_m\|_{W_m})\|_\Gamma = \|w\|_{\Gamma(W_m)}.$$

To establish the other embedding in (2.7) let

$$w = (w_m) \in (\ell_\infty(W_m), \ell_\infty(2^{-m}W_m))_\Gamma$$

and take any representation $w = u + v$ where $u = (u_m) \in \ell_\infty(W_m)$ and $v = (v_m) \in \ell_\infty(2^{-m}W_m)$. Then

$$\|w_k\|_{W_k} \leq c\big(\|u_k\|_{W_k} + \|v_k\|_{W_k}\big) \leq c\big(\|u\|_{\ell_\infty(W_m)} + 2^k \|v\|_{\ell_\infty(2^{-m}W_m)}\big).$$

This implies that $\|w_k\|_{W_k} \leq cK(2^k, w)$ and therefore

$$\|w\|_{\Gamma(W_m)} = \|(\|w_m\|_{W_m})\|_\Gamma \leq c\|(K(2^m, w))\|_\Gamma = c\|w\|_{(\ell_\infty(W_m), \ell_\infty(2^{-m}W_m))_\Gamma}.$$

This completes the proof.

□

We end this section with some examples. Before them, we recall that two functions $f, g : (0,\infty) \longrightarrow (0,\infty)$ are said to be *equivalent* ($f \sim g$) if there are positive constants c_1, c_2 such that $c_1 g(t) \leq f(t) \leq c_2 g(t)$ for all $t > 0$. A function $\rho : (0,\infty) \longrightarrow (0,\infty)$ is said to be a *function parameter* if $\rho(t)$ increases from 0 to ∞, $\rho(t)/t$ decreases from ∞ to 0 and, for every $t > 0$, $s_\rho(t) = \sup\{\rho(ts)/\rho(s) : s > 0\}$ is finite with $s_\rho(t) = o(\max(1,t))$ as $t \to 0$ and $t \to \infty$ (see [26, 28, 43]).

Example 2.5 Let $0 < q \leq \infty$ and let ρ be a function parameter. Then $\Gamma = \ell_q(1/\rho(2^m))$ is a quasi-Banach sequence lattice. Shift operators in $\ell_q(1/\rho(2^m))$ satisfy $\|\tau_k\|_{\ell_q(1/\rho(2^m)),\ell_q(1/\rho(2^m))} \leq s_\rho(2^k)$, so (2.1) is satisfied. Moreover, if $0 < p \leq q$, the quasi-norm of the Calderón transform Λ_p in $\ell_q(1/\rho(2^m))$ is bounded by the series $\left(\sum_{r=-\infty}^{\infty}\big(\min(1,2^r)s_\rho(2^{-r})\big)^p\right)^{1/p}$ which converges because for some $\delta > 0$ we have $s_\rho(t) = O(\max(t^\delta, t^{1-\delta}))$ (see [43, Proposition 1.3]). The space $\ell_q(1/\rho(2^m))$ is also (p,J)-non-trivial and K-non-trivial. The interpolation method generated by $\ell_q(1/\rho(2^m))$ is known in the literature as the *real method with a function parameter* $(A_0,A_1)_{\ell_q(1/\rho(2^m))} = (A_0,A_1)_{\rho,q}$. It has been studied in [26, 28, 43] among other papers.

Example 2.6 Let $g : (0,\infty) \longrightarrow (0,\infty)$ be a measurable function which is equivalent to a function parameter ρ and let $0 < q \leq \infty$. Then $\Gamma = \ell_q(1/g(2^m))$ is also a quasi-Banach sequence lattice. If we choose $g(t) = t^\theta(1+|\log t|)^{\mathbb{A}}$ where $0 < \theta < 1, \mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ and

$$(1+|\log t|)^{\mathbb{A}} = \begin{cases} (1-\log t)^{-\alpha_0} & \text{if} \quad 0 < t \leq 1 \\ (1+\log t)^{-\alpha_\infty} & \text{if} \quad 1 < t < \infty \end{cases}$$

then we obtain *logarithmic interpolation spaces*, studied in [20, 21, 19, 16]. Note that here it is not allow that θ takes the values 0 or 1 because we want that (2.1) and (2.6) are satisfied.

Example 2.7 Let $0 < \theta < 1$. The special case in Example 2.5 when $\rho(t) = t^\theta$ gives the classical *real interpolation method* $(A_0,A_1)_{\theta,q}$ (see [4, 44, 1, 5]).

3 Bilinear operators

Let A, B, E be quasi-Banach spaces and let $T : A \times B \longrightarrow E$ be a bilinear operator. The operator T is said to be *bounded* if

$$\|T\|_{A\times B,E} = \sup\big\{\|T(a,b)\|_E : \|a\|_A \leq 1\,,\ \|b\|_B \leq 1\big\} < \infty.$$

We write $\mathcal{B}(A \times B, E)$ for the set of all bounded bilinear operators from $A \times B$ into E.

Let $\bar{A} = (A_0,A_1), \bar{B} = (B_0,B_1), \bar{E} = (E_0,E_1)$ be quasi-Banach couples. We write $T : \bar{A} \times \bar{B} \longrightarrow \bar{E}$ to mean that T is a bounded bilinear operator $T \in \mathcal{B}(\Sigma(\bar{A}) \times \Sigma(\bar{B}), \Sigma(\bar{E}))$ such that for $j = 0,1$, the restriction of T to $A_j \times B_j$ defines a bounded bilinear operator $T \in \mathcal{B}(A_j \times B_j, E_j)$.

Next we describe the interpolation properties of bounded bilinear operators by the general real method.

Given two sequences $\xi = (\xi_m)_{m\in\mathbb{Z}}, \eta = (\eta_m)_{m\in\mathbb{Z}}$ of non-negative scalars, we define their convolution by the sequence $\xi \star \eta = (\sum_{k=-\infty}^{\infty} \xi_k \eta_{m-k})_{m\in\mathbb{Z}}$. If $0 < r \leq 1$, we write $\xi^r = (\xi_m^r)_{m\in\mathbb{Z}}$.

Theorem 3.1 *Let $\bar{A} = (A_0,A_1)$ be a quasi-Banach couple, let $\bar{B} = (B_0,B_1)$ be a p-normed quasi-Banach couple and let $\bar{E} = (E_0,E_1)$ be an r-normed quasi-Banach couple ($0 < p, r \leq 1$). Assume that Γ_0 and Γ_2 are K-non-trivial quasi-Banach sequence lattices and Γ_1 is a (p,J)-non-trivial quasi-Banach sequence lattice satisfying (2.1). Furthermore, we suppose that there is a constant $M > 0$ such that for all non-negative scalar sequences $\xi \in \Gamma_0$ and $\eta \in \Gamma_1$ we have*

$$\|(\xi^r \star \eta^r)^{1/r}\|_{\Gamma_2} \leq M\|\xi\|_{\Gamma_0}\|\eta\|_{\Gamma_1}. \tag{3.1}$$

Let $T : \bar{A} \times \bar{B} \longrightarrow \bar{E}$ and put $\|T\|_j = \|T\|_{A_j \times B_j, E_j}$, $j = 0, 1$. Then the restriction of T to $\bar{A}_{\Gamma_0;K} \times \bar{B}_{\Gamma_1;J}$ defines a bounded bilinear operator $T : \bar{A}_{\Gamma_0;K} \times \bar{B}_{\Gamma_1;J} \longrightarrow \bar{E}_{\Gamma_2;K}$ with

$$\|T\|_{\bar{A}_{\Gamma_0;K} \times \bar{B}_{\Gamma_1;J}, \bar{E}_{\Gamma_2;K}} \leq \begin{cases} 0 & \textit{if} \quad \|T\|_j = 0,\ j = 0 \textit{ or } 1 \\ C\|T\|_0 f_{\Gamma_1}(\|T\|_1/\|T\|_0) & \textit{otherwise.} \end{cases}$$

Here C is a constant independent of T.

Proof. Let $\sigma_j > \|T\|_j$, $j = 0, 1$ and choose $n \in \mathbb{Z}$ such that $2^n \leq \sigma_1/\sigma_0 < 2^{n+1}$. Take any $a \in \bar{A}_{\Gamma_0;K}$, any $u \in B_0 \cap B_1$ and $m, k \in \mathbb{Z}$. If $a = a_0 + a_1$ with $a_j \in A_j$, we get

$$\begin{aligned} K(2^m, T(a,u)) &\leq \|T(a_0,u)\|_{E_0} + 2^m \|T(a_1,u)\|_{E_1} \\ &\leq \sigma_0 \|a_0\|_{A_0} \|u\|_{B_0} + 2^{m-k-n} 2^{k+n} \sigma_1 \|a_1\|_{A_1} \|u\|_{B_1} \\ &\leq \max(\sigma_0, 2^{-n}\sigma_1)(\|a_0\|_{A_0} + 2^{m-k}\|a_1\|_{A_1}) J(2^{k+n}, u). \end{aligned}$$

