ON SO(3)-BUNDLES OVER THE WOLF SPACES

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Abstract. We study the formality of the total space of principal SU(2) and SO(3)-bundles over a Wolf space, that is a symmetric positive quaternionic Kähler manifold. We apply this to conclude that all the 3-Sasakian homogeneous spaces are formal. We also determine the principal SU(2) and SO(3)-bundles over the Wolf spaces whose total space is non-formal.

1. Introduction

A Riemannian manifold (S, g) is called 3-*Sasakian manifold* if $S \times \mathbb{R}^+$ equipped with the cone metric $g^c = t^2 g + dt^2$ is hyperkähler, and so the holonomy group of g^c is a subgroup of $\mathrm{Sp}(n+1)$, where $2n+2$ is the complex dimension of the hyperkähler cone, and so S has odd dimension $4n+3$. The hyperkähler structure on the cone $(S \times \mathbb{R}^+, g^c = t^2 g + dt^2)$ induces a 3-*Sasakian structure* on the base of the cone. In particular, the triple of complex structures on $S \times \mathbb{R}^+$ gives rise to a triple of Reeb vector fields (ξ_1, ξ_2, ξ_3) on S whose Lie brackets give a copy of the Lie algebra $\mathfrak{su}(2)$ (see section 2 for details).

A 3-Sasakian manifold (S, g) is said to be *regular* if the vector fields ξ_1, ξ_2, ξ_3 are complete and the corresponding 3-dimensional foliation is regular, so that the space of leaves is a smooth $4n$-dimensional manifold M. Ishihara and Konishi [25] noticed that the induced metric on the latter is quaternionic Kähler with positive scalar curvature, that is M is a *positive quaternionic Kähler manifold*. Conversely [30, 44], starting with a positive quaternionic Kähler manifold M, the manifold M can be recovered as the total space of a bundle naturally associated to M.

Salamon [41] proved that compact positive quaternionic Kähler manifolds are simply connected and their odd Betti numbers are zero. Important results on the topology of a compact 3-Sasakian manifold were proved by Galicki and Salamon [20], showing that the odd Betti numbers b_{2i+1} of such a manifold of dimension $4n+3$, are all zero for $0 \leq i \leq n$. Moreover, for regular compact 3-Sasakian manifolds many topological properties are known (see [9, Proposition 13.5.6 and Theorem 13.5.7]). For example, such a manifold is simply connected unless $N = \mathbb{RP}^{4n+3}$. Also, using the results of LeBrun and Salamon [31] about the topology of positive quaternionic Kähler manifolds, Boyer and Galicki [9] show interesting relations among the Betti numbers of regular compact 3-Sasakian manifolds; in particular that $b_2 \leq 1$.

In this paper we deal with homotopical properties of *3-Sasakian homogeneous spaces*. Such a space S is a 3-Sasakian manifold with a transitive action of the group of automorphisms of the Sasakian 3-structure (see section 2 for details). The 3-dimensional foliation

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on S is regular, and the space of leaves is a homogeneous positive quaternionic Kähler manifold, that is a symmetric positive quaternionic Kähler manifold [1]. Such quaternionic Kähler manifolds are given by the infinite series $\mathbb{HP}^n$, $\mathbb{G}r_2(\mathbb{C}^{n+2})$ and $\widetilde{\mathbb{G}r}_4(\mathbb{R}^{n+4})$ (the Grassmannian of oriented real 4-planes) and by the exceptional symmetric spaces of compact type

$$GI = \frac{\mathrm{G}_2}{\mathrm{SO}(4)}, \qquad FI = \frac{\mathrm{F}_4}{\mathrm{Sp}(3)\cdot\mathrm{Sp}(1)}, \qquad EII = \frac{\mathrm{E}_6}{\mathrm{SU}(6)\cdot\mathrm{Sp}(1)},$$
$$EVI = \frac{\mathrm{E}_7}{\mathrm{Spin}(12)\cdot\mathrm{Sp}(1)} \qquad \text{and} \qquad EIX = \frac{\mathrm{E}_8}{\mathrm{E}_7\cdot\mathrm{Sp}(1)},$$

which are part of the E. Cartan classification [14]. The corresponding 3-Sasakian homogeneous spaces are given in Theorem 2.8. With the exception of the sphere S^{4n+3}, which is an $\mathrm{SU}(2) = \mathrm{Sp}(1)$-bundle over the quaternionic projective space $\mathbb{HP}^n$, 3-Sasakian homogeneous spaces are principal SO(3)-bundles over the spaces listed above. The Euler class $e(S)$ of the SO(3)-bundle is $-\frac{1}{4}p_1(S)$, where $p_1(S)$ is the first Pontryagin class of the SO(3)-bundle and it is defined by the quaternionic Kähler form of the quaternionic Kähler manifold.

A simply connected manifold is formal if its rational homotopy type is determined by its rational cohomology algebra. We shall say that M is *formal* if its minimal model is formal or, equivalently, if the de Rham complex $(\Omega^*(M), d)$ of M and the algebra of the de Rham cohomology $(H^*(M), d = 0)$ have the same minimal model (see section 3 for details).

A celebrated result of Deligne, Griffiths, Morgan and Sullivan states that any compact Kähler manifold is formal [16]. In the same spirit, formality of a manifold is related to the existence of suitable geometric structures on the manifold. Amann and Kapovitch [3] have proved that positive quaternionic Kähler manifolds are formal. Many other examples of formal manifolds are known: spheres, projective spaces, compact Lie groups, symmetric spaces of compact type and flag manifolds. Nevertheless, there are examples of non-formal homogeneous spaces (see [4] and references therein).

For Sasakian manifolds, that is Riemannian manifolds whose cone metric is Kähler, the first and second authors have proved that the formality is not an obstruction to the existence of Sasakian structures even on simply connected manifolds [5]. However, in [18] it is proved that the formality allows one to distinguish 7-dimensional Sasaki-Einstein manifolds which admit 3-Sasakian structures from those which do not.

Since the aforementioned homogeneous quaternionic spaces are formal, it seems interesting to understand the formality not only of the 3-Sasakian homogeneous spaces but also of the total space of any principal $\mathrm{SU}(2) = S^3$ and any principal $\mathrm{SO}(3) = \mathbb{RP}^3$-bundle over a homogeneous positive quaternionic Kähler manifold.

For a fibration $F \to E \to B$ there are conditions on the base B and the fiber F which imply that if B is formal, then E is formal [3, 33]. Let us recall that a simply connected topological space F is called *positively elliptic*, or F_0, if it is rationally elliptic, that is if it has finite dimensional rational homotopy and cohomology, and if it has positive Euler characteristic. In this case, its rational cohomology is concentrated in even degrees only. For F_0-spaces, Halperin's conjecture states that if F is such a space, then $H^*(F, \mathbb{Q})$ has no negative degree derivations. Lupton in [33] proved that if the base space B of the fibration is simply connected, and the fiber F is F_0 and satisfies Halperin's conjecture, then the formality of B implies the formality of E (see also [3]). Lupton's result can not be applied to SO(3) nor SU(2)-bundles because the mentioned conditions on the fiber F fail.

The structure of the paper is as follows. First, in sections 2 and 3 we review some definitions and results about 3-Sasakian homogeneous spaces and models (not necessarily minimal) of SO(3) and SU(2)-fibrations. In particular, we recall the calculation of the first Pontryagin class of an SO(3)-bundle and the second Chern class of an SU(2)-bundle (see (2.1)-(2.3)).

Section 4 is devoted to the study of the formality of SU(2) and SO(3)-bundles over the complex Grassmannian $\mathbb{G}r_2(\mathbb{C}^{n+2})$. Using the concept of s-formal minimal model, introduced in [19] as an extension of formality [16], we determine the principal SU(2) and SO(3)-bundles over $\mathbb{G}r_2(\mathbb{C}^{n+2})$ whose total space is formal (Theorem 4.3 and Theorem 4.4). In Theorem 4.4 we determine the principal SU(2) and SO(3)-bundles over $\mathbb{G}r_2(\mathbb{C}^{n+2})$ whose total space is non-formal because it has a non-trivial Massey product. In particular, if n is even, we show that there are non-formal SU(2) and SO(3)-bundles over $\mathbb{G}r_2(\mathbb{C}^{n+2})$. The characterization of all these bundles is given by the Euler class of the bundle. On the other hand, we show the cohomology class of the quaternionic Kähler form on $\mathbb{G}r_2(\mathbb{C}^{n+2})$ in terms of the generators of the rational cohomology of $\mathbb{G}r_2(\mathbb{C}^{n+2})$ (Proposition 4.2). Then, from Theorem 4.3 we conclude that the 3-Sasakian homogeneous space $\mathrm{SU}(n+2)/\mathrm{S}\big(\mathrm{U}(n)\times\mathrm{U}(1)\big)$ is formal.

In section 5, proceeding in the same way as in section 4, we determine the principal SU(2) and SO(3)-bundles over the oriented real Grassmannian $\widetilde{\mathbb{G}r}_4(\mathbb{R}^{n+4})$ whose total space is formal and also those whose total space is non-formal (Theorems 5.3, 5.4 and 5.6). Nevertheless, this study is more subtle than the one made in section 4. It is due to the fact that the rational cohomology of $\widetilde{\mathbb{G}r}_4(\mathbb{R}^{n+4})$ changes depending mainly of whether n is even or odd. In any case, we show that the 3-Sasakian homogeneous space $\mathrm{SO}(n+4)/(\mathrm{SO}(n)\times\mathrm{Sp}(1))$ is formal. From this result, Theorem 4.3 and the formality of the sphere S^{4n+3} and the real projective space $\mathbb{RP}^{4n+3}$, we get that *the non-exceptional 3-Sasakian homogeneous spaces are formal*. The formality of the exceptional 3-Sasakian homogeneous spaces and, more generally, the formality of the total space of SU(2) and SO(3)-bundles over the exceptional Wolf spaces is proved in section 6.

A central point in the study of positive quaternionic Kähler manifolds is the LeBrun-Salamon conjecture [31] that says that every positive quaternionic Kähler manifold is a symmetric space. This has been proved by Hitchin in dimension 4, by Poon-Salamon in dimension ≤ 8, and by Herrera and Herrera in dimension 12. By Theorem 2.1, any compact regular 3-Sasakian manifold S is an SU(2) or SO(3)-bundle over a compact positive quaternionic Kähler manifold M. Therefore, our results prove that if S is a compact regular 3-Sasakian manifold of dimension ≤ 15, then S is formal.

2. Homogeneous 3-Sasakian manifolds

We recall the notion of homogeneous 3-Sasakian space and the classification theorem of these spaces following [6, 9, 10].

An odd dimensional Riemannian manifold (S, g) is Sasakian if its cone $(S \times \mathbb{R}^+, g^c = t^2 g + dt^2)$ is Kähler, that is the cone metric $g^c = t^2 g + dt^2$ admits a compatible integrable almost complex structure J so that $(S \times \mathbb{R}^+, g^c = t^2 g + dt^2, J)$ is a Kähler manifold. In this case the Reeb vector field $\xi = J\partial_t$ is a Killing vector field of unit length. The corresponding 1-form η defined by $\eta(X) = g(\xi, X)$, for any vector field X on S, is a contact form. Let ∇ be the Levi-Civita connection of g. The (1,1) tensor $\phi X = \nabla_X \xi$ satisfies the identities

$$\phi^2 = -\operatorname{Id} + \eta \otimes \xi, \quad g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \quad d\eta(X,Y) = 2g(\phi X, Y),$$

for vector fields X, Y on S.

A collection of three Sasakian structures on a $(4n+3)$-dimensional Riemannian manifold satisfying quaternionic-like identities form a 3-Sasakian structure. More precisely, a Riemannian manifold (S, g) of dimension $4n+3$ is called 3-Sasakian if its cone $(S \times \mathbb{R}^+, g^c = t^2 g + dt^2)$ is hyperkähler, that is the metric $g^c = t^2 g + dt^2$ admits three compatible integrable almost complex structure J_s, $s = 1, 2, 3$, satisfying the quaternionic relations, i.e., $J_1 J_2 = -J_2 J_1 = J_3$, such that $(S \times \mathbb{R}^+, g^c = t^2 g + dt^2, J_1, J_2, J_3)$ is a hyperkähler manifold. Equivalently, the holonomy group of the cone metric g^c is a subgroup of $\mathrm{Sp}(n+1)$. In this case the Reeb vector fields $\xi_s = J_s \partial_t$, $s = 1, 2, 3$, are Killing vector fields. The three Reeb vector fields ξ_s, the three 1-forms η_s and the three $(1,1)$ tensors ϕ_s, $s = 1, 2, 3$, satisfy the relations

$$\begin{aligned}
&\eta_i(\xi_j) = g(\xi_i, \xi_j) = \delta_{ij}, \\
&\phi_i \, \xi_j = -\phi_j \, \xi_i = \xi_k, \\
&\eta_i \circ \phi_j = -\eta_j \circ \phi_i = \eta_k, \\
&\phi_i \circ \phi_j - \eta_j \otimes \xi_i = -\phi_j \circ \phi_i + \eta_i \otimes \xi_j = \phi_k,
\end{aligned}$$

for any cyclic permutation (i, j, k) of $(1, 2, 3)$.

The Reeb vector fields ξ_s satisfy the following relations $g(\xi_i, \xi_j) = \delta_{ij}$ and $[\xi_i, \xi_j] = 2\xi_k$. Thus, they span an integrable 3-dimensional distribution on a 3-Sasakian manifold. Denote by $\mathcal{F}$ the 3-dimensional foliation generated by the Reeb vector fields (ξ_1, ξ_2, ξ_3).

If (S, g) is a compact 3-Sasakian manifold, then the Reeb vector fields ξ_s are complete and the leaves of the foliation $\mathcal{F}$ are compact. Hence, $\mathcal{F}$ is *quasi-regular*. The 3-Sasakian structure on S is said to be *regular* if $\mathcal{F}$ is a regular foliation.

The following theorem was first proved by Ishihara [24] in the regular case. The general version, that we recall here, was proved in [11] (see also [10]). First we recall that a $4n$-dimensional $(n > 1)$ Riemannian manifold/orbifold is quaternionic Kähler if it has holonomy group contained in $\mathrm{Sp}(n)\mathrm{Sp}(1)$, and a 4-dimensional quaternionic Kähler manifold/orbifold is a self-dual Einstein Riemannian manifold/orbifold.

Theorem 2.1 ([10])**.** *Let (S, g) be a 3-Sasakian manifold of dimension $4n+3$ such that the Reeb vector fields (ξ_1, ξ_2, ξ_3) are complete. Then the space of leaves $S/\mathcal{F}$ has the structure of a quaternionic Kähler orbifold $(\mathcal{O}, g_{\mathcal{O}})$ of dimension $4n$ such that the natural projection $\pi \colon S \to \mathcal{O}$ is a principal orbi-bundle with group $\mathrm{SU}(2)$ or $\mathrm{SO}(3)$, and π is a Riemannian orbifold submersion such that the scalar curvature of $g_{\mathcal{O}}$ is $16n(n+2)$.*

Proposition 2.2 ([10])**.** *Let (S, g) be a 3-Sasakian manifold such that the Reeb vector fields (ξ_1, ξ_2, ξ_3) are complete. Denote by $\mathcal{F}$ the canonical three dimensional foliation on S. Then,*

i) The leaves of $\mathcal{F}$ *are totally geodesic spherical space forms* $\Gamma\backslash S^3$ *of constant curvature one, where* $\Gamma \subset \mathrm{Sp}(1) = \mathrm{SU}(2)$ *is a finite subgroup.*
ii) The 3*-Sasakian structure on* S *restricts to a* 3*-Sasakian structure on each leaf.*
iii) The generic leaves are either $\mathrm{SU}(2)$ *or* $\mathrm{SO}(3)$*.*

Let (S, g) be a 3-Sasakian manifold. Then, the isometry group $\mathrm{Iso}(S, g)$ of (S, g) is non-trivial, and it has dimension ≥ 3 since each Sasakian structure has an isometry group of dimension ≥ 1. Denote by $\mathrm{Aut}(S, g) \subset \mathrm{Iso}(S, g)$ the subgroup of the isometry group which preserves the 3-Sasakian structure $(g, \xi_s, \eta_s, \phi_s; s = 1, 2, 3)$ on S. The group $\mathrm{Aut}(S, g)$ can be characterized as follows.

Lemma 2.3 ([10])**.** *Let* (S, g) *be a* 3*-Sasakian manifold, with the* 3*-Sasakian structure* $(g, \xi_s, \eta_s, \phi_s; s = 1, 2, 3)$*, and let* $f \in \mathrm{Iso}(S, g)$*. Then, the following conditions are equivalent*

i) $f_\star\xi_s = \xi_s$, $s = 1, 2, 3$;
ii) $f^\star\eta_s = \eta_s$, $s = 1, 2, 3$;
iii) $f_\star \circ \phi_s = \phi_s \circ f_\star$, $s = 1, 2, 3$;
iv) $f \in \mathrm{Aut}(S, g)$.