Taking the infimum over all possible decompositions $a = a_0 + a_1$ with $a_j \in A_j$ and having in mind the choice of n we get

$$K(2^m, T(a,u)) \leq 2\sigma_0 K(2^{m-k}, a) J(2^{k+n}, u). \tag{3.2}$$

Take $b \in \bar{B}_{\Gamma_1;J}$ and let $b = \sum_{k=-\infty}^{\infty} u_k$ any J-representation of b. Then in $\Sigma(\bar{B})$ we also have that $b = \sum_{k=-\infty}^{\infty} u_{k+n}$. Moreover, since $K_r(t, \cdot; E_0, E_1)$ is an r-norm on $\Sigma(\bar{E})$ which is equivalent to $K(t, .; E_0, E_1)$, we obtain that $K(2^m, T(a,b)) \leq C_1 \big(\sum_{k=-\infty}^{\infty} K(2^m, T(a, u_{k+n}))^r\big)^{1/r}$. Using (3.2) and (3.1), we derive that

$$\begin{aligned} \|T(a,b)\|_{\bar{E}_{\Gamma_2;K}} &\leq C_1 \|\big(\sum_{k=-\infty}^{\infty} K(2^m, T(a, u_{k+n}))^r\big)^{1/r}\|_{\Gamma_2} \\ &\leq 2C_1\sigma_0 \|\big(\sum_{k=-\infty}^{\infty} K(2^{m-k}, a)^r J(2^{k+n}, u_{k+n})^r\big)^{1/r}\|_{\Gamma_2} \\ &= 2C_1\sigma_0 \|\big(\sum_{j=-\infty}^{\infty} K(2^j, a)^r J(2^{m+n-j}, u_{m+n-j})^r\big)^{1/r}\|_{\Gamma_2} \\ &\leq 2C_1 M \sigma_0 \|(K(2^m, a))\|_{\Gamma_0} \|(J(2^{m+n}, u_{m+n}))\|_{\Gamma_1} \\ &\leq 2C_1 M \sigma_0 \|\tau_n\|_{\Gamma_1\Gamma_1} \|a\|_{\bar{A}_{\Gamma_0;K}} \|(J(2^m, u_m))\|_{\Gamma_1}. \end{aligned}$$

Since $\|\tau_n\|_{\Gamma_1,\Gamma_1} = f_{\Gamma_1}(\sigma_1/\sigma_0)$, we get that

$$\|T\|_{\bar{A}_{\Gamma_0;K} \times \bar{B}_{\Gamma_1;J}, \bar{E}_{\Gamma_2;K}} \leq C\sigma_0 f_{\Gamma_1}(\sigma_1/\sigma_0).$$

Now, if $\|T\|_j = 0$ for $j = 0$ or 1, letting $\sigma_j \to 0$ and using (2.2) we derive that $\|T\|_{\bar{A}_{\Gamma_0;K} \times \bar{B}_{\Gamma_1;J}, \bar{E}_{\Gamma_2;K}} = 0$. If $\|T\|_j \neq 0$ for $j = 0, 1$, taking $\sigma_j = (1+\varepsilon)\|T\|_j$ and letting $\varepsilon \to 0$ we conclude that $\|T\|_{\bar{A}_{\Gamma_0;K} \times \bar{B}_{\Gamma_1;J}, \bar{E}_{\Gamma_2;K}} \leq C\|T\|_0 f_{\Gamma_1}(\|T\|_1/\|T\|_0)$. □

Writing down Theorem 3.1 for $\bar{A}, \bar{B}, \bar{E}$ Banach couples, $p = r = 1$ and $\Gamma_0, \Gamma_1, \Gamma_2$ Banach sequence lattices satisfying (2.1), (2.6) and (3.1), we recover [23, Theorem 3.1].

Theorem 3.1 applies to the real method with a function parameter (Example 2.5) and the real method (Example 2.7). Previous results on interpolation of bilinear operators by the real method among quasi-Banach couples can be found in the papers by Karadzhov [30] and König [31].

Let A, B, E be quasi-Banach spaces. We say that $T \in \mathcal{B}(A \times B, E)$ is *compact* if for any bounded sets $V \subseteq A$ and $W \subseteq B$, we have that the closure of $T(V, W) = \{T(a, b) : a \in V, b \in W\}$ is compact in E. We put $\mathcal{K}(A \times B, E)$ for the collection of all compact operators from $A \times B$ into E.

It is not hard to check that compactness of $T \in \mathcal{B}(A \times B, E)$ is equivalent to the fact that $T(U_A, U_B)$ is precompact in E. Here U_A is the closed unit ball of A and U_B the corresponding ball of B. Moreover, as in the Banach case (see [3, Proposition 1]), $T \in \mathcal{B}(A \times B, E)$ is compact if, and only if, for any bounded sequences $(a_n) \subseteq A$, $(b_n) \subseteq B$, the sequence $(T(a_n, b_n))$ has a convergent subsequence.

Using the characterization of compactness by sequences, it is not hard to check that if $T \in \mathcal{K}(A \times B, E)$, E_1 is another quasi-Banach space and R is a bounded linear operator $R \in \mathcal{L}(E, E_1)$, then $RT = R \circ T \in \mathcal{K}(A \times B, E_1)$. Moreover, if A_1, B_1 are quasi-Banach spaces and R_1, R_2 are bounded linear operators $R_1 \in \mathcal{L}(A_1, A)$, $R_2 \in \mathcal{L}(B_1, B)$, then $T \circ (R_1, R_2)(a, b) = T(R_1, R_2)(a, b) = T(R_1 a, R_2 b)$ belongs to $\mathcal{K}(A_1 \times B_1, E)$. It is also clear that if $T_1, T_2 \in \mathcal{K}(A \times B, E)$ and $\alpha, \beta \in \mathbb{K}$, then $T = \alpha T_1 + \beta T_2 \in \mathcal{K}(A \times B, E)$.

Minor changes in the arguments given by Bényi and Torres [3, Proposition 3] for the Banach case, show that if $(T_n) \subseteq \mathcal{K}(A \times B, E)$ and (T_n) converges to the bounded bilinear operator $T \in \mathcal{B}(A \times B, E)$ then $T \in \mathcal{K}(A \times B, E)$. In what follows, we will use freely all these properties of compact bilinear operators.

The following results will be useful in the proof of the main interpolation theorem of the next section. We write c_E for the constant in the quasi-triangle inequality in the space E.

Lemma 3.2 *Let A, B, E, Z be quasi-Banach spaces, let D be a dense subspace of A and let V be a dense subspace of B. Assume that $T \in \mathcal{K}(A \times B, E)$ is a compact bilinear operator and let $S_n \in \mathcal{L}(E, Z)$ be a bounded linear operator for each $n \in \mathbb{N}$ such that $\sup_{n\in\mathbb{N}} \|S_n\|_{E,Z} = M < \infty$. If $\lim_{n\to\infty} \|S_n T u\|_Z = 0$ for all $u \in D \times V$, then $\lim_{n\to\infty} \|S_n T\|_{A\times B,Z} = 0$.*

P r o o f. Using compactness of T and density of D in A and of V in B, given any $\varepsilon > 0$, we can find a finite set $\{u_1, \dots, u_r\} \subseteq D \times V$ with $u_j = (a_j, b_j)$, $\|a_j\|_{A_j} \le 1$, $\|b_j\|_{B_j} \le 1$ and such that

$$T(U_A, U_B) \subseteq \bigcup_{j=1}^{r} \{Tu_j + \frac{\varepsilon}{2Mc_Z} U_E\}.$$

By the assumption on (S_n), there exists $N \in \mathbb{N}$ such that for any $n \ge N$ and any $1 \le j \le r$, we have that $\|S_n T u_j\|_Z \le \varepsilon/2c_Z$. Consequently, given any $u \in U_A \times U_B$ if we choose $1 \le j \le r$ such that $\|Tu - Tu_j\|_E \le \varepsilon/2Mc_Z$, then we obtain for $n \ge N$ that

$$\|S_n T u\|_Z \le c_Z\big(\|S_n(Tu - Tu_j)\|_Z + \|S_n T u_j\|_Z\big) \le c_Z M \|Tu - Tu_j\|_E + \varepsilon/2 \le \varepsilon.$$

□

Lemma 3.3 *Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$, $\bar{E} = (E_0, E_1)$ be quasi-Banach couples and let A, B, E be intermediate spaces with respect to $\bar{A}, \bar{B}, \bar{E}$, respectively. Assume that $T : \Sigma(\bar{A}) \times \Sigma(\bar{B}) \longrightarrow \Sigma(\bar{E})$ is bounded with $T \in \mathcal{K}(A \times B, E)$. Let X, Y be quasi-Banach spaces and let $R_n \in \mathcal{L}(X, A)$, $S_n \in \mathcal{L}(Y, B)$ such that $\sup_{n\in\mathbb{N}} \|R_n\|_{X,A} = M < \infty$, $\sup_{n\in\mathbb{N}} \|S_n\|_{Y,B} = L < \infty$ and $\lim_{n\to\infty} \|T(R_n, S_n)\|_{X\times Y, \Sigma(\bar{E})} = 0$. Then*

$$\lim_{n\to\infty} \|T(R_n, S_n)\|_{X\times Y, E} = 0.$$

P r o o f. We proceed by contradiction. Since

$$\sup_{n\in\mathbb{N}} \|T(R_n, S_n)\|_{X\times Y,E} \le ML\|T\|_{A\times B,E} < \infty,$$

if $\lim_{n\to\infty} \|T(R_n, S_n)\|_{X\times Y,E} \neq 0$ then we can find $\lambda > 0$, a subsequence (n') and vectors $(x_{n'}) \subseteq U_X$, $(y_{n'}) \subseteq U_Y$ such that

$$\lim_{n'\to\infty} \|T(R_{n'}x_{n'}, S_{n'}y_{n'})\|_E = \lambda.$$

The assumption on (R_n) and (S_n) yields that the sequence $(R_{n'}x_{n'})$ is bounded in A and $(S_{n'}y_{n'})$ in B. Compactness of $T : A \times B \longrightarrow E$ implies, passing to another subsequence if necessary, that $\big(T(R_{n''}x_{n''}, S_{n''}y_{n''})\big)$ converges to some w in E. So $\|w\|_E = \lambda > 0$ and $\big(T(R_{n''}x_{n''}, S_{n''}y_{n''})\big)$ converges also to w in $E_0 + E_1$. However, using that $\lim_{n\to\infty} \|T(R_n, S_n)\|_{X\times Y,\Sigma(\bar{E})} = 0$, we get that $\big(T(R_{n''}x_{n''}, S_{n''}y_{n''})\big) \to 0$ in $E_0 + E_1$, which contradicts that $w \neq 0$. □

4 Interpolation of compact bilinear operators

Using Lemma 2.1 the arguments in [23, Theorem 5.1] can be modified to give the following.