Definition 2.4. A 3-Sasakian manifold (S, g) is said to be a 3-*Sasakian homogeneous space* if the group $\mathrm{Aut}(S, g)$ acts transitively on S.

Proposition 2.5 ([10])**.** *Let* (S, g) *be a* 3*-Sasakian homogeneous space of dimension* $4n+3$*. Then, all leaves are diffeomorphic and* $\mathcal{O} = S/\mathcal{F}$ *is a quaternionic Kähler manifold such that the natural projection* $\pi\colon S \to \mathcal{O}$ *is a locally trivial Riemannian fibration. Moreover,* $\mathrm{Aut}(S, g)$ *passes to the quotient and acts transitively on the space of leaves* $\mathcal{O}$*.*

Note that a quaternionic Kähler manifold is not necessarily Kähler, as the name might suggest. Moreover, if $\mathcal{O}$ is a quaternionic Kähler manifold, then the scalar curvature t of $\mathcal{O}$ is constant since it is Einstein. Thus, there are three classes of examples of quaternionic Kähler manifolds corresponding to $t > 0$, $t = 0$ and $t < 0$.

Definition 2.6. A *positive quaternionic Kähler manifold* is a quaternionic Kähler manifold with complete metric and with positive scalar curvature.

We will use also the following properties of quaternionic Kähler manifolds.

Theorem 2.7 ([41])**.** *Let* M *be a compact quaternionic Kähler manifold of positive scalar curvature. Then* $\pi_1(M) = 0$ *and its odd Betti numbers are zero.*

In order to show a classification of 3-Sasakian homogeneous spaces, we recall that any homogeneous positive quaternionic Kähler manifold is a symmetric space [1]. These homogeneous manifolds are referred to as *Wolf spaces* in recognition of [45], and they are given by the $4n$-dimensional spaces of (real) dimension $4n$

$$\mathbb{HP}^n = \frac{\mathrm{Sp}(n+1)}{\mathrm{Sp}(n) \times \mathrm{Sp}(1)}, \quad \mathbb{G}r_2(\mathbb{C}^{n+2}), \quad \widetilde{\mathbb{G}r}_4(\mathbb{R}^{n+4})$$

where $\widetilde{\mathbb{G}r}_4(\mathbb{R}^{n+4})$ is the Grassmannian of oriented real 4-planes, and by the exceptional symmetric spaces

$$GI = \frac{\mathrm{G}_2}{\mathrm{SO}(4)}, \qquad FI = \frac{\mathrm{F}_4}{\mathrm{Sp}(3)\cdot\mathrm{Sp}(1)}, \qquad EII = \frac{\mathrm{E}_6}{\mathrm{SU}(6)\cdot\mathrm{Sp}(1)},$$
$$EVI = \frac{\mathrm{E}_7}{\mathrm{Spin}(12)\cdot\mathrm{Sp}(1)} \qquad \text{and} \qquad EIX = \frac{\mathrm{E}_8}{\mathrm{E}_7\cdot\mathrm{Sp}(1)},$$

of real dimension 8, 28, 40, 64 and 112, respectively. This classification and Proposition 2.5 imply the following classification theorem of 3-Sasakian homogeneous spaces.

Theorem 2.8 ([10]). *Let (S,g) be a 3-Sasakian homogeneous space. Then S is precisely one of the following homogeneous spaces:*

$$\frac{\mathrm{Sp}(k+1)}{\mathrm{Sp}(k)} \cong S^{4k+3}, \quad \frac{\mathrm{Sp}(k+1)}{\mathrm{Sp}(k)\times\mathbb{Z}_2} \cong \mathbb{RP}^{4k+3}, \quad \frac{\mathrm{SU}(n+2)}{\mathrm{S}\big(\mathrm{U}(n)\times\mathrm{U}(1)\big)}, \quad \frac{\mathrm{SO}(m+4)}{\mathrm{SO}(m)\times\mathrm{Sp}(1)},$$
$$\frac{\mathrm{G}_2}{\mathrm{Sp}(1)}, \quad \frac{\mathrm{F}_4}{\mathrm{Sp}(3)}, \quad \frac{\mathrm{E}_6}{\mathrm{SU}(6)}, \quad \frac{\mathrm{E}_7}{\mathrm{Spin}(12)}, \quad \frac{\mathrm{E}_8}{\mathrm{E}_7},$$

where $k \geq 0$, $n \geq 1$ and $m \geq 3$ (the ones in the first line are called non-exceptional, and the ones in the second line are called exceptional). For the first two cases when $k=0$, Sp(0) *is the identity group. Furthermore, the fiber F over the Wolf space is* Sp(1) *only for S^{4k+3}. In all the other cases $F = \mathrm{SO}(3)$.*

Note that Theorem 2.8 implies that any homogeneous 3-Sasakian manifold is simply connected with the exception of the real projective space. Moreover, the dimension of the 3-Sasakian homogeneous spaces corresponding to the exceptional Lie groups are (in the order given in the previous Theorem) 11, 31, 43, 67 and 115, respectively.

Let (S,g) be a 3-Sasakian regular manifold, and let (M,g_M) be the quaternionic Kähler manifold given as the space of orbits. Then, Theorem 2.1 implies that there is a principal fiber bundle $F \to S \to M$ with $F = \mathrm{SU}(2)$ or $F = \mathrm{SO}(3)$.

Suppose first that $F = \mathrm{SO}(3)$. By formulas (3.6) and (3.7) of [30], the principal fiber bundle $S \to M$ has a natural connection whose curvature can be described locally in terms of the quaternionic Kähler structure on M. Indeed, locally on M we have three complex structures J_1, J_2, J_3 (with $J_3 = J_1J_2 = -J_2J_1$) each of them compatible with the Riemannian metric g_M, and with corresponding (local) Kähler forms $\omega_1, \omega_2, \omega_3$. The SO(3)-bundle has a bundle of Lie algebras $\mathfrak{so}(3)$ with a frame e_1, e_2, e_3, and the curvature of the natural connection is given by $R = \omega_1\, e_1 + \omega_2\, e_2 + \omega_3\, e_3$. The quaternionic Kähler form is the 4-form Ω on M given as

$$\Omega = \mathrm{tr}(R\wedge R) = \omega_1\wedge\omega_1 + \omega_2\wedge\omega_2 + \omega_3\wedge\omega_3\,.$$

This form is the representative of the first Pontryagin class

$$p_1(S) = [\mathrm{tr}(R\wedge R)] = [\Omega] \in H^4(M,\mathbb{Z}). \tag{2.1}$$

In the case that the principal fiber bundle $F \to S \to M$ has fiber $F = \mathrm{SU}(2)$, we can take the associated SO(3)-bundle via the epimorphism $\mathrm{SU}(2) \to \mathrm{SO}(3) = \mathrm{SU}(2)/\mathbb{Z}_2$, equivalently quotient by the center $\mathbb{Z}_2$. In this way we obtain another 3-Sasakian manifold $S' = S/\mathbb{Z}_2$ with a fiber bundle $F' = \mathrm{SO}(3) \to S' \to M$. The SU(2)-bundle S is characterised by the Euler class $e(S)$ of the fibration, that is the second Chern class $c_2(S) \in H^4(S, \mathbb{Z})$ and it is given by

$$e(S) = c_2(S) = -\frac{1}{4}[\Omega]. \tag{2.2}$$

This can be proved as follows. Let $E \to M$ be the rank 2 complex vector bundle associated to S, then $\mathrm{End}(E) \to M$ consisting of skew-hermitian endomorphisms, is the rank 3 real vector bundle associated to S'. Then $[\Omega] = p_1(S') = p_1(\mathrm{End}(E)) = -c_2(\mathrm{End}(E) \otimes \mathbb{C}) = -c_2(\mathrm{End}_{\mathbb{C}}(E)) = -c_2(E \otimes E^*)$. To compute this, let x_1, x_2 be the Chern roots of E. So the Chern roots of $E \otimes E^*$ are $1, x_1 - x_2$ and $x_2 - x_1$. So $[\Omega] = -c_2(E \otimes E^*) = (x_1 - x_2)^2 = c_1(E)^2 - 4c_2(E) = -4c_2(E) = -4c_2(S)$ (see [32]).

Finally, for later use, we compute the Pontryagin class in the following situation. Suppose that we have a principal $F = \mathrm{SO}(3)$-bundle $S \to M$, and that there is a lifting to an $\mathrm{U}(2)$-bundle under the epimorphism $\mathrm{U}(2) \to \mathrm{SO}(3)$, say $\widetilde{S} \to M$. The rank 2 complex vector bundle $E \to M$ associated to $\widetilde{S}$ has Chern classes $c_1(E), c_2(E)$. The rank 3 real vector bundle associated to S is $\mathrm{End}(E) \to M$, consisting of skew-hermitian endomorphisms. Then,

$$p_1(S) = -c_2(E \otimes E^*) = c_1(E)^2 - 4c_2(E) = c_1(\widetilde{S})^2 - 4c_2(\widetilde{S}). \tag{2.3}$$

For the Euler class $e(S)$ of the SO(3)-bundle we have

$$e(S) = -\frac{1}{4}p_1(S). \tag{2.4}$$

3. Minimal models and formal manifolds

In this section we review some definitions and results about minimal models and Massey products on smooth manifolds (see [16, 17, 19] for more details).

We work with *differential graded commutative algebras*, or DGAs, over the field $\mathbb{R}$ of real numbers. The degree of an element a of a DGA is denoted by $|a|$. A DGA $(\mathcal{A}, d)$ is said to be *minimal* if:

(1) $\mathcal{A}$ is free as an algebra, that is $\mathcal{A}$ is the free algebra $\bigwedge V$ over a graded vector space $V = \bigoplus_i V^i$, and
(2) there is a collection of generators $\{a_\tau\}_{\tau \in I}$ indexed by some well ordered set I, such that $|a_\mu| \leq |a_\tau|$ if $\mu < \tau$ and each da_τ is expressed in terms of the previous a_μ, $\mu < \tau$. This implies that da_τ does not have a linear part.

In our context, the main example of DGA is the de Rham complex $(\Omega^*(M), d)$ of a smooth manifold M, where d is the exterior differential.

The cohomology of a differential graded commutative algebra $(\mathcal{A}, d)$ is denoted by $H^*(\mathcal{A})$. This space is naturally a DGA with the product inherited from that on $\mathcal{A}$ while the differential on $H^*(\mathcal{A})$ is identically zero. A DGA $(\mathcal{A}, d)$ is connected if $H^0(\mathcal{A}) = \mathbb{R}$, and it is 1-connected if, in addition, $H^1(\mathcal{A}) = 0$.

Morphisms between DGAs are required to preserve the degree and to commute with the differential. We say that $(\bigwedge V, d)$ is a *minimal model* of a differential graded commutative algebra $(\mathcal{A}, d)$ if $(\bigwedge V, d)$ is minimal and there exists a quasi-isomorphism, that is a morphism of differential graded algebras

$$\rho\colon\ (\bigwedge V,\, d) \longrightarrow (\mathcal{A},\, d)$$

inducing an isomorphism $\rho^*:\ H^*(\bigwedge V) \xrightarrow{\sim} H^*(\mathcal{A})$ of cohomologies. A connected differential graded algebra has a minimal model unique up to isomorphism [23] (see [16, 22, 43] for the 1-connected case).

A *minimal model* of a connected smooth manifold M is a minimal model $(\bigwedge V,\, d)$ for the de Rham complex $(\Omega^*(M),\, d)$ of differential forms on M. If M is a simply connected manifold, then the dual of the real homotopy vector space $\pi_i(M)\otimes\mathbb{R}$ is isomorphic to the space V^i of generators in degree i, for any i. The latter also happens when $i > 1$ and M is nilpotent, that is, the fundamental group $\pi_1(M)$ is nilpotent and its action on $\pi_j(M)$ is nilpotent for all $j > 1$ (see [16]).

We say that a DGA $(\mathcal{A},\, d)$ is a *model* of a manifold M if $(\mathcal{A},\, d)$ and M have the same minimal model. Thus, if $(\bigwedge V,\, d)$ is the minimal model of M, we have

$$(\mathcal{A},\, d) \xleftarrow{\ \nu\ } (\bigwedge V,\, d) \xrightarrow{\ \rho\ } (\Omega^*(M),\, d),$$

where ρ and ν are quasi-isomorphisms.

A minimal algebra $(\bigwedge V,\, d)$ is *formal* if there exists a morphism of differential algebras $\psi\colon\ (\bigwedge V,\, d) \longrightarrow (H^*(\bigwedge V),\, 0)$ inducing the identity map on cohomology. A DGA $(\mathcal{A}, d)$ is formal if its minimal model is formal. A smooth manifold M is formal if its minimal model is formal. Many examples of formal manifolds are known: spheres, projective spaces, compact Lie groups, symmetric spaces, flag manifolds, and compact Kähler manifolds. Recently, in [3] it is proved the following

Theorem 3.1 ([3])**.** *Compact positive quaternionic Kähler manifolds are formal.*

Remark 3.2. Note that there are examples of non-formal homogeneous spaces (see [4] and references therein). Amann has proved [4] that in every dimension ≥ 72, there is an irreducible simply connected compact homogeneous space which is not formal.

The formality property of a minimal algebra is characterized as follows.

Theorem 3.3 ([16])**.** *A minimal algebra $(\bigwedge V,\, d)$ is formal if and only if the space V can be decomposed into a direct sum $V = C \oplus N$ with $d(C) = 0$, d is injective on N and such that every closed element in the ideal $I(N)$ generated by N in $\bigwedge V$ is exact.*

This characterization of formality can be weakened using the concept of s-formality introduced in [19].

Definition 3.4. A minimal algebra $(\bigwedge V,\, d)$ is s-formal ($s > 0$) if for each $i \leq s$ the space V^i of generators of degree i decomposes as a direct sum $V^i = C^i \oplus N^i$, where the spaces C^i and N^i satisfy the following conditions:

1. $d(C^i) = 0$,
2. the differential map $d\colon\ N^i \longrightarrow \bigwedge V$ is injective, and
3. any closed element in the ideal $I_s = I(\bigoplus_{i\leq s} N^i)$, generated by the space $\bigoplus_{i\leq s} N^i$ in the free algebra $\bigwedge(\bigoplus_{i\leq s} V^i)$, is exact in $\bigwedge V$.

A smooth manifold M is s-formal if its minimal model is s-formal. Clearly, if M is formal then M is s-formal for every $s > 0$. The main result of [19] shows that sometimes the weaker condition of s-formality implies formality.

Theorem 3.5 ([19])**.** *Let M be a connected and orientable compact differentiable manifold of dimension $2n$ or $(2n-1)$. Then M is formal if and only if it is $(n-1)$-formal.*

One can check that any simply connected compact manifold is 2-formal. Therefore, Theorem 3.5 implies that any simply connected compact manifold of dimension at most six is formal. (This result was proved earlier in [38].)

In order to detect non-formality, instead of computing the minimal model, which is usually a lengthy process, one can use Massey products, which are obstructions to formality. The simplest type of Massey product is the triple Massey product, which is defined as follows. Let $(\mathcal{A},\, d)$ be a DGA (in particular, it can be the de Rham complex of differential forms on a smooth manifold). Suppose that there are cohomology classes $[a_i] \in H^{p_i}(\mathcal{A})$, $p_i > 0$, $1 \leq i \leq 3$, such that $a_1 \cdot a_2$ and $a_2 \cdot a_3$ are exact. Write $a_1 \cdot a_2 = da_{1,2}$ and $a_2 \cdot a_3 = da_{2,3}$. The *(triple) Massey product* of the classes $[a_i]$ is defined as

$$\langle [a_1], [a_2], [a_3] \rangle \;=\; [a_1 \cdot a_{2,3} + (-1)^{p_1+1} a_{1,2} \cdot a_3] \in \frac{H^{p_1+p_2+p_3-1}(\mathcal{A})}{[a_1] \cdot H^{p_2+p_3-1}(\mathcal{A}) + [a_3] \cdot H^{p_1+p_2-1}(\mathcal{A})}.$$

Note that a Massey product $\langle [a_1], [a_2], [a_3] \rangle$ on $(\mathcal{A},\, d)$ is zero (or trivial) if and only if there exist $\widetilde{x}, \widetilde{y} \in \mathcal{A}$ such that $a_1 \cdot a_2 = d\widetilde{x}$, $\;a_2 \cdot a_3 = d\widetilde{y}$ and $[a_1 \cdot \widetilde{y} + (-1)^{p_1+1} \widetilde{x} \cdot a_3] = 0$.

We will use also the following property (see [18] for a proof).

Lemma 3.6. *Let M be a connected differentiable manifold. Then, Massey products on M can be calculated by using any model of M.*

Moreover, we will use the following results.