Theorem 4.1 *Let Γ_0, Γ_1 be K-non-trivial quasi-Banach sequence lattices satisfying (2.1). Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ be quasi-Banach couples and let E be a quasi-Banach space. Assume that $T \in \mathcal{B}(\Sigma(\bar{A}) \times \Sigma(\bar{B}), E)$ with $T : A_j \times B_j \longrightarrow E$ compact for $j = 0$ or 1. Then $T : \bar{A}_{\Gamma_0;K} \times \bar{B}_{\Gamma_1;K} \longrightarrow E$ is also compact.*

The next result can be derived by using (2.5) and proceeding as in [23, Theorem 5.3] for the Banach case.

Theorem 4.2 *Let Γ be a (p, J)-non-trivial quasi-Banach sequence lattice ($0 < p \leq 1$) satisfying (2.1). Assume that A, B are quasi-Banach spaces and let $\bar{E} = (E_0, E_1)$ be a p-normed quasi-Banach couple. If $T \in \mathcal{B}(A \times B, \Delta(\bar{E}))$ satisfies that $T : A \times B \longrightarrow E_j$ is compact for $j = 0$ or 1, then $T : A \times B \longrightarrow \bar{E}_{\Gamma;J}$ is compact as well.*

Writing down these results for the real method with a function parameter (Example 2.5) we get the following.

Corollary 4.3 *Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ be quasi-Banach couples and let E be a quasi-Banach space. Assume that $T \in \mathcal{B}(\Sigma(\bar{A}) \times \Sigma(\bar{B}), E)$ satisfies that $T : A_j \times B_j \longrightarrow E$ is compact for $j = 0$ or 1. Then, for any $0 < q_0, q_1 \leq \infty$ and any function parameters ρ_0, ρ_1 we have that*

$$T : (A_0, A_1)_{\rho_0,q_0} \times (B_0, B_1)_{\rho_1;q_1} \longrightarrow E$$

is compact as well.

Corollary 4.4 *Let A, B are quasi-Banach spaces and let $\bar{E} = (E_0, E_1)$ be a quasi-Banach couple. Assume that $T \in \mathcal{B}(A \times B, \Delta(\bar{E}))$ satisfies that $T : A \times B \longrightarrow E_j$ is compact for $j = 0$ or 1. Then, for any $0 < q \leq \infty$ and any function parameter ρ, we have that*

$$T : A \times B \longrightarrow (E_0, E_1)_{\rho,q}$$

is also compact.

In particular, taking the function parameters as power functions, we derive the following interpolation result for the real method.

Corollary 4.5 *Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ be quasi-Banach couples and let E be a quasi-Banach space. Let $T \in \mathcal{B}(\Sigma(\bar{A}) \times \Sigma(\bar{B}), E)$ such that $T : A_j \times B_j \longrightarrow E$ is compact for $j = 0$ or 1. Then, for any $0 < q_0, q_1 \leq \infty$ and any $0 < \theta_0, \theta_1 < 1$, we have that*

$$T : (A_0, A_1)_{\theta_0,q_0} \times (B_0, B_1)_{\theta_1;q_1} \longrightarrow E$$

is compact as well.

Note that in Corollary 4.5 no relationship is assumed between parameters θ_0 and θ_1. Such freedom will be useful in the application given in Section 5.

Corollary 4.6 *Let A, B be quasi-Banach spaces and let $\bar{E} = (E_0, E_1)$ be a quasi-Banach couple. Assume that $T \in \mathcal{B}(A \times B, \Delta(\bar{E}))$ with $T : A \times B \longrightarrow E_j$ compact for $j = 0$ or 1. Then, for any $0 < q \leq \infty$ and any $0 < \theta < 1$, we have that*

$$T : A \times B \longrightarrow (E_0, E_1)_{\theta,q}$$

is also compact.

Now we prove the main result of this section.

Theorem 4.7 *Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ be p-normed quasi-Banach couples $(0 < p \leq 1)$, let $\bar{E} = (E_0, E_1)$ be an r-normed quasi-Banach couple $(0 < r \leq 1)$ and let Γ_0, Γ_1, Γ_2 be quasi-Banach sequence lattices. We suppose that Γ_0 and Γ_1 satisfy (2.6) and that Γ_2 satisfies (2.1) and (2.6) with parameter r. Assume also that the sequence spaces satisfy the condition (3.1) on convolutions. Let $T : \bar{A} \times \bar{B} \longrightarrow \bar{E}$ such that any of the restrictions $T : A_j \times B_j \longrightarrow E_j$ is compact for $j = 0$ or $j = 1$. Then $T : \bar{A}_{\Gamma_0} \times \bar{B}_{\Gamma_1} \longrightarrow \bar{E}_{\Gamma_2}$ is also compact.*

Proof. Let A_j° be the closure of $\Delta(\bar{A})$ in A_j. The couple $\overline{A^\circ} = (A_0^\circ, A_1^\circ)$ is also a p-normed quasi-Banach couple. Moreover, using the J-representation of $\bar{A}_{\Gamma_0}$, it is not difficult to check that $(A_0, A_1)_{\Gamma_0} = (A_0^\circ, A_1^\circ)_{\Gamma_0} = \overline{A^\circ}_{\Gamma_0}$. Similarly, $(B_0, B_1)_{\Gamma_1} = (B_0^\circ, B_1^\circ)_{\Gamma_1} = \overline{B^\circ}_{\Gamma_1}$. Note also that the operator T satisfies that $T : \overline{A^\circ} \times \overline{B^\circ} \longrightarrow \bar{E}$, with $T : A_j^\circ \times B_j^\circ \longrightarrow E_j$ being compact provided that $T : A_j \times B_j \longrightarrow E_j$ is so. This allows us to work with the couples $\overline{A^\circ}$, $\overline{B^\circ}$ instead of $\bar{A}$, $\bar{B}$.

For $m \in \mathbb{Z}$, consider the p-normed spaces

$$F_m = (A_0^\circ \cap A_1^\circ \,,\, J(2^m, \cdot\,; A_0^\circ, A_1^\circ)) \;, \quad G_m = (B_0^\circ \cap B_1^\circ \,,\, J(2^m, \cdot\,; B_0^\circ, B_1^\circ))$$

$$\text{and} \quad W_m = (E_0 + E_1 \,,\, K_r(2^m, \cdot\,; E_0, E_1)) \,.$$

These vector-valued sequence spaces are closely linked with the construction of the general real method. Namely, if we realized $\overline{A^\circ}_{\Gamma_0}$ by means of the J-functional, then the map $\pi(u_m) = \sum_{m=-\infty}^{\infty} u_m$ (convergence in $\Sigma(\overline{A^\circ})$) is surjective from $\Gamma_0(F_m)$ into $\overline{A^\circ}_{\Gamma_0}$ and it induces the quasi-norm $\|\cdot\|_{\overline{A^\circ}_{\Gamma_0;J}}$. Moreover, since A_j° is p-normed, we also have that $\pi : \ell_p(2^{-jm} F_m) \longrightarrow A_j^\circ$ is bounded for $j = 0, 1$. Similarly, $\pi : \Gamma_1(G_m) \longrightarrow \overline{B^\circ}_{\Gamma_1}$ is surjective, bounded, it induces the quasi-norm $\|\cdot\|_{\overline{B^\circ}_{\Gamma_1;J}}$ and $\pi : \ell_p(2^{-jm} G_m) \longrightarrow B_j^\circ$ is bounded for $j = 0, 1$. For $\bar{E}$ and $\bar{E}_{\Gamma_2}$ the relevant map is $\tau w = (\ldots, w, w, w, \ldots)$. Indeed, if we realize $\bar{E}_{\Gamma_2}$ as a K-space but replacing the K-functional by the equivalent K_r-functional, then τ is a metric injection from $\bar{E}_{\Gamma_2}$ into $\Gamma_2(W_m)$. Moreover, the restrictions $\tau : E_j \longrightarrow \ell_\infty(2^{-jm} W_m)$ are bounded for $j = 0, 1$. We have the following diagram

$$\begin{array}{c}
\ell_p(F_m) \times \ell_p(G_m) \xrightarrow{(\pi,\pi)} A_0^\circ \times B_0^\circ \xrightarrow{T} E_0 \xrightarrow{\tau} \ell_\infty(W_m) \\
\ell_p(2^{-m} F_m) \times \ell_p(2^{-m} G_m) \xrightarrow{(\pi,\pi)} A_1^\circ \times B_1^\circ \xrightarrow{T} E_1 \xrightarrow{\tau} \ell_\infty(2^{-m} W_m) \\
\hline
\Gamma_0(F_m) \times \Gamma_1(G_m) \xrightarrow{(\pi,\pi)} \overline{A^\circ}_{\Gamma_0} \times \overline{B^\circ}_{\Gamma_1} \xrightarrow{T} \bar{E}_{\Gamma_2} \xrightarrow{\tau} \Gamma_2(W_m)
\end{array}$$