Lemma 3.7 ([16])**.** *If M has a non-trivial Massey product, then M is non-formal.*

Lemma 3.8 ([18])**.** *Let M be a 7-dimensional simply connected compact manifold with $b_2(M) \leq 1$. Then, M is 3-formal and so formal.*

Minimal models of $\mathrm{SU}(2)$ **and** $\mathrm{SO}(3)$**-fibrations.** Let $F \to E \to B$ be a fibration of simply connected spaces. Let $(\mathcal{A}_B, \widetilde{d})$ be a model (not necessarily minimal) of the base B, and let $(\bigwedge V_F, d)$ be a minimal model of the fiber F. By [40], a model of E is the *KS-extension* $(\mathcal{A}_B \otimes \bigwedge V_F, D)$, where D is defined as $Db = \widetilde{d}b$, for $b \in \mathcal{A}_B$, and $Df = df + \Theta(f)$, there $f \in V_F$, and

$$\Theta : V_F \to \mathcal{A}_B$$

is called the *transgression map.* This is also true in the case that F, E and B are nilpotent spaces and the fibration is nilpotent, that is $\pi_1(B)$ acts nilpotently in the homotopy groups $\pi_j(F)$ of the fiber.

In the case that E and B are simply connected and $F = \mathrm{SU}(2) = S^3$ or $F = \mathrm{SO}(3) = \mathbb{RP}^3$, the fibration is nilpotent. Note that $\pi_1(\mathrm{SO}(3)) = \mathbb{Z}_2$ but it acts trivially on the higher homotopy groups, since the antipodal map on S^3 is homotopic to the identity. The minimal model of S^3 is $(\bigwedge u, d)$, with $|u| = 3$ and $du = 0$. Both spaces S^3 and $\mathbb{RP}^3$ are rationally homotopy equivalent, so a fibration $\mathrm{SO}(3) \to E \to B$ is a rational S^3-fibration (that is, after rationalization of the spaces, it becomes a fibration). The transgression map Θ is such that $\Theta(u) \in \mathcal{A}_B^4$ is a closed element of degree 4 defining the Euler class $e(E)$ of the fibration.

4. SU(2) and SO(3)-bundles over the complex Grassmannian $\mathbb{G}r_2(\mathbb{C}^{n+2})$

Now it is our purpose to prove the formality of all the 3-Sasakian homogeneous spaces. By Theorem 2.8 we know that, except for the sphere S^{4n+3}, such an space is the total space of an $\mathrm{SO}(3) = \mathbb{RP}^3$-bundle over a Wolf space. The SO(3)-bundles over the exceptional Wolf spaces will be treated in Section 6. For the SO(3)-bundles over the non-exceptional Wolf spaces, we note that the sphere S^{4n+3} and the projective space $\mathbb{RP}^{4n+3}$ are formal, so it is sufficient to prove the formality of the spaces $\mathrm{SU}(n+2)/\mathrm{S}\big(\mathrm{U}(n) \times \mathrm{U}(1)\big)$ and $\mathrm{SO}(n+4)/(\mathrm{SO}(n) \times \mathrm{Sp}(1))$. In this section we deal with the first case and, more generally, we study the formality of the total space of SU(2) and SO(3)-bundles over the complex Grassmannian

$$\mathbb{G}r_2(\mathbb{C}^{n+2}) = \frac{\mathrm{SU}(n+2)}{\mathrm{S}\big(\mathrm{U}(n) \times \mathrm{U}(2)\big)}.$$

Its cohomology ring is given by (see [2])

$$H^*\big(\mathbb{G}r_2(\mathbb{C}^{n+2})\big) = H^*(BT)^{W(\mathrm{U}(n)\times\mathrm{U}(2))}/H^{>0}(BT)^{W(\mathrm{U}(n+2))}, \tag{4.1}$$

where BT is the classifying space of a maximal torus of $\mathrm{U}(n+2)$, and $W(G)$ denotes the Weyl group of a Lie group G. Thus the classes of the cohomology of $\mathbb{G}r_2(\mathbb{C}^{n+2})$ are symmetric polynomials of $H^*(BT) = \mathbb{Q}[x_1, \dots, x_n, x_{n+1}, x_{n+2}]$, where each generator x_i has degree 2, for $1 \le i \le (n+2)$. Denote by y_1, y_2 the classes x_{n+1}, x_{n+2}, respectively. On one hand, $H^*(BT)^{W(\mathrm{U}(n)\times\mathrm{U}(2))}$ is generated by the symmetric polynomials τ_n of $x_1, \dots, x_n$, and by the symmetric polynomials σ_n of y_1, y_2. On the other hand, $H^{>0}(BT)^{W(\mathrm{U}(n+2))}$ is generated by the symmetric polynomials of $x_1, \dots, x_n, y_1, y_2$. This gives the relation $\sigma\,\tau = 1$, where $\sigma = 1 + \sigma_1 + \sigma_2 + \dots$ and $\tau = 1 + \tau_1 + \tau_2 + \dots$ Denote $l = y_1 + y_2$ and $x = y_1\, y_2$. Here and in what follows, $y_1 y_2$ stands for the cup product $y_1 \cup y_2$, and so on. Thus,

$$H^*\big(\mathbb{G}r_2(\mathbb{C}^{n+2})\big) = \mathbb{Q}[l, x]/(\sigma_{n+1}, \sigma_{n+2}), \tag{4.2}$$

where $|l| = 2$, $|x| = 4$, and σ_r ($r \ge 0$) is the cohomology class of degree $2r$ that is defined recursively by

$$\begin{cases} \sigma_0 = 1, \\ \sigma_1 = -l, \\ \sigma_r = -l\,\sigma_{r-1} - x\,\sigma_{r-2}, \qquad r \ge 2. \end{cases} \tag{4.3}$$

In the following Lemma, we determine the expression of σ_r in terms of the cohomology classes l, x and their cup products.

Lemma 4.1. *The cohomology class σ_r ($r \ge 0$) on $\mathbb{G}r_2(\mathbb{C}^{n+2})$ has the following expression*

$$\sigma_r = \sum_{k=0}^{\lfloor r/2 \rfloor} (-1)^{r+k} \binom{r-k}{k} l^{r-2k}\, x^k. \tag{4.4}$$

Proof. We proceed by induction on $r \geq 0$. It is clear that (4.4) is true for $r = 0, 1$. Assume that (4.4) holds for $s \leq r$. Then, using the recursive definition of σ_k given by (4.3) and the induction hypothesis we have

$$\begin{aligned}
\sigma_{r+1} &= -l\sigma_r - x\sigma_{r-1} \\
&= -l\sum_{k=0}^{\lfloor r/2\rfloor}(-1)^{r+k}\binom{r-k}{k}l^{r-2k}x^k \,-\, x\sum_{k=0}^{\lfloor (r-1)/2\rfloor}(-1)^{r-1+k}\binom{r-1-k}{k}l^{r-1-2k}x^k \\
&= -\sum_{k=0}^{\lfloor r/2\rfloor}(-1)^{r+k}\binom{r-k}{k}l^{r-2k+1}x^k - \sum_{k=0}^{\lfloor (r-1)/2\rfloor}(-1)^{r-1+k}\binom{r-1-k}{k}l^{r-1-2k}x^{k+1} \\
&= -\sum_{k=0}^{\lfloor r/2\rfloor}(-1)^{r+k}\binom{r-k}{k}l^{r-2k+1}x^k - \sum_{k=1}^{\lfloor (r-1)/2\rfloor+1}(-1)^{r+k}\binom{r-k}{k-1}l^{r+1-2k}x^k.
\end{aligned}$$

Thus,

$$\begin{aligned}
\sigma_{r+1} = &-\sum_{k=1}^{\lfloor r/2\rfloor}(-1)^{r+k}\left(\binom{r-k}{k-1}+\binom{r-k}{k}\right)l^{r-2k+1}x^k - (-1)^r\binom{r}{0}l^{r+1} \\
&+\epsilon(-1)^{r+\lfloor (r-1)/2\rfloor}\binom{r-\lfloor (r-1)/2\rfloor-1}{\lfloor (r-1)/2\rfloor}x^{\lfloor (r-1)/2\rfloor+1},
\end{aligned} \tag{4.5}$$

where $\epsilon \in \{0,1\}$, and $\epsilon = 1$ if and only if r is odd, otherwise $\epsilon = 0$. Clearly if r is odd, $\lfloor (r-1)/2\rfloor = (r-1)/2$ and $\binom{r-\lfloor (r-1)/2\rfloor-1}{\lfloor (r-1)/2\rfloor} = 1$. Moreover, it is well known that $\binom{r-k}{k-1}+\binom{r-k}{k} = \binom{r+1-k}{k}$ and $\binom{r}{0} = \binom{r+1}{0}$. Substituting these equalities into (4.5), we obtain

$$\sigma_{r+1} = \sum_{k=0}^{\lfloor (r+1)/2\rfloor}(-1)^{r+1+k}\binom{r+1-k}{k}l^{r+1-2k}x^k.$$

□

For the quaternionic Kähler form on $\mathbb{G}r_2(\mathbb{C}^{n+2})$ we have:

Proposition 4.2. *Let Ω be the quaternionic Kähler form on $\mathbb{G}r_2(\mathbb{C}^{n+2})$. Then, in terms of the generators l and x of $H^*\big(\mathbb{G}r_2(\mathbb{C}^{n+2})\big)$ given by (4.2), the de Rham cohomology class $[\Omega] \in H^4(\mathbb{G}r_2(\mathbb{C}^{n+2}))$ of Ω is*

$$[\Omega] \,=\, l^2 - 4x.$$

Proof. The cohomology class defined by Ω in $H^4(\mathbb{G}r_2(\mathbb{C}^{n+2}))$ determines the Pontryagin class of the principal SO(3)-bundle

$$\mathrm{SO}(3) = \mathrm{SU}(2)/\mathbb{Z}_2 \hookrightarrow S = \frac{\mathrm{SU}(n+2)}{\mathrm{S}\big(\mathrm{U}(n)\times\mathrm{U}(1)\big)} \longrightarrow \mathbb{G}r_2(\mathbb{C}^{n+2}) = \frac{\mathrm{SU}(n+2)}{\mathrm{S}\big(\mathrm{U}(n)\times\mathrm{U}(2)\big)},$$

which gives the natural 3-Sasakian homogeneous space S, where $\mathrm{U}(1) \subset \mathrm{U}(2)$ diagonally. We can lift this SO(3)-bundle to an U(2)-bundle by considering the complex Grassmanian $\mathbb{G}r_2(\mathbb{C}^{n+2})$ as the quotient space $\mathbb{G}r_2(\mathbb{C}^{n+2}) = \mathrm{U}(n+2)\,/\big(\mathrm{U}(n)\times\mathrm{U}(2)\big)$, and taking the principal U(2)-bundle

$$\mathrm{U}(2) \hookrightarrow \widetilde{S} = \mathrm{U}(n+2)/\mathrm{U}(n) \longrightarrow \mathbb{G}r_2(\mathbb{C}^{n+2}) = \mathrm{U}(n+2)\,/\big(\mathrm{U}(n)\times\mathrm{U}(2)\big).$$

By the description in (4.1), y_1 and y_2 are the Chern roots of $\mathrm{U}(2) < \mathrm{U}(n+2)$ embedded as the last two entries. Then, $c_1(\widetilde{S}) = y_1 + y_2 = l$ and $c_2(\widetilde{S}) = y_1 y_2 = x$. Thus, by (2.1) and (2.3),

$$[\Omega] = p_1(S) = c_1(\widetilde{S})^2 - 4c_2(\widetilde{S}) = l^2 - 4x.$$

□

Theorem 4.3. *Consider the principal F-fiber bundle $F \to S \to \mathbb{G}r_2(\mathbb{C}^{n+2})$ with $F = \mathrm{SU}(2)$ or $F = \mathrm{SO}(3)$, and Euler class $al^2 + bx$, where $a, b \in \mathbb{Q}$. If $b \neq 0$, then S is formal. In particular, the 3-Sasakian homogeneous space $\mathrm{SU}(n+2)/\mathrm{S}\big(\mathrm{U}(n) \times \mathrm{U}(1)\big)$ is formal.*

Proof. We can assume that $n \geq 2$. Indeed, if $n = 0$, then S is formal by Theorem 3.5 and, if $n = 1$, then S is formal by Lemma 3.8.

Since $\mathbb{G}r_2(\mathbb{C}^{n+2})$ is a compact positive quaternionic Kähler manifold, $\mathbb{G}r_2(\mathbb{C}^{n+2})$ is simply connected [41]. Then, according with section 3, the fibre bundle $F \to S \to \mathbb{G}r_2(\mathbb{C}^{n+2})$, with $F = \mathrm{SU}(2)$ or $F = \mathrm{SO}(3)$ and Euler class $al^2 + bx$, is a rational S^3-fibration. Thus [40], if $(\mathcal{A}, d_{\mathcal{A}})$ is a model of $\mathbb{G}r_2(\mathbb{C}^{n+2})$, we have that $(\mathcal{A} \otimes \bigwedge(u), d)$, with $|u| = 3$, $d|_{\mathcal{A}} = d_{\mathcal{A}}$ and $du = al^2 + bx$, is a model of S.

We know that $\mathbb{G}r_2(\mathbb{C}^{n+2})$ is formal since it is a symmetric space of compact type (see also Theorem 3.1). Thus, a model of $\mathbb{G}r_2(\mathbb{C}^{n+2})$ is $\big(H^*(\mathbb{G}r_2(\mathbb{C}^{n+2})), 0\big)$, where $H^*(\mathbb{G}r_2(\mathbb{C}^{n+2}))$ is the cohomology algebra of $\mathbb{G}r_2(\mathbb{C}^{n+2})$ defined by (4.2). Hence, a model of S is the differential algebra $(H^*(\mathbb{G}r_2(\mathbb{C}^{n+2})) \otimes \bigwedge(u), d)$, where u has degree 3 and $du = al^2 + bx$.

Since $b \neq 0$, $x = -(a\,/b)\, l^2$ on $H^*(S)$. Using that $\sigma_{n+1} = 0 = \sigma_{n+2}$ on $\mathbb{G}r_2(\mathbb{C}^{n+2})$, from Lemma 4.1 we have

$$l^{n+1} \sum_{k=0}^{\lfloor (n+1)/2 \rfloor} (-1)^{n+1+k} \binom{n+1-k}{k} \left(-\frac{a}{b}\right)^k = 0,$$

$$l^{n+2} \sum_{k=0}^{\lfloor (n+2)/2 \rfloor} (-1)^{n+k} \binom{n+2-k}{k} \left(-\frac{a}{b}\right)^k = 0.$$

One of the coefficients should be non-zero. If $a = 0$, it is clear (because only the summand with $k = 0$ contributes). If $a \neq 0$, we can run the recursive relations (4.3) backwards with $x = (-a/b)l^2$. If both coefficients were zero, then all $\sigma_r|_{x=(-a/b)l^2} = 0$, but this contradicts that $\sigma_0 = 1$. Therefore we have that either $l^{n+1} = 0$ or $l^{n+2} = 0$ on $H^*(S)$.

We deal first with the possibility that $l^{n+1} = 0$. Since $x = -\frac{a}{b} l^2$ on S, the cohomology of S up to the degree $2n+1$ is

$$H^0(S) = \langle 1 \rangle\,, \qquad H^{2i+1}(S) = 0\,,\ 0 \leq i \leq n, \qquad H^{2j}(S) = \langle l^j \rangle,\ 1 \leq j \leq n\,.$$

By Poincaré duality, $H^{4n+3-2j}(S) = \langle PD(l^j) \rangle$, for $1 \leq j \leq n$, where $PD(l^j)$ denotes the Poincaré dual of l^j. Therefore, the minimal model of S must be a differential graded algebra $(\bigwedge V, d)$, being $\bigwedge V$ the free algebra of the form $\bigwedge V = \bigwedge(a_2, v_{2n+1}) \otimes \bigwedge V^{\geq (2n+2)}$, where $|a_2| = 2$, $|v_{2n+1}| = 2n+1$, and d is defined by $da_2 = 0$, $dv_{2n+1} = a_2^{n+1}$. According with Definition 3.4, we get $N^j = 0$ for $j \leq 2n$, thus S is $2n$-formal. Moreover, S is $(2n+1)$-formal. In fact, take $\alpha \in I(N^{\leq 2n+1})$ a closed element in $\bigwedge V$. As $H^*(\bigwedge V) = H^*(S)$ has only non-zero cohomology in even degrees $2j$ with $1 \leq j \leq n$, and in odd degrees $4n+3-2j$ with $1 \leq j \leq n$, it must be $|\alpha| = 4n+3-2j$, where $1 \leq j \leq n$. Hence $\alpha = a_2^{n-j+1}\, v_{2n+1}$, which is not closed. So, according with Definition 3.4, S is $(2n+1)$-formal and, by Theorem 3.5, S is formal.

Suppose now that $l^{n+1} \neq 0$ but $l^{n+2} = 0$. Note that $l^{n+1} \neq 0$ if and only if $\sigma_{n+1} = \tau\,(al^2+bx)$, for some non-zero cohomology class $\tau \in H^{2n-2}(\mathbb{G}r_2(\mathbb{C}^{n+2}))$. Then $\tau\,u$ is closed on S because $d(\tau\,u) = \sigma_{n+1}$, and hence $H^{2n+1}(S) = \langle \tau\,u\rangle$. Clearly, $\tau\,u$ is not exact, since the image of the differential map d is contained in $\mathcal{A}$. Then, the minimal model of S must be a differential graded algebra $(\bigwedge V, d)$, being $\bigwedge V = \bigwedge(a_2, a_{2n+1}, v_{2n+3}) \otimes \bigwedge V^{\geq(2n+4)}$, where $|a_2| = 2$, $|a_{2n+1}| = 2n+1$, $|v_{2n+3}| = 2n+3$, and the differential d is given by $da_2 = 0 = da_{2n+1}$ and $dv_{2n+3} = a_2^{2n+2}$. According with Definition 3.4, we get $N^j = 0$ for $j \leq 2n+1$, thus the manifold S is $(2n+1)$-formal and, by Theorem 3.5, S is formal.

From (2.4), Theorem 2.8 and Proposition 4.2, the 3-Sasakian homogeneous space $S = \mathrm{SU}(n+2)\,/\mathrm{S}\big(\mathrm{U}(n)\times \mathrm{U}(1)\big)$ is the SO(3)-bundle $\mathrm{SO}(3) \to S \to \mathbb{G}r_2(\mathbb{C}^{n+2})$ with Euler class $-\frac{1}{4}(l^2-4x)$, and so S is formal. Even more, using (4.4) for σ_{n+1}, one can check that $l^{n+1} = 0$ on S. So a minimal model of S is the minimal model previously described for $l^{n+1} = 0$. □

Theorem 4.4. *A principal* SU(2) *or* SO(3)*-fiber bundle* $F \to S \to \mathbb{G}r_2(\mathbb{C}^{n+2})$ *with Euler class* al^2*, where* $a \in \mathbb{Q}$*, is formal if and only if* n *is odd or* $a = 0$*.*

Proof. If $a = 0$ then S is rationally equivalent to $S^3 \times \mathbb{G}r_2(\mathbb{C}^{n+2})$, which is formal being the product of two formal manifolds.

Suppose now that $a \neq 0$. Proceeding as in the proof of Theorem 4.3, a model of S is $(H^*(\mathbb{G}r_2(\mathbb{C}^{n+2})) \otimes \bigwedge(u), d)$, where $|u| = 3$ and $du = al^2$. The cohomology of S up to degree $2n+1$ is $H^0(S) = \langle 1\rangle$, $H^{2k}(S) = \langle l^{\epsilon}x^{\lfloor k/2\rfloor}\rangle$, $H^{2k+1}(S) = 0$, where $\epsilon = k - 2\lfloor k/2\rfloor$ and $k = 0, 1, \ldots, n$.

Suppose that n is odd. Since the cohomology class σ_{n+1} is zero in $\mathbb{G}r_2(\mathbb{C}^{n+2})$, the explicit expression of σ_{n+1} in Lemma 4.1 implies that $x^{(n+1)/2} = 0$ in S. Thus, the minimal model of S must be a differential graded algebra $(\bigwedge V, d)$ where $\bigwedge V = \bigwedge(a_2, a_4) \otimes \bigwedge(v_3, v_{2n+1}) \otimes \bigwedge V^{\geq 2n+2}$, where $|a_i| = i$ with $i = 2, 4$, $|v_j| = j$ with $j = 3, 2n+1$, and the differential map is defined by $da_i = 0$, $dv_3 = a_2^2$ and $dv_{2n+1} = a_4^{(n+1)/2}$. Now take α a closed element in the ideal generated by $I(N^{\leq 3})$ in $\bigwedge V$. Then, α is of the form $\alpha = a_2^p\,v_3$ which is not closed, for any integer number $p \geq 1$. Therefore S is 3-formal, and so S is $2n$-formal because the spaces N^j are zero for $4 \leq j \leq 2n$. Let us prove that S is $(2n+1)$-formal. Let α be an element of the ideal $I(N^{\leq 2n+1})$, and let us suppose that α is closed and homogeneous. Then, $|\alpha| > (2n+1)$ must be odd by the cohomology of S. Thus, α is of the form $\alpha = P_1\,v_3 + P_2\,v_{2n+1}$, where $P_1, P_2 \in \bigwedge(a_2, a_4)$. The equality $d\alpha = 0$ implies that there exists $P \in \bigwedge(a_2, a_4)$ such that $P_1 = P\,a_4^{(n+1)/2}$ and $P_2 = -P\,a_2^2$. Hence $\alpha = d(P\,v_3\,v_{2n+1})$ is exact. Therefore S is $(2n+1)$-formal, and by Theorem 3.5 it is formal.

If n is even, the explicit expression of σ_{n+1} in Lemma 4.1 implies that $l\,x^{n/2} = 0$ on S since $\sigma_{n+1} = 0$. Then, $\langle l, l, x^{n/2}\rangle$ defines a triple Massey product. By Lemma 3.6, we compute this Massey product in our model. Here, $l^2 = du$ and

$$l\,x^{n/2} = d\left(\sum_{k=0}^{\lfloor (n-1)/2\rfloor} (-1)^{n+k}\binom{n+1-k}{k} l^{n-1-2k}x^k u\right),$$

by Lemma 4.1. So the triple Massey product $\langle l, l, x^{n/2}\rangle = \xi\, u$, where

$$\xi = x^{n/2} - \sum_{k=0}^{\lfloor (n-1)/2\rfloor} (-1)^{n+k} \binom{n+1-k}{k} l^{n-1-2k} x^k.$$

Let us see that $\xi\, u$ is not exact. For $\xi\, u$ to be exact in the model $\mathcal{A}\otimes \bigwedge(u)$ we need that $\xi = 0$ in $\mathcal{A} = H^*(\mathbb{G}r_2(\mathbb{C}^{n+2}))$, due to the image of the differential map is contained in $\mathcal{A}$. Since $|\xi| = 2n$, it cannot be a combination of σ_{n+1} and σ_{n+2}, of degrees $2n+2$ and $2n+4$, respectively. Hence, ξ does not belong to the ideal $(\sigma_{n+1}, \sigma_{n+2})$ and thus it is non-zero. Therefore, S is non-formal. □

Remark 4.5. These results can be extended to the family $S(\mathbf{p})$ of 3-Sasakian manifolds for $\mathbf{p} = (p_1, p_2, \dots, p_{n+2}) \in \mathbb{Z}^{n+2}_{>0}$ introduced in [11]. They are defined as $\mathrm{U}(n+2)/(\mathrm{U}(n)\times \mathrm{U}(1))$, where the action of $\mathrm{U}(n)\times \mathrm{U}(1)$ on $\mathrm{U}(n+2)$ is defined by

$$((B,\lambda),A) \mapsto \begin{pmatrix} \lambda^{p_1} & & \\ & \ddots & \\ & & \lambda^{p_{n+2}} \end{pmatrix} \cdot A \cdot \begin{pmatrix} B & 0 \\ 0 & I_2 \end{pmatrix}.$$

In particular, this family includes the homogeneous space of Theorem 4.3 by letting $\mathbf{p} = (1,1,\dots,1) \in \mathbb{Z}^{n+2}$. If $\mathbf{p} \neq (1,\dots,1)$, the manifold $S(\mathbf{p})$ is inhomogeneous.

A description of their cohomology with integer coefficients is in [11, Theorem E]:

$$H^*(S(\mathbf{p}),\mathbb{Z}) \cong \left(\frac{\mathbb{Z}[b_2]}{\langle b_2^{n+2}\rangle} \otimes E[f_{2n+3}]\right) / \langle \sigma_{n+1}(\mathbf{p}) b_2^{n+1}, f_{2n+3}\, b_2^{n+1}\rangle,$$

where $|b_2| = 2$ and $|f_{2n+3}| = 2n+3$, and $\sigma_{n+1}(\mathbf{p})$ denotes the $(n+1)$-th elementary symmetric polynomial on $\mathbf{p}$. The first relation implies that $S(\mathbf{p})$ are not homotopic equivalent [11, Corollary 8.1]. Nevertheless, if we take rational coefficients, we have $b_2^{n+1} = 0$ in $H^*(S(\mathbf{p}))$ for any $\mathbf{p}$. Hence, the cohomology ring of $S(\mathbf{p})$ with rational coefficients depends only on the length of $\mathbf{p}$. In particular, Theorem 4.2 implies that they are formal.

5. SU(2) and SO(3)-bundles over the oriented real Grassmannian $\widetilde{\mathbb{G}r}_4(\mathbb{R}^{n+4})$

In this section we show that the 3-Sasakian homogeneous space $\mathrm{SO}(n+4)/(\mathrm{SO}(n)\times \mathrm{Sp}(1))$, where $n\geq 3$, is formal. More generally, we study the formality of the total space of SU(2) and SO(3)-bundles over the oriented real Grassmannian manifold

$$\widetilde{\mathbb{G}r}_4(\mathbb{R}^{n+4}) = \frac{\mathrm{SO}(n+4)}{\mathrm{SO}(n)\times \mathrm{SO}(4)}.$$

To make explicit the cohomology ring of $\widetilde{\mathbb{G}r}_4(\mathbb{R}^{n+4})$, we distinguish the case when n is even and the case when n is odd.

5.1. n **is even.** Take $n = 2m \geq 4$. Then, the cohomology of $\widetilde{\mathbb{G}r}_4(\mathbb{R}^{n+4})$ is given by (see [2])

$$H^*\big(\widetilde{\mathbb{G}r}_4(\mathbb{R}^{n+4})\big) = H^*(BT)^{W(\mathrm{SO}(n)\times \mathrm{SO}(4))}/H^{>0}(BT)^{W(\mathrm{SO}(n+4))}$$

where BT is the classifying space of a maximal torus T in $\mathrm{SO}(n+4)$, and $W(G)$ denotes the Weyl group of the Lie group G. So the cohomology classes on $\widetilde{\mathbb{G}r}_4(\mathbb{R}^{n+4})$ can be viewed as symmetric polynomials of elements in $H^*(BT) = \mathbb{Q}[x_1, x_2, \dots, x_m, x_{m+1}, x_{m+2}]$, where

x_i has degree 2, for $1 \leq i \leq (m+2)$. If we denote by y_1 and y_2 the classes x_{m+1} and x_{m+2}, respectively, the symmetric polynomials $\tilde{\tau}_k$ of $x_1^2, \ldots, x_m^2$ and the symmetric polynomials $\tilde{\sigma}_k$ of y_1^2, y_2^2, for $k \geq 0$, are the generators of $H^*(BT)^{W(\mathrm{SO}(n)\times\mathrm{SO}(4))}$. As n and 4 are even, $x_1 \cdots x_m$ and $y_1 y_2$ are also invariant by the action of the group $W(\mathrm{SO}(n) \times \mathrm{SO}(4))$. On the other hand, the symmetric polynomials σ_k of $x_1^2, \ldots, x_m^2, y_1^2, y_2^2$ and $x_1 \cdots x_m \, y_1 \, y_2$ are invariant by $W(\mathrm{SO}(n+4))$. These are the generators of $H^{>0}(BT)^{W(\mathrm{SO}(n+4))}$ with the relations $(x_1 \cdots x_m)\,(y_1 y_2) = 0$ and $\widetilde{\tau}\,\widetilde{\sigma} = 1$, where $\widetilde{\tau} = 1 + \widetilde{\tau}_1 + \widetilde{\tau}_2 + \ldots$ and $\widetilde{\sigma} = 1 + \widetilde{\sigma}_1 + \widetilde{\sigma}_2 + \ldots$. Hence, we can take $l = y_1^2 + y_2^2$, $x = y_1\, y_2$ and $z = x_1\, x_2 \cdots x_m$ as the generators of the cohomology of $\widetilde{\mathbb{G}r}_4(\mathbb{R}^{n+4})$, and so

$$H^*\big(\widetilde{\mathbb{G}r}_4(\mathbb{R}^{n+4})\big) = \mathbb{Q}[l, x, z]/(xz, z^2 - \widetilde{\sigma}_m, \widetilde{\sigma}_{m+1}), \tag{5.1}$$

where $|l| = |x| = 4$, $|z| = 2m$ and $\widetilde{\sigma}_r$ $(r \geq 0)$ is the cohomology class of degree $4r$ that is defined recursively by

$$\begin{cases} \widetilde{\sigma}_0 = 1, \\ \widetilde{\sigma}_1 = -l, \\ \widetilde{\sigma}_r = -l\widetilde{\sigma}_{r-1} - x^2\widetilde{\sigma}_{r-2}, \qquad r \geq 2\,. \end{cases} \tag{5.2}$$

A similar proof to the one made for Lemma 4.1 shows

$$\widetilde{\sigma}_r = \sum_{k=0}^{\lfloor r/2 \rfloor} (-1)^{r+k} \binom{r-k}{k} l^{r-2k} x^{2k}, \qquad r \geq 0\,. \tag{5.3}$$

Proposition 5.1. *Let Ω be the quaternionic Kähler form on $\widetilde{\mathbb{G}r}_4(\mathbb{R}^{n+4})$ with $n = 2m \geq 4$. Then, in terms of the generators l, x, z of $H^*(\widetilde{\mathbb{G}r}_4(\mathbb{R}^{n+4}))$ given by (5.1), the de Rham cohomology class $[\Omega] \in H^4(\widetilde{\mathbb{G}r}_4(\mathbb{R}^{n+4}))$ is $[\Omega] = l + 2x$.*

Proof. By (2.1), the cohomology class $[\Omega]$ equals the Pontryagin class of the SO(3)-fiber bundle which gives the 3-Sasakian homogeneous manifold $S = \mathrm{SO}(n+4)/(\mathrm{SO}(n)\times\mathrm{Sp}(1))$. To make the description more explicit, recall that the double cover of SO(4) is Spin(4) $\cong$ $\mathrm{SU}(2)_+ \times \mathrm{SU}(2)_-$, where we label the two copies of SU(2) according to whether they act on $\bigwedge^2_\pm \mathbb{R}^4$ of the standard representation $\mathbb{R}^4$ of SO(4). By projecting onto $\mathrm{SU}(2)_+$, we have a map $\mathrm{Spin}(4) \to \mathrm{SU}(2)_+$, which descends to a map $\mathrm{SO}(4) \to \mathrm{SU}(2)_+/\mathbb{Z}_2 \cong \mathrm{SO}(3)$. The kernel is $\mathrm{SU}(2)_- < \mathrm{SO}(4)$. This gives the fibration

$$\frac{\mathrm{SO}(4)}{\mathrm{SU}(2)_-} \cong \mathrm{SO}(3) \hookrightarrow S = \frac{\mathrm{SO}(n+4)}{\mathrm{SO}(n)\times\mathrm{SU}(2)_-} \longrightarrow \widetilde{\mathbb{G}r}_4(\mathbb{R}^{n+4}) = \frac{\mathrm{SO}(n+4)}{\mathrm{SO}(n)\times\mathrm{SO}(4)},$$

that determines the 3-Sasakian manifold S.

To compute $p_1(S)$, consider the universal real oriented rank 4-bundle $V \to \widetilde{\mathbb{G}r}_4(\mathbb{R}^{n+4})$. By the description of the cohomology (5.1), the roots corresponding to V are y_1, y_2. The two real rank 3-bundles $\bigwedge^2_+ V, \bigwedge^2_- V$ are $\mathrm{SO}(3) = \mathrm{SU}(2)/\mathbb{Z}_2$-bundles, associated to the two morphisms $\mathrm{SO}(4) \to \mathrm{SU}(2)_\pm/\mathbb{Z}_2$. The roots of $\bigwedge^2_+ V$ are $1, y_1+y_2, -y_1-y_2$, and the roots of $\bigwedge^2_- V$ are $1, y_1 - y_2, -y_1 + y_2$. Let us see this: we take $V = L_1 \oplus L_2$, where L_1, L_2 are $\mathrm{SO}(2) = \mathrm{U}(1)$-bundles with $y_1 = c_1(L_1)$, $y_2 = c_1(L_2)$. Then $\bigwedge^2_+ \otimes \mathbb{C} = \bigwedge^{2,0} \oplus \mathbb{C}\omega \oplus \bigwedge^{0,2} \cong \mathbb{C} \oplus (L_1 \otimes L_2) \oplus (\overline{L}_1 \otimes \overline{L}_2)$. The other case is similar.

The bundle associated to S is $\bigwedge_+^2 V$. So

$$[\Omega] = p_1(S) = p_1(\bigwedge_+^2 V) = -c_2(\bigwedge_+^2 V \otimes \mathbb{C}) = (y_1+y_2)^2 = l + 2x,$$

since $l = y_1^2 + y_2^2$ and $x = y_1 y_2$. Note that $p_1(\bigwedge_-^2 V) = (y_1 - y_2)^2 = l - 2x$, and that the cohomology $H^*(\widetilde{\mathbb{G}r}_4(\mathbb{R}^{n+4}))$ is invariant under $x \mapsto -x$. □

The following result will be useful for our purposes.