Let $\widehat{T} = \tau T(\pi, \pi)$. The properties of the maps π and τ yield that a necessary and sufficient condition for $T : \overline{A^\circ}_{\Gamma_0} \times \overline{B^\circ}_{\Gamma_1} \longrightarrow \bar{E}_{\Gamma_2}$ to be compact is that

$$\widehat{T} : \Gamma_0(F_m) \times \Gamma_1(G_m) \longrightarrow \Gamma_2(W_m) \quad \text{is compact.} \tag{4.1}$$

Since the sequences $(F_m), (G_m)$ are formed by p-normed quasi-Banach spaces, we have that the couples $\overline{\ell_p(F)} = (\ell_p(F_m), \ell_p(2^{-m}F_m))$ and $\overline{\ell_p(G)} = (\ell_p(G_m), \ell_p(2^{-m}G_m))$ are p-normed quasi-Banach couples. Moreover, the couple $\overline{\ell_\infty(W)} = (\ell_\infty(W_m), \ell_\infty(2^{-m}W_m))$ is an r-normed quasi-Banach couple. These three couples are the relevant couples to work with $\widehat{T}$. In fact, the operator $\widehat{T}$ belongs to $\mathcal{B}(\overline{\ell_p(F)} \times \overline{\ell_p(G)}, \overline{\ell_\infty(W)})$ and Lemma 2.4 shows that $\Gamma_0(F_m)$, $\Gamma_1(G_m)$ and $\Gamma_2(W_m)$ are interpolation spaces with respect to the couples $\overline{\ell_p(F)}$, $\overline{\ell_p(G)}$ and $\overline{\ell_\infty(W)}$, respectively.

The following families of projections are useful in order to show the compactness of $\widehat{T}$ in (4.1). For $n \in \mathbb{N}$, put

$$\begin{aligned} R_n(z_m) &= (\dots, 0, 0, z_{-n}, z_{-n+1}, \dots, z_{n-1}, z_n, 0, 0, \dots), \\ R_n^+(z_m) &= (\dots, 0, 0, z_{n+1}, z_{n+2}, z_{n+3}, \dots), \\ R_n^-(z_m) &= (\dots, z_{-n-3}, z_{-n-2}, z_{-n-1}, 0, 0, \dots). \end{aligned}$$

All these maps belong to $\mathcal{L}\left(\overline{\ell_p(F)}, \overline{\ell_p(F)}\right)$ and $R_n \in \mathcal{L}\left(\Sigma(\overline{\ell_p(F)}), \Delta(\overline{\ell_p(F)})\right)$. One can easily check that they satisfy the following conditions:

(i) Each one of the maps R_n, R_n^+, R_n^- has quasi-norm 1 acting from $\ell_p(F_m)$ into $\ell_p(F_m)$, from $\ell_p(2^{-m}F_m)$ into $\ell_p(2^{-m}F_m)$ and from $\Gamma_0(F_m)$ into $\Gamma_0(F_m)$.

(ii) The identity operator I on $\Sigma(\overline{\ell_p(F)})$ can be decomposed as $I = R_n + R_n^+ + R_n^-$, $n \in \mathbb{N}$.

(iii) For each $n \in \mathbb{N}$, projections $R_n : \ell_p(F_m) \longrightarrow \ell_p(2^{-m}F_m)$ and $R_n : \ell_p(2^{-m}F_m) \longrightarrow \ell_p(F_m)$ are bounded with

$$\|R_n\|_{\ell_p(F_m), \ell_p(2^{-m}F_m)} = 2^n = \|R_n\|_{\ell_p(2^{-m}F_m), \ell_p(F_m)}.$$

Moreover $R_n^+ : \ell_p(F_m) \longrightarrow \ell_p(2^{-m}F_m)$ and $R_n^- : \ell_p(2^{-m}F_m) \longrightarrow \ell_p(F_m)$ are also bounded with

$$\|R_n^+\|_{\ell_p(F_m), \ell_p(2^{-m}F_m)} = 2^{-(n+1)} = \|R_n^-\|_{\ell_p(2^{-m}F_m), \ell_p(F_m)}.$$

Similar sequences of projections can be defined on the couples $\overline{\ell_p(G)}$ and $\overline{\ell_\infty(W)}$. We call them S_n, S_n^+, S_n^- and P_n, P_n^+, P_n^-, respectively. They satisfy the corresponding versions of (i), (ii) and (iii).

Next we split the operator $\widehat{T}$ as in the Banach case [24, Theorem 3.1] and we work with each piece with the help of results of Section 3 and Theorems 4.1 and 4.2.

Using (ii), for $n \in \mathbb{N}$ we obtain $\widehat{T} = P_n\widehat{T} + P_n^+\widehat{T} + P_n^-\widehat{T}$. Moreover, $P_n^-\widehat{T} = P_n^-\widehat{T}(R_n + R_n^+ + R_n^-\, , S_n + S_n^+ + S_n^-)$. Whence

$$\begin{aligned} \widehat{T} &= P_n\widehat{T} + P_n^-\widehat{T}(R_n, S_n) + P_n^+\widehat{T} + P_n^-\widehat{T}(R_n^+, S_n) + P_n^-\widehat{T}(R_n, S_n^+) \\ &\quad + P_n^-\widehat{T}(R_n^+, S_n^+) + P_n^-\widehat{T}(R_n^-, S_n^+) + P_n^-\widehat{T}(R_n^-, S_n^-) \\ &\quad + P_n^-\widehat{T}(R_n, S_n^-) + P_n^-\widehat{T}(R_n^+, S_n^-) + P_n^-\widehat{T}(R_n^-, S_n). \end{aligned}$$

Suppose that $T : A_1 \times B_1 \longrightarrow E_1$ is compact. The case when we have compactness in $T : A_0 \times B_0 \longrightarrow E_0$ is similar. We are going to check that acting from $\Gamma_0(F_m) \times \Gamma_1(G_m)$ into $\Gamma_2(W_m)$ the operators $P_n\widehat{T}$ and $P_n^-\widehat{T}(R_n, S_n)$ are compact. Then we will show that the remaining nine operators have norms converging to zero as $n \to \infty$. This will show that $\widehat{T}$ in (4.1) is the limit of a sequence of compact bilinear operators and it is therefore compact.

Having in mind the corresponding property to (iii) for P_n and Lemma 2.4, we can factorize $P_n\widehat{T}$ by means of the diagram

$$\ell_p(F_m)\times\ell_p(G_m) \quad \xrightarrow{\widehat{T}} \quad \ell_\infty(W_m) \xrightarrow{P_n} \Delta(\overline{\ell_\infty(W)}) \hookrightarrow \Gamma_2(W_m).$$

We also have

$$\ell_p(2^{-m}F_m)\times\ell_p(2^{-m}G_m) \quad \xrightarrow{\widehat{T}} \quad \ell_\infty(2^{-m}W_m) \xrightarrow{P_n} \Delta(\overline{\ell_\infty(W)}) \hookrightarrow \Gamma_2(W_m)$$

and this last operator is compact because compactness of $T : A_1 \times B_1 \longrightarrow E_1$ yields that $\widehat{T} : \ell_p(2^{-m}F_m) \times \ell_p(2^{-m}G_m) \longrightarrow \ell_\infty(2^{-m}W_m)$ is compact. Using the diagram

$$\begin{tikzcd}
\ell_p(F_m)\times\ell_p(G_m) \arrow[dr, "P_n\widehat{T}"] & \\
& \Gamma_2(W_m) \\
\ell_p(2^{-m}F_m)\times\ell_p(2^{-m}G_m) \arrow[ur, "P_n\widehat{T}"'] &
\end{tikzcd}$$

and applying Theorem 4.1 and Lemma 2.4, compactness of $P_n\widehat{T} : \Gamma_0(F_m)\times\Gamma_1(G_m) \longrightarrow \Gamma_2(W_m)$ follows.

Since

$$\Gamma_0(F_m)\times\Gamma_1(G_m) \hookrightarrow \Sigma\big(\overline{\ell_p(F)}\big)\times\Sigma\big(\overline{\ell_p(G)}\big) \xrightarrow{(R_n,S_n)} \Delta\big(\overline{\ell_p(F)}\big)\times\Delta\big(\overline{\ell_p(G)}\big)$$

for the operator $P_n^-\widehat{T}(R_n,S_n)$ we can use the following diagram

$$\begin{tikzcd}
& \ell_p(F_m)\times\ell_p(G_m) \arrow[r, "P_n^-\widehat{T}"] & \ell_\infty(W_m) \\
\Gamma_0(F_m)\times\Gamma_1(G_m) \arrow[ur, "{(R_n,S_n)}"] \arrow[dr, "{(R_n,S_n)}"'] & & \\
& \ell_p(2^{-m}F_m)\times\ell_p(2^{-m}G_m) \arrow[r, "P_n^-\widehat{T}"] & \ell_\infty(2^{-m}W_m)
\end{tikzcd}$$

with $P_n^-\widehat{T}(R_n,S_n) : \Gamma_0(F_m)\times\Gamma_1(G_m) \longrightarrow \ell_\infty(2^{-m}W_m)$ being compact. Whence, according to Theorem 4.2 and Lemma 2.4, we derive that $P_n^-\widehat{T}(R_n,S_n) : \Gamma_0(F_m)\times\Gamma_1(G_m) \longrightarrow \Gamma_2(W_m)$ is also compact.

Next we show that the norm of $P_n^+\widehat{T}$ tends to 0 as $n\to\infty$. Consider the operator $P_n^+\tau T : A_1^\circ \times B_1^\circ \longrightarrow \ell_\infty(2^{-m}W_m)$. Using the corresponding property to (iii) for P_n^+, given any $a\in\Delta(\bar{A})$ and $b\in\Delta(\bar{B})$, we have that

$$\|P_n^+\tau T(a,b)\|_{\ell_\infty(2^{-m}W_m)} \le 2^{-(n+1)}\|\tau T(a,b)\|_{\ell_\infty(W_m)} \to 0 \quad \text{as} \quad n\to\infty.$$

Then, applying Lemma 3.2, we get that $\lim_{n\to\infty}\|P_n^+\tau T\|_{A_1^\circ\times B_1^\circ,\ell_\infty(2^{-m}W_m)} = 0$ and consequently

$$\lim_{n\to\infty}\|P_n^+\widehat{T}\|_{\ell_p(2^{-m}F_m)\times\ell_p(2^{-m}G_m),\ell_\infty(2^{-m}W_m)} = 0.$$

Using now Theorem 3.1 and having in mind properties (2.3) and (2.2) of f_{Γ_1}, we obtain that

$$\begin{aligned}
&\|P_n^+\widehat{T}\|_{\Gamma_0(F_m)\times\Gamma_1(G_m),\Gamma_2(W_m)} \\
&\qquad \le C_1\,\|P_n^+\widehat{T}\|_{\ell_p(F_m)\times\ell_p(G_m),\ell_\infty(W_m)}\, f_{\Gamma_1}\left(\frac{\|P_n^+\widehat{T}\|_{\ell_p(2^{-m}F_m)\times\ell_p(2^{-m}G_m),\ell_\infty(2^{-m}W_m)}}{\|P_n^+\widehat{T}\|_{\ell_p(F_m)\times\ell_p(G_m),\ell_\infty(W_m)}}\right) \\
&\qquad \le C_2\|P_n^+\widehat{T}\|_{\ell_p(F_m)\times\ell_p(G_m),\ell_\infty(W_m)} f_{\Gamma_1}\left(\frac{1}{\|P_n^+\widehat{T}\|_{\ell_p(F_m)\times\ell_p(G_m),\ell_\infty(W_m)}}\right) \\
&\qquad\qquad \times f_{\Gamma_1}\left(\|P_n^+\widehat{T}\|_{\ell_p(2^{-m}F_m)\times\ell_p(2^{-m}G_m),\ell_\infty(2^{-m}W_m)}\right) \\
&\qquad \le C_3\, f_{\Gamma_1}\left(\|P_n^+\widehat{T}\|_{\ell_p(2^{-m}F_m)\times\ell_p(2^{-m}G_m),\ell_\infty(2^{-m}W_m)}\right) \longrightarrow 0 \quad \text{when} \quad n\to\infty.
\end{aligned}$$

For the operator $P_n^-\widehat{T}(R_n^+,S_n)$, factorization

$$\ell_p(F_m)\times\ell_p(G_m) \xrightarrow{(R_n^+,S_n)} \ell_p(2^{-m}F_m)\times\ell_p(2^{-m}G_m) \xrightarrow{\widehat{T}} \ell_\infty(2^{-m}W_m) \xrightarrow{P_n^-} \ell_\infty(W_m)$$

and (iii) give that

$$\begin{aligned}
&\|P_n^-\widehat{T}(R_n^+,S_n)\|_{\ell_p(F_m)\times\ell_p(G_m),\ell_\infty(W_m)} \\
&\qquad\qquad \le \|R_n^+\|_{\ell_p(F_m),\ell_p(2^{-m}F_m)}\,\|S_n\|_{\ell_p(G_m),\ell_p(2^{-m}G_m)} \\
&\qquad\qquad\qquad \times\|\widehat{T}\|_{\ell_p(2^{-m}F_m)\times\ell_p(2^{-m}G_m),\ell_\infty(2^{-m}W_m)}\,\|P_n^-\|_{\ell_\infty(2^{-m}W_m),\ell_\infty(W_m)} \\
&\qquad\qquad \le 2^{-(n+1)}\,2^n\,\|T\|_{A_1^\circ\times B_1^\circ,E_1}\,2^{-(n+1)} \longrightarrow 0 \quad \text{as} \quad n\to\infty.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\|P_n^-\widehat{T}(R_n^+,S_n)\|_{\ell_p(2^{-m}F_m)\times\ell_p(2^{-m}G_m),\ell_\infty(2^{-m}W_m)} \\
&\qquad\qquad \le \|R_n^+\|_{\ell_p(2^{-m}F_m),\ell_p(2^{-m}F_m)}\,\|S_n\|_{\ell_p(2^{-m}G_m),\ell_p(2^{-m}G_m)} \\
&\qquad\qquad\qquad \times\|\widehat{T}\|_{\ell_p(2^{-m}F_m)\times\ell_p(2^{-m}G_m),\ell_\infty(2^{-m}W_m)}\,\|P_n^-\|_{\ell_\infty(2^{-m}W_m),\ell_\infty(2^{-m}W_m)} \\
&\qquad\qquad \le \|T\|_{A_1^\circ\times B_1^\circ,E_1}.
\end{aligned}$$

Hence, it follows from Theorem 3.1 and properties (2.3) and (2.2) of f_{Γ_1} that

$$\lim_{n\to\infty}\|P_n^-\widehat{T}(R_n^+,S_n)\|_{\Gamma_0(F_m)\times\Gamma_1(G_m),\Gamma_2(W_m)} = 0.$$

Operators $P_n^-\widehat{T}(R_n,S_n^+)$ and $P_n^-\widehat{T}(R_n^+,S_n^+)$ can be treated similarly.

Next we consider the operator $P_n^-\widehat{T}(R_n^-,S_n^+) = P_n^-\tau T(\pi R_n^-,\pi S_n^+)$. For the quasi-norm of $T(\pi R_n^-,\pi S_n^+)$ acting from $\ell_p(2^{-m}F_m)\times\ell_p(2^{-m}G_m)$ into E_0+E_1 we have that

$$\begin{aligned}
&\|T(\pi R_n^-,\pi S_n^+)\|_{\ell_p(2^{-m}F_m)\times\ell_p(2^{-m}G_m),\Sigma(\bar{E})} \\
&\qquad\qquad \le \|T\|_{\Sigma(\overline{A^\circ})\times\Sigma(\overline{B^\circ}),\Sigma(\bar{E})}\|\pi R_n^-\|_{\ell_p(2^{-m}F_m),\Sigma(\overline{A^\circ})}\|\pi S_n^+\|_{\ell_p(2^{-m}G_m),\Sigma(\overline{B^\circ})} \\
&\qquad\qquad \le \|T\|_{\Sigma(\overline{A^\circ})\times\Sigma(\overline{B^\circ}),\Sigma(\bar{E})}\|\pi\|_{\ell_p(F_m),\Sigma(\overline{A^\circ})}\|R_n^-\|_{\ell_p(2^{-m}F_m),\ell_p(F_m)} \\
&\qquad\qquad\qquad \times\|\pi\|_{\ell_p(2^{-m}G_m),\Sigma(\overline{B^\circ})}\|S_n^+\|_{\ell_p(2^{-m}G_m),\ell_p(2^{-m}G_m)} \\
&\qquad\qquad \le 2^{-(n+1)}\|T\|_{\Sigma(\overline{A^\circ})\times\Sigma(\overline{B^\circ}),\Sigma(\bar{E})} \longrightarrow 0 \quad \text{as} \quad n\to\infty.
\end{aligned}$$

Since $T: A_1^\circ\times B_1^\circ \longrightarrow E_1$ is compact, Lemma 3.3 yields that

$$\lim_{n\to\infty}\|T(\pi R_n^-,\pi S_n^+)\|_{\ell_p(2^{-m}F_m)\times\ell_p(2^{-m}G_m),E_1} = 0.$$

Therefore, $\lim_{n\to\infty} \|P_n^- \widehat{T}(R_n^-, S_n^+)\|_{\ell_p(2^{-m}F_m)\times\ell_p(2^{-m}G_m),\ell_\infty(2^{-m}W_m)} = 0$. For the norm of the other restriction we get $\|P_n^- \widehat{T}(R_n^-, S_n^+)\|_{\ell_p(F_m)\times\ell_p(G_m),\ell_\infty(W_m)} \le \|T\|_{A_0^\circ\times B_0^\circ, E_0}$. Consequently, using again Theorem 3.1 and the properties of f_{Γ_1} we derive that

$$\lim_{n\to\infty} \|P_n^- \widehat{T}(R_n^-, S_n^+)\|_{\Gamma_0(F_m)\times\Gamma_1(G_m),\Gamma_2(W_m)} = 0.$$

For the remaining operators $P_n^- \widehat{T}(R_n^-, S_n^-)$, $P_n^- \widehat{T}(R_n, S_n^-)$, $P_n^- \widehat{T}(R_n^+, S_n^-)$ and $P_n^- \widehat{T}(R_n^-, S_n)$ we can proceed similarly as with $P_n^- \widehat{T}(R_n^-, S_n^+)$ and show that their norms converge also to 0 as $n \to \infty$. This completes the proof. □

For the Banach case, that is, when $\bar{A}, \bar{B}, \bar{E}$ are Banach couples, $p = r = 1$ and $\Gamma_0, \Gamma_1, \Gamma_2$ are Banach sequence lattices satisfying (2.1), (2.6) and (3.1), then Theorem 4.7 recovers [24, Theorem 3.1]. In particular, in the Banach case Theorem 4.7 improves [23, Theorem 5.8].