Lemma 5.2. *Consider the cohomology class $\tilde\sigma_r$ of degree $2r$ ($r \geq 0$), defined by (5.2), as a polynomial $\tilde\sigma_r = \tilde\sigma_r(l,x)$ in l and x, and let $a, b \in \mathbb{Q}$, $(a,b) \neq (0,0)$. Then $\tilde\sigma_r$ factors through $al + bx$ if and only if one of the following statements holds:*

(1) r is odd and $b = 0$;
(2) $r \equiv 2 \pmod 3$ and $|a| = |b|$.

Proof. In order to determine the linear factors $al + bx$ of $\widetilde\sigma_r(l,x)$ for any r, we consider the generating function $\sum_{r=0}^\infty \widetilde\sigma_r(l,x)t^r$. The recursive definition (5.2) of $\widetilde\sigma_r(l,x)$ can be rewritten as the equality $(1 + lt + x^2t^2)(\widetilde\sigma_0 + \widetilde\sigma_1 t + \widetilde\sigma_2 t^2 + \cdots) = 1$, so

$$\sum_{r=0}^\infty \widetilde\sigma_r(l,x)t^r = \frac{1}{1 + lt + x^2t^2}. \tag{5.4}$$

Next, we are going to calculate the values $a, b \in \mathbb{Q}$ such that $al + bx$ divides $\widetilde\sigma_r(l,x)$. First, let us consider the case $b = 0$. Clearly, al divides $\widetilde\sigma_r(l,x)$ if and only if $\widetilde\sigma_r(0,x) = 0$. Then, the expansion series on $x\,t$ of (5.4) becomes

$$\sum_{r=0}^\infty \widetilde\sigma_r(0,x)t^r = \frac{1}{1+(xt)^2} = \sum_{r=0}^\infty (-1)^r (xt)^{2r}.$$

Hence

$$\widetilde\sigma_r(0,x) = \begin{cases} (-1)^{r/2}x^r & \text{if } r \text{ is even,} \\ 0 & \text{if } r \text{ is odd.} \end{cases}$$

Therefore, al divides $\widetilde\sigma_r(l,x)$ if and only if r is odd. This is the condition (1) of the statement of this Lemma.

Assume $b \neq 0$. Let $\lambda = \frac{a}{b} \in \mathbb{Q}$. Clearly, $al + bx$ divides $\widetilde\sigma_r(l,x)$ if and only if $\widetilde\sigma_r(l, -\lambda l) = 0$. We will determine the values of $\lambda \in \mathbb{Q}$ such that $\widetilde\sigma_r(l,-\lambda l) = 0$. First, we deal with $\lambda = 0, \pm\frac{1}{2}$.

- If $\lambda = 0$, the expansion series on lt of (5.4) becomes

$$\sum_{r=0}^\infty \widetilde\sigma_r(l,0)t^r = \frac{1}{1+lt} = \sum_{r=0}^r (-1)^r l^r t^r,$$

  which implies that $\sigma_r(l,0) = (-1)^r l^r$, and so bx does not divide $\widetilde\sigma_r(l,x)$ for any $r \geq 0$.
- If $\lambda = \pm\frac{1}{2}$, the expansion series on lt of (5.4) yields

$$\sum_{r=0}^\infty \widetilde\sigma_r(l, \pm\tfrac{1}{2}l)t^r = \frac{1}{1 + lt + \frac{1}{4}(lt)^2} = \frac{1}{(1+\frac{1}{2}lt)^2} = \sum_{r=0}^\infty (-1)^r \frac{r+1}{2^r}(lt)^r.$$

This implies that $\widetilde{\sigma}_r(l, \pm\frac{1}{2}l) = (-1)^r \frac{r+1}{2^r} l^r$. Thus, if $2|a| = |b|$, then $al + bx$ does not divide $\widetilde{\sigma}_r(l, x)$ for any $r \geq 0$.

Now, consider $\lambda \neq 0, \pm\frac{1}{2}$. As before, we compute an explicit expression of $\widetilde{\sigma}_r(l, -\lambda l)$ by means of (5.4). We notice that (5.4) can be written as follows

$$\sum_{r=0}^{\infty} \widetilde{\sigma}_r(l, -\lambda l) t^r = \frac{1}{1 + lt + \lambda^2 (lt)^2} = \frac{1}{(\alpha - \beta)\lambda^2} \left(\frac{1}{lt - \alpha} - \frac{1}{lt - \beta} \right),$$

where $\alpha, \beta \in \mathbb{C}$ such that

$$\alpha + \beta = -\frac{1}{\lambda^2} \text{ and } \alpha \cdot \beta = \frac{1}{\lambda^2}. \tag{5.5}$$

This can be done because we are taking $\lambda \neq 0, \pm\frac{1}{2}$. Moreover, by (5.5), we have $\alpha - \beta = 0$ if and only if $\lambda = \pm\frac{1}{2}$. We write each fraction as a geometric power series and arrange this as power series on lt:

$$\begin{aligned}
\sum_{r=0}^{\infty} \widetilde{\sigma}_r(l, -\lambda l) t^r &= \frac{1}{(\alpha - \beta)\lambda^2} \left(\frac{1}{lt - \alpha} - \frac{1}{lt - \beta} \right) \\
&= \frac{1}{(\alpha - \beta)\lambda^2} \left(\frac{1}{\beta} \sum_{r=0}^{\infty} \left(\frac{lt}{\beta} \right)^n - \frac{1}{\alpha} \sum_{r=0}^{\infty} \left(\frac{lt}{\alpha} \right)^r \right) \\
&= \sum_{r=0}^{\infty} \frac{1}{(\alpha - \beta)\lambda^2} \left(\frac{1}{\beta^{r+1}} - \frac{1}{\alpha^{r+1}} \right) (lt)^r \\
&= \sum_{r=0}^{\infty} \frac{(\alpha/\beta)^{r+1} - 1}{\alpha^{r+1}(\alpha - \beta)\lambda^2} l^r t^r.
\end{aligned}$$

Hence,

$$\widetilde{\sigma}_r(l, -\lambda l) = \frac{(\alpha/\beta)^{r+1} - 1}{\alpha^{r+1}(\alpha - \beta)\lambda^2} l^r, \tag{5.6}$$

and $\widetilde{\sigma}_r(l, -\lambda l) = 0$ if and only if α/β is a $(r+1)$th root of unity. Note that once we fix a value for α/β, the equations (5.5) determine $\lambda \in \mathbb{Q}$ up to sign. Then, we look for α/β which comes from $\alpha, \beta \in \mathbb{C}$ satisfying (5.5), and $k \geq 1$ such that α/β is a primitive kth root of unity. We distinguish two possibilities, namely $\alpha/\beta \in \mathbb{Q}$ and $\alpha/\beta \notin \mathbb{Q}$.

Assume $\alpha/\beta \in \mathbb{Q}$. In such a case, α/β is a root of unity if and only if $\alpha/\beta = \pm 1$.

- $\alpha/\beta = 1$ is equivalent to $\alpha - \beta = 0$, which corresponds to the case $\lambda = \pm\frac{1}{2}$ discussed previously;
- $\alpha/\beta = -1$ is equivalent to $\alpha + \beta = 0$. But this contradicts the first equality in (5.5).

If $\alpha/\beta \notin \mathbb{Q}$, we check that α/β is a primitive kth root of unity by means of its minimal polynomial over $\mathbb{Q}$. More concretely, by uniqueness of the minimal polynomials, α/β is a primitive kth root of unity if and only its minimal polynomial is equal to the kth cyclotomic polynomial $\Phi_k(X)$. Therefore, α/β is a $(r+1)$th root of unity if and only if its minimal polynomial is equal to a kth cyclotomic polynomial for some k divisor of $r+1$ or, equivalently, $r + 1 \equiv 0 \pmod{k}$.

Now, since $\alpha/\beta \notin \mathbb{Q}$, its minimal polynomial over $\mathbb{Q}$ is of degree greater than or equal to 2. Taking into account (5.5) we have that

$$-\left(\frac{\alpha}{\beta}+\frac{\beta}{\alpha}\right)=-\frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}=-\frac{1}{\lambda^2}+2, \qquad \frac{\alpha}{\beta}\cdot\frac{\beta}{\alpha}=1$$

are rational numbers, and so α/β is a root of the polynomial $Q_\lambda(X)=X^2+\left(2-\frac{1}{\lambda^2}\right)X+1$. Thus $Q_\lambda(X)$ must be the minimal polynomial of α/β because it is of the lowest degree and $\mathbb{Q}$-irreducible. Now, let us check when $Q_\lambda(X)$ is equal to a cyclotomic polynomial. Since $\deg Q_\lambda(X)=2$, the only possibilities are the cyclotomic polynomials $\Phi_k(X)$ of degree 2, that is, $\Phi_k(X)$ with $k=3,4$ and 6. We discuss each of these three cases:

- For $k=3$, the equality $Q_\lambda(X)=\Phi_3(X)=X^2+X+1$ implies $\frac{a}{b}=\lambda=1$ or $\frac{a}{b}=\lambda=-1$. This means that α/β is the 3rd root of unity and, by (5.6), $\widetilde{\sigma}_r(l,-\lambda l)=0$, for $r+1\equiv 0 \pmod 3$ and $\lambda=\pm 1$. Therefore, if $|a|=|b|$, $al+bx$ divides $\widetilde{\sigma}_r(l,x)$ when $r\equiv 2 \pmod 3$, which is the condition (2) of the statement of this Lemma.
- For $k=4$, the equality $Q_\lambda(X)=\Phi_4(X)=X^2+1$ implies $\lambda=\frac{\sqrt{2}}{2}\notin\mathbb{Q}$. But this is not possible because λ must be a rational number.
- For $k=6$, the equality $Q_\lambda(X)=\Phi_6(X)=X^2-X+1$ implies $\lambda=-\frac{\sqrt{3}}{3}\notin\mathbb{Q}$, which is not possible because λ must be a rational number.

Finally, for the remaining values of $\lambda\in\mathbb{Q}$ (that is, $\lambda\neq -1$, and $\lambda\neq 0,\pm\frac{1}{2}$ which were discussed previously as specific cases), the minimal polynomial $Q_\lambda(X)$ of α/β is not a cyclotomic polynomial, and so α/β is not a root of unity. By (5.6), $\widetilde{\sigma}_r(l,-\lambda l)\neq 0$, for any $r\geq 0$. Therefore, $al+bx$ with $a/b=\lambda$ does not divide $\widetilde{\sigma}_r(l,x)$ for $r\geq 0$. □

To prove the formality of $\mathrm{SO}(n+4)/(\mathrm{SO}(n)\times\mathrm{Sp}(1))$ with $n=2m\geq 4$ we study firstly the case when $n=2m\geq 6$.

Theorem 5.3. *Consider the principal F-fiber bundle $F\to S\to \widetilde{\mathbb{G}r}_4(\mathbb{R}^{n+4})$ with $n=2m\geq 6$, $F=\mathrm{SU}(2)$ or $F=\mathrm{SO}(3)$ and Euler class $e(S)=al+bx$, where $a,b\in\mathbb{Q}$. Then S is not formal if and only if one of the following statements is satisfied:*

(1) m is odd, $a\neq 0$ and $b=0$;
(2) $m\equiv 2 \pmod 3$ and $|a|=|b|\neq 0$.

In particular, for $n=2m\geq 6$, the homogeneous 3-Sasakian manifold $\mathrm{SO}(n+4)/(\mathrm{SO}(n)\times\mathrm{Sp}(1))$ is formal.

Proof. Since $\widetilde{\mathbb{G}r}_4(\mathbb{R}^{n+4})$ is a compact positive quaternionic Kähler manifold, $\widetilde{\mathbb{G}r}_4(\mathbb{R}^{n+4})$ is simply connected [41]. Then, according with section 3, the fibre bundle $F\to S\to \widetilde{\mathbb{G}r}_4(\mathbb{R}^{n+4})$, with $F=\mathrm{SU}(2)$ or $F=\mathrm{SO}(3)$ and Euler class $al+bx$, is a rational S^3-fibration. Thus [40], if $(\mathcal{A},d_{\mathcal{A}})$ is a model of $\widetilde{\mathbb{G}r}_4(\mathbb{R}^{n+4})$, a model of S is $(\mathcal{A}\otimes\bigwedge(u),d)$ with $|u|=3$, $d|_{\mathcal{A}}=d_{\mathcal{A}}$ and $du=al+bx$. On the other hand, $\widetilde{\mathbb{G}r}_4(\mathbb{R}^{n+4})$ is formal by Theorem 3.1, and hence a model of this space is $(H^*(\widetilde{\mathbb{G}r}_4(\mathbb{R}^{n+4})),0)$. Therefore, a model of S is the differential graded algebra

$$\Big(H^*(\widetilde{\mathbb{G}r}_4(\mathbb{R}^{n+4}))\otimes\bigwedge(u),d\Big), \tag{5.7}$$

where $|u|=3$ and $du=al+bx$.

We show first that if one of the conditions (1) or (2) stated in this Theorem is satisfied, then S is non-formal. By Lemma 3.6, we know that Massey products on a manifold M can be computed by using any model for M. Let us prove that the triple Massey product $\langle x, z, z\rangle$ is defined on S and it is non-zero by using the model of S given by (5.7). By (5.1) and (5.7), $x\,z = 0$ on S because $x\,z = 0$ in $H^*\big(\widetilde{\mathbb{G}r}_4(\mathbb{R}^{n+4})\big)$. Moreover, by Lemma 5.2, each of conditions (1) and (2) implies that $\widetilde{\sigma}_m = (al+bx)\,\tau$, for some non-zero cohomology class τ of degree $4m-4$ on $\widetilde{\mathbb{G}r}_4(\mathbb{R}^{n+4})$ such that τ only depends of l and x. Thus, taking into account (5.7), $\widetilde{\sigma}_m = d(u\,\tau)$ on S. So, by (5.1), $z^2 = d(\tau u)$ on S. Therefore, the triple Massey product $\langle x, z, z\rangle$ is defined on S and

$$\langle x, z, z\rangle = x\,\tau\,u.$$

Let us see that $x\,\tau\,u$ is not exact. The element $x\,\tau\,u$ is exact in the model (5.7) of S if $x\,\tau = 0$ in $H^*(\widetilde{\mathbb{G}r}_4(\mathbb{R}^{n+4}))$, because the image of the differential map is contained in $H^*(\widetilde{\mathbb{G}r}_4(\mathbb{R}^{n+4}))$. But $x\,\tau = 0$ in $H^*(\widetilde{\mathbb{G}r}_4(\mathbb{R}^{n+4}))$ if and only if $x\,\tau$ belongs to the ideal $(x\,z, z^2-\widetilde{\sigma}_m, \widetilde{\sigma}_{m+1})$. Let us to show that this is not possible. Since $4m = |x\,\tau| < |\widetilde{\sigma}_{m+1}| = 4m+4$, $x\,\tau$ would be a combination of xz and $z^2-\widetilde{\sigma}_m$. We know that $x\,\tau$ is a combination of l and x because τ only depends of l and x. On the other hand, any combination of $x\,z$ and $z^2-\widetilde{\sigma}_m$ that is independent of z, it must be of degree greater than or equal $4m+4$. Hence, $x\,\tau$ does not belong to the ideal $(x\,z, z^2-\widetilde{\sigma}_m, \widetilde{\sigma}_{m+1})$ and thus $x\,\tau$ is non-zero. Therefore, $\langle x, z, z\rangle$ is a non-trivial Massey product on S and, by Lemma 3.7, S is non-formal.

Conversely, we must prove that if S is non-formal, then one of conditions (1) or (2) is satisfied. But this is equivalent to prove that S is formal if $a = 0$ or $a \neq 0$ but $\widetilde{\sigma}_m$ on S is non-zero (that is, $\widetilde{\sigma}_m$ does not factorize by $al + bx$). We study each of these two cases separately.