Applying Theorem 4.7 to the case of the real method with a function parameter described in Example 2.5, we get the following result.

Theorem 4.8 *Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ be quasi-Banach couples and let $\bar{E} = (E_0, E_1)$ be an r-normed quasi-Banach couple $(0 < r \le 1)$. Suppose that ρ_0, ρ_1, ρ_2 are function parameters such that for some constant $C > 0$ we have*

$$\rho_0(t)\rho_1(s) \le C\rho_2(ts) \quad t, s > 0. \tag{4.2}$$

Let $0 < q_0, q_1 \le \infty$ and write

$$\frac{1}{q} = \begin{cases} \frac{1}{q_0} + \frac{1}{q_1} - \frac{1}{r} & \text{if} \quad q_0, q_1 \ge r, \\ \frac{1}{\max(q_0,q_1)} & \text{if} \quad q_0 < r \text{ or } q_1 < r. \end{cases}$$

If $T : \bar{A} \times \bar{B} \longrightarrow \bar{E}$ and $T : A_j \times B_j \longrightarrow E_j$ is compact for $j = 0$ or 1, then

$$T : (A_0, A_1)_{\rho_0,q_0} \times (B_0, B_1)_{\rho_1,q_1} \longrightarrow (E_0, E_1)_{\rho_2,q}$$

is also compact.

Proof. Assume first that $q_0, q_1 \ge r$, so $r/q + 1 = r/q_0 + r/q_1$. If $\xi = (\xi_m) \in \ell_{q_0}(1/\rho_0(2^m))$, $\eta = (\eta_m) \in \ell_{q_1}(1/\rho_1(2^m))$ are non-negative scalar sequences, then according to (4.2) and Young's inequality we obtain

$$\begin{aligned} \|(\xi^r \star \eta^r)^{1/r}\|_{\ell_q(1/\rho_2(2^m))} &= \Big(\sum_{m=-\infty}^{\infty} \big(\sum_{k=-\infty}^{\infty} \xi_k^r \eta_{m-k}^r / \rho_2(2^m)^r \big)^{q/r} \Big)^{1/q} \\ &\le C \Big(\sum_{m=-\infty}^{\infty} \big(\sum_{k=-\infty}^{\infty} (\xi_k/\rho_0(2^k))^r (\eta_{m-k}/\rho_1(2^{m-k}))^r \big)^{q/r} \Big)^{1/q} \\ &\le C \left\| \big(\xi_m/\rho_0(2^m)\big)^r \right\|_{\ell_{q_0/r}}^{1/r} \left\| \big(\eta_m/\rho_1(2^m)\big)^r \right\|_{\ell_{q_1/r}}^{1/r} \\ &= C\|\xi\|_{\ell_{q_0}(1/\rho_0(2^m))} \|\eta\|_{\ell_{q_1}(1/\rho_1(2^m))}. \end{aligned}$$

This shows that inequality (3.1) holds. Hence, the result follows from Theorem 4.7.

Suppose now that $q_0 < r$ or $q_1 < r$. Then either

$$(a) \qquad q_1 = \max(q_0, q_1) = q \quad \text{and} \quad q_0 < r,$$

or

$$(b) \qquad q_0 = \max(q_0, q_1) = q \quad \text{and} \quad q_1 < r.$$

If (a) holds, then $1/q = 1/q_0 + 1/q_1 - 1/q_0$. Moreover, since the couple (E_0, E_1) is r-normed, it is also q_0-normed. Therefore, we are in the situation considered before and the result follows. If (b) holds, we can proceed similarly. □

If we choose $\rho_0(t) = \rho_1(t) = \rho_2(t) = t^\theta$ with $0 < \theta < 1$ then we get the following result for the real method.

Theorem 4.9 *Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ be quasi-Banach couples and let $\bar{E} = (E_0, E_1)$ be an r-normed quasi-Banach couple $(0 < r \le 1)$. Let $0 < \theta < 1$, $0 < q_0, q_1 \le \infty$ and let $0 < q \le \infty$ satisfying that*

$$\frac{1}{q} = \begin{cases} \frac{1}{q_0} + \frac{1}{q_1} - \frac{1}{r} & \textit{if} \quad q_0, q_1 \ge r, \\ \frac{1}{\max(q_0,q_1)} & \textit{if} \quad q_0 < r \textit{ or } q_1 < r. \end{cases}$$

If $T : \bar{A} \times \bar{B} \longrightarrow \bar{E}$ and $T : A_j \times B_j \longrightarrow E_j$ is compact for $j = 0$ or 1, then

$$T : (A_0, A_1)_{\theta,q_0} \times (B_0, B_1)_{\theta,q_1} \longrightarrow (E_0, E_1)_{\theta,q}$$

is compact as well.

Remark 4.10 Assumption $T : \bar{A} \times \bar{B} \longrightarrow \bar{E}$ implies that

$$T \in \mathcal{B}\big(\Sigma(\bar{A}) \times \Sigma(\bar{B}), \Sigma(\bar{E})\big) \text{ and that } T \in \mathcal{B}(A_j \times B_j, E_j),\ j = 0, 1.$$

Hence, there is a constant $M > 0$ such that

$$\|T(a, b)\|_{E_j} \le M\|a\|_{A_j}\|b\|_{B_j},\ a \in \Delta(\bar{A}),\ b \in \Delta(\bar{B}),\ j = 0, 1. \tag{4.3}$$

In applications sometimes we have (4.3) but we do not have that $T \in \mathcal{B}\big(\Sigma(\bar{A}) \times \Sigma(\bar{B}), \Sigma(\bar{E})\big)$. However, if we have (4.3) and some extra information on the operator T then there is a certain replacement for the assumption $T \in \mathcal{B}\big(\Sigma(\bar{A}) \times \Sigma(\bar{B}), \Sigma(\bar{E})\big)$. Next we show it.

Suppose that (4.3) holds. Let $a \in \Delta(\bar{A})$, $b \in \Delta(\bar{B})$ with $b = b_0 + b_1$ and $b_j \in B_j$, $j = 0, 1$, then $b_j \in \Delta(\bar{B})$ and so

$$\|T(a, b)\|_{\Sigma(\bar{E})} \le \|T(a, b_0)\|_{E_0} + \|T(a, b_1)\|_{E_1} \le MJ(1, a)\big(\|b_0\|_{B_0} + \|b_1\|_{B_1}\big).$$

This yields that

$$\|T(a, b)\|_{\Sigma(\bar{E})} \le M\|a\|_{A_0\cap A_1}\|b\|_{B_0+B_1}.$$

If $\Delta(\bar{B})$ is dense in B_0 and B_1, then T may be uniquely extended to a bounded bilinear operator

$$T : \Delta(\bar{A}) \times \Sigma(\bar{B}) \longrightarrow \Sigma(\bar{E}). \tag{4.4}$$

Similarly, if $\Delta(\bar{A})$ is dense in A_0 and A_1, it follows from (4.3) that T has a unique extension to a bounded bilinear operator

$$T : \Sigma(\bar{A}) \times \Delta(\bar{B}) \longrightarrow \Sigma(\bar{E}).$$

Next we follow an idea of Janson [29]. Let $0 < \theta_j, \mu_j < 1$, $0 < q_j, r_j < \infty$, $j = 0, 1, 2$. Suppose that the operator T satisfies that

$$\|T(a,b)\|_{\bar{E}_{\theta_2,q_2}} \leq M_1 \|a\|_{\bar{A}_{\theta_0,q_0}} \|b\|_{\bar{B}_{\theta_1,q_1}},\ a \in \Delta(\bar{A}),\ b \in \Delta(\bar{B}), \tag{4.5}$$

and

$$\|T(a,b)\|_{\bar{E}_{\mu_2,r_2}} \leq M_2 \|a\|_{\bar{A}_{\mu_0,r_0}} \|b\|_{\bar{B}_{\mu_1,r_1}},\ a \in \Delta(\bar{A}),\ b \in \Delta(\bar{B}). \tag{4.6}$$

Estimates (4.5) and (4.6) may be deduced from (4.3) by means of the bilinear interpolation theorem for the real method provided parameters $\theta_j, \mu_j, q_j, r_j$ satisfy suitable conditions. Assume in addition that we have the following extra information on T

$$\|T(a,b)\|_{E_0+E_1} \leq M_3 \|a\|_{\bar{A}_{\theta_0,q_0}} \|b\|_{\bar{B}_{\mu_1,r_1}},\ a \in \Delta(\bar{A}),\ b \in \Delta(\bar{B}), \tag{4.7}$$

$$\|T(a,b)\|_{E_0+E_1} \leq M_4 \|a\|_{\bar{A}_{\mu_0,r_0}} \|b\|_{\bar{B}_{\theta_1,q_1}},\ a \in \Delta(\bar{A}),\ b \in \Delta(\bar{B}). \tag{4.8}$$