- Suppose that $a \neq 0$ and $\widetilde{\sigma}_m \neq 0$. Clearly, $l = -(b\,/a)\,x$ in $H^*(S)$ since $a \neq 0$ and, by (5.7), $du = al + bx$. Then, (5.3) implies that the cohomology class $\widetilde{\sigma}_m$ on S has the following expression
$$\widetilde{\sigma}_m = x^m \sum_{k=0}^{\lfloor m/2\rfloor} (-1)^{m+k} \binom{m-k}{k} \left(-\frac{b}{a}\right)^{m-k}.$$
Thus the cohomology class z^2 on S is a multiple of the cohomology class x^m because, by (5.1), z^2 and $\widetilde{\sigma}_m$ are the same cohomology class on $\widetilde{\mathbb{G}r}_4(\mathbb{R}^{n+4})$ and so on S. (Note that if m is odd, the condition $\widetilde{\sigma}_m \neq 0$ implies $b \neq 0$, but if m is even then b may be 0.) Moreover, the class x^{m+1} is zero on S since it is a multiple of $z(xz) - x(z^2 - \bar{\sigma}_m)$, which is zero on $\widetilde{\mathbb{G}r}_4(\mathbb{R}^{n+4})$ by (5.1). Taking into account (5.1) and the model of S given by (5.7), we have that the non-zero cohomology groups of S up to the degree $2n+1$ are
$$H^0(S) = \langle 1\rangle\,, \qquad H^{4k}(S) = \langle x^k\rangle, \quad 0 \leq k \leq m, \quad \text{and} \quad 4k \neq 2m,$$
$$H^{2m}(S) = \begin{cases} \langle x^{m/2}, z\rangle, & \text{if } m \text{ is even,} \\ \langle z\rangle, & \text{if } m \text{ is odd.} \end{cases}$$
Therefore, the minimal model of S must be a differential graded algebra $(\bigwedge V, d)$, being $\bigwedge V$ the free algebra of the form $\bigwedge V = \bigwedge(a_4, a_{2m}, v_{2m+3}, v_{4m-1}) \otimes \bigwedge V^{\geq(2n+2)}$,

where $|a_i| = i$, for $i = 4, 2m$, $|v_j| = j$, for $j = 2m+3, 4m-1$, and d is defined by $da_i = 0$, $dv_{2m+3} = a_4\, a_{2m}$ and $dv_{4m-1} = a_4^m - a_{2m}^2$. Let us prove that S is $(4m+1)$-formal. Any element $\alpha \in I(N^{\leq 4m+1})$ is of the form $\alpha = P_1\, v_{2m+3} + P_2\, v_{4m-1} + P_3\, v_{2m+3}\, v_{4m-1}$, where $P_1, P_2, P_3 \in \bigwedge(a_4, a_{2m})$. If α is a closed element of even degree, then $\alpha = P_3\, v_{2m+3}\, v_{4m-1}$. But $d\alpha = 0$ implies $P_3 = 0$, and so α is trivially exact. If α is a closed element of odd degree, then $\alpha = P_1\, v_{2m+3} + P_2\, v_{4m-1}$. The condition $d\alpha = 0$ implies $P_1 = P\,(a_{2m}^2 - a_4^m)$ and $P_2 = -P\, a_4\, a_{2m}$, for some $P \in \bigwedge(a_4, a_{2m})$. Hence, $\alpha = d(P\, v_{2m+3}\, v_{4m-1})$ is exact. This proves that S is $(4m+1)$-formal and, by Theorem 3.5, S is formal.

- Consider the case $a = 0$, so $du = bx$. If $b = 0$, then S is clearly formal because S is rationally equivalent to $S^3 \times \widetilde{\mathbb{G}r}_4(\mathbb{R}^{n+4})$, which is formal being the product of two formal manifolds.

  Suppose now that $a = 0$ and $b \neq 0$. The condition $du = bx$ and (5.3) imply that the element $\widetilde{\sigma}_m$ of the model of S given by (5.7) has the following expression
  $$\widetilde{\sigma}_m = (-1)^m l^m + \text{ elements which are exact in (5.7)}.$$
  Thus the cohomology class z^2 on S is $\pm\, l^m$ because, by (5.1), z^2 and $\widetilde{\sigma}_m$ are the same cohomology class on $\widetilde{\mathbb{G}r}_4(\mathbb{R}^{n+4})$. Similarly, using again (5.3), the condition $du = bx$ on the model of S, and taking into account that, by (5.1), $\widetilde{\sigma}_{m+1} = 0$ on $\widetilde{\mathbb{G}r}_4(\mathbb{R}^{n+4})$, one can check that l^{m+1} is the zero on S. the non-zero cohomology groups of even degree of S up to degree $(2n+1)$ are
  $$H^0(S) = \langle 1\rangle\,, \qquad H^{4j}(S) = \langle l^j\rangle\,,\ \ 1 \leq j \leq m,$$
  $$H^{2m+4k}(S) = \langle l^k\, z\rangle\,,\ \ 0 \leq k \leq \lfloor m/2\rfloor\,,$$
  if m is odd; and
  $$H^0(S) = \langle 1\rangle\,, \qquad H^{4k}(S) = \begin{cases} \langle l^k\rangle, & \text{if } 0 \leq k < \frac{m}{2},\\ \langle l^k, l^{k-(m/2)} z\rangle, & \text{if } \frac{m}{2} \leq k \leq m, \end{cases}$$
  if m is even. But in both cases, S has non-zero cohomology groups of odd degree up to degree $2n+1$, namely
  $$H^{2m+3+4k}(S) = \langle l^k z\, u\rangle,$$
  for $0 \leq k < \lfloor m/2\rfloor$. Therefore, the minimal model of S must be a differential graded algebra $(\bigwedge V, d)$, being $\bigwedge V$ the free algebra of the form $\bigwedge V = \bigwedge(a_4, a_{2m}, a_{2m+3}, v_{4m-1}) \otimes V^{\geq(2n+2)}$, where $|a_i| = i$ $(i = 4, 2m, 2m+3)$, $|v_{4m-1}| = 4m-1$, $da_4 = da_{2m} = da_{2m+3} = 0$ and $dv_{4m-1} = a_{2m}^2 - a_4^m$. Clearly, S is $(4m+1)$-formal, and hence it is formal by Theorem 3.5.

Let us consider the 3-Sasakian homogeneous space $S = \mathrm{SO}(n+4)\,/(\mathrm{SO}(n) \times \mathrm{Sp}(1))$ with $n = 2m \geq 6$. From (2.4), Theorem 2.8 and Proposition 5.1, S is the SO(3)-bundle $\mathrm{SO}(3) \to S \to \widetilde{\mathbb{G}r}_4(\mathbb{R}^{n+4})$ with Euler class $-\frac{1}{4}(l + 2x)$. Hence, none of the conditions (1) and (2) stated in this Theorem is satisfied because now $a = -\frac{1}{4} \neq -\frac{1}{2} = b$. Thus, S is formal. Note that the minimal model of S is the minimal model previously described for $a \neq 0$ and $\widetilde{\sigma}_m \neq 0$. □

Now, in order to show that that the 3-Sasakian homogeneous space $\mathrm{SO}(n+4)\,/(\mathrm{SO}(n) \times \mathrm{Sp}(1))$ is formal, for $n = 2m \geq 4$, only remains to prove that this space is formal for $n = 4$.

For this, note that (5.1) implies that, for $n = 4$, the cohomology ring of $\widetilde{\mathbb{G}r}_4(\mathbb{R}^8)$ is

$$H^*(\widetilde{\mathbb{G}r}_4(\mathbb{R}^8)) = \mathbb{Q}[l, x, z]/(xz, z^2 - l^2 + x^2, l^3 - 2lx^2), \tag{5.8}$$

where $|x| = |y| = |z| = 4$.

Theorem 5.4. *Let $F = \mathrm{SO}(3), \mathrm{SU}(2)$. Consider a fiber bundle $F \to S \to \widetilde{\mathbb{G}r}_4(\mathbb{R}^8)$ with Euler class $e(S) = al + bx + cz$, where $a, b, c \in \mathbb{Q}$. Then, S is not formal if and only if one of the following statements is satisfied:*

(1) $|a| = |b| \neq 0$ and $|a| \neq |c|$,
(2) $|a| = |c| \neq 0$ and $|a| \neq |b|$.

In particular, the homogeneous 3-Sasakian manifold $\mathrm{SO}(8)/(\mathrm{SO}(4) \times \mathrm{Sp}(1))$ is formal.

Proof. Since $\widetilde{\mathbb{G}r}_4(\mathbb{R}^8)$ is simply connected and formal, proceeding as in the proof of Theorem 5.3, a model of S is

$$\left(H^*(\widetilde{\mathbb{G}r}_4(\mathbb{R}^8)) \otimes \bigwedge(u), d\right), \tag{5.9}$$

where $|u| = 3$ and $du = al + bx + cz \in H^4(\widetilde{\mathbb{G}r}_4(\mathbb{R}^8))$.

In order to prove that if one of the conditions (1) or (2) stated in this Theorem is satisfied, then S is non-formal, we consider first a new basis of the cohomology ring $H^*(\mathbb{G}r_4(\mathbb{R}^8))$. For this we proceed as follows. In any of the cases (1) or (2), we have $(b, c) \neq (0, 0)$. We can assume that $a, b, c \geq 0$, and so $b + c > 0$ since $(b, c) \neq (0, 0)$. (In fact, it is sufficient to change l by $-l$ if $a < 0$, or x by $-x$ if $b < 0$, or z by $-z$ if $c < 0$.) Now we define the basis ξ_0, ξ_1, ξ_2 of $H^4(\mathbb{G}r_4(\mathbb{R}^8))$ by

$$\xi_0 = al + bx + cz, \qquad \xi_1 = l - x + z, \qquad \xi_2 = -l - x + z.$$

Hence, ξ_0, ξ_1 and ξ_2 generate the cohomology ring $H^*(\mathbb{G}r_4(\mathbb{R}^8))$, and the relations given by (5.8) can be written in terms of ξ_0, ξ_1 and ξ_2 as

$$\begin{gathered} \xi_1\xi_2, \qquad \frac{1}{4(b+c)^2}\left((a-b)(a+c)\xi_1^2 + (a+b)(a-c)\xi_2^2\right) + \beta_4\xi_0, \qquad \text{and} \\ \frac{2(a-c)^2 - (b+c)^2}{8(b+c)^2}\left(\xi_1^3 - \xi_2^3\right) + \beta_8\xi_0, \end{gathered} \tag{5.10}$$

where β_4 and β_8 are elements of the cohomology ring (5.8) of degree $|\beta_i| = i$, with

$$\beta_4 = \frac{1}{(b+c)^2}\xi_0 + \frac{-2a+b-c}{2(b+c)^2}\xi_1 + \frac{2a+b-c}{2(b+c)^2}\xi_2\,. \tag{5.11}$$

According with the model of S given by (5.9), we have now $du = \xi_0$. Then, from (5.10), $(a-b)(a+c)\xi_1^2 + (a+b)(a-c)\xi_2^2$ and $\xi_1^3 - \xi_2^3$ are exact in the model (5.9). So, the non-zero cohomology groups of S up to degree 12 are:

$$H^4(S) = \langle \xi_1, \xi_2 \rangle, \qquad H^8(S) = \begin{cases} \langle \xi_1^2 \rangle, & \text{if } a \neq b \text{ and } a \neq c, \\ \langle \xi_1^2 \rangle, & \text{if } a = b \neq 0 \text{ and } a \neq c, \\ \langle \xi_2^2 \rangle, & \text{if } a = c \neq 0 \text{ and } a \neq b, \\ \langle \xi_1^2, \xi_2^2 \rangle, & \text{if } a = b = c \text{ or } a = b = 0 \text{ or } a = c = 0, \end{cases} \tag{5.12}$$

and

$$H^{12}(S) = \begin{cases} \langle \xi_1^3 \rangle, & \text{if } a = b = c, \text{ or } a = b = 0 \text{ or } a = c = 0 \\ 0, & \text{otherwise.} \end{cases}$$

We may determine the odd degree cohomology groups by applying the Poincaré duality. If $a = b = c$ or $a = b = 0$ or $a = c = 0$, $H^7(S)$ is generated by the Poincaré dual $PD(\xi_1^3)$ of the non-zero cohomology class $\xi_1^3 \in H^{12}(S)$.

To prove that S is non-formal if the condition (1) is satisfied, we apply Lemma 3.6 to compute a non-trivial triple Massey product in the model (5.9). Since $a, b, c \geq 0$, the condition (1) becomes $a = b \neq 0$ and $a \neq c$. From (5.10), we have that $\xi_1 \xi_2$ is zero. Moreover, the second class in (5.10) yields $\xi_2^2 = -d\left(\frac{2(b+c)^2}{b(b-c)}\beta_4 u\right)$ since $du = \xi_0$ on S. Therefore, the triple Massey product $\langle \xi_2, \xi_2, \xi_1 \rangle$ is defined and, using (5.11), we have

$$\langle \xi_2, \xi_2, \xi_1 \rangle = -\frac{2(b+c)^2}{b(b-c)}\beta_4\, u\, \xi_2 = \frac{1}{b(b-c)}\left(-2\xi_0 - (3b-c)\xi_2 + (b+c)\xi_1\right)\xi_1\, u,$$

that is the class $\frac{(b+c)}{b(b-c)}\xi_1^2\, u \in H^{11}(S)$ since ξ_0 defines the zero cohomology class on S. Thus, the triple Massey product $\langle \xi_2, \xi_2, \xi_1 \rangle$ is non-trivial because $b \neq 0$ and $a = b \neq c$. Hence S is non-formal by Lemma 3.7.

A similar argument shows that if condition (2) is satisfied, then S is non-formal. In fact, since $a, b, c \geq 0$, condition (2) is equivalent to $a = c \neq 0$ and $a \neq b$. Then, from (5.10), $\xi_1 \xi_2$ is zero and $\xi_1^2 = d\left(\frac{2(b+c)^2}{c(b-c)}\beta_4 u\right)$ is exact in the model (5.9). Thus, the triple Massey product $\langle \xi_1, \xi_1, \xi_2 \rangle$ is defined and by (5.11) we have

$$\langle \xi_1, \xi_1, \xi_2 \rangle = \frac{2(b+c)^2}{c(b-c)}\beta_4\, u\, \xi_2 = \frac{1}{c(b-c)}\left(2\xi_0 + (b-3c)\xi_1 + (b+c)\xi_2\right)\xi_2\, u.$$

This is the non-zero class $\frac{(b+c)}{c(b-c)}\xi_2^2\, u \in H^{11}(S)$. Hence $\langle \xi_1, \xi_1, \xi_2 \rangle$ is non-trivial and, by Lemma 3.7, S is non-formal.

Now, we prove the converse statement, that is, if S is non-formal, then one of the conditions (1) or (2) is satisfied. This is equivalent to prove that S is formal if none of these conditions (1) and (2) is satisfied. But this happens in one of the following cases:

- If $a \neq b$ and $a \neq c$, the minimal model of S is a differential graded algebra $(\bigwedge V, d)$, being $\bigwedge V = \bigwedge(a_4, b_4, v_7, w_7) \otimes \bigwedge V^{\geq 10}$ and $|a_4| = |b_4| = 4$, $|v_7| = |w_7| = 7$, and $da_4 = 0$, $db_4 = 0$, $dv_7 = a_4 b_4$, $dw_7 = (a-b)(a+c)a_4^2 + (a+b)(a-c)b_4^2$. In order to prove the formality, let us consider a closed element $\alpha \in I(N^{\leq 9})$. Then $\alpha = P_1 v_7 + P_2 w_7 + P_3\, v_7 w_7$, where $P_1, P_2, P_3 \in \bigwedge(a_4, b_4)$. If α is of even degree, $\alpha = P_3\, v_7 w_7$. Clearly, $d\alpha = 0$ implies $P_3 = 0$, and hence α is trivially exact. If α is of odd degree, then $\alpha = P_1 v + P_2 w$. The condition $d\alpha = 0$ implies that there exists $P \in \bigwedge(a_4, b_4)$ such that $P_1 = P((a-b)(a+c)a_4^2 + (a+b)(a-c)b_4^2)$ and $P_2 = -P\, a_4 b_4$. Thus, $\alpha = d(P\, v_7 w_7)$ is exact. This proves the 9-formality and therefore S is formal by Theorem 3.5.
- For the case $a = b = c$ or $a = b = 0$ or $a = c = 0$, recall that $H^7(S)$ is generated by $PD(\xi_1^3)$ the Poincaré dual of ξ_1^3. Then, the minimal model of S is a differential graded algebra $(\bigwedge V, d)$, being $\bigwedge V = \bigwedge(a_4, b_4, c_7, v_7) \otimes \bigwedge V^{\geq 10}$ where

$|a_4| = |b_4| = 4$, $|c_7| = |v_7| = 7$, and $da_4 = db_4 = dc_7 = 0$ and $dv_7 = a_4 b_4$. Clearly, S is 9-formal and, by Theorem 3.5, S is formal.