Since $q_j, r_j < \infty$, then $\Delta(\bar{A})$ is dense in $\bar{A}_{\theta_0,q_0}$ and $\bar{A}_{\mu_0,r_0}$ (see [4, Theorem 3.4.2/(b) and page 66]), and $\Delta(\bar{B})$ is dense in $\bar{B}_{\theta_1,q_1}$ and $\bar{B}_{\mu_1,r_1}$. Proceeding as we have done to established (4.4), it follows from (4.5) and (4.7) that T may be uniquely extended to a bounded bilinear operator

$$T : \bar{A}_{\theta_0,q_0} \times \big(\bar{B}_{\theta_1,q_1} + \bar{B}_{\mu_1,r_1}\big) \longrightarrow E_0 + E_1. \tag{4.9}$$

On the other hand, by (4.6) and (4.8), T has a unique extension to a bounded bilinear operator

$$T : \bar{A}_{\mu_0,r_0} \times \big(\bar{B}_{\theta_1,q_1} + \bar{B}_{\mu_1,r_1}\big) \longrightarrow E_0 + E_1. \tag{4.10}$$

Finally, from (4.9) and (4.10), it follows that T may be uniquely extended to a bounded bilinear operator

$$T : \big(\bar{A}_{\theta_0,q_0} + \bar{A}_{\mu_0,r_0}\big) \times \big(\bar{B}_{\theta_1,q_1} + \bar{B}_{\mu_1,r_1}\big) \longrightarrow E_0 + E_1,$$

which may be used as a replacement for the assumption $T \in \mathcal{B}\big(\Sigma(\bar{A}) \times \Sigma(\bar{B}), \Sigma(\bar{E})\big)$.

5 Compact bilinear operators among L_p spaces

Let (Ω, μ) be a σ-finite measure space. We denote by $\mathcal{M}(\mu)$ the collection of all (equivalence classes of) measurable functions f on Ω which are finite almost everywhere. We endow $\mathcal{M}(\mu)$ with the topology of convergence in measure on each measurable set of finite measure. In this way, $\mathcal{M}(\mu)$ is a metrizable topological vector space.

For $0 < p \leq \infty$, we let $L_p(\Omega)$ be the usual *Lebesgue space*. Given $0 < p < \infty$ and $0 < q \leq \infty$, the *Lorentz space* $L_{p,q}(\Omega)$ is defined to be the set of all (equivalence classes of) measurable functions f on Ω which have a finite quasi-norm

$$\|f\|_{L_{p,q}(\Omega)} = \Big(\int_0^{\mu(\Omega)} \big(t^{1/p} f^*(t)\big)^q \frac{dt}{t} \Big)^{1/q}$$

(the integral should be replaced by the supremum if $q = \infty$). Here f^* stands for the non-increasing rearrangement of f. When $p = q$ we have $L_p(\Omega) = L_{p,p}(\Omega)$. The Lebesgue spaces $L_p(\Omega)$ and the Lorentz spaces $L_{p,q}(\Omega)$ are continuously embedded in $\mathcal{M}(\mu)$.

If $0 < p < \infty$ it turns out that

$$K(t, f; L_p(\Omega), L_\infty(\Omega)) \sim \Big(\int_0^{t^p} \big(f^*(t)\big)^p dt \Big)^{1/p}.$$

Moreover, for $0 < q \leq \infty,\ 0 < r_0 \neq r_1 \leq \infty,\ 0 < \theta < 1$ and $1/r = (1-\theta)/r_0 + \theta/r_1$, then we have

$$\big(L_{r_0}(\Omega), L_{r_1}(\Omega)\big)_{\theta,q} = L_{r,q}(\Omega) \quad \text{(equivalent quasi-norms)}$$

(see [4, Theorems 5.2.1 and 5.3.1] or [44, 1.18.6]).

The following interpolation result is a consequence of Theorem B_1 in the paper [8] by Calderón and Zygmund.

Theorem 5.1 *Let (Ω_k, μ_k) be σ-finite measure spaces for $k = 0, 1, 2$. Suppose $1 \leq p_j, q_j \leq \infty$ and $0 < r_j \leq \infty, j = 0, 1$. Let $0 < \theta < 1$ and put $1/p = (1-\theta)/p_0 + \theta/p_1,\ 1/q = (1-\theta)/q_0 + \theta/q_1$ and $1/r = (1-\theta)/r_0 + \theta/r_1$. Assume that $p \neq \infty, q \neq \infty$ and let*

$$T : \big(L_{p_0}(\Omega_0) + L_{p_1}(\Omega_0)\big) \times \big(L_{q_0}(\Omega_1) + L_{q_1}(\Omega_1)\big) \longrightarrow \big(L_{r_0}(\Omega_2) + L_{r_1}(\Omega_2)\big)$$

be a bounded bilinear operator such that for $j = 0, 1$ the restriction

$$T : L_{p_j}(\Omega_0) \times L_{q_j}(\Omega_1) \longrightarrow L_{r_j}(\Omega_2)$$

is bounded with quasi-norm M_j. Then

$$T : L_p(\Omega_0) \times L_q(\Omega_1) \longrightarrow L_r(\Omega_2)$$

is also bounded with quasi-norm $M \leq M_0^{1-\theta} M_1^{\theta}$.

Next we are going to establish a reinforced version of this result.

If $D \subseteq \Omega_2$ is a μ_2-measurable set, we put P_D for the linear operator defined by $P_D f = \chi_D f$.

Theorem 5.2 *Let (Ω_k, μ_k) be σ-finite measure spaces for $k = 0, 1, 2$. Suppose $1 \leq p_j, q_j \leq \infty$ and $0 < r_j \leq \infty, j = 0, 1$. Let $0 < \theta < 1$ and put $1/p = (1-\theta)/p_0 + \theta/p_1,\ 1/q = (1-\theta)/q_0 + \theta/q_1$ and $1/r = (1-\theta)/r_0 + \theta/r_1$. Suppose that $p \neq \infty, q \neq \infty$ and let*

$$T : \big(L_{p_0}(\Omega_0) + L_{p_1}(\Omega_0)\big) \times \big(L_{q_0}(\Omega_1) + L_{q_1}(\Omega_1)\big) \longrightarrow L_{r_0}(\Omega_2) + L_{r_1}(\Omega_2)$$

be a bounded bilinear operator such that for $j = 0, 1$ the restriction

$$T : L_{p_j}(\Omega_0) \times L_{q_j}(\Omega_1) \longrightarrow L_{r_j}(\Omega_2)$$

is bounded. Assume, in addition, that $r_0 \neq \infty$ and that

$$T : L_{p_0}(\Omega_0) \times L_{q_0}(\Omega_1) \longrightarrow L_{r_0}(\Omega_2) \quad \textit{is compact}.$$

Then

$$T : L_p(\Omega_0) \times L_q(\Omega_1) \longrightarrow L_r(\Omega_2) \quad \textit{is also compact}.$$

P r o o f. Let U_{L_p} be the closed unit ball of L_p and let U_{L_q} be the corresponding ball in L_q. Our aim is to show that $T\big(U_{L_p}, U_{L_q}\big) = \{T(f,g) : f \in U_{L_p}, g \in U_{L_q}\}$ is relatively compact set in $L_r(\Omega_2)$. Since $r_0 < \infty$, we also have $r < \infty$. Then, according to [35, Lemma I.1.1] or [1, page 31], the set $T\big(U_{L_p}, U_{L_q}\big)$ is relatively compact in $L_r(\Omega_2)$ if, and only if, the following two properties hold:

(a) $\lim_{\mu_2(D) \to 0} \|P_D T\|_{L_p \times L_q, L_r} = 0$.

(b) $T\big(U_{L_p}, U_{L_q}\big)$ is relatively compact in $\mathcal{M}(\mu_2)$.

Let $D \subseteq \Omega_2$ be any μ_2-measurable set. Since $\|P_D T f\|_{L_{r_j}} \le \|Tf\|_{L_{r_j}}$, we have that

$$P_D T : \big(L_{p_0}(\Omega_0), L_{p_1}(\Omega_0)\big) \times \big(L_{q_0}(\Omega_1), L_{q_1}(\Omega_1)\big) \longrightarrow \big(L_{r_0}(\Omega_2), L_{r_1}(\Omega_2)\big).$$

Moreover $\lim_{\mu_2(D)\to 0} \|P_D T\|_{L_{p_0}\times L_{q_0}, L_{r_0}} = 0$ because $T : L_{p_0}(\Omega_0) \times L_{q_0}(\Omega_1) \longrightarrow L_{r_0}(\Omega_2)$ is compact. Hence, using Theorem 5.1, we derive that

$$\begin{aligned}\|P_D T\|_{L_p\times L_q, L_r} &\le \|P_D T\|^{1-\theta}_{L_{p_0}\times L_{q_0}, L_{r_0}} \|P_D T\|^{\theta}_{L_{p_1}\times L_{q_1}, L_{r_1}} \\ &\le \|P_D T\|^{1-\theta}_{L_{p_0}\times L_{q_0}, L_{r_0}} \|T\|^{\theta}_{L_{p_1}\times L_{q_1}, L_{r_1}} \longrightarrow 0 \text{ as } \mu_2(D)\to 0.\end{aligned}$$

This establishes (a).