- For $b = c = 0$. If $a = 0$ then the fibration S is rationally equivalent to $S^3 \times \widetilde{\mathbb{G}r}_4(\mathbb{R}^8)$, which is formal being the product of two formal manifolds. Then, assume that $a \neq 0$. Since $du = al$, $H^4(S) = \langle x, z \rangle$. The relations of (5.8) yield the equalities $xz = 0$ and $x^2 + z^2 = \frac{1}{a} l du$ in the model (5.9), so $H^8(S) = \langle x^2 \rangle$ and $H^{4k}(S) = 0$ for $k \geq 3$. Then, the minimal model of S is a differential graded algebra $(\bigwedge V, d)$, being $\bigwedge V$ the free algebra of the form $\bigwedge V = \bigwedge(a_4, b_4, v_7, w_7) \otimes \bigwedge V^{\geq 10}$, where $|a_4| = |b_4| = 4$, $|v_7| = |w_7| = 7$, and $da_4 = 0$, $db_4 = 0$, $dv_7 = a_4 b_4$, $dw_7 = a_4^2 + b_4^2$. To prove the formality, take a closed element $\alpha \in I(N^{\leq 9})$. Then, $\alpha = P_1 v_7 + P_2 w_7 + P_3\, v_7 w_7$, where $P_1, P_2, P_3 \in \bigwedge(a_4, b_4)$. If α is of even degree, $\alpha = P_3\, v_7 w_7$. The equality $d\alpha = 0$ implies $P_3 = 0$, and hence $\alpha = 0$ is exact. If α is of odd degree, $\alpha = P_1 v_7 + P_2 w_7$. Since $d\alpha = 0$, there exists $P \in \bigwedge(\alpha, \beta)$ such that $P_1 = P(a_4^2 + b_4^2)$ and $P_2 = -P\, a_4 b_4$. Thus, $\alpha = d(P\, v_7 w_7)$ is exact. This proves that S is 9-formal and, by Theorem 3.5, S is formal.

From (2.4), Theorem 2.8 and Proposition 5.1, we know that the 3-Sasakian homogeneous space $S = \mathrm{SO}(8)/(\mathrm{SO}(4) \times \mathrm{Sp}(1))$ is the SO(3)-bundle $\mathrm{SO}(3) \to S \to \widetilde{\mathbb{G}r}_4(\mathbb{R}^8)$ with Euler class $-\frac{1}{4}(l + 2x)$. Then, $(a, b, c) = \left(-\frac{1}{4}, -\frac{1}{2}, 0\right)$ and so none of the conditions (1) and (2) stated in the Theorem is satisfied. Thus, S is formal. The minimal model of S is the minimal model described before for the case $a \neq b$ and $a \neq c$. $\square$

5.2. n **is odd.** If $n = 2m + 1 \geq 3$, the cohomology of $\widetilde{\mathbb{G}r}_4(\mathbb{R}^{n+4})$ is

$$H^*(\widetilde{\mathbb{G}r}_4(\mathbb{R}^{n+4})) = H^*(BT)^{W(\mathrm{SO}(n) \times \mathrm{SO}(4))} / H^{>0}(BT)^{W(\mathrm{SO}(n+4))},$$

where BT is the classifying space of a maximal torus T of $\mathrm{SO}(n+4)$ and $W(G)$ denotes the Weyl group of a Lie group G. Then, the cohomology of $\widetilde{\mathbb{G}r}_4(\mathbb{R}^{n+4})$ are invariant polynomials of $H^*(BT) = \mathbb{Q}[x_1, \ldots, x_{m+2}]$, where $|x_i| = 2$, for $1 \leq i \leq (n+2)$. Denote by y_1, y_2 the classes x_{m+1}, x_{m+2}, respectively. On one hand, $H^*(BT)^{W(\mathrm{SO}(n) \times \mathrm{SO}(4))}$ is generated by the symmetric polynomials $\tilde{\tau}_k$ of $x_1^2, \ldots, x_m^2$ and the symmetric polynomials $\widetilde{\sigma}_k$ of y_1^2, y_2^2, and also by $y_1 y_2$. On the other hand, $H^{>0}(BT)^{W(\mathrm{SO}(n+4))}$ is generated by the symmetric polynomials σ_k of $x_1^2, \ldots, x_m^2, y_1^2, y_2^2$. This gives the relation $\widetilde{\sigma} \cdot \tilde{\tau} = 1$ between $\widetilde{\sigma}_k$ and $\tilde{\tau}_k$, where $\widetilde{\sigma} = 1 + \widetilde{\sigma}_1 + \widetilde{\sigma}_2 + \ldots$ and $\widetilde{\tau} = 1 + \tilde{\tau}_1 + \widetilde{\tau}_2 + \ldots$ Denote by l and x the classes $l = y_1^2 + y_2^2$ and $x = y_1\, y_2$. We have [2]

$$H^*(\widetilde{\mathbb{G}r}_4(\mathbb{R}^{n+4})) = \mathbb{Q}[l, x]/(\widetilde{\sigma}_{m+1}, \widetilde{\sigma}_{m+2}), \tag{5.13}$$

where $|l| = |x| = 4$, and $\widetilde{\sigma}_r$ ($r \geq 0$) is the cohomology class of degree $4r$ defined recursively as in (5.2). Note that $\widetilde{\sigma}_r$ also satisfies (5.3).

The same proof as that given for Proposition 5.1 allows us to prove the following:

Proposition 5.5. *Let Ω be the quaternionic Kähler form on $\widetilde{\mathbb{G}r}_4(\mathbb{R}^{n+4})$ with $n = 2m+1 \geq 3$. Then, in terms of the generators l and x of $H^*(\widetilde{\mathbb{G}r}_4(\mathbb{R}^{n+4}))$ given by (5.13), the de Rham cohomology class $[\Omega] \in H^4(\widetilde{\mathbb{G}r}_4(\mathbb{R}^{n+4}))$ is $[\Omega] = l + 2x$.*

Theorem 5.6. *Consider a principal F-fiber bundle $F \to S \to \widetilde{\mathbb{G}r}_4(\mathbb{R}^{n+4})$ with $n = 2m + 1 \geq 3$, $F = \mathrm{SU}(2)$ or $F = \mathrm{SO}(3)$ and Euler class $e(S) = al + bx$, where $a, b \in \mathbb{Q}$.*

Then, S is formal. In particular, if $n = 2m+1 \geq 3$, the 3-Sasakian homogeneous space $\mathrm{SO}(n+4)/(\mathrm{SO}(n)\times \mathrm{Sp}(1))$ *is formal.*

Proof. Since $\widetilde{\mathbb{G}r}_4(\mathbb{R}^{n+4})$ is simply connected and formal, proceeding as in the proof of Theorem 5.3, we have that a model of S is the differential graded algebra

$$(H^*(\widetilde{\mathbb{G}r}_4(\mathbb{R}^{n+4}))\otimes \bigwedge(u), d), \tag{5.14}$$

where $|u| = 3$ and $du = al + bx$.

If $a = 0 = b$, then S is rationally equivalent to $S^3 \times \widetilde{\mathbb{G}r}_4(\mathbb{R}^{n+4})$, which is formal being the product of two formal manifolds.

Suppose now that $a \neq 0$ and $b = 0$. Then, if m is odd, using that $\widetilde{\sigma}_{m+1} = 0$ on $\widetilde{\mathbb{G}r}_4(\mathbb{R}^{n+4})$, from (5.3) we have

$$\pm\, x^{m+1} + \text{ elements which are exact in (5.14)} = 0$$

on S. This means that $x^{m+1} = 0$ on S. Then, since $du = al$ by (5.14), the cohomology of S up to the degree $2n+1 = 4m+3$ is $H^0(S) = \langle 1\rangle$, $H^{2i+1}(S) = 0$, for $0 \leq i \leq 4m+3$, and $H^{4j}(S) = \langle x^j\rangle$, for $1 \leq j \leq m$. Thus, the minimal model of S must be a differential graded algebra $(\bigwedge V, d)$, being $\bigwedge V$ the free algebra of the form $\bigwedge V = \bigwedge(a_4, v_{4m+3}) \otimes V^{\geq(4m+4)}$, where $|a_4| = 4$, $|v_{4m+3}| = 4m+3$, $da_4 = 0$ and $dv_{4m+3} = a_4^{m+1}$. Now, take $\alpha \in I(N^{\leq 4m+3})$ a closed element in $\bigwedge V$. Then $\alpha = a_4^p\, v_{4m+3}$ which is not closed, for any integer number $p \geq 1$. So, according with Definition 3.4, S is $(4m+3)$-formal and, by Theorem 3.5, S is formal.

Moreover, if $a \neq 0$ and $b = 0$, but m is even, using that $\widetilde{\sigma}_{m+2} = 0$ on $\widetilde{\mathbb{G}r}_4(\mathbb{R}^{n+4})$, from (5.3) we have

$$\pm\, x^{m+2} + \text{ elements which are exact in (5.14)} = 0,$$

that is $x^{m+2} = 0$ on S. Furthermore, by Lemma 5.2, $\widetilde{\sigma}_{m+1} = \tau\cdot(al)$ for some non-zero class of degree $4m$ on $\mathbb{G}r_4(\mathbb{R}^{n+4})$. Thus, taking into account that $d(\tau\, u) = \widetilde{\sigma}_{m+1}$ that is zero on $\mathbb{G}r_4(\mathbb{R}^{n+4})$, $\tau\, u$ is a closed element on S. The cohomology of S up to the degree $2n+1 = 4m+3$ is $H^0(S) = \langle 1\rangle$, $H^{2i+1}(S) = 0$, for $0 \leq i \leq 2m$, $H^{4j}(S) = \langle x^j\rangle$, for $1 \leq j \leq m$, and $H^{4m+3}(S) = \langle \tau\, u\rangle$. Hence the minimal model of S must be a differential graded algebra $(\bigwedge V, d)$, being $\bigwedge V$ the free algebra of the form $\bigwedge V = \bigwedge(a_4, a_{4m+3}) \otimes V^{\geq(4m+4)}$, where $|a_4| = 4$, $|a_{4m+3}| = 4m+3$ and the differential is defined by $da_4 = 0 = da_{4m+3}$. So $N^j = 0$ for $j \leq (4m+3)$. Thus, according with Definition 3.4, the manifold S is $(4m+3)$-formal and, by Theorem 3.5, S is formal.

If $a = 0$ and $b \neq 0$, proceeding as in the previous case we have that $l^{m+1} = 0$ or $l^{m+1} \neq 0$ but $l^{m+2} = 0$ on S. If $l^{m+1} = 0$, then the minimal model of S is the one given in the previous case when $x^{m+1} = 0$, and so S is formal. If $l^{m+1} \neq 0$ but $l^{m+2} = 0$ on S, the minimal model of S is the one given in the case $a \neq 0$, $b = 0$ and $x^{m+2} = 0$. Thus S is formal.

Suppose that $a \neq 0$ and $b \neq 0$. Then, $l = -\frac{b}{a}x$ on S since $a \neq 0$ and $du = al + bx$ by (5.14). Using that $\widetilde{\sigma}_{m+1} = 0 = \widetilde{\sigma}_{m+2}$ on $\widetilde{\mathbb{G}r}_4(\mathbb{R}^{n+4})$, from (5.3) we have that $x^{m+1} = 0$ or $x^{m+1} \neq 0$ but $x^{m+2} = 0$ on S. If $x^{m+1} = 0$, the minimal model of S is the one given in the case $a \neq 0$, $b = 0$ and $x^{m+1} = 0$, and so S is formal. When $x^{m+1} \neq 0$ but $x^{m+2} = 0$, the minimal model of S is the one given in the case $a \neq 0$, $b = 0$ and $x^{m+2} = 0$. Therefore, S is formal.

Now consider the 3-Sasakian homogeneous space $S = \mathrm{SO}(n+4)/(\mathrm{SO}(n) \times \mathrm{Sp}(1))$ with $n = 2m+1 \geq 3$. From (2.4), Theorem 2.8 and Proposition 5.5, S is the SO(3)-bundle $\mathrm{SO}(3) \to S \to \widetilde{\mathbb{G}r}_4(\mathbb{R}^{n+4})$ with Euler class $-\frac{1}{4}(l + 2x)$, and so S is formal. Indeed, a minimal model of S is the minimal model previously described for $a \neq 0$ and $b \neq 0$. □

6. SU(2) and SO(3)-bundles over the exceptional Wolf spaces

Here we prove that the exceptional 3-Sasakian homogeneous spaces appearing in Theorem 2.8 are all formal. They are principal SO(3)-bundles over the exceptional Wolf spaces. We study each of these spaces separately. We also show that the total space of SU(2) and SO(3)-bundles over the exceptional Wolf spaces are formal.

6.1. **The Wolf space GI.** First we consider the 8-dimensional homogeneous quaternionic Kähler manifold

$$GI = \frac{\mathrm{G}_2}{\mathrm{SO}(4)}.$$

The rational cohomology ring of $\mathrm{G}_2/\mathrm{SO}(4)$ is given by [8, 27]

$$H^*(GI) = \mathbb{Q}[x]/(x^3), \tag{6.1}$$

where x has degree 4.

Theorem 6.1. *The 11-dimensional 3-Sasakian homogeneous space $S = \mathrm{G}_2/\mathrm{Sp}(1)$ is formal.*

Proof. The space $S = \mathrm{G}_2/\mathrm{Sp}(1)$ is the total space of the SO(3)-bundle $\mathrm{SO}(3) \to S \to GI$ with Pontryagin class given by the integral cohomology class of the quaternionic Kähler 4-form Ω on GI. This must be (a non-zero multiple of) the class x in $H^4(GI)$. By Theorem 2.7, $GI = \mathrm{G}_2/\mathrm{SO}(4)$ is simply connected, and so $\mathrm{SO}(3) \to S \to GI$ is a rational fibration with rational fiber S^3. Thus, according with section 3, if $(\mathcal{A}, d_{\mathcal{A}})$ is a model of GI, then $(\mathcal{A} \otimes \bigwedge(u), d)$, with $|u| = 3$, $d|_{\mathcal{A}} = d_{\mathcal{A}}$ and $du = x$, is a model of S. Furthermore, $GI = \mathrm{G}_2/\mathrm{SO}(4)$ is formal because it is a symmetric space (see also Theorem 3.1). Hence a model of GI is the DGA $(H^*(GI), 0)$. Then, a model of S is the DGA $(H^*(GI) \otimes \bigwedge(u), d)$, where $|u| = 3$ and $du = x$.

By (6.1) the unique non-zero de Rham cohomology groups of GI are

$$H^0(GI) = \langle 1 \rangle, \quad H^4(GI) = \langle x \rangle, \quad H^8(GI) = \langle x^2 \rangle.$$

Therefore, the cohomology of S is

$$H^0(S) = 1, \quad H^i(S) = 0, \quad 1 \leq i \leq 10, \quad H^{11}(S) = x^3 u.$$

Then, the minimal model of S must be a differential graded algebra $(\bigwedge V, d)$, where $V^j = 0$, and so $C^j = 0 = N^j$, for $1 \leq j \leq 10$. In particular, $N^j = 0$ for $j \leq 5$. Thus, according with Definition 3.4, the manifold S is 5-formal and, by Theorem 3.5, S is formal. □

6.2. **The Wolf space FI.** Now we consider the 28-dimensional homogeneous quaternionic Kähler manifold

$$FI = \frac{\mathrm{F}_4}{\mathrm{Sp}(3)\cdot\mathrm{Sp}(1)}.$$

Its rational cohomology is given by [27]

$$H^*(FI) = \mathbb{Q}[x,y,z]/(x^3-12xy+8z, xz-3y^2, y^3-z^2), \tag{6.2}$$

where $|x|=4$, $|y|=8$ and $|z|=12$.

Theorem 6.2. *The* 31*-dimensional* 3*-Sasakian homogeneous space* $S=\mathrm{F}_4/\mathrm{Sp}(3)$ *is formal.*

Proof. The space $S=\mathrm{F}_4/\mathrm{Sp}(3)$ is the total space of the SO(3)-bundle $\mathrm{SO}(3)\to S\to FI$ with Pontryagin class given by the cohomology class of the quaternionic Kähler 4-form which is (a non-zero multiple of) x. As $FI=\mathrm{F}_4/(\mathrm{Sp}(3)\cdot\mathrm{Sp}(1))$ is simply connected and formal, a model for S is given by the DGA $(H^*(FI)\otimes\bigwedge(u),d)$ with $|u|=3$ and $du=x\in H^4(FI)$.

By (6.2) the unique non-zero de Rham cohomology groups of FI are

$$H^0(FI)=\langle 1\rangle,\quad H^4(FI)=\langle x\rangle,\quad H^8(FI)=\langle x^2,y\rangle,\quad H^{12}(FI)=\langle x^3,xy\rangle,$$
$$H^{16}(FI)=\langle x^4,y^2\rangle,\quad H^{20}(FI)=\langle x^2y,xy^2\rangle,\quad H^{24}(FI)=\langle x^6,xy^2\rangle,\quad H^{28}(FI)=\langle x^7\rangle.$$

Therefore, the unique non-zero cohomology groups of S are

$$H^0(S)=1,\quad H^8(S)=\langle y\rangle\,,\quad H^{23}(S)=\langle PD(y)\rangle\,,\quad H^{31}(S)=\langle x^7u\rangle\,,$$

where $PD(y)$ is the Poincaré dual of y. Then the minimal model of S must be a differential graded algebra $(\bigwedge V,d)$, being $\bigwedge V$ the free algebra of the form $\bigwedge V=\bigwedge(a_8,v_{15})\otimes\bigwedge V^{\geq 16}$, where $|a_8|=8$, $|v_{15}|=15$, $da_8=0$ and $dv_{15}=a_8^2$. According with Definition 3.4, S is 14-formal because $N^j=0$, for $j\leq 14$. Moreover, S is 15-formal. In fact, take $\alpha\in I(N^{\leq 15})$ a closed element in $\bigwedge V$. As $H^*(\bigwedge V)=H^*(S)$ has only non-zero cohomology in degrees $0,8,23$ and 31, it must be $|\alpha|=23,31$. If $|\alpha|=23$ then $\alpha=a_8\,v_{15}$ which is not closed, and if $|\alpha|=31$ then $\alpha=a_8^2\,v_{15}$ which is not closed either. So, according with Definition 3.4, S is 15-formal and, by Theorem 3.5, S is formal. □

6.3. **The Wolf space EII.** For the 40-dimensional homogeneous quaternionic Kähler manifold

$$EII = \frac{\mathrm{E}_6}{\mathrm{SU}(6)\cdot\mathrm{Sp}(1)}\,,$$

we know that its rational cohomology is [26]

$$H^*(EII)=\mathbb{Q}[x,y,z,t]/(R_{12},R_{16},R_{18},R_{24}), \tag{6.3}$$

where $|x|=4$, $|y|=6$, $|z|=8$, $|t|=12$, and

$$R_{12}=y^2-8t-6zx+x^3,\qquad R_{16}=x^4+12xt-6x^2z-3z^2, \tag{6.4}$$
$$R_{18}=yt,\qquad R_{24}=t^2+z^3-\tfrac{3}{2}xzt.$$

Theorem 6.3. *The* 3*-Sasakian homogeneous space* $S=\mathrm{E}_6/\mathrm{SU}(6)$ *of dimension* 43 *is formal.*

Proof. The space $S = \mathrm{E}_6/\mathrm{SU}(6)$ is the total space of the SO(3)-bundle $\mathrm{SO}(3) \to S \to EII$ with Pontryagin class given by the quaternionic 4-form which is (a non-zero multiple of) x. Since $EII = \mathrm{E}_6/(\mathrm{SU}(6)\cdot \mathrm{Sp}(1))$ is simply connected and formal, a model for S is the DGA $(H^*(EII)\otimes \bigwedge(u), d)$, where $|u| = 3$ and $du = x \in H^4(EII)$.

By (6.3) and the relations given in (6.4), the 20 first de Rham cohomology groups of EII are

$$\begin{aligned} &H^0(EII) = \langle 1\rangle, && H^4(EII) = \langle x\rangle, && H^6(EII) = \langle y\rangle,\\ &H^8(EII) = \langle x^2, z\rangle, && H^{10}(EII) = \langle xy\rangle, && H^{12}(EII) = \langle x^3, xz, y^2\rangle,\\ &H^{14}(EII) = \langle x^2y, yz\rangle, && H^{16}(EII) = \langle x^4, x^2z, xy^2\rangle, && H^{18}(EII) = \langle x^3y, xyz\rangle,\\ &H^{20}(EII) = \langle x^5, x^3z, x^2y^2, y^2z\rangle, \end{aligned}$$

and $H^{2i+1}(EII) = 0$, for $0 \le i \le 9$. Therefore, the 21 first de Rham cohomology groups of S are

$$\begin{aligned} &H^0(S) = 1, && H^6(S) = \langle y\rangle, && H^8(S) = \langle z\rangle,\\ &H^{12}(S) = \langle y^2\rangle, && H^{14}(S) = \langle yz\rangle, && H^{20}(S) = \langle y^2z\rangle, \end{aligned}$$

and $H^{2i+1}(S) = 0$, for $0 \le i \le 10$.

Then the minimal model of S must be a differential graded algebra $(\bigwedge V, d)$, being $\bigwedge V = \bigwedge(a_6, a_8, v_{15}, v_{17}) \otimes \bigwedge V^{\ge 22}$, where $|a_6| = 6$, $|a_8| = 8$, $|v_{15}| = 15$, $|v_{17}| = 17$, $da_6 = 0 = da_8$, $dv_{15} = a_8^2$ and $dv_{17} = a_6^3$. According with Definition 3.4, we have $N^j = 0$, for $j \le 14$, thus the manifold S is 14-formal. Let us see that it is 21-formal. For this, it is sufficient to prove that S is 17-formal because $N^j = 0$, for $18 \le j \le 21$. Take $\alpha \in I(N^{\le 17})$ a closed element in $\bigwedge V$. As $H^*(\bigwedge V) = H^*(S)$ has only non-zero cohomology in degrees $0, 6, 8, 12, 14, 20, 23, 29, 31, 35, 37$ and 43, it must be $|\alpha| = 23, 29, 31, 35, 37, 43$, that is α has odd degree. If $|\alpha| = 29$, then $\alpha = a_6\, a_8\, v_{15}$ which is not closed, and if $|\alpha| = 31$, then $\alpha = a_6\, a_8\, v_{17}$ which is not closed either. In the other cases, α is of the form $\alpha = P_1\, v_{15} + P_2\, v_{17}$, where $P_1, P_2 \in \bigwedge(a_6, a_8)$. Now, $d\alpha = 0$ implies that $P_1 = P\, a_6^3$ and $P_2 = -P\, a_8^2$, for some $P \in \bigwedge(a_6, a_8)$. So $\alpha = d(P\, v_{15}\, v_{17})$ is exact. Thus any closed element in the ideal $I(N^{\le 21})$ is exact and hence S is 21-formal. By Theorem 3.5, the manifold S is formal. □

6.4. **The Wolf space EVI.** Now we consider the 64-dimensional homogeneous quaternionic Kähler manifold

$$EVI = \frac{E_7}{\mathrm{Spin}(12)\cdot \mathrm{Spin}(1)}.$$

Its rational cohomology ring is [36]

$$H^*(EVI) = \mathbb{Q}[x, y, z]/(R_1, R_2, R_3), \tag{6.5}$$

where $|x| = 4$, $|y| = 8$, $|z| = 12$, and

$$\begin{aligned} R_1 &= -2x^4y + 2x^3z - 3xyz + \tfrac{1}{8}y^3 + 3z^2,\\ R_2 &= x^7 - x^5y + 4x^4z - \tfrac{3}{2}x^3y^2 - \tfrac{3}{8}xy^3 + 3xz^2 - \tfrac{3}{4}y^2z,\\ R_3 &= 4x^7y + 3x^5y^2 + 8x^4yz + x^3y^3 + 4x^3z^2 + 6x^2y^2z + \tfrac{3}{16}xy^4 + \tfrac{3}{8}y^3z + 8z^3. \end{aligned} \tag{6.6}$$

Theorem 6.4. *The 3-Sasakian homogeneous space $S = \mathrm{E}_7/\mathrm{Spin}(12)$ of dimension 67 is formal.*

Proof. The space $S = \mathrm{E}_7/\mathrm{Spin}(12)$ is the total space of the SO(3)-bundle $\mathrm{SO}(3) \to S \to EVI$ with Pontryagin class given by the the quaternionic 4-form which is (a non-zero multiple of) x. Since $EVI = E_7/(\mathrm{Spin}(12)\cdot \mathrm{Spin}(1))$ is simply connected and formal, a model of S is given by $(H^*(EVI)\otimes \bigwedge(u), d)$, where $|u| = 3$ and $du = x \in H^4(EVI)$.

By (6.5) and the relations (6.6), the unique non-zero de Rham cohomology groups of EVI up to degree 33 are

$$H^0(EVI) = \langle 1\rangle, \qquad H^4(EVI) = \langle x\rangle,$$
$$H^8(EVI) = \langle x^2, y\rangle, \qquad H^{12}(EVI) = \langle x^3, xy, z\rangle,$$
$$H^{16}(EVI) = \langle x^4, x^2y, y^2, xz\rangle, \qquad H^{20}(EVI) = \langle x^5, x^3y, xy^2, x^2z, yz\rangle,$$
$$H^{24}(EVI) = \langle x^6, x^4y, x^2y^2, y^3, x^3z, xyz\rangle, \qquad H^{28}(EVI) = \langle x^7, x^5y, x^3y^2, xy^3, x^4z, x^2yz\rangle,$$
$$H^{32}(EVI) = \langle x^8, x^6y, x^4y^2, x^2y^3, y^4, x^5z, x^3yz\rangle.$$

Therefore, the 32-first non-zero de Rham cohomology groups of S are

$$H^0(S) = \langle 1\rangle, \quad H^8(S) = \langle y\rangle, \quad H^{12}(S) = \langle z\rangle, \quad H^{16}(S) = \langle y^2\rangle,$$
$$H^{20}(S) = \langle yz\rangle, \quad H^{24}(S) = \langle y^3\rangle, \quad H^{32}(S) = \langle y^4\rangle,$$

and $H^{2i+1}(S) = 0$ for $0 \le i \le 16$. The minimal model of S must be the differential graded algebra $(\bigwedge V, d)$ where $\bigwedge V = \bigwedge(a_8, a_{12}, v_{23}, v_{27}) \otimes \bigwedge V^{\geq 34}$, where $|a_8| = 8$, $|a_{12}| = 12$, $|v_{23}| = 23$ and $|v_{27}| = 27$, and the differential d is given by $da_i = 0$ ($i = 8, 12$), $dv_{23} = a_8^3 + 24a_{12}^2$ and $dv_{27} = a_8^2\, a_{12}$. Thus, the manifold S is 22-formal because the space $N^j = 0$, for $j \le 22$. Moreover, S is 33-formal. Take $\alpha \in I(N^{\le 33})$. Since $H^*(\bigwedge V) = H^*(S)$ has only non-zero cohomology in degrees $0, 8, 12, 16, 20, 24, 32, 35, 43, 47, 51, 55, 59$ and 67, the degree of α must be $|\alpha| = 35, 43, 47, 51, 55, 59, 67$, and so α has not component in $v_{23}\, v_{27}$. Therefore, α is of the form $\alpha = P_1\, v_{23} + P_2\, v_{27}$, where P_1 and P_2 live in the subalgebra $\bigwedge(a_8, a_{12})$ of $\bigwedge V^{\le 33}$. The equality $d\alpha = 0$ implies $P_1 = P\, dv_{27}$ and $P_2 = -P\, dv_{23}$, for some $P \in \bigwedge(a_8, a_{12})$. Hence, $\alpha = d(P\, v_{23}\, v_{27})$ which proves that any closed element in the ideal $I(N^{\le 33})$ is exact. By Definition 3.4, S is 33-formal and, by Theorem 3.5, S is formal. □

6.5. **The Wolf space EIX.** We consider the homogeneous quaternionic Kähler manifold

$$EIX = \frac{\mathrm{E}_8}{\mathrm{E}_7 \cdot \mathrm{Sp}(1)}.$$

By [42] (see also [39]) its rational cohomology ring is

$$H^*(EIX) = \mathbb{Q}[x_4, x_{12}, x_{20}]/(x_{12}^4, x_{20}^2), \tag{6.7}$$

where $|x_i| = i$, with $i = 4, 12, 20$.

Theorem 6.5. *The 3-Sasakian homogeneous space $S = \mathrm{E}_8/\mathrm{E}_7$ of dimension 115 is formal.*

Proof. The space $S = \mathrm{E}_8/\mathrm{E}_7$ is the total space of the SO(3)-bundle $\mathrm{SO}(3) \to S \to EIX$ with Pontryagin class given by the cohomology class of the quaternionic 4-form, which is (a non-zero multiple of) x_4. Since $EIX = \mathrm{E}_8/(\mathrm{E}_7\cdot \mathrm{Sp}(1))$ is simply connected and formal, a model of S is the differential algebra $(H^*(EIX) \otimes \bigwedge(u), d)$ with $|u| = 3$ and $du = x_4$.

Using the cohomology algebra in (6.7) (which this time we do not write explicitly because it is very long, and easy to do for the reader) and the model for S, we get easily that the non-zero de Rham cohomology groups of S up to degree 58 are

$$\begin{array}{llll} H^0(S) = \langle 1\rangle, & H^{12}(S) = \langle x_{12}\rangle, & H^{20}(S) = \langle x_{20}\rangle & H^{24}(S) = \langle x_{12}^2\rangle, \\ H^{32}(S) = \langle x_{12}x_{20}\rangle, & H^{36}(S) = \langle x_{12}^3\rangle, & H^{44}(S) = \langle x_{12}^2 x_{20}\rangle & H^{56}(S) = \langle x_{12}^3 x_{20}\rangle. \end{array}$$

By Poincaré duality, there exist elements $x_i \in H^i(S)$, $i = 59, 71, 79, 83, 91, 95, 103, 115$, such that (see [37])

$$x_{12}^3 x_{20} x_{59} = x_{12}^2 x_{20} x_{71} = x_{12}^3 x_{79} = x_{12} x_{20} x_{83} = x_{12}^2 x_{91} = x_{20} x_{95} = x_{12} x_{103} = x_{115},$$

and

$$\begin{array}{lll} x_{71} = x_{12} x_{59}\,, & x_{79} = x_{20} x_{59}\,, & x_{83} = x_{12}^2 x_{59}\,, \\ x_{91} = x_{12} x_{20} x_{59}\,, & x_{103} = x_{12}^2 x_{20} x_{59}\,, & x_{115} = x_{12}^3 x_{20} x_{59}. \end{array}$$

Then the minimal model of S must be a differential graded algebra $(\bigwedge V, d)$, with $\bigwedge V = \bigwedge(a_{12}, a_{20}, v_{39}, v_{47}) \otimes \bigwedge V^{\geq 59}$, where $|a_i| = i$ for $i = 12, 20$, $|v_j| = j$ for $j = 39, 47$, and d is given by $da_{12} = 0 = da_{20}$, $dv_{39} = a_{20}^2$ and $dv_{47} = a_{12}^4$. According with Definition 3.4, the manifold S is 38-formal because $N^j = 0$, for $j \leq 38$. To prove that S is 57-formal it is sufficient to prove that S is 47-formal since $N^j = 0$, for $48 \leq j \leq 57$. Let $\alpha \in I(N^{\leq 47})$ be a closed element in $\bigwedge V$. Since $H^*(\bigwedge V) = H^*(S)$ has only non-zero cohomology in degrees $0, 12, 20, 24, 32, 36, 44, 56, 59, 71, 79, 83, 91, 95, 103$ and 115, it must be $|\alpha| = 59, 71, 79, 83, 91, 95, 103, 115$. (Note that $|\alpha| \neq 44, 56$ because $|v_{39}| = 39$, $|v_{47}| = 47$ and $b_{2i+1}(S) = 0$, for $0 \leq i \leq 28$.) For each of these cases, α is given as follows: $\alpha = \lambda_1\, a_{20}\, v_{39} + \mu_1\, a_{12}\, v_{47}$ if $|\alpha| = 59$; $\alpha = \lambda_2\, a_{12}\, a_{20}\, v_{39} + \mu_2\, a_{12}^2\, v_{47}$ if $|\alpha| = 71$; $\alpha = \lambda_3\, a_{20}^2\, v_{39} + \mu_3\, a_{12}\, a_{20}\, v_{47}$ if $|\alpha| = 79$; $\alpha = \lambda_4\, a_{12}^2\, a_{20}\, v_{39} + \mu_4\, a_{12}^3\, v_{47}$ if $|\alpha| = 83$; $\alpha = \lambda_5\, a_{12}\, a_{20}^2\, v_{39} + \mu_5\, a_{12}^2\, a_{20}\, v_{47}$ if $|\alpha| = 91$; $\alpha = \lambda_6\, a_{12}^3\, a_{20}\, v_{39} + \mu_6\, a_{12}^4\, v_{47}$ if $|\alpha| = 95$; $\alpha = \lambda_7\, a_{12}^2\, a_{20}^2\, v_{39} + \mu_7\, a_{12}^3\, a_{20}\, v_{47}$ if $|\alpha| = 103$; and $\alpha = \lambda_8\, a_{12}^3\, a_{20}^2\, v_{39} + \mu_8\, a_{12}^4\, a_{20}\, v_{47}$ if $|\alpha| = 115$, where $\lambda_i, \mu_j \in \mathbb{R}$. One can check that α is not closed in any of these cases. Then Definition 3.4 implies that the manifold S is 47-formal. Hence S is 57-formal, and by Theorem 3.5, S is formal. □

Proposition 6.6. *Let $F = \mathrm{SU}(2)$ or $\mathrm{SO}(3)$, and let $F \to S \to B$ be a principal fiber bundle, where $B = GI, FI, EII, EVI$ or EIX. Then S is formal.*

Proof. The principal fiber bundle $F \to S \to B$, with B being one of the exceptional Wolf spaces $B = GI, FI, EII, EVI$ or EIX, is such that in all these cases, $H^4(B)$ is one-dimensional, generated by some $x \in H^4(B)$. Then the Euler class of S is $e(S) = a\,x$. If $a = 0$, the fibration is rationally a product and hence S is formal. If $a \neq 0$, then the fibration is rationally the same as the one considered in each of the previous subsections. Therefore S is formal as it has been computed above. □

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