In order to check (b), take $0 < \varepsilon < 1$ and let $s = \min(r_0, r_1, 1, \varepsilon p, \varepsilon q)$. Then the couple $\big(L_{r_0}(\Omega_2), L_{r_1}(\Omega_2)\big)$ is s-normed. Whence, Theorem 4.9 yields that the restrictions

$$T : \big(L_{p_0}(\Omega_0), L_{p_1}(\Omega_0)\big)_{\theta,s} \times \big(L_{q_0}(\Omega_1), L_{q_1}(\Omega_1)\big)_{\theta,\infty} \longrightarrow \big(L_{r_0}(\Omega_2), L_{r_1}(\Omega_2)\big)_{\theta,\infty} \tag{5.1}$$

$$T : \big(L_{p_0}(\Omega_0), L_{p_1}(\Omega_0)\big)_{\theta,\infty} \times \big(L_{q_0}(\Omega_1), L_{q_1}(\Omega_1)\big)_{\theta,s} \longrightarrow \big(L_{r_0}(\Omega_2), L_{r_1}(\Omega_2)\big)_{\theta,\infty} \tag{5.2}$$

are compact. The target space being

$$\big(L_{r_0}(\Omega_2), L_{r_1}(\Omega_2)\big)_{\theta,\infty} = \begin{cases} L_{r,\infty}(\Omega_2) & \text{if} \quad r_0 \ne r_1\,, \\ L_r(\Omega_2) & \text{if} \quad r_0 = r_1 = r\,. \end{cases}$$

Choose $0 < \eta_0, \eta_1 < 1$ such that $1/p = (1-\eta_0)/s$ and $1/q = \eta_1/s$. According to [4, Theorem 5.2.4], we get

$$\Big(\big(L_{p_0}(\Omega_0), L_{p_1}(\Omega_0)\big)_{\theta,s}, \big(L_{p_0}(\Omega_0), L_{p_1}(\Omega_0)\big)_{\theta,\infty}\Big)_{\eta_0,p} = \big(L_{p_0}(\Omega_0), L_{p_1}(\Omega_0)\big)_{\theta,p} = L_p(\Omega_0)$$

and

$$\Big(\big(L_{q_0}(\Omega_1), L_{q_1}(\Omega_1)\big)_{\theta,\infty}, \big(L_{q_0}(\Omega_1), L_{q_1}(\Omega_1)\big)_{\theta,s}\Big)_{\eta_1,q} = \big(L_{q_0}(\Omega_1), L_{q_1}(\Omega_1)\big)_{\theta,q} = L_q(\Omega_1).$$

Now applying Corollary 4.5 to restrictions (5.1) and (5.2) and having in mind the previous reiteration formulae, we conclude that $T : L_p(\Omega_0)\times L_q(\Omega_1) \longrightarrow L_{r,\infty}(\Omega_2)$ is compact. Therefore, $T\big(U_{L_p}, U_{L_q}\big)$ is relatively compact in $L_{r,\infty}(\Omega_2)$ and so it is also relatively compact in $\mathcal{M}(\mu_2)$. This proves (b) and completes the proof.

□

6 Compactness of bilinear commutators of Calderón-Zygmund operators

In this final section we work with the measure space $(\Omega,\mu) = (\mathbb{R}^n, dx)$. For this reason we drop the measure space in the notation for function spaces.

By a *bilinear Calderón-Zygmund operator* T we mean a bounded bilinear operator $T : L_p \times L_q \longrightarrow L_r$ where $1 < p, q < \infty$, $1/r = 1/p + 1/q$, such that there exits a kernel $K(x,y,z)$ defined away of the diagonal $x = y = z$ such that

$$|K(x,y,z)| \le c\,\frac{1}{(|x-y| + |x-z|)^{2n}},$$

$$|\nabla K(x,y,z)| \le c\,\frac{1}{(|x-y| + |x-z|)^{2n+1}}\,,$$

and

$$T(f,g)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x,y,z) f(y) g(z)\, dydz\,, \quad x \notin \operatorname{supp} f \cap \operatorname{supp} g$$

where f, g are bounded functions with compact support. See the paper by Grafakos and Torres [25] and the references given there.

Consider the following *bilinear commutators*

$$\begin{cases} [T,b]_1(f,g) = T(bf,g) - bT(f,g), \\ [T,b]_2(f,g) = T(f,bg) - bT(f,g), \\ \left[[T,b_1]_1, b_2\right]_2(f,g) = [T,b_1]_1(f,b_2 g) - b_2[T,b_1]_1(f,g). \end{cases} \tag{6.1}$$

where the functions b, b_1, b_2 belongs to CMO, the closure in BMO of the space of C^∞ functions with compact support.

Let S be any of the bilinear commutators in (6.1). It has been shown by Lerner et al [36] and Pérez et al [41] that $S : L_p \times L_q \longrightarrow L_r$ is bounded for $1 < p, q < \infty$ and $1/r = 1/p + 1/q$, so $1/2 < r < \infty$. Bényi and Torres [3, Theorem 1] have established compactness of S provided $1 \leq r < \infty$. Next we use the interpolation results of the previous sections to show that $S : L_p \times L_q \longrightarrow L_r$ is also compact if $1/2 < r < 1$.

Theorem 6.1 *Let T be a bilinear Calderón-Zygmund operator, let $b, b_1, b_2 \in CMO$ and let S be any of the bilinear commutators defined in (6.1). If $1 < p, q < \infty$, $1/2 < r < 1$ and $1/p + 1/q = 1/r$, then*

$$S : L_p \times L_q \longrightarrow L_r \quad \textit{is compact.}$$

Proof. Take $0 < \varepsilon < \min(1 - 1/2r,\, 1 - 1/p,\, 1 - 1/q)$ and put

$$r_1 = (1-\varepsilon)r\,, \quad p_1 = (1-\varepsilon)p\,, \quad q_1 = (1-\varepsilon)q\,.$$

Then $1/2 < r_1 < r < 1$, $1 < p_1 < p$, $1 < q_1 < q$ and $1/p_1 + 1/q_1 = 1/r_1$. Hence, according to [36, 41], $S : L_{p_1} \times L_{q_1} \longrightarrow L_{r_1}$ is bounded.

Choose $m \in \mathbb{N}$ such that $mr > 1$ and write

$$r_0 = mr > 1\,, \quad p_0 = mp > p\,, \quad q_0 = mq > q\,.$$

Again $1/p_0 + 1/q_0 = 1/r_0$ and, since $r_0 > 1$, it follows from [3, Theorem 1] that $S : L_{p_0} \times L_{q_0} \longrightarrow L_{r_0}$ is compact.

Next we show that S may be uniquely extended to a bounded bilinear operator

$$S : \left(L_{p_0} + L_{p_1}\right) \times \left(L_{q_0} + L_{q_1}\right) \longrightarrow L_{r_0} + L_{r_1}\,.$$

Put $1/s_0 = 1/p_0 + 1/q_1$ and $1/s_1 = 1/p_1 + 1/q_0$. Then

$$S : L_{p_0} \times L_{q_1} \longrightarrow L_{s_0} \text{ and } S : L_{p_1} \times L_{q_0} \longrightarrow L_{s_1}$$

are bounded. By our choices for parameters, we have that $1/r_0 < 1/s_0 < 1/r_1$, so there is $0 < \eta_0 < 1$ such that $1/s_0 = (1-\eta_0)/r_0 + \eta_0/r_1$. Hence $L_{s_0} \hookrightarrow L_{r_0} + L_{r_1}$. On the other hand, since $1/r_0 < 1/s_1 < 1/r_1$, there is $0 < \eta_1 < 1$ such that $1/s_1 = (1-\eta_1)/r_0 + \eta_1/r_1$. Whence $L_{s_1} \hookrightarrow L_{r_0} + L_{r_1}$. Consequently, the following restrictions are bounded

$$S : L_{p_0} \times L_{q_0} \longrightarrow L_{r_0} + L_{r_1}\,,$$

$$S : L_{p_1} \times L_{q_1} \longrightarrow L_{r_0} + L_{r_1}\,,$$

$$S : L_{p_0} \times L_{q_1} \longrightarrow L_{r_0} + L_{r_1}\,,$$

$$S : L_{p_1} \times L_{q_0} \longrightarrow L_{r_0} + L_{r_1} .$$

Now, proceeding as in Remark 4.10, we get that S has a unique extension to a bounded bilinear operator

$$S : \big(L_{p_0} + L_{p_1}\big) \times \big(L_{q_0} + L_{q_1}\big) \longrightarrow L_{r_0} + L_{r_1}$$

as we claimed.

Next we choose $0 < \theta < 1$ such that $1/r = (1-\theta)/r_0 + \theta/r_1$. That is to say, satisfying that $1 = (1-\theta)/m + \theta/(1-\varepsilon)$. Then we also have that $1/p = (1-\theta)/p_0 + \theta/p_1$ and $1/q = (1-\theta)/q_0 + \theta/q_1$. Since

$$S : \big(L_{p_0}, L_{p_1}\big) \times \big(L_{q_0}, L_{q_1}\big) \longrightarrow \big(L_{r_0}, L_{r_1}\big)$$

with $S : L_{p_0} \times L_{q_0} \longrightarrow L_{r_0}$ compactly and $r_0 \neq \infty$, applying Theorem 5.2 we conclude that

$$S : L_p \times L_q \longrightarrow L_r \quad \text{compactly.}$$

$\square$

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